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Sharper ABC-based bounds for congruent polynomials

par Daniel J. Bernstein

Résumé. Agrawal, Kayal, et Saxena ont récemment introduit une nouvelle méthode pour montrer qu'un entier est premier. La vitesse de cette méthode dépend des minorations prouvées pour la taille du semi-groupe multiplicatif engendré par plusieurs polynômes modulo un autre polynôme h. Voloch a trouvé une application du théoreme ABC de Stothers et Mason dans ce contexte: sous de petites hypothèses, des polynômes distincts A, B, C de degré au plus $1.2 \deg h - 0.2 \deg \operatorname{rad} ABC$ ne peuvent pas être tous congrus modulo h. Nous présentons deux améliorations de la partie combinatoire de l'argument de Voloch. La première amélioration augmente $1.2 \deg h - 0.2 \deg \operatorname{rad} ABC$ en $2 \deg h - \deg \operatorname{rad} ABC$. La deuxième amélioration est une généralisation à $A_1, \ldots, A_m$ de degré au plus $((3m-5)/(3m-7)) \deg h - (6/(3m-7)m) \deg \operatorname{rad} A_1 \cdots A_m$, avec $m \geq 3$.

Abstract. Agrawal, Kayal, and Saxena recently introduced a new method of proving that an integer is prime. The speed of the Agrawal-Kayal-Saxena method depends on proven lower bounds for the size of the multiplicative semigroup generated by several polynomials modulo another polynomial h. Voloch pointed out an application of the Stothers-Mason ABC theorem in this context: under mild assumptions, distinct polynomials A, B, C of degree at most $1.2 \deg h - 0.2 \deg \operatorname{rad} ABC$ cannot all be congruent modulo h. This paper presents two improvements in the combinatorial part of Voloch's argument. The first improvement moves the degree bound up to $2 \deg h - \deg \operatorname{rad} ABC$. The second improvement generalizes to $m \geq 3$ polynomials $A_1, \ldots, A_m$ of degree at most $((3m-5)/(3m-7)) \deg h - (6/(3m-7)m) \deg \operatorname{rad} A_1 \cdots A_m$.

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1. Introduction

Fix a nonconstant univariate polynomial h over a field k. Assume that the characteristic of k is at least $3\deg h - 1$. The main theorem of this paper, Theorem 2.3, states that if $m \geq 3$ distinct polynomials $A_1, \ldots, A_m$ are all congruent modulo h and coprime to h then

$$\max\{\deg A_1, \ldots, \deg A_m\} > \frac{3m-5}{3m-7}\deg h - \frac{6}{(3m-7)m}\deg\operatorname{rad} A_1 \cdots A_m.$$

As usual, $\operatorname{rad} X$ means the largest monic squarefree divisor of X, i.e., the product of the monic irreducibles dividing X. If $\deg\operatorname{rad} A_1 \cdots A_m < (m/3)\deg h$ then the bound in Theorem 2.3 is better than the obvious bound $\max\{\deg A_1, \ldots, \deg A_m\} > \deg h - 1$.

For example, if distinct polynomials A, B, C are congruent modulo h and coprime to h then $\max\{\deg A, \deg B, \deg C\} > 2\deg h - \deg\operatorname{rad} ABC$. No better bound is possible in this level of generality: if $h = x^{10} - 1$, $A = x^{20}$, $B = x^{10}$, and $C = 1$ then $\operatorname{rad} ABC = \operatorname{rad} x^{30} = x$ so $2\deg h - \deg\operatorname{rad} ABC = 19$.

The proof relies on the Stothers-Mason ABC theorem. Analogous bounds in the number-field case follow from the ABC conjecture.

Previous work. Voloch in [3] proved that $\max\{\deg A, \deg B, \deg C\} > 1.2\deg h - 0.2\deg\operatorname{rad} ABC$. This paper improves Voloch's result in two ways:

- This paper is quantitatively stronger, in the interesting case that $\deg\operatorname{rad} ABC < \deg h$.
- This paper applies to larger values of m.

