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# Local distinction, quadratic base change and automorphic induction for $\mathrm{GL}_n$

par Nadir MATRINGE

Résumé. Ce titre sophistiqué dissimule un exercice élémentaire sur la théorie de Clifford pour les sous-groupes d'indice deux et les représentations auto-duales ou conjuguées-duales. Appliqué aux représentations du groupe Weil–Deligne $W'_F$ d'un corps local non archimédien $F$, puis interprété en termes de représentations de $\mathrm{GL}_n(F)$ via correspondance de Langlands locale lorsque $F$ est de caractéristique nulle, l'exercice en question établit divers énoncés concernant le comportememnt de différents types de distinction sous changement de base et induction automorphe quadratiques. Lorsque $F$ est de caractéristique résiduelle non 2, en combinant un des résultats simples obtenus ici avec la trivialité des valeurs centrales de facteurs epsilon des représentations de $W'_F$ conjuguées-orthogonales ([8]), nous retrouvons sans faire appel à la correspondance de Langlands locale un résultat de Serre sur la parité du conducteur d'Artin de ces représentations ([23]). D'autre part, nous discutons cette parité pour les représentations symplectiques à l'aide de la correspondance de Langlands locale et de la conjecture dite de Prasad et Takloo-Bighash.

Abstract. Behind this sophisticated title hides an elementary exercise on Clifford theory for index two subgroups and self-dual or conjugate-dual representations. When applied to semi-simple representations of the Weil–Deligne group $W'_F$ of a non Archimedean local field $F$, and further translated in terms of representations of $\mathrm{GL}_n(F)$ via the local Langlands correspondence when $F$ has characteristic zero, it yields various statements concerning the behaviour of different types of distinction under quadratic base change and automorphic induction. When $F$ has residual characteristic different from 2, combining of one of the simple results that we obtain with the tiviality of conjugate-orthogonal root numbers ([8]), we recover without using the LLC a result of Serre on the parity of the Artin conductor of orthogonal representations of $W'_F$ ([23]). On the other hand we discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture.

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## Introduction

Let $E/F$ be a separable quadratic extension of non Archimedean local fields. Then thanks to the known local Langlands correspondence for $\mathrm{GL}_n(E)$ and $\mathrm{GL}_n(F)$, one has a base change map $\mathrm{BC}_F^E$ from the set of isomorphism classes of irreducible representations of $\mathrm{GL}_n(F)$ to that of $\mathrm{GL}_n(E)$, and an automorphic induction map $\mathrm{AI}_E^F$ from the set or isomorphism classes of irreducible representations of $\mathrm{GL}_n(E)$ to that of $\mathrm{GL}_{2n}(F)$. A typical statement proved in this note (for $F$ of characteristic zero) is that if $\pi$ is a generic unitary representation of $\mathrm{GL}_n(F)$ with orthogonal Langlands parameter (orthogonal in short), then $\mathrm{BC}_F^E(\pi)$ is orthogonal and $\mathrm{GL}_n(F)$-distinguished, and that the converse holds if $\pi$ is a discrete series (see Corollary 3.2 for the general statement). Corollary 3.2 is itself a translation via the LLC of our main result which concerns representations of the Weil–Deligne group of $F$ (Proposition 3.1). Another lucky application of Proposition 3.1 is that the result of [23] on the parity of Artin conductors of representations of the Weil–Deligne group of $F$ is a consequence of that in [6] on root numbers of orthogonal representations, when $F$ has odd residual characteristic, as we show in Corollary 4.1. We also discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture in Corollary 4.4.

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## 1. Notation, definitions and basic facts about self-dual and conjugate-dual representations

For $K$ a non Archimedean local field we denote by $W_K$ the Weil group of $K$ (see [26]), and by $W'_K = W_K \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group of $K$. By a representation of $W_K$ we mean a finite dimensional smooth complex representation of $W_K$. By a representation of $W'_K$ we mean a representation which is a direct sum of representations of the form $\phi \otimes S$, where $\phi$ is an irreducible representation of $W_K$ and $S$ is an irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$. We sometimes abbreviate "$\phi$ is a representation of $W'_K$" as "$\phi \in \mathrm{Rep}(W'_K)$". We denote by $\phi^\vee \in \mathrm{Rep}(W'_K)$ the dual of $\phi \in \mathrm{Rep}(W'_K)$.

For the following facts on self-dual and conjugate-dual representations of $W'_K$, we refer to [8, Section 3]. We recall that a representation $\phi$ of $W'_K$

is self-dual if and only if there exists on $\phi \times \phi$ a $W'_K$-invariant bilinear form $B$ which is non degenerate: we will say that $B$ is $W'_K$-*bilinear* (which in particular means non degenerate). If moreover $B$ is alternate, we say that $B$ is $(W'_K, -1)$-*bilinear* in which case we say that $\phi$ is *symplectic* or $(-1)$-*self-dual*, whereas if $B$ is symmetric, and we say that $B$ is $(W'_K, 1)$-*bilinear* in which case we say that $\phi$ is *orthogonal* or 1-*self-dual*. If $\phi$ is irreducible and self-dual, then there is up to nonzero scaling a unique $W'_K$-bilinear form on $\phi \times \phi$, which is either $(W'_K, -1)$-bilinear or $(W'_K, 1)$-bilinear, but not both.

