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An a priori bound for rational functions on the Berkovich projective line

par Yûsuke OKUYAMA

Résumé. On établit une majoration locale a priori pour la dynamique d'une fraction rationnelle f de degré > 1 sur la droite projective de Berkovich sur un corps algébriquement clos de caractéristique quelconque et complet pour une norme non archimédienne non triviale. On en déduit un résultat d'équidistribution pour des cibles mobiles vers la mesure d'équilibre (ou la mesure canonique) μ_f de f, sous condition que f n'a pas de bonnes réductions potentielles. Cela répond en partie à une question posée par Favre et Rivera-Letelier. On obtient aussi un résultat d'équidistribution pour la distribution moyenne de valeurs des dérivées des polynômes itérés.

Abstract. We establish a local a priori bound on the dynamics of a rational function f of degree > 1 on the Berkovich projective line over an algebraically closed field of arbitrary characteristic that is complete with respect to a non-trivial and non-archimedean absolute value, and deduce an equidistribution result for moving targets towards the equilibrium (or canonical) measure μ_f of f, under the no potentially good reduction condition. This partly answers a question posed by Favre and Rivera-Letelier. We also obtain an equidistribution on the averaged value distribution of the derivatives of the iterated polynomials.

1. Introduction

Let K be an algebraically closed field of arbitrary characteristic that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is some compactification of the classical projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$ (see [5]) and is canonically regarded as a tree in the sense of Jonsson [13, Definition 2.2], the topology of which coincides with the Gelfand topology of P^1.

The action on $\mathbb{P}^1$ of a rational function $h \in K(z)$ canonically extends to a continuous action on P^1. If in addition $\deg h > 0$, then this extended continuous action of h is also open and surjective, preserves both $\mathbb{P}^1$ and

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$\mathsf{P}^1 \setminus \mathbb{P}^1$, and satisfies $\#(h^{-1}(\mathcal{S})) \le \deg h$ for every $\mathcal{S} \in \mathsf{P}^1$. More precisely, the local degree function $z \mapsto \deg_z h : \mathbb{P}^1 \to \{1, \dots, \deg h\}$ of h also canonically extends to an upper semicontinuous function $\mathcal{S} \mapsto \deg_{\mathcal{S}}(h) : \mathsf{P}^1 \to \{1, \dots, \deg h\}$ so that $\sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} \deg_{\mathcal{S}'}(h) = \deg h$ for every $\mathcal{S} \in \mathsf{P}^1$, and induces the pullback action h^* of h on the space of all Radon measures on P^1; letting $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at each point $\mathcal{S} \in \mathsf{P}^1$, $h^*\delta_{\mathcal{S}} = \sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} (\deg_{\mathcal{S}'}(h))\delta_{\mathcal{S}'}$ on P^1. For the details, see e.g. [1, 4, 9].

By the seminal Baker–Rumely [1], Chambert-Loir [7], and Favre–Rivera-Letelier [9], for every $f \in K(z)$ of degree $d > 1$, there is a unique f-equilibrium (or canonical) measure μ_f on P^1. This μ_f is a probability Radon measure on P^1, has no masses on polar subsets in P^1, and satisfies the f-balanced property

$$f^*\mu_f = d \cdot \mu_f \quad \text{on } \mathsf{P}^1$$

and in particular the f-invariance $f_*\mu_f = \mu_f$ on P^1, and is f-ergodic. Moreover, the equidistribution for iterated pullbacks of points

$$\lim_{n \to \infty} \frac{(f^n)^*\delta_{\mathcal{S}}}{d^n} = \mu_f \quad \text{weakly on } \mathsf{P}^1 \tag{1.1}$$

holds for every $\mathcal{S} \in \mathsf{P}^1$ outside the (classical) exceptional set of f

$$E(f) := \left\{ a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(a) < \infty \right\};$$

if $\operatorname{char} K = 0$, then $\#(E(f)) \le 2$. In general, $E(f)$ is at most countable and $\mathbb{P}^1 \setminus E(f)$ is dense in $\mathbb{P}^1$.

Our aim is to contribute to the study of a local a priori bound for the proximity of the dynamics of f on P^1 to a given non-constant $g \in K(z)$, which is closely related to the equidistribution towards μ_f of the roots in $\mathbb{P}^1$ of the algebraic equation $f^n = g$ as $n \to \infty$, for any non-constant $g \in K(z)$.

1.1. A local a priori bound of the dynamics of f. Recall that the absolute value $|\cdot|$ is said to be non-trivial if $|K| \not\subset \{0, 1\}$ and that the absolute value $|\cdot|$ is said to be non-archimedean if the *strong triangle inequality*

$$|z + w| \le \max\{|z|, |w|\} \quad \text{for any } z, w \in K$$

holds. The (normalized) chordal metric $[z, w]_{\mathbb{P}^1}$ on $\mathbb{P}^1 = K \cup \{\infty\}$ (the notation is adopted from Nevanlinna's and Tsuji's books [15, 21]) is in particular written as

$$[z, w]_{\mathbb{P}^1} = \frac{|z - w|}{\max\{1, |z|\} \max\{1, |w|\}} (\le 1)$$

for any $z, w \in K = \mathbb{P}^1 \setminus \{\infty\}$. The projective transformation group on $\mathbb{P}^1$ and the isometry group on $(\mathbb{P}^1, [z, w]_{\mathbb{P}^1})$ are identified with $\mathrm{PGL}(2, K)$ and

$\mathrm{PGL}(2, \mathcal{O}_K)$, respectively, where

$$\mathcal{O}_K := \{z \in K : |z| \le 1\}$$

is the ring of K-integers.

For a polynomial $p(z) = \sum_{j=0}^{N} a_j z^j \in \mathcal{O}_K[z]$, the polynomial $\widetilde{p}(\zeta) := \sum_{j=0}^{N} \widetilde{a_j}\zeta^j \in k[\zeta]$ is called the coefficient reduction of p modulo $\mathfrak{m}_K$, where $\mathfrak{m}_K := \{z \in K : |z| < 1\}$ is the unique maximal ideal of $\mathcal{O}_K$ and $k := \mathcal{O}_K/\mathfrak{m}_K$ is the residue field of K, and for each $a \in \mathcal{O}_K$, $\widetilde{a} \in k$ is the residue class of a modulo $\mathfrak{m}_K$. Similarly, the coefficient reduction $\widetilde{P}(\zeta_0, \zeta_1) \in k[\zeta_0, \zeta_1]$ modulo $\mathfrak{m}_K$ of a polynomial $P \in \mathcal{O}_K[z_0, z_1]$ is defined by reducing the coefficients of P modulo $\mathfrak{m}_K$.

Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$. Writing as $f(z) = F_1(1, z)/F_0(1, z)$, where $F = (F_0, F_1) \in (\mathcal{O}_K[z_0, z_1]_d)^2 \setminus (\mathfrak{m}_K[z_0, z_1]_d)^2$ is called a *minimal* non-degenerate homogeneous lift of f, the reduction $\widetilde{f} \in k(\zeta)$ of f modulo $\mathfrak{m}_K$ is defined by

$$(\widetilde{F_1}(1, \zeta)/H(1, \zeta))/(\widetilde{F_0}(1, \zeta)/H(1, \zeta)),$$

where $H(\zeta_0, \zeta_1) := \gcd(\widetilde{F_0}, \widetilde{F_1})(\neq 0)$ in $k[\zeta_0, \zeta_1]$; so $\deg \widetilde{f} \le \deg f$ in general (see e.g. Kawaguchi–Silverman [14, Definition 2]). We say f has a good reduction if $\deg \widetilde{f} = \deg f$, and say f has no potentially good reductions unless the conjugation $\gamma \circ f \circ \gamma^{-1}$ of f under some $\gamma \in \mathrm{PGL}(2, K)$ has a good reduction. It is known that f *has no potentially good reductions if and only if* $\mu_f(\{\mathcal{S}\}) = 0$ *for any* $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$ (see, e.g., [1, Corollary 10.33]).

Our principal result is the following local a priori bound of the dynamics of f for moving targets, under the condition that f has no potentially good reductions.