Application. Inside the unit group $(k[x]/h)^*$ consider the subgroup G generated by $\{x - s : s \in S\}$, where $S \subseteq k$ and $0 \notin h(S)$. The Agrawal-Kayal-Saxena primality-proving method requires a lower bound on $\#G$ for groups G of this type, typically with $\#S = \deg h$. The primality-proving method becomes faster as the lower bound on $\#G$ increases, as discussed in [1, Section 7].

This paper shows that

$$\#G \geq \frac{1}{m-1}\binom{\lfloor((3m-5)/(3m-7))\deg h - (6/(3m-7)m)\#S\rfloor + \#S}{\#S}$$

for any $m \geq 3$. Indeed, the binomial coefficient is the number of products of powers of $\{x - s\}$ in $k[x]$ of degree at most

$$\lfloor((3m-5)/(3m-7))\deg h - (6/(3m-7)m)\#S\rfloor;$$

m distinct such products cannot all have the same image modulo h.

In particular, if $\#S = \deg h$, then $\#G \geq \frac{1}{3}\binom{\lfloor 2.1 \deg h \rfloor}{\deg h} \approx 4.27689^{\deg h}$. Compare this to the bound $\#G \geq \binom{2\deg h - 1}{\deg h} \approx 4^{\deg h}$ obtained from a degree bound of $\deg h - 1$. Note that the improvement requires $m > 3$.

Different methods from [3] produce a lower bound around $5.828^{\deg h}$, so the ABC-based techniques in [3] and in this paper have not yet had an impact on the speed of primality proving. However, I suspect that these techniques have not yet reached their limits.

2. Proofs

Theorem 2.1. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let A, B, C be distinct nonzero elements of $k[x]$. If $\gcd\{A, B, C\} = 1$ and $A \equiv B \equiv C \pmod{h}$ then $\max\{\deg A, \deg B, \deg C\} > 2\deg h - \deg \operatorname{rad} ABC$.*

Proof. Permute A, B, C so that $\deg A = \max\{\deg A, \deg B, \deg C\}$.

The nonzero polynomial $A-B$ is a multiple of h, so $\deg A \geq \deg(A-B) \geq \deg h > 0$; thus $\deg \operatorname{rad} ABC > 0$.

If $\deg A \geq 2\deg h$ then $\deg A > 2\deg h - \deg \operatorname{rad} ABC$; done.

Define $U = (B - C)/h$, $V = (C - A)/h$, and $W = (A - B)/h$. Then $U \neq 0$; $V \neq 0$; $W \neq 0$; U, V, W each have degree at most $\deg A - \deg h$; and $UA + VB + WC = 0$. Define $D = \gcd\{UA, VB, WC\}$.

If $\deg D = \deg UA$ then UA divides VB, WC; so A divides VWA, VWB, VWC; so A divides $\gcd\{VWA, VWB, VWC\} = VW$; but $VW \neq 0$, so $\deg A \leq \deg VW \leq 2(\deg A - \deg h)$; so $\deg A \geq 2\deg h$; done.

Assume from now on that $\deg D < \deg UA$ and that $\deg A \leq 2\deg h - 1$. Then $\deg(UA/D)$ is between 1 and $2\deg A - \deg h \leq 3\deg h - 2$; so the derivative of UA/D is nonzero. Also $UA/D + VB/D + WC/D = 0$, and $\gcd\{UA/D, VB/D, WC/D\} = 1$. By Theorem 3.1 below, $\deg(UA/D) < \deg \operatorname{rad}((UA/D)(VB/D)(WC/D)) = \deg \operatorname{rad}(UVWABC/D^3)$.

The proof follows Voloch up to this point. Voloch next observes that D divides $\gcd\{UVWA, UVWB, UVWC\} = UVW \gcd\{A, B, C\} = UVW$. I claim that more is true: $D \operatorname{rad}(UVWABC/D^3)$ divides $UVW \operatorname{rad} ABC$.