Now suppose that $L/K$ is a separable quadratic extension so that $W_L$ has index two in $W_K$, and fix $s \in W_K - W_L$. For $\phi$ a representation of $W'_L$, we denote by $\phi^s$ the representation of $W'_L$ defined as $\phi^s := \phi(s.s^{-1})$. We say that $\phi$ is $L/K$-*dual* or *conjugate-dual* if $\phi^s \simeq \phi^\vee$. The representation $\phi \in \mathrm{Rep}(W'_L)$ is conjugate-dual if and only if there is on $\phi \times \phi$ a non-degenerate bilinear form $B$ such that

$$B(w.x, sws^{-1}.y) = B(x, y)$$

for all $(w, x, y)$ in $W'_L \times \phi \times \phi$. We say that such a bilinear form $B$ is $L/K$-*bilinear* (this in particular means non degenerate). If moreover there is $\varepsilon \in \{\pm 1\}$ such that $B$ satisfies

$$B(x, s^2.y) = \varepsilon B(y, x)$$

for all $(x, y)$ in $\phi \times \phi$ we say that $B$ is $(L/K, \varepsilon)$-*bilinear*, in which case we say that $\phi$ is $(L/K, \varepsilon)$-*dual* or *conjugate-symplectic* if $\varepsilon = -1$ and *conjugate-orthogonal* if $\varepsilon = 1$. All the definitions above do not depend on the choice of $s$. When $\phi$ is $L/K$-dual and also irreducible, then there is up to nonzero scaling a unique $L/K$-bilinear form on $\phi \times \phi$, which is either $(L/K, -1)$-bilinear or $(L/K, 1)$-bilinear, but not both.

## 2. Preliminary results

**2.1. Clifford–Mackey theory for index two subgroups.** We refer to [5, Section 3] for the following standard results.

**Theorem 2.1.** *Let $G$ be a finite group, $H$ a finite subgroup of index 2, $s \in G - H$, and let $\eta : G \to \{\pm 1\}$ be the nontrivial character of $G$ trivial on $H$.*

- *For $\phi$ a (finite dimensional complex) representation of $H$ which is irreducible, the representation $\mathrm{Ind}_H^G(\phi)$ is irreducible if and only if $\phi^s \not\simeq \phi$, which is also equivalent to the fact that $\phi$ does not extend to $G$. If it is reducible then $\phi$ extends to $G$, and if $\widetilde{\phi}$ is such an extension, then $\eta \otimes \widetilde{\phi}$ is the only other extension different from $\widetilde{\phi}$, and $\mathrm{Ind}_H^G(\phi) \simeq \widetilde{\phi} \oplus (\eta \otimes \widetilde{\phi})$.*
- *An irreducible representation $\phi'$ of $G$ restricts to $H$ either irreducibly, or breaks into two irreducible pieces, and the second case*

*occurs if and only if* $\phi' \simeq \eta \otimes \phi'$*, which is also equivalent to* $\phi' = \mathrm{Ind}_H^G(\phi)$ *for* $\phi$ *an irreducible representation of* $H$ *such that* $\phi^s \simeq \phi$*.*

For $E/F$ a separable quadratic extension of non Archimedean local fields, we denote by $\eta_{E/F} : W'_F \to \{\pm 1\}$ the nontrivial character of $W'_F$ trivial on $W'_E$. Theorem 2.1 has the following corollary.

**Corollary 2.2.** *Let* $E/F$ *be a separable quadratic extension of non Archimedean local fields, and fix* $s \in W_F - W_E$*.*

- *For* $\phi_E \in \mathrm{Rep}(W'_E)$ *an irreducible representation, the representation* $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ *is irreducible if and only if* $\phi_E^s \not\simeq \phi_E$*, which is also equivalent to the fact that* $\phi_E$ *does not extend to* $W'_F$*. If it is reducible then* $\phi_E$ *extends to* $W'_F$*, and if* $\phi_F$ *is such an extension, then* $\eta_{E/F} \otimes \phi_F$ *is the only other extension different from* $\phi_F$*, and* $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E) \simeq \phi_F \oplus (\eta_{E/F} \otimes \phi_F)$*.*
- *An irreducible representation* $\phi_F$ *of* $W'_F$ *restricts to* $W'_E$ *either irreducibly, or breaks into two irreducible pieces, and the second case occurs if and only if* $\phi_F \simeq \eta_{E/F} \otimes \phi_F$*, which is also equivalent to* $\phi_F \simeq \mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ *for* $\phi_E$ *and irreducible representation of* $W'_E$ *such that* $\phi_E^s \simeq \phi_E$*.*

*Proof.* We recall that by [4, 28.6], if $\alpha_K$ is an irreducible representation of $W_K$ for $K$ local and non Archimedean, then there exists an unramified character $\chi_K$ of $W_K$ such that $\chi_K \otimes \alpha_K$ has co-finite kernel.