Theorem 1. *Let K be an algebraically closed field of arbitrary characteristic that is complete with respect to a non-trivial and non-archimedean absolute value. Then for every rational function $f \in K(z)$ on $\mathbb{P}^1$ of degree $d > 1$ having no potentially good reductions, every rational function $g \in K(z)$ on $\mathbb{P}^1$ of degree > 0, and every non-empty open subset D in $\mathbb{P}^1$, we have*

$$\lim_{n \to \infty} \frac{\sup_{w \in D} \log[f^n(w), g(w)]_{\mathbb{P}^1}}{d^n + \deg g} = 0. \tag{1.2}$$

A *global* a priori lower bound (4.2) below, or equivalently the equality (1.2) for $D = \mathbb{P}^1$, holds no matter whether f has no potentially good reductions, and Theorem 1 localizes this global one. The argument in the proof of Theorem 1 is similar to those in Buff–Gauthier [6] and Gauthier [11], using a domination principle (Bedford–Taylor [3]; see also Bedford–Smillie [2, p. 77]) from pluripotential theory. We note that if $g \equiv a$

on $\mathbb{P}^1$ for some $a \in \mathbb{P}^1$, then the equality (1.2) in Theorem 1 still holds unless $a \in E(f)$.

1.2. Equidistribution towards μ_f for moving targets. The equality (1.2) is more simply written as $\lim_{n\to\infty}(\sup_{w\in D}\log[f^n(w), g(w)]_{\mathbb{P}^1})/d^n = 0$, but the $d^n + \deg g$ in (1.2) is also natural in that for every $g \in K(z)$ of degree > 0, whenever $f^n \not\equiv g$, the degree of the effective divisor $[f^n = g]$ on $\mathbb{P}^1$ defined by all the roots in $\mathbb{P}^1$ of the algebraic equation $f^n = g$ taking into account their multiplicities equals $d^n + \deg g$ (for more details on the root divisor $[f^n = g]$ on $\mathbb{P}^1$, see Subsection 2.3 below). For $n \gg 1$, this effective divisor $[f^n = g]$ on $\mathbb{P}^1$ is canonically regarded as a (purely atomic) positive Radon measure on P^1 and is written as

$$[f^n = g] = \sum_{a\in\mathbb{P}^1}(\mathrm{ord}_a[f^n = g])\delta_a \quad \text{on } \mathsf{P}^1,$$

so that the averaged $[f^n = g]/(d^n + \deg g)$ is (regarded as) a probability Radon measure on P^1.

The following equidistribution towards μ_f for moving targets is an application of Theorem 1, and partly answers the question posed by Favre–Rivera-Letelier [9, after Théorème B].

Theorem 2. *Let K be an algebraically closed field of arbitrary characteristic that is complete with respect to a non-trivial and non-archimedean absolute value. Then for every $f \in K(z)$ of degree $d > 1$ having no potentially good reductions and every $g \in K(z)$ of degree > 0, we have*

$$\lim_{n\to\infty}\frac{[f^n = g]}{d^n + \deg g} = \mu_f \quad \textit{weakly on } \mathsf{P}^1. \tag{1.3}$$

In [9, Théorème B], the authors established the weak convergence (1.3) in the case of $\mathrm{char}\, K = 0$ (even no matter whether f has no potentially good reductions) and asked about the situation in the $\mathrm{char}\, K > 0$ case. In Theorem 2, in the $\mathrm{char}\, K > 0$ case, the no potentially good reduction assumption can be relaxed but cannot be omitted (e.g., $f(z) = z + z^p$ and $g(z) = z$ where $p = \mathrm{char}\, K > 0$, as pointed out in [9, après Théorème B]). More specifically, the difference between the proofs of Theorem 2 and Favre–Rivera-Letelier's [9, Théorème B] is caused by the fact that when $\mathrm{char}\, K > 0$, no geometric structure theorems are known on *quasiperiodicity domains*, which are subsets of Berkovich *domaines singuliers* (appearing as V in the proof of Theorem 1).

1.3. Value distribution of the sequence of the first order derivatives of iterated polynomials. By an argument similar to that in the proof of Theorem 2, we also show the following, based on the computation in [18] (for the $\mathbb{C}$ case, see [18, 12]).

Theorem 3. *Let K be an algebraically closed field of characteristic 0 that is complete with respect to a non-trivial and non-archimedean absolute value, and let $f \in K[z]$ be a polynomial of degree $d > 1$ having no potentially good reductions. Then for every $a \in K$,*

$$\lim_{n\to\infty} \frac{((f^n)')^*\delta_a}{d^n - 1} = \mu_f \quad \textit{weakly on } \mathsf{P}^1. \tag{1.4}$$

In Theorem 3, the value $a = \infty \in \mathbb{P}^1$ is excluded since it is clear that for every $n \in \mathbb{N}$, $((f^n)')^*\delta_\infty/(d^n - 1) = \delta_\infty (\neq \mu_f)$ on P^1. It does not seem to be known whether we could remove the assumption that f has no potentially good reductions in Theorem 3 (assuming $a \neq 0$). In [19], the higher order generalization of Theorem 3 is established, by a more involved argument.

1.4. Organization of the article. In Section 2, we recall background on the topology, potential theory, and dynamics on the Berkovich projective line. In Section 3, we show a lemma, which plays a key role in the proof of Theorems 1 and 3. In Sections 4, 5, and 6, we show Theorems 1, 2, and 3, respectively.

2. Background

2.1. Berkovich projective line P^1. For the full generality of Berkovich analytic spaces, see [5], and for the details on P^1, see [1, 9]. As a set, the Berkovich affine line $\mathsf{A}^1 = \mathsf{A}^1(K)$ is the set of all multiplicative seminorms on $K[z]$ which restrict to $|\cdot|$ on $K(\subset K[z]$ naturally). We write an element of A^1 like $\mathcal{S}$, and denote it by $[\,\cdot\,]_{\mathcal{S}}$ as a multiplicative seminorm on $K[z]$. Under this convention, A^1 is equipped with the weakest topology (the Gelfand topology) such that for any $\phi \in K[z]$, the function $\mathsf{A}^1 \ni \mathcal{S} \mapsto [\phi]_{\mathcal{S}} \in \mathbb{R}_{\geq 0}$ is continuous, and then A^1 is a locally compact, uniquely arcwise connected, and Hausdorff topological space.

A subset B in K is called a *K-closed disk* if

$$B = \{z \in K : |z - a| \leq r\} \quad \text{for some } a \in K \text{ and some } r \geq 0.$$

For any K-closed disks B, B', *if $B \cap B' \neq \emptyset$, then either $B \subset B'$ or $B' \subset B$* by the strong triangle inequality. The Berkovich representation [5] asserts that *any element $\mathcal{S} \in \mathsf{A}^1$ is induced by a non-increasing and nesting sequence (B_n) of K-closed disks B_n in that*

$$[\phi]_{\mathcal{S}} = \inf_{n\in\mathbb{N}} \sup_{z\in B_n} |\phi(z)| \quad \text{for any } \phi \in K[z].$$

In particular, each point in $K = \mathbb{P}^1 \setminus \{\infty\}$ or, more generally, each K-closed disk is regarded as an element of A^1; the *Gauss* (*or canonical*) *point* $\mathcal{S}_{\text{can}} \in \mathsf{A}^1 \setminus K$ is represented by the K-closed disk $\mathcal{O}_K$.

We will need some details on the topology of the Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$, so let us introduce P^1 as an "$\mathbb{R}$-"tree in the sense of

Jonsson [13, Definition 2.2] as follows. Any $[\,\cdot\,]_{\mathcal{S}} \in \mathsf{A}^1$ extends to the function $K(z) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for any $\phi = \phi_1/\phi_2 \in K(z)$ where $\phi_1, \phi_2 \in K[z]$ are coprime, we have $[\phi]_{\mathcal{S}} = [\phi_1]_{\mathcal{S}}/[\phi_2]_{\mathcal{S}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, and we also regard $\infty \in \mathbb{P}^1$ as the function $[\,\cdot\,]_\infty : K(z) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for every $\phi \in K(z)$, $[\phi]_\infty = |\phi(\infty)| \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. As a set, we define $\mathsf{P}^1 := \mathsf{A}^1 \cup \{\infty\}$, which is also equipped with a (partial) order $\leq_\infty$ so that for any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, $\mathcal{S} \leq_\infty \mathcal{S}'$ if and only if $[\,\cdot\,]_{\mathcal{S}} \leq_\infty [\,\cdot\,]_{\mathcal{S}'}$ on $K[z]$. For any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, if $\mathcal{S} \leq_\infty \mathcal{S}'$, then we first set $[\mathcal{S}, \mathcal{S}'] = [\mathcal{S}', \mathcal{S}] := \{\mathcal{S}'' \in \mathsf{P}^1 : \mathcal{S} \leq_\infty \mathcal{S}'' \leq_\infty \mathcal{S}'\}$, and in general, there is a unique point, say, $\mathcal{S} \wedge_\infty \mathcal{S}' \in \mathsf{P}^1$ such that $[\mathcal{S}, \infty] \cap [\mathcal{S}', \infty] = [\mathcal{S} \wedge_\infty \mathcal{S}', \infty]$, and we set

