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On genera containing non-split Eichler orders over function fields

par Luis ARENAS-CARMONA et Claudio BRAVO

Résumé. Le théorème de Grothendieck–Birkhoff établit que tout faisceaux vectoriel de dimension finie sur la droite projective $\mathbb{P}^1$ se scinde en somme de faisceaux vectoriels unidimensionnels (fibrés en droites). Il peut être reformulé en termes d'ordres comme l'énoncé que tous les $\mathbb{P}^1$-ordres maximaux se scindent. Ceci est utile, car les ordres scindés jouent un rôle important dans le calcul des graphes quotients. Dans ce travail, on étudie dans quelle mesure ce résultat se généralise aux $\mathbb{P}^1$-ordres d'Eichler, lorsque le corps de base $\mathbb{F}$ est fini. Pour être précis, on caractérise, d'une part, les genres des ordres d'Eichler contenant uniquement des ordres scindés et, d'autre part, les genres ne contenant qu'un nombre fini de classes d'isomorphie non scindées. La méthode développée ici nous permet également de calculer les graphes quotients pour certains sous-groupes de $\mathrm{PGL}_2(\mathbb{F}[t])$ d'intérêt arithmétique.

Abstract. Grothendieck–Birkhoff Theorem states that every finite dimensional vector bundle over the projective line $\mathbb{P}^1$ splits as the sum of one dimensional vector bundles (line bundles). This can be rephrased, in terms of orders, as stating that all maximal $\mathbb{P}^1$-orders in a matrix algebra split. This is useful, since split orders play an important role when computing quotient graphs. In this work we study the extent to which this result can be generalized to Eichler $\mathbb{P}^1$-orders when the base field $\mathbb{F}$ is finite. To be precise, we characterize both the genera of Eichler orders containing only split orders and the genera containing only a finite number of non-split isomorphism classes. The method developed here also allows us to compute quotient graphs for some subgroups of $\mathrm{PGL}_2(\mathbb{F}[t])$ of arithmetical interest.

1. Introduction

Split orders in the 4-dimensional matrix algebra $\mathbb{M}_2(k)$, where k is a local field, were characterized by Hijikata in [13, §2.2]. By definition, an order in $\mathbb{M}_2(k)$ is split if it contains an isomorphic copy of the ring $\mathcal{O}_k \times \mathcal{O}_k$, where

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$\mathcal{O}_k$ is the ring of integers in k, or equivalently, if it is a conjugate of

$$\mathfrak{E}(I,J)=\begin{pmatrix}\mathcal{O}_k & I\\ J & \mathcal{O}_k\end{pmatrix}=\mathcal{O}_kE_{1,1}\oplus\mathcal{O}_kE_{2,2}\oplus IE_{1,2}\oplus JE_{2,1},$$

for some pair (I,J) of fractional ideals, where $\{E_{i,j}|1\leq i,j\leq 2\}$ is the canonical basis of $\mathbb{M}_2(k)$. Hijikata proved these to be either maximal orders or intersections of two different maximal orders. These are local properties, and in fact, for any global field K, and for any ring $\mathcal{O}_S\subseteq K$ of S-integers, i.e., elements that are integral outside a nonempty finite set S of places that includes the archimedean places if any, global split $\mathcal{O}_S$-orders in $\mathbb{M}_2(K)$ share the same characterization. This is, for example, a consequence of the fact that the ring $\mathcal{O}_k\times\mathcal{O}_k$ is non-selective for any genus of Eichler orders [4, Ex. 5.5].

When K is a global function field, i.e., the field of rational functions on a smooth irreducible projective curve X over a finite field $\mathbb{F}$, we define (full) X-orders in $\mathbb{M}_2(K)$ as sheaves of rings for which the stalk at the generic point is $\mathbb{M}_2(K)$ [10]. This is usually regarded as the case $S=\varnothing$ in the theory of orders, and this point of view has been fruitful in the past to study quotients of Bruhat–Tits trees by groups of arithmetical interest (cf. [5]). The preceding characterization fails in this setting, as one would expect, giving the absence of a Strong Approximation Theorem with respect to the empty set. However, we do have a result in this direction, although a significantly more specific one. This is essentially Grothendieck–Birkhoff Theorem [8, Thm. 2.1], which implies the following statement (cf. Section 2):