For the first part of the first point, write $\phi_E = \alpha_E \otimes S$, and suppose first that $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is irreducible. Twist $\mathrm{Ind}_{W_E}^{W_F}(\alpha_E)$ by an unramified character $\chi_F$ so that $\mathrm{Ind}_{W_E}^{W_F}(\mathrm{Res}_{W_F}^{W_E}(\chi_F) \otimes \alpha_E)$ has a co-finite kernel (hence $\mathrm{Res}_{W_E}^{W_F}(\chi_F) \otimes \alpha_E$ has co-finite kernel as well, as it has to be trivial on $W_E \cap \mathrm{Ker}(\mathrm{Ind}_{W_E}^{W_F}(\mathrm{Res}_{W_E}^{W_F}(\chi_E) \otimes \alpha_E)))$. Because $\mathrm{Res}_{W_E}^{W_F}(\chi_F)^s = \mathrm{Res}_{W_E}^{W_F}(\chi_F)$, one deduces from Theorem 2.1 applied to $\mathrm{Res}_{W_E}^{W_F}(\chi_E) \otimes \alpha_E$ that $\alpha_E^s \not\simeq \alpha_E$ and that $\alpha_E$ does not extend. This implies the same statements for $\phi_E$. Conversely if $\phi_E^s \not\simeq \phi_E$, then the same holds for $\alpha_E$. Take $\chi_E$ unramified such that $\chi_E \otimes \alpha_E$ has co-finite kernel, and $\chi_F$ any unramified extension of $\chi_E$ to $W_F$. Then $\mathrm{Ind}_{W_E}^{W_F}(\alpha_E) = \chi_F^{-1} \otimes \mathrm{Ind}_{W_E}^{W_F}(\chi_E \otimes \alpha_E)$ is irreducible by Theorem 2.1, and so is $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E) = \mathrm{Ind}_{W_E}^{W_F}(\alpha_E) \otimes S$. The second part of the first point is similar, using an unramified character $\chi_F$ of $W_F$ such that $\chi_E \otimes \phi_E$ has cofinite kernel (just take such a $\chi_E$ and extend it to an unramified character of $W_F$).

The proof of the second point is similar. □

We will tacitly use the above corollary from now on.

**2.2. Distinction and LLC for $\mathrm{GL}_n$.** Let $F$ be a non Archimedean local field, we denote by LLC the local Langlands correspondence ([10, 11, 17]). For any $n \geq 1$, it restricts as a bijection from the set of isomorphism classes of $n$-dimensional representations of $W'_F$ to that of (smooth and complex) irreducible representations of $\mathrm{GL}_n(F)$. If $E/F$ is a quadratic extension, and $\pi = \mathrm{LLC}(\phi_F)$ for $\phi_F$ a representation of $W'_F$, we set $\mathrm{BC}_F^E(\pi) = \mathrm{LLC}(\mathrm{Res}_{W'_E}^{W'_F}(\phi_F))$ (the *quadratic base change* of $\pi$), whereas if $\tau = \mathrm{LLC}(\phi_E)$ for $\phi_E$ a representation of $W'_E$, we set $\mathrm{AI}_E^F(\tau) = \mathrm{LLC}(\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E))$ (the *quadratic automorphic induction* of $\tau$). For $\pi$ a representation of $\mathrm{GL}_n(F)$, we denote by $\pi^\vee$ its dual. If $\pi$ is irreducible, we call it a *discrete series* representation if it has a matrix coefficient $c$ such that $|\chi \otimes c|^2$ is integrable on $\mathrm{GL}_n(F)/F^\times.I_n$ (with respect to any Haar measure on the group $\mathrm{GL}_n(F)/F^\times.I_n$) for some character $\chi$ of $\mathrm{GL}_n(F)$. A representation $\phi$ of $W'_F$ is irreducible if and only if $\mathrm{LLC}(\phi)$ is a discrete series.

Let $N_n(F)$ be the subgroup of $\mathrm{GL}_n(F)$ of upper triangular unipotent matrices, and let $\psi$ be a non trivial character of $F$, which in turn defines a character $\widetilde{\psi} : u \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n})$ of $N_n(F)$. We say that an irreducible representation $\pi$ of $\mathrm{GL}_n(F)$ is generic if $\mathrm{Hom}_{N_n(F)}(\pi, \widetilde{\psi}) \neq \{0\}$ and this does not depend on the choice of $\psi$. Genericity can be read on the Langlands parameter from [30, Theorem 9.7] (one way to state it is that $\mathrm{LLC}(\phi)$ is generic if and only if the adjoint L factor of $\phi$ is holomorphic at $s = 1$). From this one easily deduces the direct implications of the following proposition, the converse implications being special cases of [20, Theorem 9.1].

**Proposition 2.3.**

- *Let $\pi$ be an irreducible representation of $\mathrm{GL}_n(F)$. If $\mathrm{BC}_F^E(\pi)$ is generic, then $\pi$ is generic, and conversely if $\pi$ is generic unitary, then $\mathrm{BC}_F^E(\pi)$ is generic (unitary).*
- *Let $\tau$ be an irreducible representation of $\mathrm{GL}_n(E)$. If $\mathrm{AI}_E^F(\tau)$ is generic, then $\tau$ is generic, and conversely if $\tau$ is generic unitary, then $\mathrm{AI}_E^F(\tau)$ is generic (unitary).*