$$[\mathcal{S}, \mathcal{S}'] := [\mathcal{S}, \mathcal{S} \wedge_\infty \mathcal{S}'] \cup [\mathcal{S} \wedge_\infty \mathcal{S}', \mathcal{S}']$$

and call it the interval between $\mathcal{S}, \mathcal{S}'$. For any $\mathcal{S} \in \mathsf{P}^1$, let us introduce the coset $T_{\mathcal{S}}\mathsf{P}^1 := (\mathsf{P}^1 \setminus \{\mathcal{S}\})/\sim$, where for every $\mathcal{S}', \mathcal{S}'' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$, we say $\mathcal{S}' \sim \mathcal{S}''$ if $[\mathcal{S}, \mathcal{S}'] \cap [\mathcal{S}, \mathcal{S}''] \neq \{\mathcal{S}\}$ or equivalently if $[\mathcal{S}, \mathcal{S}'] \cap [\mathcal{S}, \mathcal{S}''] = [\mathcal{S}, \mathcal{S}''']$ for some (unique) $\mathcal{S}''' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$. An element of $T_{\mathcal{S}}\mathsf{P}^1$ represented by an element $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$ is denoted by $\overrightarrow{\mathcal{S}\mathcal{S}'}$. We call an element of $T_{\mathcal{S}}\mathsf{P}^1$ a direction of P^1 at $\mathcal{S}$ and write it like $\boldsymbol{v}$, and also denote it by $U(\boldsymbol{v})$ as a subset in $\mathsf{P}^1 \setminus \{\mathcal{S}\}$; for every $a \in \mathbb{P}^1$, $\#(T_a\mathsf{P}^1) = 1$.

We equip P^1 with the weak (or *observer*) topology having the quasi-open basis $\{U(\boldsymbol{v}) : \mathcal{S} \in \mathsf{P}^1, \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\}$ or equivalently having the open basis consisting of all *simple domains* in P^1 defined below. This topological space P^1 coincides with the one-point compactification of A^1, both $\mathbb{P}^1$ and $\mathsf{P}^1 \setminus \mathbb{P}^1$ are dense in P^1, the set $\{U(\boldsymbol{v}) : \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\} \cong T_{\mathcal{S}}\mathsf{P}^1$ coincides with the set of all components of $\mathsf{P}^1 \setminus \{\mathcal{S}\}$, and for any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, $[\mathcal{S}, \mathcal{S}']$ is the unique arc in P^1 between $\mathcal{S}$ and $\mathcal{S}'$. Here, a domain in P^1 is a topological domain in P^1, that is, a non-empty, connected, and open subset in P^1. The following two kinds of domains in P^1 are important; a non-empty subset in P^1 written as the intersection of at most finitely many elements of $\{U(\boldsymbol{v}) : \mathcal{S} \in \mathsf{P}^1$ satisfying $\#T_{\mathcal{S}}\mathsf{P}^1 > 1, \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\}$ is a domain in P^1, and is called a *simple domain* in P^1 following Baker–Rumely [1]. For a non-empty open subset U in P^1, a connected component of U is a domain in P^1, and is called a component of U for simplicity. We show the following.

Lemma 2.1. *For any domains U, V in P^1, if $U \cap V = \emptyset$ and $(\partial U) \cap \partial V \neq \emptyset$, then $(\partial U) \cap \partial V$ is a singleton, say, $\{\mathcal{S}_0\}$ in $\mathsf{P}^1 \setminus \mathbb{P}^1$, and moreover, there are distinct $\boldsymbol{u}, \boldsymbol{v} \in T_{\mathcal{S}_0}\mathsf{P}^1$ such that $U \subset U(\boldsymbol{u})$ and $V \subset U(\boldsymbol{v})$.*

Proof. Let U, V be domains in P^1 satisfying $U \cap V = \emptyset$ and $(\partial U) \cap \partial V \neq \emptyset$, and fix $\mathcal{S}_0 \in (\partial U) \cap \partial V$. Then $U \subset U(\boldsymbol{u})$ and $V \subset U(\boldsymbol{v})$ for some $\boldsymbol{u}, \boldsymbol{v} \in T_{\mathcal{S}_0}\mathsf{P}^1$ since both U and V are connected. We claim that $\boldsymbol{u} \neq \boldsymbol{v}$ or equivalently that $U(\boldsymbol{u}) \cap U(\boldsymbol{v}) = \emptyset$; for, otherwise, $\boldsymbol{u} = \boldsymbol{v}$, and then since $\boldsymbol{u} = \overrightarrow{\mathcal{S}_0\mathcal{S}}$ and $\boldsymbol{v} = \overrightarrow{\mathcal{S}_0\mathcal{S}'}$ for some $\mathcal{S} \in U$ and some $\mathcal{S}' \in V$, recalling the connectedness of U, V and the definition of directions, we must have $\emptyset \neq ([\mathcal{S}_0, \mathcal{S}] \cap [\mathcal{S}_0, \mathcal{S}']) \setminus \{\mathcal{S}_0\} \subset U \cap V$. This contradicts $U \cap V = \emptyset$. Once the claim is at our disposal, the proof is complete since $(\partial U) \setminus \{\mathcal{S}_0\} \subset U(\boldsymbol{u})$ and $(\partial V) \setminus \{\mathcal{S}_0\} \subset U(\boldsymbol{v})$. $\square$

2.2. Potential theory on P^1. For a potential theory on P^1 including the fully general study of harmonic analysis on P^1, i.e., harmonic/subharmonic functions and the Laplacians on open subsets in P^1, see Baker–Rumely [1, §7 and §8, and §5] (and also Thuillier [20] on more general Berkovich curves).

Let $\|(z_0, z_1)\| = \max\{|z_0|, |z_1|\}$ be the maximal norm on K^2, $(z_0, z_1) \wedge (w_0, w_1) = z_0 w_1 - z_1 w_0$ be the exterior product of K^2 (which would not be confused with $\mathcal{S} \wedge \mathcal{S}' \in \mathsf{P}^1$ for $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$), and $\pi : K^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ be the canonical projection so that $\pi(1, z) = z$ for every $z \in K$ and that $\pi(0,1) = \infty$, following the convention in the book [10]. The (normalized) chordal metric $[z, w]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ is defined as

$$[z, w]_{\mathbb{P}^1} := \frac{|Z \wedge W|}{\|Z\| \cdot \|W\|} \leq 1, \quad z, w \in \mathbb{P}^1, \tag{2.1}$$

where $Z \in \pi^{-1}(z), W \in \pi^{-1}(w)$; the topology on $(\mathbb{P}^1, [z, w]_{\mathbb{P}^1})$ coincides with the relative topology of $\mathbb{P}^1$ as a subset of P^1.

This chordal metric $[z, w]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ extends to an upper semicontinuous and separately continuous function $(\mathcal{S}, \mathcal{S}') \mapsto [\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$, which still satisfies $0 \leq [\mathcal{S}, \mathcal{S}']_{\mathrm{can}} \leq 1$ on $\mathsf{P}^1 \times \mathsf{P}^1$ and is invariant under the (extended) $\mathrm{PGL}(2, \mathcal{O}_K)$-action to P^1. This function $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1 \times \mathsf{P}^1$ is called the generalized *Hsia kernel* on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ (see [1, §4] and [9, §4] for more details. The notations $[z, w]_{\mathbb{P}^1}$ and $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ would not be confused with that of an interval $[\mathcal{S}, \mathcal{S}'] \subset \mathsf{P}^1$). In particular, the absolute value $|\cdot| = [\,\cdot\,, 0]_{\mathbb{P}^1} / [\,\cdot\,, \infty]_{\mathbb{P}^1}$ on K extends to the function

$$|\cdot| = [\,\cdot\,, 0]_{\mathrm{can}} / [\,\cdot\,, \infty]_{\mathrm{can}} : \mathsf{P}^1 \to \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