Theorem GB. *Every maximal X-order in $\mathbb{M}_2(K)$ is split when X is the projective line $\mathbb{P}^1$.*

There is also a finiteness result that can be regarded as a partial generalization of the preceding statement to an arbitrary smooth projective curve defined over a finite field. It follows easily from [5, §1, Thm. **S**] and [5, §1, Thm. 1.2] (cf. Section 3 below):

Finiteness Theorem. *If X is an arbitrary smooth projective curve over a finite field, all but finitely many isomorphism classes of maximal X-orders in $\mathbb{M}_2(K)$ contain only split orders.*

The purpose of the present work is to study the extent to which similar results apply to Eichler orders, i.e., intersections of two maximal orders. The main results of the paper, namely Theorem 3.1 and Theorem 3.2, provide an answer to this question. Our interest in the subject arises from the fact that split orders play a significant role in the computation of quotient graphs, which are a powerful tool for the study of some arithmetically significant matrix groups. This can be seen in some of our previous work [5], or in the work of R. Köhl, B. Mühlherr and K. Struyve [14]. Either reference describes a quotient graph of the form $G\backslash\mathfrak{t}(K_\infty)$, for an arithmetic group of the form $G=\mathrm{GL}_2(A)$, where $A=\mathcal{O}_{\{P_\infty\}}\subseteq K$ is the ring of functions that are regular outside a single place $P_\infty\in X$, and $\mathfrak{t}(K_\infty)$ is the Bruhat–Tits tree of SL_2 over the completion K_∞ of K at P_∞. See Section 5 for details. A classical example of such results is due to Serre [28, Ch. II, §2.3]:

Theorem S. *The quotient graph $G\backslash\mathfrak{t}(K_\infty)$ is combinatorially finite, i.e., it is the result of attaching a finite set of infinite half lines, called cusps, to a certain finite graph Y. The set of such cusps can be indexed by the elements of the Picard group* $\mathrm{Pic}(A)=\mathrm{Pic}(X)/\langle\overline{P_\infty}\rangle$ *provided redundant cusps are avoided.*

This structural result was used by Serre to study the group G through Bass–Serre Theory, which is developed in the same book. Thus, he extends a classical work of Nagao (cf. [24]), on the existence of a decomposition of the group of matrices with polynomial coefficients $\mathrm{GL}_2(\mathbb{F}[t])$ as an amalgamation of simpler groups. Quotient graphs create the natural setting for the study of the structure of related groups. Serre described the quotient graphs precisely, whenever the degree of P_∞ fails to exceed 4. The results in [5] and [14] mentioned above generalized their computations, and also results by W. Mason and A. Schweizer in [19], [20] and [23]. The latter authors have applied this theory to the study of non-congruence subgroups in Drinfeld modular groups [21], [22]. S. Takahashi has computed some quotient graphs for elliptic curves in [31], and the case of orders on a division algebra has also been studied [26]. The arithmetic groups mentioned in all of the aforementioned references are closely related to normalizers of maximal orders. However, the general theory requires only working with orders that are maximal at P_∞, so these computations can be extended to a much larger family of orders, including Eichler orders. As far as we are aware, the present work is the first attempt to extend this type of results in such direction.

We expect that the tools we have developed here inspire future study on the structure of these groups. In fact, the theory introduced here to describe Eichler orders can be used to characterize quotient graphs or, equivalently, fundamental regions for some congruence subgroups of G. We illustrate that in the last section of this paper. Theorem 3.3 below can bee seen as a partial refinement of [27, §3.3, Lem. 8], since a more explicit description of the fundamental region is given for the corresponding group.

More generally, the study of actions on buildings, in particular Bruhat–Tits Buildings or BTB's, plays a similar role for arithmetic groups in global function fields to that of the upper half plain for subgroups of $\mathrm{PSL}_2(\mathbb{R})$ [9, §2]. In current literature, there exist a few significant results on the quotient structure for these actions. As an example, we have some characterizations, such as [30, Thm. 1] and [18, Thm. 2.1], of fundamental regions for groups of $\mathbb{F}[t]$-points in some particular K-reductive groups, such as SL_n. Both [30] and [18] contain results on group structure that are deduced from the description of the quotient complex or fundamental region. A much more general result along these lines can be found in [11, Prop. 13.6]. We have reasons to believe that the interplay between orders and quotient buildings will help to shed additional light on these structures in the future, for some particular groups, specially when looking for descriptions of these quotient sets that are more explicit than the result in [11].