We denote by $\widetilde{\mathrm{GL}_n}(F)$ the double cover of $\mathrm{GL}_n(F)$ defined for example in [15, Section 2.1]. Following [15] we call a map $\gamma : F^\times \to \mathbb{C}^\times$ a pseudo-character if it satisfies $\gamma(xy) = \gamma(x)\gamma(y)(x,y)_2^{\lfloor n/2 \rfloor}$ for all $x$ and $y$ in $F^\times$, where $( \ , \ )_2$ is the Hilbert symbol of $F^\times$. For $\gamma$ a pseudo-character of $F^\times$ we denote by $\theta_{1,\gamma}$ the corresponding Kazhdan–Patterson exceptional representation of $\widetilde{\mathrm{GL}_n}(F)$ as in [15, Section 2.5]. We say that an irreducible representation $\pi$ of $\mathrm{GL}_n(F)$ is $\Theta_F$*-distinguished* if there exist pseudo-characters $\gamma$ and $\gamma'$ of $F^\times$ such that $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\theta_{1,\gamma} \otimes \theta_{1,\gamma'}, \pi^\vee) \neq \{0\}$ (where $\theta_{1,\gamma} \otimes \theta_{1,\gamma'}$ indeed factors through $\mathrm{GL}_n(F)$ so that the defintion makes sense).

When $n$ is even, we denote by $S_n(F)$ the Shalika subgroup of $\mathrm{GL}_n(F)$ consisting of matrices of the form $s(g,x) = \mathrm{diag}(g,g)\begin{pmatrix} I_{n/2} & x \\ & I_{n/2}\end{pmatrix}$ for $g \in \mathrm{GL}_{n/2}(F)$ and $x \in \mathcal{M}_{n/2}(F)$, and for $\psi$ a non trivial character of $F$, we denote by $\Psi$ the character of $S_n(F)$ defined by $\Psi(s(g,x)) = \psi(\mathrm{tr}(x))$. We say that an irreducible representation $\pi$ of $\mathrm{GL}_n(F)$ is $\Psi_F$*-distinguished* if $n$ is even and $\mathrm{Hom}_{S_n(F)}(\pi,\Psi) \neq \{0\}$ for some non trivial character $\psi$ of $F$. This does not depend on the choice of $\psi$.

Finally if $E/F$ is quadratic separable, identifying $\eta_{E/F}$ to the character of $F^\times$ trivial on $N_{E/F}(E^\times)$ via local class field theory, we say that an irreducible representation $\tau$ of $\mathrm{GL}_n(E)$ is $\mathbb{1}_{E/F}$*-distinguished* if $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\tau,\mathbb{1}) \neq \{0\}$ and $\eta_{E/F}$*-distinguished* if $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\tau,\eta_{E/F}\circ \det) \neq \{0\}$.

The following theorem follows from [1, 3, 12, 13, 14, 15, 16, 18, 19, 29]. Parts of it are known to hold when $F$ is of positive characteristic and odd residual characteristic ([2, Appendix A]).

**Theorem 2.4.** *Suppose that $F$ has characteristic zero.*

- *Let $\pi = \mathrm{LLC}(\phi_F)$ be a generic representation of $\mathrm{GL}_n(F)$, then $\phi_F$ is symplectic if and only $\pi$ is $\Psi_F$-distinguished, whereas $\phi_F$ is orthogonal if and only if $\pi$ is $\Theta_F$-distinguished.*
- *Let $\tau = \mathrm{LLC}(\phi_E)$ be a generic representation of $\mathrm{GL}_n(E)$, then $\phi_E$ is conjugate-symplectic if and only $\tau$ is $\eta_{E/F}$-distinguished, whereas $\phi_E$ is conjugate-orthogonal if and only if $\tau$ is $\mathbb{1}_{E/F}$-distinguished.*

**2.3. A reminder on epsilon factors.** Let $K'/K$ be a finite separable extension of non Archimedean local fields. We denote by $\varpi_K$ a uniformizer of $K$ and by $P_K$ the maximal ideal of the ring of integers $O_K$ of $K$. If $\psi$ is a non trivial character of $K$, we denote by $\psi_{K'}$ the character $\psi \circ \mathrm{tr}_{K'/K}$. We call *the conductor of* $\psi$ and write $d(\psi)$ for the smallest integer $d$ such that $\psi$ is trivial on $P_K^d$. When $K'/K$ *is unramified*, it follows from [27, Chapter 8, Corollary 3] that

$$d(\psi_{K'}) = d(\psi). \tag{2.1}$$

Similarly if $\chi$ is a character of $W'_K$ identified by local class field theory with a character of $K^*$, we call the Artin conductor of $\chi$ the integer $a(\chi)$ equal to zero if $\chi$ is unramified, or equal to the smallest integer $a$ such that $\chi$ is trivial on $1 + P_K^a$ if $\chi$ is ramified. More generally one can define the Artin conductor $a(\phi)$ (which is an integer) of any representation $\phi$ of $W'_K$, see [26, 3.4.5] when $\phi$ is a representation of $W_K$ and [9, Section 2.2, (10)] in general. The Artin conductor is additive:

$$a(\phi \oplus \phi') = a(\phi) + a(\phi')$$

for $\phi$ and $\phi'$ in $\mathrm{Rep}(W'_K)$. If $\phi$ is a representation of $W'_K$, and $\psi$ is a non trivial character of $K$, we refer to [26, 3.6.4] and [4, 31.3] or [9, Section 2.2] for the definition of the root number $\epsilon(1/2, \phi, \psi)$. One then defines the Langlands $\lambda$-constant:

$$\lambda(K'/K, \psi) = \frac{\epsilon(1/2, \mathrm{Ind}_{W_{K'}}^{W_K}(\mathbb{1}_{W_K}), \psi)}{\epsilon(1/2, \mathbb{1}_{W'_K}, \psi_{K'})}.$$

For $a \in K^\times$, we set $\psi_a = \psi(a.\ )$. These constants enjoy the following list of properties, which we will freely use later in the paper.

