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Kronecker-Weber via Stickelberger

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Résumé. Nous donnons une nouvelle démonstration du théorème de Kronecker et Weber fondée sur la théorie de Kummer et le théorème de Stickelberger.

Abstract. In this note we give a new proof of the theorem of Kronecker-Weber based on Kummer theory and Stickelberger's theorem.

Introduction

The theorem of Kronecker-Weber states that every abelian extension of $\mathbb{Q}$ is cyclotomic, i.e., contained in some cyclotomic field. The most common proof found in textbooks is based on proofs given by Hilbert [2] and Speiser [7]; a routine argument shows that it is sufficient to consider cyclic extensions of prime power degree p^m unramified outside p, and this special case is then proved by a somewhat technical calculation of differents using higher ramification groups and an application of Minkowski's theorem, according to which every extension of $\mathbb{Q}$ is ramified. In the proof below, this not very intuitive part is replaced by a straightforward argument using Kummer theory and Stickelberger's theorem.

In this note, ζ_m denotes a primitive m-th root of unity, and "unramified" always means unramified at all finite primes. Moreover, we say that a normal extension K/F

- is of type (p^a, p^b) if $\text{Gal}(K/F) \simeq (\mathbb{Z}/p^a\mathbb{Z}) \times (\mathbb{Z}/p^b\mathbb{Z})$;
- has exponent m if $\text{Gal}(K/F)$ has exponent m.

1. The Reduction

In this section we will show that it is sufficient to prove the following special case of the Kronecker-Weber theorem (it seems that the reduction to extensions of prime degree is due to Steinbacher [8]):

Proposition 1.1. *The maximal abelian extension of exponent p that is unramified outside p is cyclic: it is the subfield of degree p of $\mathbb{Q}(\zeta_{p^2})$.*

The corresponding result for the prime $p = 2$ is easily proved:

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Proposition 1.2. *The maximal real abelian* 2*-extension of* $\mathbb{Q}$ *with exponent* 2 *and unramified outside* 2 *is cyclic: it is the subfield* $\mathbb{Q}(\sqrt{2})$ *of* $\mathbb{Q}(\zeta_8)$.

Proof. The only quadratic extensions of $\mathbb{Q}$ that are unramified outside 2 are $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{2})$. □

The following simple observation will be used repeatedly below:

Lemma 1.3. *If the compositum of two cyclic* p*-extensions* K, K' *is cyclic, then* $K \subseteq K'$ *or* $K' \subseteq K$.

Now we show that Prop. 1.1 implies the corresponding result for extensions of prime power degree:

Proposition 1.4. *Let* $K/\mathbb{Q}$ *be a cyclic extension of odd prime power degree* p^m *and unramified outside* p. *Then* K *is cyclotomic.*

Proof. Let K' be the subfield of degree p^m in $\mathbb{Q}(\zeta_{p^{m+1}})$. If $K'K$ is not cyclic, then it contains a subfield of type (p,p) unramified outside p, which contradicts Prop. 1.1. Thus $K'K$ is cyclic, and Lemma 1.3 implies that $K = K'$. □

Next we prove the analog for $p = 2$:

Proposition 1.5. *Let* $K/\mathbb{Q}$ *be a cyclic extension of degree* 2^m *and unramified outside* 2. *Then* K *is cyclotomic.*

Proof. If $m = 1$ we are done by Prop. 1.2. If $m \geq 2$, assume first that K is nonreal. Then $K(i)/K$ is a quadratic extension, and its maximal real subfield M is cyclic of degree 2^m by Prop. 1.2. Since $K/\mathbb{Q}$ is cyclotomic if and only if M is, we may assume that K is totally real.

Now let K' be the the maximal real subfield of $\mathbb{Q}(\zeta_{2^{m+2}})$. If $K'K$ is not cyclic, then it contains three real quadratic fields unramified outside 2, which contradicts Prop. 1.2. Thus $K'K$ is cyclic, and Lemma 1.3 implies that $K = K'$. □

Now the theorem of Kronecker-Weber follows: first observe that abelian groups are direct products of cyclic groups of prime power order; this shows that it is sufficient to consider cyclic extensions of prime power degree p^m. If $K/\mathbb{Q}$ is such an extension, and if $q \neq p$ is ramified in $K/\mathbb{Q}$, then there exists a cyclic cyclotomic extension $L/\mathbb{Q}$ with the property that $KL = FL$ for some cyclic extension $F/\mathbb{Q}$ of prime power degree in which q is unramified. Since K is cyclotomic if and only if F is, we see that after finitely many steps we have reduced Kronecker-Weber to showing that cyclic extensions of degree p^m unramified outside p are cyclotomic. But this is the content of Prop. 1.4 and 1.5.