Hijikata's characterization has been generalized to higher dimensional algebras in the local setting by Shemanske in [29] via BTB's. Also, BTB's play a significant role in the study of the selectivity problem, i.e., understanding when a commutative order embeds into all, or just into some, of the orders in a particular genus [12], [15], [16]. This problem arises naturally from questions regarding spectral properties of hyperbolic varieties [32], [17], and has been intensively studied in the quaternionic case, where the relevant buildings are trees.

2. Conventions on vector bundles

We start by recalling some basic facts on bundles and lattices. In all that follows, we let $\mathcal{O}_X$ denote the structure sheaf of a smooth irreducible projective curve X over a finite field $\mathbb{F}$. We can assume that $\mathbb{F}$ is algebraically closed in K, so that $\mathcal{O}_X(X) = \mathbb{F}$. We do so in the sequel. By an n-dimensional X-lattice Λ, we mean a locally free sheaf of $\mathcal{O}_X$-modules of rank n. In particular, for every open set $U \subseteq X$, the group $\Lambda(U)$ is a lattice over the Dedekind domain $\mathcal{O}_X(U)$, in the classical sense (cf. [25, §81 A]). The group $\Lambda(X)$ of global sections is always a finite dimensional vector space over the field $\mathbb{F}$. One important example of an X-lattice is the sheaf of sections of a vector bundle, which we usually identify with the bundle itself. By a lattice in a K-vector space V, we mean a lattice Λ together with a fix injection of the generic fiber $\Lambda \otimes_{\mathcal{O}_X} K$ into V. When such injection is an isomorphism, we say a full lattice. The same convention applies to bundles. When $V = K^n$, such an isomorphism can be made explicit by choosing n diferent K-linearly independent sections over some afine subset $U_0 \subset X$, which we identify with the canonical basis. This induces an identification of the $\mathcal{O}_X(U)-$module of U-sections $\Lambda(U)$ with a subset of K^n, for an arbitrary open set $U \subseteq X$. An isomorphism of this type always exists for n-dimensional vector bundles. Furthermore, two vector bundles are isomorphic if the corresponding lattices satisfy an identity of the form $T\Lambda = \Lambda'$, where T is an element in the general linear group $\mathrm{GL}_n(K)$. At the sheaf level, the preceeding notations mean $T\Lambda(U) = \Lambda'(U)$ for every open set U. We adopt similar conventions for other explicit vector spaces and linear maps. An X-order $\mathfrak{R}$ in a K-algebra $\mathfrak{A}$ is an X-lattice in $\mathfrak{A}$ whose group of sections $\mathfrak{R}(U)$ is a ring, for any open subset U. If the lattice is full we say a full order, e.g., the structure sheaf $\mathcal{O}_X$ is a full X-order in K. We let $\mathfrak{R}$, $\mathfrak{D}$ and $\mathfrak{E}$ denote full X-orders in $\mathbb{M}_2(K)$ in all that follows.

Every full X-lattice in the space K is the sheaf of sections $\mathfrak{L}^B$ of the line bundle defined by some divisor B, namely:

$$\mathfrak{L}^B(U) = \left\{ f \in K \middle| \operatorname{div}(f)|_U + B|_U \geq 0 \right\},$$

for every open subset $U \subseteq X$. These are usually called invertible bundles, and have been extensively studied in existing literature. They can be seen either as the projective equivalent of ideals, or as a multiplicative version of divisors, as ilustrated by the following properties:

(1) Two divisors B and D are linearly equivalent if and only if $\mathfrak{L}^B$ and $\mathfrak{L}^D$ are isomorphic as line bundles,
(2) for any pair (B, D) of divisors, we have $\mathfrak{L}^B\mathfrak{L}^D = \mathfrak{L}^{B+D}$,
(3) we have $\mathfrak{L}^B(U) \subseteq \mathfrak{L}^D(U)$, for all open sets U, precisely when $B \leq D$ and
(4) $\mathfrak{L}^{\operatorname{div}(g)} = g^{-1}\mathcal{O}_X$.