(1) $\epsilon(1/2, \phi \oplus \phi', \psi) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \phi', \psi)$ where $\phi'$ is another representation of $W'_K$ ([26, (3.4.2)]).
(2) $\epsilon(1/2, \phi, \psi_a) = \det(\phi(a))\epsilon(1/2, \phi, \psi)$ ([26, (3.6.6)]).
(3) $\epsilon(1/2, \phi, \psi)^2 = \det(\phi)(-1)$ when $\phi$ is self-dual ([9, Section 2.3, (11)]).
(4) If $d(\psi) = 0$ and $\mu$ is an unramified character of $K^*$, it follows from [9, Section 2.3, (9)] that:

$$\epsilon(1/2, \mu \otimes \phi, \psi) = \mu(\varpi_K^{a(\phi)})\epsilon(1/2, \phi, \psi).$$

(5) If $K'/K$ is quadratic with $K$ of characteristic not 2, $\delta \in \ker(\mathrm{tr}_{K'/K}) \setminus \{0\}$, and $\phi$ is a $K'/K$-orthogonal representation of $W'_{K'}$, then by [8, Proposition 5.2] (generalizing [7, Theorem 3]):

$$\epsilon(1/2, \phi, \psi_{K'}) = \det(\phi)(\delta).$$

(6) If $\phi_{K'}$ is an $r$-dimensional representation of $W'_{K'}$, then

$$\epsilon(1/2, \mathrm{Ind}_{W'_{K'}}^{W'_K}(\phi_{K'}), \psi) = \lambda(K'/K, \psi)^r \epsilon(1/2, \phi_{K'}, \psi_{K'})$$

([4, (30.4.2)]).

When applied to a $K'/K$ quadratic and $\phi_{K'} = \mathrm{Res}_{W'_{K'}}^{W'_K}(\phi)$ for $\phi$ a representation of $W'_K$, one gets

$$\epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{K'/K} \otimes \phi, \psi) = \lambda(K'/K, \psi)^r \epsilon\Big(1/2, \mathrm{Res}_{W'_{K'}}^{W'_K}(\phi), \psi_{K'}\Big)$$

(7) If $K'/K$ is unramified with $[K'/K] = n$:

$$\lambda(K'/K, \psi) = (-1)^{d(\psi)(n-1)}$$

(for example [21] and (2), together with Equation (2.1)). In particular if $d(\psi) = 0$ then

$$\lambda(K'/K, \psi) = 1.$$

## 3. Distinction, base change, and automorphic induction

From now on $E/F$ is a separable quadratic extension of non Archimedean local fields. Our main result is the following proposition, and we notice that half of its first point is [8, Lemma 3.5 (i)].

**Proposition 3.1.**

(1) *Let $\phi_E$ be a semi-simple representation of $W'_E$ which is either $\varepsilon$-self-dual or $(E/F,\varepsilon)$-dual, then $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is $\varepsilon$-selfudal.*

(2) *Conversely if $\phi_E$ is irreducible and $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is $\varepsilon$-self-dual:*

  (a) *if $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is irreducible, i.e. $\phi_E^s \not\simeq \phi_E$, then either $\phi_E$ is $\varepsilon$-self-dual or $(E/F,\varepsilon)$-dual, but not both together,*

  (b) *if $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is reducible, i.e. $\phi_E^s \simeq \phi_E$, then $\phi_E$ is both $\varepsilon$-self-dual and $(E/F,\varepsilon)$-dual.*

(3) *Let $\phi_F$ be a semi-simple representation of $W'_F$ which is $\varepsilon$-self-dual, then $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual and $(E/F,\varepsilon)$-dual.*

(4) *Conversely, if $\phi_F$ is irreducible and $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual and $(E/F,\varepsilon)$-dual then $\phi_F$ is $\varepsilon$-self-dual.*

*Proof.*

(1). First suppose that $B_E$ is a $(E/F,\varepsilon)$-bilinear form on $\phi_E$. Write an element $v$ (resp. $v'$) in $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ under the form $v = x + s^{-1}.y$ (resp. $v' = x' + s^{-1}.y'$) for $x,\ x',\ y,\ y'$ in $\phi_E$, and set

$$B_F(v,v') = B_E(x,y') + \varepsilon B_E(x',y).$$

Then $B_F$ is $W'_E$-invariant because $B_E$ is $(W'_E,\varepsilon)$-conjugate (it is non-degenerate because so is $B_E$). Finally

$$B_F(s.v,s.v') = B_E(y,s^2.x') + \varepsilon B_E(y',s^2.x) = \varepsilon B_E(x',y) + B_E(x,y') = B_F(v,v').$$

Similarly if $B_E$ is $(W'_E,\varepsilon)$-bilinear, then one checks that

$$B_F(x+s^{-1}y, x'+s^{-1}.y') = B_E(x,x') + B_E(y,y')$$

defines a $(W'_F,\varepsilon)$-bilinear form on $\phi_F$.

(2). Suppose that $\phi_E$ is irreducible and that $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is $\varepsilon$-self-dual with $(W'_F,\varepsilon)$-bilinear form $B_F$.

