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A monogenic Hasse-Arf theorem

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Résumé. On étend le théorème de Hasse–Arf de la classe des extensions résiduellement séparables des anneaux de valuation discrète complets à la classe des extensions monogènes.

Abstract. I extend the Hasse–Arf theorem from residually separable extensions of complete discrete valuation rings to monogenic extensions.

Let B/A be a finite extension of henselian discrete valuation rings which is generically Galois with group G, that is, for which the corresponding extension of fraction fields is Galois with group G. For $\sigma \in G - \{1\}$, let $I_B(\sigma)$ be the ideal of B generated by $(\sigma - 1)B$ and let $i_B(\sigma)$ be the length of the B-module $B/I_B(\sigma)$.

For any finite dimensional complex representation $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$, we define the naive Artin conductor exactly as we do when B/A is residually separable, i.e., when the extension of residue fields is separable:

$$\mathrm{ar}_{\mathrm{n}}(\rho) = e_{B/A}^{-1} \sum_{\sigma \neq 1} [\dim(V) - \mathrm{trace}(\rho(\sigma))] i_B(\sigma).$$

By looking at real parts, it is immediate that this is a non-negative rational number, and when B/A is residually separable, the Hasse-Arf theorem [3, VI §2] tells us that it is also an integer.

In [4], De Smit shows that most of the classical ramification-theoretic properties of residually separable extensions B/A hold in the slightly more general, "monogenic" case where we require only that B is generated as an A-algebra by one element. The purpose of this note is to show that the Hasse–Arf theorem also holds in this context.

Partial results in this direction were obtained by Spriano [5]. A proof of the Hasse-Arf theorem in equal characteristic that is strong enough to cover monogenic extensions was outlined at the 1999 Luminy conference on ramification theory. It was based on a technical analysis of a refinement [2, 3.2.2] of Kato's refined Swan conductor [1], but since then, an elementary reduction to the classical Hasse-Arf theorem has been found.

The contents of this paper are contained in my dissertation (U.C. Berkeley, 2000), which was written under the direction of Hendrik Lenstra.

Proposition 1. *Let B/A be a finite generically separable extension of henselian discrete valuation rings. Then the following are equivalent.*

(i) There exists an $x \in B$ such that $B = A[x]$.

(ii) The second exterior power $\Omega^2_{B/A}$ of the module of relative Kähler differentials is zero.

(iii) There is a henselian discrete valuation ring A' that is finite over the maximal unramified subextension A^{nr} of B/A such that $e_{A'/A^{\mathrm{nr}}} = 1$ and B'/A' is a residually separable extension of discrete valuation rings, where $B' = A' \otimes_{A^{\mathrm{nr}}} B$.

Proof. De Smit [4, 4.2] shows that (i) follows from (ii). For any A' as in (iii), we have $B' \otimes_B \Omega^2_{B/A} \cong B' \otimes_B \Omega^2_{B/A^{\mathrm{nr}}} \cong \Omega^2_{B'/A'} = 0$, so (iii) implies (ii). Now we show (i) implies (iii).

Assume, as we may, that $A = A^{\mathrm{nr}}$, and let l/k denote the residue extension of B/A. Take some $x \in B$ such that $B = A[x]$ and let $\bar{x}$ denote the image of x in l. Let $g(X) \in A[X]$ be a monic lift of the minimal polynomial $X^q - a$ of $\bar{x}$ over k. Since the maximal ideal of B is generated by that of A and $g(x)$, we may assume that $g(x)$ generates the maximal ideal of B. Then modulo the maximal ideal of B, we have $g(X + x) \equiv X^q + x^q - a \equiv X^q$, so $g(X + x)$ is an Eisenstein polynomial with coefficients in B. Now let A' be the discrete valuation ring $A[X]/(g(X))$. Then

$$B' = A' \otimes_A B \cong B[X]/(g(X)) \cong B[X]/(g(X + x))$$

is a discrete valuation ring which has the same residue field as B and, hence, A'. □

Proposition 2. *Let B/A be a finite extension of henselian discrete valuation rings that is generically Galois with group G, and let $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ be a finite dimensional representation of G. If A'/A is a finite extension of henselian discrete valuation rings such that $B' = A' \otimes_A B$ is a discrete valuation ring, then we have $\mathrm{ar}_{\mathrm{n}}(\rho') = e_{A'/A}\mathrm{ar}_{\mathrm{n}}(\rho)$, where ρ' is ρ viewed as a representation of the generic Galois group of the extension B'/A'.*

Proof. For $\sigma \in G - \{1\}$, we have $I_{B'}(\sigma) = A' \otimes_A I_B(\sigma) = B' \otimes_B I_B(\sigma)$, so

$$\begin{aligned} i_{B'}(\sigma) &= \mathrm{length}_{B'}(B'/I_{B'}(\sigma)) = \mathrm{length}_{B'}(B' \otimes_B B/I_B(\sigma)) \\ &= e_{B'/B}\mathrm{length}_B(B/I_B(\sigma)) = e_{B'/B} i_B(\sigma). \end{aligned}$$

Thus

$$\mathrm{ar}_{\mathrm{n}}(\rho') = e_{B'/B}\frac{e_{B/A}}{e_{B'/A'}}\mathrm{ar}_{\mathrm{n}}(\rho) = e_{A'/A}\mathrm{ar}_{\mathrm{n}}(\rho).$$

□

Corollary 3. *Let B/A be a finite monogenic extension of henselian discrete valuation rings that is generically Galois with group G, and let $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ be a finite dimensional representation of G. Then $\mathrm{ar}_{\mathrm{n}}(\rho)$ is an integer.*

Proof. Restricting to the maximal unramified subextension of B/A does not change the naive Artin conductor or the monogeneity of the extension. So assume B/A is residually purely inseparable. Now just apply the previous proposition with A' taken as in the first proposition and then use the classical Hasse-Arf theorem. □

Remark. One can define a naive Swan conductor [1, 6.7] as well. It also is an integer in the monogenic case but simply because it agrees with the naive Artin conductor whenever B/A is monogenic and not residually separable. It is not, however, a good invariant even in the monogenic case: it is a consequence of results outlined at the Luminy conference that in the (monogenic) equal-characteristic case, the naive Swan conductor of a faithful, one-dimensional representation agrees with Kato's Swan conductor if and only if either B/A is residually separable or $e_{B/A} = 1$, whereas for general monogenic extensions in equal-characteristic, the naive Artin conductor of a one-dimensional representation is equal to a non-logarithmic, "Artin-type" variant of Kato's Swan conductor.

References

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