Note that the product $\mathfrak{L}^B\mathfrak{L}^D$ in (2) is defined locally, on open sets U, by the relation $(\mathfrak{L}^B\mathfrak{L}^D)(U) = \mathfrak{L}^B(U)\mathfrak{L}^D(U)$. With this definition, $\mathfrak{L}^B\mathfrak{L}^D$ is isomorphic to the tensor product $\mathfrak{L}^B \otimes_{\mathcal{O}_X} \mathfrak{L}^D$. In higher dimensions, we adopt similar conventions for scalar products or any other bilinear maps.

An X-lattice (bundle) is called split if it is isomorphic to a direct sum of one-dimensional lattices (invertible bundles). We are specifically interested in split two dimensional lattices, i.e., lattices of the form $\Lambda \cong \mathfrak{L}_1 \times \mathfrak{L}_2$, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are invertible bundles. This must be understood as an $\mathcal{O}_X$-module isomorphism. We often make this isomorphism explicit by the choice of a basis. A basis $\{e_1, e_2\}$ is call a splitting basis for a full X-bundle Λ in K^2 whenever $\Lambda = \mathfrak{L}_1 e_1 \oplus \mathfrak{L}_2 e_2$, for some pair $(\mathfrak{L}_1, \mathfrak{L}_2)$ of invertible bundles. Note that a full bundle in K^2 is split precisely when it has a splitting basis. We often say that a given basis splits or diagonalizes a bundle or lattice in this sense.

For every full X-lattice Λ in K^2, there is a corresponding maximal order $\mathfrak{D}_\Lambda = \mathcal{E}nd_{\mathcal{O}_X}(\Lambda)$ in the matrix algebra $\mathbb{M}_2(K)$. It is defined, on open sets U, by the following relation:

$$\mathfrak{D}_\Lambda(U) = \left\{ a \in \mathbb{M}_2(K) \middle| a\Lambda(U) \subseteq \Lambda(U) \right\}.$$

Every maximal order in the algebra $\mathbb{M}_2(K)$ has this form. Whenever the canonical basis $\{e_1, e_2\}$ splits Λ, namely

$$\Lambda = \mathfrak{L}^B e_1 \oplus \mathfrak{L}^C e_2 = \begin{pmatrix} \mathfrak{L}^B \\ \mathfrak{L}^C \end{pmatrix},$$

the corresponding order is $\mathfrak{D}_\Lambda = \left(\begin{smallmatrix} \mathcal{O}_X & \mathfrak{L}^{B-C} \\ \mathfrak{L}^{C-B} & \mathcal{O}_X \end{smallmatrix}\right)$. In particular, the projectors $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ belong to the ring of global sections $\mathfrak{D}_\Lambda(X)$. More generally, an order $\mathfrak{E}$ in $\mathbb{M}_2(K)$ is called split whenever is conjugate to any order of the form $\mathfrak{E}(\mathfrak{L}_1, \mathfrak{L}_2) = \left(\begin{smallmatrix} \mathcal{O}_X & \mathfrak{L}_1 \\ \mathfrak{L}_2 & \mathcal{O}_X \end{smallmatrix}\right)$, where $(\mathfrak{L}_1, \mathfrak{L}_2)$ is a pair of invertible bundles. An order is split precisely when it has a non-trivial idempotent global section. Such orders are split as four-dimensional lattices, but the converse does not hold in general. However, $\mathfrak{D}_\Lambda$ is split precisely when Λ is split as an X-lattice.

The X-lattice $\mathfrak{L}^D\Lambda$, for any divisor D, is referred to as a multiple of Λ. Two lattices define the same maximal order precisely when they are multiples of each other. A basis splits a lattice precisely when it splits every multiple, which makes maximal orders the natural context for the study of split lattices. In this sense, Theorem GB can be seen as a particular case of the following well-known result:

Grothendieck–Birkhoff Theorem ([8, Thm. 2.1])**.** *Every vector bundle over $\mathbb{P}^1$ is a direct product of one dimensional bundles.*

An Eichler order in $\mathbb{M}_2(K)$ is an order of the form $\mathfrak{E}_{\Lambda,\Lambda'} = \mathfrak{D}_\Lambda \cap \mathfrak{D}_{\Lambda'}$, for a pair of lattices (Λ, Λ'), or equivalently, any intersection of two maximal orders. Thus defined, the order $\mathfrak{E}_{\Lambda,\Lambda'}$ splits precisely when there exists a basis splitting the lattices Λ and Λ' simultaneously. It follows easily from Hijikata's local characterization that split orders are Eichler, as being Eichler is a local property, but the converse is not always true. It follows from the results in this work that non-split Eichler orders exist for every curve X. This is hardly surprising for geometry experts, as split bundles are a thin subset of the moduli space for curves of higher genus.