  (a) If $\phi_E^s \not\simeq \phi_E$, because $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ is self-dual then either $\phi_E$ is self-dual, or $\phi_E^s \simeq \phi_E^\vee$ but not both together. In the first case, say that

$\phi_E$ is $\varepsilon'$-self-dual, then so is $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ by (1), but then $\varepsilon' = \varepsilon$ by irreducibility of $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$. If $\phi_E^s \simeq \phi_E^\vee$ we conclude in a similar manner.

(b) If $\phi_E^s \simeq \phi_E$ then $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E) \simeq \phi \oplus \eta_{E/F} \otimes \phi$ for $\phi$ extending $\phi_E$, and $\phi \not\simeq \eta_{E/F} \otimes \phi$. Because $\phi \not\simeq \eta_{E/F} \otimes \phi$ there are two disjoint cases. The first is when $\phi$ is self-dual, in which case $\phi \perp \eta_{E/F} \otimes \phi$ and $B_F$ restricts non trivially to $\phi \times \phi$ (and $\eta_{E/F} \otimes \phi \times \eta_{E/F} \otimes \phi$). Then $\phi_E$ is $\varepsilon$-dual and $(\varepsilon, s)$-dual by (3). Otherwise $\phi^\vee \simeq \eta_{E/F} \otimes \phi$ and $B_F$ is zero on $\phi \times \phi$ and $\eta_{E/F} \otimes \phi \times \eta_{E/F} \otimes \phi$. In this case there is up to scaling a unique $W'_F$-invariant bilinear form on $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$, namely $B_F$. Because $\phi_E^\vee \simeq \phi_E$ (by restricting the relation $\phi^\vee \simeq \eta_{E/F} \otimes \phi$ to $W'_E$), $\phi_E$ must be $\varepsilon'$-self-dual, hence $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$ as well by (1), but then we have $\varepsilon' = \varepsilon$ by multiplicity one of $W'_F$-invariant bilinear form on $\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$. Moreover because $\phi_E^s = \phi_E$ the parameter $\phi_E$ is also $(\varepsilon'', s)$-self-dual and by (1) again we deduce that $\varepsilon'' = \varepsilon$.

(3). Let $B_F$ be a $(W'_F, \varepsilon)$-bilinear form on $\phi_F$, then it remains a $(W'_E, \varepsilon)$-bilinear on $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$, and on the other hand

$$B_E(x, y) = B_F(x, s^{-1}.y)$$

is an $(E/F, \varepsilon)$-bilinear form on $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$.

(4). We suppose that $\phi_F$ is irreducible and that $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual and also $(E/F, \varepsilon)$-dual. There are two cases to consider.

First if $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is irreducible, then denote by $B_E$ the $(W'_E, \varepsilon)$-bilinear form on $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$. Now set $D_E(x, y) = B_E(x, s^{-1}.y)$ for $x, y \in \mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$. Clearly $D_E$ is $E/F$-bilinear, but by irreducibility $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ affords at most one such form up to scalar, hence $D_E$ must be $(E/F, \varepsilon)$-bilinear. This implies that for $x$ and $y$ in $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ one has

$$B_E(s.x, s.y) = D_E(s.x, s^2.y) = \varepsilon D_E(y, s.x) = \varepsilon B_E(y, x) = B_E(x, y).$$

All in all, when $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is irreducible we deduce that $B_E$ is in fact $W_F$-invariant hence that $\phi_F$ is $\varepsilon$-self-dual.

It remains to treat the case where $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is reducible. In this case it is of the form $\phi_E \oplus s^{-1}.\phi_E$ where $\phi_E$ is an irreducible of $W'_E$ such that $\phi_E^s \not\simeq \phi_E$ and $\phi_F = \mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)$. First because $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual, then the

$(W'_E,\varepsilon)$-bilinear form $B_E$ on $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ either induces an isomorphism $\phi_E^s \simeq \phi_E^\vee$ or $\phi_E \perp s^{-1}.\phi_E$ for $B_E$. Similarly the $(E/F,\varepsilon)$-bilinear form $C_E$ on $\mathrm{Res}_{W'_E}^{W'_F}(\phi_F)$ either induces an isomorphism $(\phi_E^s)^\vee \simeq \phi_E^s \Longleftrightarrow \phi_E \simeq \phi_E^\vee$ or $\phi_E \perp s^{-1}.\phi_E$ for $C_E$. Suppose that $B_E$ induces an isomorphism $\phi_E^s \simeq \phi_E^\vee$, then one must have $\phi_E \perp s^{-1}.\phi_E$ for $C_E$ because $\phi_E \not\simeq \phi_E^s \simeq \phi_E^\vee$. This implies that $C_E$ induces an $(E/F,\varepsilon)$-bilinear form on $\phi_E$ and by point (1) we deduce that $\phi_F$ is $\epsilon$-self-dual. On the other hand if $\phi_E \perp s^{-1}.\phi_E$ for $B_E$ then $B_E$ induces an $(W'_E,\varepsilon)$-bilinear form on $\phi_E$ and $\phi_F$ is $\varepsilon$-self-dual again by point (1). □

*Supposing that $F$ has characteristic zero*, we translate Proposition 3.1 via the LLC, in view of the results recalled in Section 2.2. For this we denote by $\sigma$ the Galois conjugation of $E/F$ and its extension to $\mathrm{GL}_n(E)$, and set $\tau^\sigma = \tau \circ \sigma$ for any representation of $\mathrm{GL}_n(E)$.