We say a function $g : \mathsf{P}^1 \to \mathbb{R} \cup \{-\infty\}$ is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic if there is a probability Radon measure μ_g on P^1 such that

$$g = \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}']_{\mathrm{can}} \mu_g(\mathcal{S}') + \mathrm{const.} \quad \text{on } \mathsf{P}^1,$$

which belongs to $\mathrm{BDV}(\mathsf{P}^1)$ and is not only upper semicontinuous on P^1 but also continuous on any closed interval in P^1; then

$$\Delta g = \mu_g - \delta_{\mathcal{S}_{\mathrm{can}}} \quad \text{on } \mathsf{P}^1$$

(see Favre–Rivera-Letelier [9, §2.4], and also [1, §5.8 and §6.3] for more details including that on the class $\mathrm{BDV}(\mathsf{P}^1)$). Here and below

$$\Delta = \Delta_{\mathsf{P}^1}$$

is the Laplacian on P^1 (in [1] the opposite sign convention on Δ is adopted). For every $\mathcal{S}' \in \mathsf{P}^1$, $\log[\cdot, \mathcal{S}']_{\mathrm{can}}$ on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic; indeed,

$$\log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} = \int_{\mathsf{P}^1} \log[\mathcal{S}, \cdot]_{\mathrm{can}} \delta_{\mathcal{S}'} \quad \text{on } \mathsf{P}^1.$$

In particular, the function $\log[\cdot, \infty]_{\mathrm{can}} = -\log\max\{1, |\cdot|\}$ on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic and satisfies

$$\Delta(\log\max\{1, |\cdot|\}) = \delta_{\mathcal{S}_{\mathrm{can}}} - \delta_\infty \quad \text{on } \mathsf{P}^1. \tag{2.2}$$

Instead of giving the definition of a harmonic/subharmonic function on an open subset in P^1, we recall the facts that

- every harmonic function h on a simple domain W in P^1 extends to a continuous function on $\overline{W}$, and coincides with *the Poisson integral* $\mathcal{S} \mapsto \int_{\partial W} (h|(\partial W))(\,\cdot\,)\mu_{\mathcal{S},W}(\,\cdot\,)$ *on* W, where the family $(\mu_{\mathcal{S},W})_{\mathcal{S}\in W}$ of probability Radon measures on P^1 supported on ∂W is the Poisson–Jensen (or harmonic) measure associated to W (for the details, see [1, §7.3], [20, §3]),
- *an* $\mathbb{R} \cup \{-\infty\}$*-valued function* ϕ *on an open subset* U *in* P^1 *is subharmonic if and only if it is domination subharmonic* in that ϕ is upper semicontinuous on U and $\not\equiv -\infty$ on each component of U, and $\phi \le h$ on W for every simple domain $W \Subset U$ and every harmonic function h on W satisfying $\phi \le h$ on ∂W (for the details, see [1, §8.2]); then indeed $\phi(\,\cdot\,) \le \int_{\partial W} (\phi|(\partial W))\mu_{\cdot,W}$ on W,

and that a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function g on P^1 is subharmonic on $\mathsf{P}^1 \setminus \{\mathcal{S}_{\mathrm{can}}\}$, and then the function $g + \log\max\{1, |\cdot|\}$ on P^1 is subharmonic on A^1.

We also recall the following (non-archimedean) *version of* (Hörmander's version of) *Hartogs's lemma* for a (uniformly upper bounded) sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1.

Theorem 2.2 ([9, Proposition 2.18], [1, Proposition 8.57])**.** *Let* (g_n) *be a sequence of* $\delta_{\mathcal{S}_{\mathrm{can}}}$*-subharmonic functions on* P^1 *and suppose that* (g_n) *is uniformly bounded from above on* P^1*. Then unless* $\lim_{n\to\infty} g_n = -\infty$ *uniformly on* P^1*, there are a sequence* (n_j) *in* $\mathbb{N}$ *tending to* $+\infty$ *as* $j \to \infty$ *and a* $\delta_{\mathcal{S}_{\mathrm{can}}}$*-subharmonic function* ϕ_∞ *on* P^1 *such that*

(1) $\lim_{j\to\infty} g_{n_j} = \phi_\infty$ *(pointwisely) on* $\mathsf{P}^1 \setminus \mathbb{P}^1$ *and that*
(2) $\limsup_{j\to\infty} \sup_K (g_{n_j} - g) \le \sup_K (\phi_\infty - g)$ *for every compact subset* K *in* P^1 *and every continuous function* g *on* K*.*

2.3. Dynamics of rational functions on P^1. Let $h \in K(z)$ be of degree > 0. A non-degenerate homogeneous lift of h is an ordered pair $H = (H_0, H_1) \in (K[z_0, z_1]_{\deg h})^2$, which is unique up to multiplication in K^*, such that $\pi \circ H = h \circ \pi$ on $K^2 \setminus \{(0,0)\}$ (and that $H^{-1}(0,0) = \{(0,0)\}$).

From now on, such an H is called a lift of h, for simplicity. Then the function

$$T_H := \log \|H\| - (\deg h) \cdot \log \| \cdot \|$$

on $K^2 \setminus \{0\}$ descends to $\mathbb{P}^1$ and in turn extends continuously to P^1 so that

$$\Delta T_H = h^* \delta_{\mathcal{S}_{\mathrm{can}}} - (\deg h) \cdot \delta_{\mathcal{S}_{\mathrm{can}}} \quad \text{on } \mathsf{P}^1$$

(see, e.g., [16, Definition 2.8]), and $T_H/(\deg h)$ on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic.

Let $f \in K(z)$ be of degree $d > 1$, and fix a lift F of f. Then for every $n \in \mathbb{N}$, F^n is a lift of f^n and $\deg(f^n) = d^n$. There is the uniform limit

$$g_F := \lim_{n \to \infty} \frac{T_{F^n}}{d^n} \quad \text{on } \mathsf{P}^1, \tag{2.3}$$

which is continuous on P^1, is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic on P^1, and satisfies

$$\Delta g_F = \mu_f - \delta_{\mathcal{S}_{\mathrm{can}}} \quad \text{on } \mathsf{P}^1 \tag{2.4}$$

(see [1, §10], [9, §6.1]). We call g_F the *dynamical Green function of F on* P^1, and note that for every $n \in \mathbb{N}$, $g_{F^n} = g_F$ and $\mu_{f^n} = \mu_f$ on P^1.

Fix also $g \in K(z)$ of degree > 0, and fix a lift G of g. For every $n \in \mathbb{N}$, if $f^n \not\equiv g$, then there is a sequence $(q_j)_{j=1}^{d^n + \deg g}$ in $K^2 \setminus \{(0,0)\}$ such that the homogeneous polynomial $F^n \wedge G \in K[z_0, z_1]_{d^n + \deg g}$ factors as

$$F^n(Z) \wedge G(Z) = \prod_{j=1}^{d^n + \deg g} (Z \wedge q_j), \quad Z \in K^2,$$

and the root divisor $[f^n = g]$ is well defined as the effective divisor on $\mathbb{P}^1$ of degree $d^n + \deg g$ such that for every $w \in \mathbb{P}^1$,

$$\mathrm{ord}_w [f^n = g] = \#\{j \in \{1, \dots, d^n + \deg g\} : \pi(q_j) = w\}.$$

Moreover, for every $n \in \mathbb{N}$, if $f^n \not\equiv g$, then the function $z \mapsto [f^n(z), g(z)]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ extends continuously to a function

$$\mathcal{S} \mapsto [f^n, g]_{\mathrm{can}}(\mathcal{S})$$

on P^1 so that the function $(\log [f^n, g]_{\mathrm{can}}(\,\cdot\,) + T_{F^n} + T_G)/(d^n + \deg g)$ on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic and satisfies both

$$\Delta \frac{\log [f^n, g]_{\mathrm{can}}(\,\cdot\,) + T_{F^n} + T_G}{d^n + \deg g} = \frac{[f^n = g]}{d^n + \deg g} - \delta_{\mathcal{S}_{\mathrm{can}}} \quad \text{on } \mathsf{P}^1 \tag{2.5}$$

and $0 \le [f^n, g]_{\mathrm{can}}(\,\cdot\,) \le 1$ on P^1 ([16, Proposition 2.9 and Remark 2.10]); the function $[f^n, g]_{\mathrm{can}}(\,\cdot\,)$ on P^1 does not always coincide with the evaluation function $\mathcal{S} \mapsto [\mathcal{S}', \mathcal{S}'']_{\mathrm{can}}$ at $(\mathcal{S}', \mathcal{S}'') = (f^n(\mathcal{S}), g(\mathcal{S})) \in (\mathsf{P}^1)^2$.