(In other words: If $d = \min\{u + a, v + b, w + c\}$ and $\min\{a, b, c\} = 0$ then $d + [u + v + w + a + b + c > 3d] \leq u + v + w + [a + b + c > 0]$. Proof: Without loss of generality assume $a = 0$. Then $d \leq u \leq u + v + w$. If $d < u + v + w$ then $d + [\cdots] \leq d + 1 \leq u + v + w \leq u + v + w + [\cdots]$ as claimed. If $a + b + c > 0$ then $d + [\cdots] \leq u + v + w + 1 = u + v + w + [\cdots]$ as claimed. Otherwise $u + v + w + a + b + c = d \leq 3d$ so $d + [u + v + w + a + b + c > 3d] = d \leq u + v + w \leq u + v + w + [\cdots]$ as claimed.)

Thus $\deg UA < \deg(D \operatorname{rad}(UVWABC/D^3)) \leq \deg(UVW \operatorname{rad} ABC)$. Hence $\deg A < \deg(VW \operatorname{rad} ABC) \leq 2(\deg A - \deg h) + \deg \operatorname{rad} ABC$; i.e., $\deg A > 2\deg h - \deg \operatorname{rad} ABC$ as claimed. $\square$

Theorem 2.2. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let A, B, C be distinct nonzero elements of $k[x]$. If $\gcd\{A, B, C\}$ is coprime to h and $A \equiv B \equiv C \pmod h$ then*

$$\begin{aligned}\max\{&\deg A, \deg B, \deg C\}\\ &> 2\deg h - \deg\operatorname{rad} A - \deg\operatorname{rad} B - \deg\operatorname{rad} C\\ &\qquad + \deg\operatorname{rad}\gcd\{A, B\} + \deg\operatorname{rad}\gcd\{A, C\} + \deg\operatorname{rad}\gcd\{B, C\}.\end{aligned}$$

Proof. Write $G = \gcd\{A, B, C\}$. Then G is coprime to h, so $A/G \equiv B/G \equiv C/G \pmod h$. By Theorem 2.1,

$$\begin{aligned}\max\left\{\deg\frac{A}{G}, \deg\frac{B}{G}, \deg\frac{C}{G}\right\} &> 2\deg h - \deg\operatorname{rad}\frac{ABC}{GGG}\\ &\geq 2\deg h - \deg\operatorname{rad} ABC.\end{aligned}$$

Furthermore, $\deg G \geq \deg\operatorname{rad} G = \deg\operatorname{rad} ABC - \deg\operatorname{rad} A - \deg\operatorname{rad} B - \deg\operatorname{rad} C + \deg\operatorname{rad}\gcd\{A, B\} + \deg\operatorname{rad}\gcd\{A, C\} + \deg\operatorname{rad}\gcd\{B, C\}$ by inclusion-exclusion. Add. □

Theorem 2.3. *Let k be a field. Let h be a positive-degree element of the polynomial ring $k[x]$. Assume that $1, 2, 3, \ldots, 3\deg h - 2$ are invertible in k. Let S be a finite subset of $k[x] - \{0\}$, with $\#S \geq 3$. If each element of S is coprime to h, and all the elements of S are congruent modulo h, then*

$$\max\{\deg A : A \in S\} > \frac{3\#S - 5}{3\#S - 7}\deg h - \frac{6}{(3\#S - 7)\#S}\deg\operatorname{rad}\prod_{A\in S} A.$$

For example, $\max\{\deg A : A \in S\} > 1.4\deg h - 0.3\deg\operatorname{rad}\prod_{A\in S} A$ if $\#S = 4$, and $\max\{\deg A : A \in S\} > 1.25\deg h - 0.15\deg\operatorname{rad}\prod_{A\in S} A$ if $\#S = 5$.