**Corollary 3.2.**

(1) *Let $\tau$ be an irreducible representation of $\mathrm{GL}_n(E)$ such that $\mathrm{AI}_E^F(\tau)$ is generic (for example $\tau$ generic unitary). If $\tau$ is either $\Theta_E$-distinguished or $\mathbb{1}_{E/F}$-distinguished, then $\mathrm{AI}_E^F(\tau)$ is $\Theta_F$-distinguished, whereas if $\tau$ is either $\Psi_E$-distinguished or $\eta_{E/F}$-distinguished, then $\mathrm{AI}_E^F(\tau)$ is $\Psi_F$-distinguished.*

(2) *Conversely if $\tau$ is a discrete series representation of $\mathrm{GL}_n(E)$.*
   - (a) *Suppose that $\mathrm{AI}_E^F(\tau)$ is $\Psi_F$-distinguished:*
     - (i) *if $\mathrm{AI}_E^F(\tau)$ is a discrete series, i.e. if $\tau^\sigma \not\simeq \tau$, then either $\tau$ is $\Psi_E$-distinguished or $\eta_{E/F}$-distinguished, but not both together,*
     - (ii) *if $\mathrm{AI}_E^F(\tau)$ is not a discrete series, i.e. $\tau^\sigma \simeq \tau$, then $\tau$ is both $\Psi_E$-distinguished and $\eta_{E/F}$-distinguished.*
   - (b) *Suppose that $\mathrm{AI}_E^F(\tau)$ is $\Theta_F$-distinguished:*
     - (i) *if $\mathrm{AI}_E^F(\tau)$ is a discrete series, i.e. $\tau^\sigma \not\simeq \tau$, then either $\tau$ is $\Theta_F$-distinguished or $\mathbb{1}_{E/F}$-distinguished, but not both together,*
     - (ii) *if $\mathrm{AI}_E^F(\tau)$ is not a discrete series, i.e. $\tau^\sigma \simeq \tau$, then $\tau$ is both $\Theta_E$-distinguished and $\mathbb{1}_{E/F}$-distinguished.*

(3) *Let $\pi$ be an irreducible representation of $\mathrm{GL}_n(F)$ such that $\mathrm{BC}_F^E(\pi)$ is generic (for example $\pi$ generic unitary). If $\pi$ is $\Theta_F$-distinguished, then $\mathrm{BC}_F^E(\pi)$ is $\Theta_E$-distinguished and $\mathbb{1}_{E/F}$-distinguished, whereas if $\pi$ is $\Psi_F$-distinguished, then $\mathrm{BC}_F^E(\pi)$ is $\Psi_E$-distinguished and $\eta_{E/F}$-distinguished.*

(4) *Conversely suppose that $\pi$ is a discrete series. If $\mathrm{BC}_F^E(\pi)$ is $\Theta_E$-distinguished and $\mathbb{1}_{E/F}$-distinguished, then $\pi$ is $\Theta_F$-distinguished,*

*whereas if* $\mathrm{BC}_F^E(\pi)$ *is* $\Psi_E$*-distinguished and* $\eta_{E/F}$*-distinguished, then* $\pi$ *is* $\Psi_F$*-distinguished.*

## 4. Parity of the Artin conductor of self-dual representations

In this section $F$ is again a non Archimedean local field. First, using [8, Proposition 5.2] (which is itself a quick but non trivial consequence of a difficult result of Deligne [6] on root numbers of orthogonal representations), we quickly recover in odd residual characteristic from Proposition 3.1 (3) the following result due to Serre [23] (the result in question also holds in even residual characteristic by [23]). In other words we show that the result of [6] implies that of [23] for non Archimedean local fields of odd residual characteristic.

**Corollary 4.1** (of Proposition 3.1, [23])**.** *Let* $\phi$ *be an orthogonal representation of* $W'_F$*. We have the following congruence of Artin conductors:* $a(\phi) = a(\det(\phi))[2]$*.*

*Proof.* As we said the result is true for $F$ of any residual characteristic, and we recover it in this proof for $F$ of residual characteristic different from 2. Let $E$ be the unramified quadratic extension of $F$, and take $\psi$ a character of $F$ of conductor zero. We have according to Section 2.3, Points (6) and (7)

$$\epsilon\Big(1/2, \mathrm{Res}_{W'_E}^{W'_F}(\phi), \psi_E\Big) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi). \tag{4.1}$$