Example 2.1. A consequence of Hijikata's characterization of local split orders is the following: For every pair of lattices Λ and Λ' in k^2, there exists a basis $\{e_1, e_2\}$ for which $\Lambda = I_1 e_1 \oplus I_2 e_2$ and $\Lambda' = J_1 e_1 \oplus J_2 e_2$, for suitable ideals $I_1, I_2, J_1, J_2 \subseteq \mathcal{O}_k$. In other words, there is a basis splitting both lattices simultaneously. This also holds for arbitrary Dedekind domains, and it is the foundation of the theory of invariant factors for lattices (cf. [25, §81D]). Similarly, in the present context, characterizing split Eichler orders solves the problem of determining whether there is a common basis splitting two given lattices in K^2, or equivalently, whether a common change of variables can take a pair of vector bundles into a split form simultaneously.

3. Main results

Write $|X|$ for the set of closed points in X. As we recall in Section 4 below, any full lattice Λ in a fix vector space V is completely determined by the set $\left\{\widehat{\Lambda}_P \subseteq V_P \middle| P \in |X|\right\}$ of all its local completions $\widehat{\Lambda}_P$. Equivalently, two orders in the same space are equal if and only if they coincide locally at all places. One could hope, for this reason, that a similar property would allow us to classify conjugacy classes locally. This is not so, but this line of reasoning leads to the concept of genera. A genus is a maximal set of locally isomorphic orders. Equivalently, two orders are in the same genus if their completions at all local places are conjugate. Class Field Theory has been used to classify orders in a genus, it allows to split a genus into spinor genera. A spinor genus, in a given genus, is a maximal subset whose lattices are isomorphic over all[1] affine subsets of X. We recall part of this theory in Section 4, where a more technical, but equivalent, definition of spinor genus is given. For a full account, we refer the reader to [2]. Orders in a spinor genus are classified via quotient graphs. We recall this theory in Section 5, but we refer the reader to [5] for a full account on this subject.

A full description of the relation between the spinor genus of an Eichler order and those of the maximal orders containing it is given in [7, §6]. We just need to recall, for our purposes, that the genus of an Eichler order $\mathfrak{E}$ is determined by its level. At a local place P, the level of $\mathfrak{E}$ is the natural distance, in the Bruhat–Tits tree (cf. Section 5), between the unique pair of maximal orders whose intersection is the completion $\widehat{\mathfrak{E}}_P$. In the global context, the level of an Eichler order $\mathfrak{E}_{\Lambda,\Lambda'}$ is an effective divisor $D = D(\mathfrak{D}_\Lambda, \mathfrak{D}_{\Lambda'})$ defined in terms of these local distances (cf. Section 4). It can also be characterized by the following property:

> For every affine open set $U \subseteq X$, we have isomorphisms of $\mathcal{O}_X(U)$-modules
> $$\mathfrak{D}_\Lambda(U)/\mathfrak{E}_{\Lambda,\Lambda'}(U) \cong \mathfrak{D}_{\Lambda'}(U)/\mathfrak{E}_{\Lambda,\Lambda'}(U) \cong \mathcal{O}_X(U)/\mathfrak{L}^{-D}(U).$$

At this point, we are ready to state the main results of this work:

Theorem 3.1. *Let X be an arbitrary smooth projective curve over a finite field, and let D be an effective divisor on X. Then, the following statements are equivalent:*

- *Only finitely many conjugacy classes of Eichler orders of level D contain non-split orders.*
- *D is multiplicity free, i.e., D is a sum of different closed points.*