Proof. Define $d = \max\{\deg A : A \in S\}$ and $e = \deg\operatorname{rad}\prod_{A\in S} A$. Then

$$\begin{aligned}d &> 2\deg h - \deg\operatorname{rad} A - \deg\operatorname{rad} B - \deg\operatorname{rad} C\\ &\qquad + \deg\operatorname{rad}\gcd\{A, B\} + \deg\operatorname{rad}\gcd\{A, C\} + \deg\operatorname{rad}\gcd\{B, C\}\end{aligned}$$

for any distinct $A, B, C \in S$ by Theorem 2.2. Average this inequality over all choices of A, B, C to see that $d > 2\deg h - 3\operatorname{avg}_A \deg\operatorname{rad} A + 3\operatorname{avg}_{A\neq B}\deg\operatorname{rad}\gcd\{A, B\}$. On the other hand, $e \geq \#S\operatorname{avg}_A\deg\operatorname{rad} A - \binom{\#S}{2}\operatorname{avg}_{A\neq B}\deg\operatorname{rad}\gcd\{A, B\}$ by inclusion-exclusion, so

$$d + \frac{3}{\#S}e > 2\deg h - \frac{3\#S - 9}{2}\operatorname{avg}_{A\neq B}\deg\operatorname{rad}\gcd\{A, B\}.$$

Note that $3\#S - 9 \geq 0$ since $\#S \geq 3$.

One can bound each term $\deg\operatorname{rad}\gcd\{A, B\}$ by the simple observation that $A/\gcd\{A, B\}$ and $B/\gcd\{A, B\}$ are distinct congruent polynomials of degree at most $d - \deg\gcd\{A, B\}$; thus $d - \deg\gcd\{A, B\} \geq \deg h$, so $\deg\operatorname{rad}\gcd\{A, B\} \leq d - \deg h$. Hence

$$d + \frac{3}{\#S}e > 2\deg h - \frac{3\#S - 9}{2}(d - \deg h);$$

i.e., $d > ((3\#S - 5)/(3\#S - 7))\deg h - (6/(3\#S - 7)\#S)e$. □

3. Appendix: the ABC theorem

Theorem 3.1 is a typical statement of the Stothers-Mason ABC theorem, included in this paper for completeness. The proof given here is due to Noah Snyder; see [2].

Theorem 3.1. *Let k be a field. Let A, B, C be nonzero elements of the polynomial ring $k[x]$ with $A + B + C = 0$ and $\gcd\{A, B, C\} = 1$. If $\deg A \geq \deg\operatorname{rad} ABC$ then $A' = 0$.*

In fact, $A' = B' = C' = 0$. As usual, X' means the derivative of X; the relevance of derivatives is that $X/\operatorname{rad} X$ divides X'.

Proof. Note that $\gcd\{A, B\} = \gcd\{A, B, -(A + B)\} = \gcd\{A, B, C\} = 1$. By the same argument, $\gcd\{A, C\} = 1$ and $\gcd\{B, C\} = 1$.

$C/\operatorname{rad} C$ divides both C and C', so it divides $C'B - CB'$. Similarly, $B/\operatorname{rad} B$ divides $C'B - CB'$. Furthermore, $C' = -(A' + B')$, so $C'B - CB' = -(A' + B')B + (A + B)B' = AB' - A'B$; thus $A/\operatorname{rad} A$ divides $C'B - CB'$.

The ratios $A/\operatorname{rad} A, B/\operatorname{rad} B, C/\operatorname{rad} C$ are pairwise coprime, so their product $ABC/\operatorname{rad} ABC$ divides $C'B - CB'$. But by hypothesis

$$\deg \frac{ABC}{\operatorname{rad} ABC} = \deg ABC - \deg\operatorname{rad} ABC \geq \deg BC > \deg(C'B - CB');$$

so $C'B - CB' = 0$; so $AB' - A'B = 0$; so A divides $A'B$; but A and B are coprime, so A divides A'; but $\deg A > \deg A'$, so $A' = 0$. □

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Daniel J. Bernstein
Department of Mathematics, Statistics, and Computer Science (M/C 249)
The University of Illinois at Chicago
Chicago, IL 60607–7045
E-mail : `djb@cr.yp.to`