Now denoting by $q$ the residual cardinality of $F$, let $u$ be an element of order $q^2 - 1$ in $E^\times$, so that $\delta := u^{(q+1)/2}$ does not belong to $F$ but $\Delta := \delta^2$ belongs to $F$. Note that the image of $\Delta$ generates $O_F^\times/1 + P_F$. Then $\epsilon(1/2, \mathrm{Res}_{W'_E}^{W'_F}(\phi), \psi_{E,\delta}^{-1}) = 1$ by Proposition 3.1 (3) and Section 2.3 (5), hence

$$\begin{aligned}\epsilon\Big(1/2, \mathrm{Res}_{W'_E}^{W'_F}(\phi), \psi_E\Big) &= \det\Big(\mathrm{Res}_{W'_E}^{W'_F}(\phi)\Big)(\delta)\\ &= \det(\phi)(N_{E/F}(\delta)) = \det(\phi)(-\Delta)\end{aligned}$$

thanks to Section 2.3 (2). Now observe that $\det(\phi)$ is quadratic as $\phi$ is self-dual, but because $q$ is odd it is trivial on $1 + P_F$, hence it has conductor 0 or 1, and it is of conductor zero if and only if $\det(\phi)(\Delta) = 1$, hence $\det(\phi)(\Delta) = (-1)^{a(\det(\phi))}$, so

$$\epsilon\Big(1/2, \mathrm{Res}_{W'_E}^{W'_F}(\phi), \psi_E\Big) = (-1)^{a(\det(\phi))}\det(\phi)(-1).$$

Now $\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi) = (-1)^{a(\phi)}\epsilon(1/2, \phi, \psi)$ thanks to Section 2.3 (4), hence Section 2.3 (3) implies the following:

$$\begin{aligned}&\epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi)\\ &\qquad = (-1)^{a(\phi)}\epsilon(1/2, \phi, \psi)^2 = (-1)^{a(\phi)}\det(\phi)(-1).\end{aligned}$$

The result now follows from Equation (4.1). □

One can legitimately ask about the parity of the Artin conductor of symplectic representations of $W'_F$. The answer seems much more complicated, and one way to adress it is via the LLC, using the so called Prasad and Takloo-Bighash conjecture, which is now a theorem when $F$ has characteristic zero and residual characteristic different from 2 ([22, 24, 25, 28]). To this end we recall that for $E/F$ a separable quadratic extension, then the matrix algebra $\mathcal{M}_n(E)$ embeds uniquely up to $\mathrm{GL}_{2n}(F)$-conjugacy into $\mathcal{M}_{2n}(F)$ as an $F$-subalgebra by the Skolem–Noether theorem. We fix such an embedding, which in turn gives rise to an embedding of $\mathrm{GL}_n(E)$ into $\mathrm{GL}_{2n}(F)$. We then say that an irreducible representation $\pi$ of $\mathrm{GL}_{2n}(F)$ is $\mathbb{1}^{E/F}$-distinguished if and only if $\mathrm{Hom}_{\mathrm{GL}_n(E)}(\pi, \mathbb{1}) \neq \{0\}$. We recall the following theorem, which is a consequence of one part of the Prasad and Takloo-Bighash conjecture.

**Theorem 4.2** ([22, 25, 28])**.** *Suppose that $F$ has characteristic zero and residual characteristic different from 2. If $\phi$ is an irreducible symplectic representation of $W'_F$ of dimension $2n$, then*

$$\epsilon\Big(1/2, \phi \otimes \mathrm{Ind}_{W'_E}^{W'_F}(\mathbb{1})\Big) = \eta_{E/F}(-1)^n$$

*if $\mathrm{LLC}(\phi)$ is $\mathbb{1}^{E/F}$-distinguished and*

$$\epsilon\Big(1/2, \phi \otimes \mathrm{Ind}_{W'_E}^{W'_F}(\mathbb{1})\Big) = -\eta_{E/F}(-1)^n$$

*otherwise.*

**Remark 4.3.** In the statement above, as the determinant of a symplectic representation is equal to 1, we suppressed the dependence of the root number $\epsilon(1/2, \phi \otimes \mathrm{Ind}_{W'_E}^{W'_F}(\mathbb{1}), \psi)$ on the non-trivial additive character $\psi$ of $F$.

As an immediate corollary we obtain the following result on the parity of Artin conductors of symplectic representations.

**Corollary 4.4.** *Suppose that $F$ has characteristic zero and residual characteristic different from 2, denote by $E$ the unramified quadratic extension of $F$, and let $\phi$ be an irreducible symplectic representation of $W'_F$. Then $a(\phi)$ is even if and only if $\mathrm{LLC}(\phi)$ is $\mathbb{1}^{E/F}$-distinguished.*

*Proof.* It easily follows, along the lines of the proof of Corollary 4.1, from Theorem 4.2, noting that $\eta_{E/F}(-1) = 1$. □

**Remark 4.5.** A general symplectic representation $\phi$ of $W'_F$ being a direct sum of the form $\oplus_{i=1}^r \phi_i \oplus_{j=1}^s (\phi'_j \oplus {\phi'_j}^\vee)$ for $\phi_i$ irreducible symplectic and $\phi'_j$ irreducible, we deduce the parity of $a(\phi)$ from Corollary 4.4 and such a decomposition. Namely, by Corollary 4.1 $a(\phi'_j \oplus {\phi'}_j^{\vee}) \equiv 0[2]$. Hence setting $\epsilon_i \in \{\pm 1\}$ being equal to 1 if and only if $\mathrm{LLC}(\phi_i)$ is $\mathbb{1}^{E/F}$-distinguished, we deduce by additivity of the Artin conductor that $(-1)^{a(\phi)} = \prod_{i=1}^{r} \epsilon_i$.

**Remark 4.6.** Looking at it from another angle, one sees that a symplectic discrete series representation of $\mathrm{GL}_{2n}(F)$ is $\mathbb{1}^{E/F}$-distinguished ($E/F$ unramified) if and only if it has even conductor.

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