2.4. Fatou–Julia decomposition of P^1. Let $f\in K(z)$ be of degree $d>1$. The Berkovich Julia set $\mathsf{J}(f)$ of f is defined by $\operatorname{supp}\mu_f$, so is non-empty and compact in P^1, and satisfies $f^{-1}(\mathsf{J}(f))=\mathsf{J}(f)$, and the Berkovich Fatou set $\mathsf{F}(f)$ of f is defined by $\mathsf{P}^1\setminus\mathsf{J}(f)$, and also satisfies $f^{-1}(\mathsf{F}(f))=\mathsf{F}(f)$.

Lemma 2.3. *Let D be an open subset in P^1 such that, for some sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j\to\infty$ and some $g\in K(z)$, $\lim_{j\to\infty}f^{n_j}=g$ uniformly, as mappings $D\cap\mathbb{P}^1\to(\mathbb{P}^1,[z,w])$. Then $D\subset\mathsf{F}(f)$.*

Proof. Suppose to the contrary that there is $\mathcal{S}_0\in D\cap\mathsf{J}(f)$. Then for any open neighborhood $D'\subset D$ of $\mathcal{S}_0$, by (1.1) applied to each $\mathcal{S}\in\mathbb{P}^1\setminus E(f)$, $\liminf_{j\to\infty}f^{n_j}(D'\cap\mathbb{P}^1)(=\bigcap_{N\in\mathbb{N}}\bigcup_{j\geq N}f^{n_j}(D'\cap\mathbb{P}^1))$ contains the dense subset $\mathbb{P}^1\setminus E(f)$ in $\mathbb{P}^1$, and in turn the closure of $g(D'\cap\mathbb{P}^1)$ in $\mathbb{P}^1$ contains $\mathbb{P}^1\setminus E(f)$ under the uniform convergence assumption. This is impossible since $\mathbb{P}^1\setminus g(D'\cap\mathbb{P}^1)$ contains a non-empty open subset in $\mathbb{P}^1$ if D' is small enough. $\square$

We call a component of $\mathsf{F}(f)$ a Berkovich Fatou component of f. We note that f maps a Berkovich Fatou component V of f properly to a Berkovich Fatou component U of f (so in particular $f(\partial V)=\partial U$), and that the preimage under f of a Berkovich Fatou component of f is the union of (at most d) Berkovich Fatou components of f. A Berkovich Fatou component W of f is said to be cyclic (under f) if $f^p(W)=W$ for some $p\in\mathbb{N}$, and then the minimal such p is called the exact period of W (under f). A cyclic Berkovich Fatou component W of f having the exact period, say, $p\in\mathbb{N}$ is called a Berkovich *domaine singulier* of f if $f^p:W\to W$ is injective (following Fatou [8, §28]); then in particular $f^{-1}(W)\neq W$ since $d>1$.

3. A key lemma

Lemma 3.1. *Let $f\in K(z)$ be of degree $d>1$ and have no potentially good reductions. Then*

(i) *for any Berkovich Fatou component U of f, we have $\partial U\neq\mathsf{J}(f)$ if $f^{-1}(U)\neq U$, and moreover,*
(ii) *for every cyclic Berkovich Fatou component W of f satisfying $f^{-1}(W)\neq W$, we have $\mu_f(\partial U)=0$ for every component U of $\bigcup_{n\in\mathbb{N}\cup\{0\}}f^{-n}(W)$.*

Proof. (i). Let U be a Berkovich Fatou component of f, and suppose to the contrary both $f^{-1}(U)\neq U$ and $\partial U=\mathsf{J}(f)$. Pick a component V of $f^{-1}(U)\setminus U$, which is also a Berkovich Fatou component of f. Then $U\cap V=\emptyset$ and $\partial V\subset\mathsf{J}(f)$, the latter in which yields $(\partial U)\cap\partial V=\mathsf{J}(f)\cap\partial V=\partial V\neq\emptyset$. Hence by Lemma 2.1, there is $\mathcal{S}_0\in\mathsf{P}^1\setminus\mathbb{P}^1$ such that

$$\partial V=(\partial U)\cap\partial V=\{\mathcal{S}_0\}\subset\partial U=f(\partial V)=\{f(\mathcal{S}_0)\},$$

and then $\operatorname{supp}\mu_f = \mathsf{J}(f) = \partial U = \{f(\mathcal{S}_0)\} = \{\mathcal{S}_0\}$. In particular, we must have $\mu_f(\{\mathcal{S}_0\}) = 1 > 0$, which contradicts the assumption that f has no potentially good reductions.

(ii). Pick a cyclic Berkovich Fatou component W of f having the exact period $p \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ and any distinct components U, V of $f^{-pn}(W)$, by Lemma 2.1, $(\partial U) \cap \partial V$ is either $\emptyset$ or a singleton in $\mathsf{P}^1 \setminus \mathbb{P}^1$, the latter of which is still a μ_f-null set under the assumption that f has no potentially good reductions. Hence for every $n \in \mathbb{N}$, also by the f-invariance of μ_f and $f^p(W) = W$, we compute

$$\begin{aligned}\mu_f(\partial W) = \mu_f(f^{-pn}(\partial W)) &= \sum_{U\colon \text{a component of } f^{-pn}(W)} \mu_f(\partial U) \\ &= \mu_f(\partial W) + \sum_{U\colon \text{a component of } f^{-pn}(W) \text{ other than } W} \mu_f(\partial U),\end{aligned}$$

which first concludes $\mu_f(\partial U) = 0$ for every component U of $f^{-pn}(W)$ other than W. In particular, $\mu_f(\bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-pn}(\partial W)) = \mu_f(\partial W)$, which with $f^p(W) = W$ and the f^p-ergodicity of $\mu_{f^p} = \mu_f$ yields $\mu_f(\partial W) \in \{0, 1\}$. If $f^{-1}(W) \neq W$, then there is a component U of $f^{-1}(W)$. By an argument similar to the above (involving Lemma 2.1), we also have $\mu_f((\partial U) \cap \partial W) = 0$ (under the assumption that f has no potentially good reductions). Suppose to the contrary that $\mu_f(\partial W) = 1$. Then $\mu_f(\partial U) = \mu_f((\partial U) \setminus \partial W) + \mu_f((\partial U) \cap \partial W) = 0 + 0 = 0$, and in turn by the f-balanced property of μ_f and $f(\partial U) = \partial W$, we indeed have $\mu_f(\partial W) \leq d \cdot \mu_f(\partial U) = 0$. This is a contradiction. $\square$

4. Proof of Theorem 1

Let $f \in K(z)$ be of degree $d > 1$. Suppose that there are $g \in K(z)$ of degree > 0 and a non-empty open subset D in $\mathbb{P}^1$ such that (1.2) in Theorem 1 does not hold, or equivalently, replacing D with some domain in P^1, suppose that there are $g \in K(z)$ of degree > 0, a domain D in P^1, and a sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j \to \infty$ such that a uniform negativity

$$\lim_{j\to\infty} \frac{\sup_{\mathcal{S}\in D} \log[f^{n_j}, g]_{\mathrm{can}}(\mathcal{S})}{d^{n_j} + \deg g} < 0 \tag{4.1}$$

on D holds. Then $D \subset \mathsf{F}(f)$ by Lemma 2.3.

Let U be the Berkovich Fatou component of f containing D. Since $\deg g > 0$, $g(D)$ is open in P^1 and, taking a subsequence of (n_j) if necessary, we have both $\lim_{j\to\infty}(n_{j+1} - n_j) = +\infty$ and

$$\lim_{j\to\infty} f^{n_{j+1}-n_j} = \mathrm{Id}_{g(D)\cap\mathbb{P}^1}$$

uniformly, as mappings $g(D) \cap \mathbb{P}^1 \to (\mathbb{P}^1, [z,w]_{\mathbb{P}^1})$ (see e.g. [17, Proof of Lemma 10.1 (i)] or [16, Proof of Lemma 4.7]). Then we also have $g(D) \subset \mathsf{F}(f)$ by Lemma 2.3, and in turn, the Berkovich Fatou component V of f containing $g(D)$ is cyclic under f and has the exact period, say, $p \in \mathbb{N}$, and there exists $N \in \mathbb{N} \cup \{0\}$ such that $f^{n_N}(U) = V$. Moreover, by Rivera-Letelier's counterpart of Fatou's classification of cyclic Berkovich Fatou components ([9, Proposition 2.16 and its *esquisse de démonstration*], see also [4, §9.3 and §9,4]), V is a Berkovich *domaine singulier* of f (so $f^{-1}(V) \neq V$).