Theorem 3.2. *Let D be an effective divisor on the projective line $\mathbb{P}^1$, defined over a finite field. If $D \in \{0, P_1, P_1 + P_2\}$, for some pair (P_1, P_2) of points of degree 1, then every Eichler order of level D splits. No other divisor on $\mathbb{P}^1$ has this property.*

The latter result is a partial generalization of Grothendieck–Birkhoff Theorem. In proving these results, we rely heavily on the computation of

[1] In generalizations to quaternion division algebras, those affine sets containing the complement of the ramification locus of the algebra must be excluded.

quotients of local Bruhat–Tits trees by suitable arithmetic groups. In fact, these quotients are themselves interesting due to the fact that the structure of the acting groups can be recovered from them, plus some suitable local information on the vertex and edge stabilizers.

To make our final statement precise, we recall that the group

$$\mathrm{PGL}_2(\mathbb{F}[t]) \subseteq \mathrm{PGL}_2(K_\infty)$$

acts naturally via Moebius transformation on the Bruhat–Tits tree for the completion at infinity $K_\infty = \mathbb{F}((t^{-1}))$ of $\mathbb{F}(t)$, which can be interpreted as the tree $\mathfrak{g}$ described in [1, §4], whose vertices are the closed balls in K_∞. We call $\mathfrak{g}$ the Ball-tree in the sequel. There is a canonical bijection between the ends of the Ball-tree and the elements in the set $\mathbb{P}^1(K_\infty)$ of K_∞-points of the projective line. We use a technical definition of fundamental domain that can involve half edges, see Section 5 for details.

Theorem 3.3. *Let $N = (t-\lambda_1)\cdots(t-\lambda_n)$ be a square-free polynomial with all its roots in $\mathbb{F}$. Let $\mathfrak{s}$ be the smallest subtree containing the ends 0, ∞ and $1/M$, for every proper monic divisor M of N. Then the Hecke congruence subgroup*

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}[t]) \middle| c \equiv 0 \pmod{N} \right\}$$

has a fundamental domain of the form $\mathfrak{s} \cup \mathfrak{f}$ for a finite graph $\mathfrak{f}$.

4. Completions and spinor genera

This section lists the basic facts about spinor genera and spinor class fields that are needed in the sequel. Further details on this topic can be found in [2] or [3]. In all that follows we use the words order and lattice with the meaning of X-order and X-lattice, respectively. When classical orders or lattices over Dedekind domains are considered (in the sense defined in [25, §81A]), we refer to them as affine lattices since all Dedekind domains mention henceforth are of the form $\mathcal{O}_X(U)$ for an affine open subset $U \subset X$.

In all that follows, we consider the set $|X|$ of closed points in a smooth projective curve X with a function field $K = K(X)$. For every closed point P in $|X|$, we write K_P for the completion of K at P. We write ν_P for the valuation at P, and $\widehat{\mathcal{O}}_P = \{x \in K_P | \nu_P(x) \geq 0\}$ for the ring of local integers. We also use $x \mapsto |x|_P$ for the absolute value. We define the adele ring $\mathbb{A}$ by the formula

$$\mathbb{A} = \mathbb{A}_X = \left\{ a = (a_P)_P \in \prod_{P \in |X|} K_P \middle| \sharp\{P | a_P \notin \widehat{\mathcal{O}}_P\} < \infty \right\},$$

where $\sharp$ denotes cardinality. It contains the subring $\mathcal{O}_{\mathbb{A}} = \prod_{P\in|X|} \widehat{\mathcal{O}}_P$ of integral ideles. The ring $\mathbb{A}$ is assumed to be endowed with the adelic topology, which has a basis of open sets of the form $x + y\mathcal{O}_{\mathbb{A}}$ with y invertible and x arbitrary [33, §IV.1]. We denote $V_{\mathbb{A}}$ for the adelization $V \otimes_K \mathbb{A} \cong \mathbb{A}^{\dim_K V}$ of any finite dimensional vector space V, and endow it with product adelic topology. We canonically identify $\mathbb{A}$ with the adelization $K_{\mathbb{A}}$ and the ring $\mathrm{End}_{\mathbb{A}}(V_{\mathbb{A}})$ of $\mathbb{A}$-linear maps with the adelization $\big(\mathrm{End}_K(V)\big)_{\mathbb{A}}$. Given an arbitrary X-lattice Λ, we write $\widehat{\Lambda}_P \subseteq V_P$ for the $\widehat{\mathcal{O}}_P$-module generated by $\Lambda(U)$, for any affine Zarisky open neighborhood $U \subset X$ of P. Thus defined, $\widehat{\Lambda}_P$ depends on the sheaf Λ, but not on the neighborhood U. It is an open and compact subgroup of V_P.