Since this argument can be found in all the proofs based on the Hilbert-Speiser approach (see e.g. Greenberg [1] or Marcus [6]), we need not repeat the details here.

2. Proof of Proposition 1.1

Let $K/\mathbb{Q}$ be a cyclic extension of prime degree p and unramified outside p. We will now use Kummer theory to show that it is cyclotomic. For the rest of this article, set $F = \mathbb{Q}(\zeta_p)$ and define $\sigma_a \in G = \text{Gal}(F/\mathbb{Q})$ by $\sigma_a(\zeta_p) = \zeta_p^a$ for $1 \le a < p$.

Lemma 2.1. *The Kummer extension $L = F(\sqrt[p]{\mu})$ is abelian over $\mathbb{Q}$ if and only if for every $\sigma_a \in G$ there is a $\xi \in F^\times$ such that $\sigma_a(\mu) = \xi^p \mu^a$.*

For the simple proof, see e.g. Hilbert [3, Satz 147] or Washington [9, Lemma 14.7].

Let $K/\mathbb{Q}$ be a cyclic extension of prime degree p and unramified outside p. Put $F = \mathbb{Q}(\zeta_p)$ and $L = KF$; then $L = F(\sqrt[p]{\mu})$ for some nonzero $\mu \in \mathcal{O}_F$, and L/F is unramified outside p.

Lemma 2.2. *Let $\mathfrak{q}$ be a prime ideal in F with $(\mu) = \mathfrak{q}^r \mathfrak{a}$, $\mathfrak{q} \nmid \mathfrak{a}$; if $p \nmid r$ and $L/\mathbb{Q}$ is abelian, then $\mathfrak{q}$ splits completely in $F/\mathbb{Q}$.*

Proof. Let σ be an element of the decomposition group $Z(\mathfrak{q}|q)$ of $\mathfrak{q}$. Since $L/\mathbb{Q}$ is abelian, we must have $\sigma_a(\mu) = \xi^p \mu^a$. Now $\sigma_a(\mathfrak{q}) = \mathfrak{q}$ implies $\mathfrak{q}^r \parallel \xi^p \mu^a$, and this implies $r \equiv ar \bmod p$; but $p \nmid r$ show that this is possible only if $a = 1$. Thus $\sigma_a = 1$, and $\mathfrak{q}$ splits completely in $F/\mathbb{Q}$. □

In particular, we find that $(1-\zeta) \nmid \mu$. Since L/F is unramified outside p, prime ideals $\mathfrak{p} \nmid p$ must satisfy $\mathfrak{p}^{bp} \parallel \mu$ for some integer b. This shows that $(\mu) = \mathfrak{a}^p$ is the p-th power of some ideal $\mathfrak{a}$. From $(\mu) = \mathfrak{a}^p$ and the fact that $L/\mathbb{Q}$ is abelian we deduce that $\sigma_a(\mathfrak{a})^p = \mathfrak{a}^{pa}\xi^p$, where $\sigma_a(\zeta_p) = \zeta_p^a$. Thus $\sigma_a(c) = c^a$ for the ideal class $c = [\mathfrak{a}]$ and for every a with $1 \le a < p$. Now we invoke Stickelberger's Theorem (cf. [4] or [5, Chap. 11]) to show that $\mathfrak{a}$ is principal:

Theorem 2.3. *Let $F = \mathbb{Q}(\zeta_p)$; then the Stickelberger element*

$$\theta = \sum_{a=1}^{p-1} a\sigma_a^{-1} \in \mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$$

annihilates the ideal class group $\text{Cl}(F)$.

From this theorem we find that $1 = c^\theta = \prod \sigma_a^{-1}(c)^a = c^{p-1} = c^{-1}$, hence $c = 1$ as claimed. In particular $\mathfrak{a} = (\alpha)$ is principal. This shows that $\mu = \alpha^p \eta$ for some unit η, hence $L = F(\sqrt[p]{\eta})$. Now write $\eta = \zeta^t \varepsilon$ for some unit ε in the maximal real subfield of F. Since ε is fixed by complex conjugation σ_{-1} and since $L/\mathbb{Q}$ is abelian, we see that $\zeta^{-t}\varepsilon = \sigma_{-1}(\mu) = \xi^p \mu^{-1}$, hence $\zeta^{-t}\varepsilon = \xi^p \zeta^{-t}\varepsilon^{-1}$. Thus ε is a p-th power, and we find $\mu = \zeta^t$. But this implies that $L = \mathbb{Q}(\zeta_{p^2})$, and Prop. 1.1 is proved.

Since every cyclotomic extension is ramified, we get the following special case of Minkowski's theorem as a corollary:

Corollary 2.4. *Every solvable extension of* $\mathbb{Q}$ *is ramified.*

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