We have the uniform upper bound $\sup_{n\in\mathbb{N}} \sup_{\mathcal{S}\in\mathsf{P}^1} \log[f^n, g]_{\mathrm{can}}(\mathcal{S}) \leq 0$ (from $[f^n, g]_{\mathrm{can}}(\,\cdot\,) \leq 1$). We also claim the global lower bound

$$\limsup_{j\to\infty} \frac{\sup_{\mathcal{S}\in\mathsf{P}^1} \log[f^{n_j}, g]_{\mathrm{can}}(\mathcal{S})}{d^{n_j} + \deg g} \geq 0 > -\infty; \tag{4.2}$$

for, by $\mathsf{J}(f) = \operatorname{supp} \mu_f \neq \emptyset$, there is a domain D' in P^1 such that $D' \cap \mathsf{J}(f) \neq \emptyset$ and that $\overline{g(D')} \neq \mathsf{P}^1$. Then by the density of $\mathbb{P}^1 \setminus E(f)$ in $\mathbb{P}^1$, there is a point $z_0 \in (\mathsf{P}^1 \setminus \overline{g(D')}) \cap (\mathbb{P}^1 \setminus E(f))$, so that by (1.1), we have $f^{-n_j}(z_0) \cap (D' \cap \mathbb{P}^1) \neq \emptyset$ for $j \gg 1$. In particular, $\liminf_{j\to\infty} \sup_{\mathcal{S}\in D'} [f^{n_j}, g]_{\mathrm{can}}(\mathcal{S}) \geq \inf_{\overline{g(D')}\cap\mathbb{P}^1} [z_0, \,\cdot\,]_{\mathbb{P}^1} > 0$, and the claim holds.

Hence recalling that for every $n \in \mathbb{N}$, $(\log[f^n, g]_{\mathrm{can}}(\,\cdot\,) + T_{F^n} + T_G)/(d^n + \deg g)$ is a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1, by the (uniform) convergence (2.3) and a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (see (1) in Theorem 2.2), taking a subsequence of (n_j) if necessary, there is a function $\phi : \mathsf{P}^1 \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$ such that

$$\lim_{j\to\infty} \frac{\log[f^{n_j}, g]_{\mathrm{can}}(\,\cdot\,)}{d^{n_j} + \deg g} = \phi \quad \text{(pointwisely) on } \mathsf{P}^1 \setminus \mathbb{P}^1$$

and that $\phi + g_F$ is a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1 (see [9, §3.4] for a similar computation). Then the function $\phi = (\phi + g_F) - g_F$ is upper semicontinuous on P^1, so $\{\phi < 0\}$ is open in P^1.

Under the uniform negativity assumption (4.1), we have

$$U \cap \{\phi < 0\} \supset D \setminus \mathbb{P}^1 \neq \emptyset.$$

We claim that $\{\phi < 0\} \subset \mathsf{F}(f)$; for, if there exists $\mathcal{S}_0 \in \mathsf{J}(f) \cap \{\phi < 0\}$, then for any open neighborhood $D' \Subset \{\phi < 0\}$ of $\mathcal{S}_0$, by the uniform convergence (2.3), a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (see (2) in Theorem 2.2), and the upper semicontinuity of ϕ, we must have

$$\begin{aligned}\limsup_{j\to\infty}\sup_{D'}\frac{\log[f^{n_j},g]_{\mathrm{can}}(\,\cdot\,)}{d^{n_j}+\deg g}\\ &=\limsup_{j\to\infty}\sup_{D'}\left(\frac{\log[f^n,g]_{\mathrm{can}}(\,\cdot\,)+T_{F^n}+T_G}{d^n+\deg g}-\frac{T_{F^n}+T_G}{d^n+\deg g}\right)\\ &\le\sup_{D'}\bigl((\phi+g_F)-g_F\bigr)=\sup_{D'}\phi<0.\end{aligned}$$

Then $D'\subset\mathsf{F}(f)$ by Lemma 2.3, which contradicts $\mathcal{S}_0\in\mathsf{J}(f)\cap D'$. Hence the claim holds, and in particular

$$\phi\equiv 0\quad\text{on }\mathsf{J}(f),\ \text{so on }\partial U.$$

From now on, we assume in addition that $\infty\in f^{-1}(U)\setminus U(\subset\mathsf{F}(f)$, so in particular that $U\Subset\mathsf{P}^1\setminus\{\infty\})$ by some $\mathrm{PGL}(2,\mathcal{O}_K)$-conjugation of f,g simultaneously if necessary, without loss of generality. Set

$$\psi:=\begin{cases}\phi & \text{on }U\\ 0 & \text{on }\mathsf{P}^1\setminus U\end{cases}:\mathsf{P}^1\to\mathbb{R}_{\le 0}\cup\{-\infty\},$$

so in particular $\phi\le\psi$ on P^1. We claim that the function

$$\widetilde{\psi}:=\psi+g_F+\log\max\{1,|\cdot|\}:\mathsf{P}^1\to\mathbb{R}\cup\{\pm\infty\}$$

is domination subharmonic on $\mathsf{A}^1=\mathsf{P}^1\setminus\{\infty\}$ (for the domination subharmonicity, which is equivalent to the subharmonicity, of an $\mathbb{R}\cup\{-\infty\}$-valued function on an open subset in P^1, see Subsection 2.2); for, $\widetilde{\psi}\not\equiv-\infty$ on A^1, and moreover, $\widetilde{\psi}$ is upper semicontinuous on A^1 since the function

$$(\phi+g_F)+\log\max\{1,|\cdot|\}$$

is subharmonic so upper semicontinuous on A^1, $g_F+\log\max\{1,|\cdot|\}$ is continuous on A^1, and $\psi=0=\phi$ on ∂U. The function $\widetilde{\psi}$ is subharmonic on $\mathsf{A}^1\setminus\partial U$ since so are the functions $(\phi+g_F)+\log\max\{1,|\cdot|\}$ and $g_F+\log\max\{1,|\cdot|\}$ on $U\subset\mathsf{A}^1$ and on $\mathsf{A}^1\setminus\overline{U}$, respectively. Pick any simple domain $W\Subset\mathsf{A}^1$ and any harmonic function h on W, which continuously extends to $\overline{W}$, such that $\widetilde{\psi}\le h$ on ∂W. It remains to show that

$$M:=\max_{\overline{W}}(\widetilde{\psi}-h)\le 0,$$

where the existence of M is by the upper semicontinuity of $\widetilde{\psi}-h$ on A^1. First, if $(\widetilde{\psi}-h)(\mathcal{S}_0)=M$ at some $\mathcal{S}_0\in W\setminus\partial U$, then for any simple domain $W'\Subset W\setminus\partial U$ containing $\mathcal{S}_0$, we have $M=(\widetilde{\psi}-h)(\mathcal{S}_0)\le\int_{\partial(W')}(\widetilde{\psi}-h)\mu_{\mathcal{S}_0,W'}\le M$ by the domination subharmonicity of $\widetilde{\psi}$ on $\mathsf{A}^1\setminus\partial U$ and the harmonicity of h on W (see Subsection 2.2). In particular, $(\widetilde{\psi}-h)|(\partial(W')) \equiv M$, and in turn, increasing W' if necessary, $\widetilde{\psi}-h$ attains M at some point in $(\partial W)\cup\partial U$ by the upper semicontinuity of $\widetilde{\psi}-h$ on A^1. Next, if $\widetilde{\psi}-h$ attains M at some $\mathcal{S}_0 \in W\cap\partial U$, then we have

$$\begin{aligned}M &= (\widetilde{\psi}-h)(\mathcal{S}_0) = (\phi+g_F+\log\max\{1,|\cdot|\}-h)(\mathcal{S}_0)\\ &\le \int_{\partial W}(\phi+g_F+\log\max\{1,|\cdot|\}-h)\mu_{\mathcal{S}_0,W} \le \int_{\partial W}(\widetilde{\psi}-h)\mu_{\mathcal{S}_0,W} \le M\end{aligned}$$

by $\psi=0=\phi$ on ∂U, the domination subharmonicity of the function $(\phi+g_F)+\log\max\{1,|\cdot|\}$ on A^1 and the harmonicity of h on W (see Subsection 2.2), and $\phi\le\psi$ on P^1. In particular, $(\widetilde{\psi}-h)|(\partial W)\equiv M$. Consequently, in any case, $\widetilde{\psi}-h$ attains its maximum M on $\overline{W}$ at some point in ∂W, where $\widetilde{\psi}-h\le 0$. Hence the claim holds.