The following results are well known for affine lattices, and their extension to the sheaf context is straightforward:

(1) If Λ and Λ' are two lattices in the same vector space V, we have $\widehat{\Lambda}_P = \widehat{\Lambda}'_P$ for all but finitely many places P,

(2) if we have $\widehat{\Lambda}_P = \widehat{\Lambda}'_P$ for every place $P \in |X|$, then we have $\Lambda = \Lambda'$, and

(3) if we have a family $\{\widehat{\Lambda}''(P)\}_{P\in|X|}$, where each $\widehat{\Lambda}''(P) \subseteq V_P$ is an $\widehat{\mathcal{O}}_P$-lattice, that satisfies the following coherence condition:

There is a lattice Λ satisfying $\widehat{\Lambda}''(P) = \widehat{\Lambda}_P$ for almost all $P \in |X|$,

then there exists a lattice Λ'' satisfying $\widehat{\Lambda}''(P) = \widehat{\Lambda}''_P$ for all $P \in |X|$.

The last property above can be used to build lattices that differ from a given lattice, in a controlled fashion, at any finite set of places. In particular, it allow us to study P-variants, i.e., lattices that differ from a given lattice in a unique place P. Similar statements hold for orders. We also use the notation $\Lambda_{\mathbb{A}} = \prod_{P\in|X|} \widehat{\Lambda}_P$, and call it the adelization of the lattice. Such adelizations can be characterized as the open and compact $\mathcal{O}_{\mathbb{A}}$-submodules of $V_{\mathbb{A}}$. In particular, for each X-lattice Λ and each $a \in \mathrm{GL}_2(\mathbb{A}) = \mathrm{End}_{\mathbb{A}}(V_{\mathbb{A}})^*$, we can define the adelic image $L = a\Lambda$ as the unique lattice satisfying $L_{\mathbb{A}} = a\Lambda_{\mathbb{A}}$. Note that L inherit each local property of Λ that is preserved coordinatewise by a. In particular, any adelic image, under a conjugation, of a full order is a full order. The same apply to maximal orders. The genus of a full order $\mathfrak{R}$ is defined by $\mathrm{Gen}(\mathfrak{R}) = \{a\mathfrak{R}a^{-1} | a \in \mathrm{GL}_2(\mathbb{A})\}$. For instance, the set of maximal X-orders is a genus [4].

We define the local distance d_P as follows: If there is a basis where two orders $\mathfrak{D}_P$ and $\mathfrak{D}'_P$ take the form

$$\widehat{\mathfrak{D}}_P = \begin{pmatrix} \widehat{\mathcal{O}}_P & \widehat{\mathcal{O}}_P \\ \widehat{\mathcal{O}}_P & \widehat{\mathcal{O}}_P \end{pmatrix} \quad \text{and} \quad \widehat{\mathfrak{D}}'_P = \begin{pmatrix} \widehat{\mathcal{O}}_P & \pi_P^d \widehat{\mathcal{O}}_P \\ \pi_P^{-d} \widehat{\mathcal{O}}_P & \widehat{\mathcal{O}}_P \end{pmatrix},$$

where $\pi_P \in K_P$ is a uniformizing parameter, we set $d_P(\widehat{\mathfrak{D}}_P, \widehat{\mathfrak{D}}'_P) = d$. The Eichler order $\mathfrak{D} \cap \mathfrak{D}'$ can be defined locally by $\widehat{\mathfrak{E}}_P = \widehat{\mathfrak{D}}_P \cap \widehat{\mathfrak{D}}'_P = (\mathfrak{D} \cap \mathfrak{D}')\widehat{}_P$. The local distance d above is, by definition, the local level of the Eichler order. It is well known that the local Eichler order $\widehat{\mathfrak{E}}_P$ uniquely determines the set $\{\widehat{\mathfrak{D}}_P, \widehat{\mathfrak{D}}'_P\}$. Two local Eichler order of the same level are conjugate and conversely. In particular, for a global Eichler order $\mathfrak{E}$, the genus $\mathrm{Gen}(\mathfrak{E})$ is the set of orders whose local levels coincide everywhere with those of $\mathfrak{E}$. This motivates the definition of a divisor-valued global distance

$$D(\mathfrak{D}, \mathfrak{D}') = \sum_{P \in |X|} d_P(\widehat{\mathfrak{D}}_P, \widehat{\mathfrak{D}}'_P) P \geq 0.$$