Once the claim is at our disposal, by the subharmonicity of $\widetilde{\psi}$ on A^1 and the identity $\widetilde{\psi}\equiv g_F+\log\max\{1,|\cdot|\}$ near $\infty\in\mathsf{F}(f)$, $\Delta\psi$ exists on P^1 and

$$\Delta\psi+\mu_f=\Delta\widetilde{\psi}+\delta_\infty$$

is a probability Radon measure on P^1. By the definition of ψ, we have $\Delta\psi=0$ on $\mathsf{P}^1\setminus\overline{U}$, or equivalently

$$\Delta\psi+\mu_f=\mu_f \quad \text{on } \mathsf{P}^1\setminus\overline{U}.$$

We also recall that $U\subset\mathsf{P}^1\setminus\mathsf{J}(f)=\mathsf{P}^1\setminus(\operatorname{supp}\mu_f)$.

Now suppose to the contrary that f has no potentially good reductions. Then $\mu_f(\partial U)=0$ by $f^{-1}(V)\neq V$ and Lemma 3.1 (ii), so we have

$$\begin{aligned}(\Delta\psi+\mu_f)(\overline{U}) &= 1-(\Delta\psi+\mu_f)(\mathsf{P}^1\setminus\overline{U}) = 1-\mu_f(\mathsf{P}^1\setminus\overline{U})\\ &= \mu_f(\overline{U})=\mu_f(U)+\mu_f(\partial U)=0.\end{aligned}$$

Consequently, $\Delta\psi+\mu_f=\mu_f$ on P^1, or equivalently, $\Delta\psi=0$ on P^1.

Hence ψ must be constant on $\mathsf{P}^1\setminus\mathbb{P}^1$, which with $\psi=0$ on $\mathsf{P}^1\setminus U$ and $\mathsf{P}^1\setminus U\supset f^{-1}(U)\setminus U\neq\emptyset$ yields $\psi\equiv 0$ on $\mathsf{P}^1\setminus\mathbb{P}^1$, so $\phi\equiv 0$ on $U\setminus\mathbb{P}^1$. This contradicts $U\cap\{\phi<0\}\neq\emptyset$ since both U and $\{\phi<0\}$ are open. □

5. Proof of Theorem 2

Let $f\in K(z)$ be of degree $d>1$ and $g\in K(z)$ be of degree >0, and fix a lift F of f. By (2.3), (2.4), (2.5), and the continuity of the Laplacian Δ, the equidistribution (1.3) in Theorem 2 would follow from

$$\lim_{n\to\infty}\frac{\log[f^n,g]_{\mathrm{can}}(\,\cdot\,)}{d^n+\deg g}=0 \quad \text{(pointwisely) on } \mathsf{P}^1\setminus\mathbb{P}^1. \tag{1.3'}$$

Unless (1.3′) holds, by an argument similar to that in Section 4 involving a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (see (1) in Theorem 2.2), there exist a sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j \to \infty$ and a function $\phi : \mathsf{P}^1 \to \mathbb{R}_{\le 0} \cup \{-\infty\}$ such that

$$\lim_{j\to\infty} \frac{\log[f^{n_j}, g]_{\mathrm{can}}(\,\cdot\,)}{d^{n_j} + \deg g} = \phi \quad \text{(pointwisely) on } \mathsf{P}^1 \setminus \mathbb{P}^1$$

and that $\phi + g_F$ is a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1. Then ϕ is upper semicontinuous on P^1, and $\{\phi < 0\}$ is non-empty and open in P^1.

For any domain $D' \Subset \{\phi < 0\}$, by the uniform convergence (2.3) and an argument similar to that in Section 4 involving a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (see (2) in Theorem 2.2), we must have

$$\limsup_{j\to\infty} \sup_{\mathcal{S}\in D'} \frac{\log[f^{n_j}, g]_{\mathrm{can}}(\mathcal{S})}{d^{n_j} + \deg g} \le \sup_{D'} \phi < 0.$$

This is impossible if f has no potentially good reductions, by Theorem 1. $\square$

6. Proof of Theorem 3

For a while, K is still of arbitrary characteristic. For every polynomial $h \in K[z]$ of degree > 0, we have $|h| = h^*|\cdot|$ on P^1, and the function $(\log|h|)/(\deg h) - \log\max\{1, |\cdot|\}$ on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic and satisfies

$$\Delta(\log|h|) = h^*\Delta\log|\cdot| = h^*\delta_0 - (\deg h)\cdot\delta_\infty$$

on P^1 (for the functoriality of Δ, see [1, §5]).

Let $f \in K[z]$ be a polynomial of degree $d > 1$.

Fact 6.1. The Berkovich filled-in Julia set

$$\mathsf{K}(f) := \Big\{\mathcal{S} \in \mathsf{P}^1 : \limsup_{n\to\infty} |f^n(\mathcal{S})| < \infty\Big\} \subset \mathsf{A}^1$$

of f is the complement in P^1 of the Berkovich (immediate) basin of attraction

$$\mathsf{I}_\infty(f) := \Big\{\mathcal{S} \in \mathsf{P}^1 : \lim_{n\to\infty} f^n(\mathcal{S}) = \infty\Big\}$$

of f associated to the superattracting fixed point ∞ of f. Both $\mathsf{K}(f)$ and $\mathsf{I}_\infty(f)$ are totally invariant under f. Moreover,

$$\lim_{n\to\infty} f^n = \infty \quad \text{locally uniformly on } \mathsf{I}_\infty(f),$$

$\mathsf{F}(f) = \mathsf{I}_\infty(f) \cup (\operatorname{int}\mathsf{K}(f))$, and $\mathsf{J}(f) = \operatorname{supp}\mu_f = \partial\mathsf{K}(f) = \partial\mathsf{I}_\infty(f)$.

Fix a canonical lift

$$F(z_0, z_1) := \big(z_0^d, z_0^d f(z_1/z_0)\big) \in (K[z_0, z_1]_d)^2$$

of the polynomial $f \in K[z]$, and let us define the escaping rate function g_f of f on P^1 by

$$g_f := g_F + \log\max\{1, |\cdot|\} = \lim_{n\to\infty} \frac{\log\max\{1, |f^n|\}}{d^n} : \mathsf{P}^1 \to \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

and then $\mathsf{K}(f) = \{g_f = 0\}$, $\mathsf{I}_\infty(f) = \{g_f > 0\}$, and

$$\Delta g_f = \mu_f - \delta_\infty \quad \text{on } \mathsf{P}^1.$$

Let $C(f)$ be the set of all (classical) critical points of f in $\mathbb{P}^1$. We note that $\infty \in E(f) \subset C(f) \subset \mathbb{P}^1$.

From now on, we assume that $\operatorname{char} K = 0$. Then we also note that f' is a polynomial in $K[z]$ of degree $d - 1 > 0$, that

$$\operatorname{supp}(\Delta \log|f'|) = \operatorname{supp}((f')^*\delta_0 - (f')^*\delta_\infty) = C(f),$$

and that

$$\#(C(f) \setminus \{\infty\}) \leq ((f')^*\delta_0)(\mathsf{P}^1) = d - 1 < \infty.$$

Fact 6.2. For every $a \in K$ and every $n \in \mathbb{N}$, the functions

$$\frac{\log|(f^n)' - a|}{d^n - 1} - g_f \quad \text{and} \quad \frac{\log\max\{1, |(f^n)'|\}}{d^n - 1} - g_f$$

on P^1 are harmonic and bounded on some punctured open neighborhood of ∞, and extend harmonically near ∞ in P^1 (see e.g. [1, §7]).

Let us show Theorem 3. As in [18], we begin with the following.

Lemma 6.3. *On $\mathsf{I}_\infty(f) \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f) \setminus \{\infty\})$,*

$$\frac{\log|(f^n)'|}{d^n - 1} - g_f = O\Big(\frac{n}{d^n - 1}\Big) \quad \text{as } n \to \infty \tag{6.1}$$

locally uniformly. Moreover, there is $C = C_f > 0$ such that for every $n \in \mathbb{N}$,

$$\frac{\log\max\{1, |(f^n)'|\}}{d^n - 1} - g_f \leq \frac{C \cdot n}{d^n - 1} \quad \text{on } \mathsf{P}^1. \tag{6.2}$$

Proof. Set $a_d = a_d(f) := \lim_{K \ni z \to \infty} f(z)/z^d \in K \setminus \{0\}$. For every $n \in \mathbb{N}$, from a direct computation (involving the chain rule for $(f^n)'$, (2.1), and

$g_f \circ f = d \cdot g_f$ on P^1) as in the proof of [18, Lemma 3.2], we have

$$\begin{aligned}(6.3)\quad &\frac{\log|(f^n)'|}{d^n-1} - g_f\\ &= \frac{1}{d^n-1}\int_{\mathsf{A}^1}\left(\sum_{j=0}^{n-1}\log[f^j(\,\cdot\,),\mathcal{S}]_{\mathrm{can}}\right)(\Delta\log|f'|)(\mathcal{S})\\ &\quad - \frac{d-1}{d^n-1}\sum_{j=0}^{n-1}(f^j)^*(g_f+\log[\cdot,\infty]_{\mathrm{can}})\\ &\quad + \left(-\int_{\mathsf{A}^1}\log[\mathcal{S},\infty]_{\mathrm{can}}(\Delta\log|f'|)(\mathcal{S}) + \log|d| + \log|a_d|\right)\frac{n}{d^n-1}\end{aligned}$$

on P^1, and then noting that $\operatorname{supp}((\Delta\log|f'|)|\mathsf{A}^1) = C(f)\setminus\{\infty\} \Subset \mathsf{P}^1\setminus\{\infty\}$ and the continuity of the function

$$g_F = g_f - \log\max\{1,|\cdot|\} = g_f + \log[\cdot,\infty]_{\mathrm{can}} \quad \text{on } \mathsf{P}^1,$$

we have the locally uniform estimate (6.1) on $\mathsf{I}_\infty(f)\setminus\bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f)\setminus\{\infty\})$. Noting also that $[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \le 1$ and that $-g_f \le 0$ on P^1 and setting

$$\begin{aligned}C = C_f &:= (d-1)\cdot\sup_{\mathsf{P}^1}|g_F|\\ &\quad + (d-1)\cdot\sup_{w\in C(f)\setminus\{\infty\}}\left|\log[w,\infty]_{\mathbb{P}^1}\right| + \left|\log|d| + \log|a_d|\right| \in \mathbb{R}_{>0},\end{aligned}$$

we also have (6.2) from (6.3). □

Lemma 6.4. *If f has no potentially good reductions, then for every $a \in K$,*

$$\lim_{n\to\infty}\left(\frac{\log|(f^n)'-a|}{d^n-1} - g_f\right) = 0 \quad \textit{(pointwisely) on } \mathsf{P}^1\setminus\mathbb{P}^1.$$

Proof. Fix $a \in K$. By the upper estimate (6.2) (and (2.1) and $[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \le 1$), for every $n \in \mathbb{N}$, we have

$$\begin{aligned}(6.4)\quad &\frac{\log|(f^n)'-a|}{d^n-1} - g_f\\ &\left(= \frac{\log[(f^n)',a]_{\mathrm{can}}}{d^n-1} + \left(\frac{\log\max\{1,|(f^n)'|\}}{d^n-1} - g_f\right) + \frac{\log\max\{1,|a|\}}{d^n-1}\right)\\ &\le 0 + \frac{C_f\cdot n}{d^n-1} + \frac{\log\max\{1,|a|\}}{d^n-1}\end{aligned}$$

on P^1, and by $|(f^n)'-a| = |(f^n)'|\cdot|1 - a/(\prod_{j=1}^n(f'\circ f^{n-j}))|$ on P^1, $\lim_{\mathcal{S}\to\infty}|f'(\mathcal{S})| = +\infty$, $\lim_{n\to\infty} f^n = \infty$ on $\mathsf{I}_\infty(f)$, and the estimate (6.1) (pointwisely) on $\mathsf{I}_\infty(f)\setminus\mathbb{P}^1$, we have

$$(6.5)\qquad \lim_{n\to\infty}\left(\frac{\log|(f^n)'-a|}{d^n-1} - g_f\right) = \lim_{n\to\infty}\left(\frac{\log|(f^n)'|}{d^n-1} - g_f\right) = 0$$

(pointwisely) on $\mathsf{I}_\infty(f)\setminus\mathbb{P}^1$.

Hence noting that for every $n\in\mathbb{N}$, the function

$$\left(\frac{\log|(f^n)'-a|}{d^n-1}-g_f\right)+g_F=\frac{\log|(f^n)'-a|}{d^n-1}-\log\max\{1,|\cdot|\}$$

on P^1 is $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic, by a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (see (1) in Theorem 2.2), there are a sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j\to\infty$ and a function $\phi:\mathsf{P}^1\to\mathbb{R}\cup\{-\infty\}$ such that

$$\lim_{j\to\infty}\left(\frac{\log|(f^{n_j})'-a|}{d^{n_j}-1}-g_f\right)=\phi\quad\text{(pointwisely) on }\mathsf{P}^1\setminus\mathbb{P}^1$$

and that $\phi+g_F$ is a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1. Then ϕ is upper semicontinuous on P^1, and $\{\phi<0\}$ is open in P^1.

By (6.4), we indeed have $\phi\le 0$ on $\mathsf{P}^1\setminus\mathbb{P}^1$, and in turn on P^1 by the continuity of $\phi=(\phi+g_F)-g_F$ on any closed interval in P^1. Moreover, by (6.5), we have $\phi\equiv 0$ on $\mathsf{I}_\infty(f)\setminus\mathbb{P}^1$ and in turn on $\overline{\mathsf{I}_\infty(f)}$ also by the upper semicontinuity of ϕ on P^1. In particular, $\{\phi<0\}\subset\mathsf{P}^1\setminus\overline{\mathsf{I}_\infty(f)}$.

Let us see $\{\phi<0\}=\emptyset$, which will complete the proof; for, suppose to the contrary that $\{\phi<0\}\neq\emptyset$. Then there is a Berkovich Fatou component U of f, which is other than $\mathsf{I}_\infty(f)$, such that $U\cap\{\phi<0\}\neq\emptyset$.

Then $\phi\equiv 0$ on ∂U (since $\partial U\subset\partial\mathsf{I}_\infty(f)$ and $\phi\equiv 0$ on $\overline{\mathsf{I}_\infty(f)}$), and there is $\mathcal{S}_0\in\mathsf{P}^1\setminus\mathbb{P}^1$ such that

$$\partial U=\{\mathcal{S}_0\}$$

by Lemma 2.1 and $\partial\mathsf{I}_\infty(f)=\mathsf{J}(f)\supset\partial U\neq\emptyset$.

The argument in the rest of the proof of Lemma 6.4 is similar to that in the latter half of the proof of Theorem 1. Setting

$$\psi:=\begin{cases}\phi & \text{on } U\\ 0 & \text{on } \mathsf{P}^1\setminus U\end{cases}:\mathsf{P}^1\to\mathbb{R}_{\le 0}\cup\{-\infty\},$$

the function $\psi+g_f$ is domination subharmonic so subharmonic on A^1, which with $\psi+g_f\equiv g_f$ near ∞ implies that $\Delta\psi$ exists on P^1 and that

$$\Delta\psi+\mu_f=\Delta(\psi+g_f)+\delta_\infty$$

is a probability Radon measure on P^1 and restricts to μ_f on $\mathsf{P}^1\setminus\overline{U}$. If in addition f have no potentially good reductions, then $\mu_f(\partial U)=\mu_f(\{\mathcal{S}_0\})=0$, and in turn $\Delta\psi+\mu_f=\mu_f$ so $\Delta\psi=0$ on P^1, and finally we must have $\psi\equiv 0$ on $\mathsf{P}^1\setminus\mathbb{P}^1$, so $\phi\equiv 0$ on $U\setminus\mathbb{P}^1$. This contradicts $U\cap\{\phi<0\}\neq\emptyset$. $\square$

If f has no potentially good reductions, then the weak convergence (1.4) in Theorem 3 follows from Lemma 6.4, the equality

$$\Delta\left(\frac{\log|(f^n)'-a|}{d^n-1}-g_f\right)=\frac{((f^n)')^*\delta_a}{d^n-1}-\mu_f \quad \text{on } \mathsf{P}^1,$$

and the continuity of the Laplacian Δ. $\square$

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