In particular, the level $\lambda(\mathfrak{E}_{\Lambda,\Lambda'})$ of the Eichler order $\mathfrak{E}_{\Lambda,\Lambda'}$ defined at the end of Section 2 is $D = D(\mathfrak{D}_\Lambda, \mathfrak{D}_{\Lambda'})$. We write $\mathbb{O}_D = \mathrm{Gen}(\mathfrak{E})$ for any Eichler order $\mathfrak{E}$ of level D. Note that this genus depends only on D. We call it the genus of Eichler orders of level D.

Let $J_X = \mathbb{A}^*$ be the idele group of X, and consider the coordinate-wise determinant $\det : \mathrm{GL}_2(\mathbb{A}) \to J_X$. The spinor genus of a full order $\mathfrak{R}$ in $\mathbb{M}_2(K)$ is defined by

$$\mathrm{Spn}(\mathfrak{R}) = \left\{ (bc)\mathfrak{R}(bc)^{-1} \middle| b \in \mathrm{GL}_2(K), c \in \mathbb{M}_2(\mathbb{A}), \det(c) = 1_{\mathbb{A}} \right\}.$$

In [2, §2], we gave the following characterization of spinor genera (cf. Remark 4.2):

Lemma 4.1. *For any two full orders $\mathfrak{R}$ and $\mathfrak{R}'$ in $\mathbb{M}_2(K)$, in the same genus, we have $\mathfrak{R}' \in \mathrm{Spn}(\mathfrak{R})$ precisely when the two rings $\mathfrak{R}(U)$ and $\mathfrak{R}'(U)$ are conjugate for any affine open subset $U \subseteq X$.*

The class field $\Sigma = \Sigma(\mathbb{O})$ corresponding to the group $K^* H(\mathfrak{R}) \subseteq J_X$, where

$$H(\mathfrak{R}) = \{\det(a) | a \in \mathbb{M}_2(\mathbb{A}), a\mathfrak{R}a^{-1} = \mathfrak{R}\},$$

is called the spinor class field of $\mathfrak{R}$, and it depends only on the genus $\mathbb{O} = \mathrm{gen}(\mathfrak{R})$. We also use the notation $\Sigma_D = \Sigma(\mathbb{O}_D)$. The spinor class field classifies spinor genera in a genus, in the sense that there is a well defined distance map $\rho : \mathbb{O} \times \mathbb{O} \to \mathrm{Gal}(\Sigma/K)$, satisfying $\rho(\mathfrak{R}, \mathfrak{R}') = [\det(a), \Sigma/K]$, where $t \mapsto [t, \Sigma/K]$ is the Artin map on ideles, whenever $a \in \mathrm{GL}_2(\mathbb{A})$ is an element satisfying $\mathfrak{R}' = a\mathfrak{R}a^{-1}$. This distance map has the additional properties

$$\rho(\mathfrak{R}, \mathfrak{R}'') = \rho(\mathfrak{R}, \mathfrak{R}')\rho(\mathfrak{R}', \mathfrak{R}''), \qquad \rho(\mathfrak{R}, \mathfrak{R}') = \mathrm{Id}_\Sigma \iff \mathfrak{R}' \in \mathrm{Spin}(\mathfrak{R}),$$

for any three orders $\mathfrak{R}, \mathfrak{R}', \mathfrak{R}'' \in \mathbb{O}$. When $\mathbb{O} = \mathbb{O}_0$ is the genus of maximal orders, the distance $\rho = \rho_0$ satisfies the formula $\rho_0(\mathfrak{D}, \mathfrak{D}') = [[D(\mathfrak{D}, \mathfrak{D}'), \Sigma_0/K]]$, which relates it to the divisor-valued distance D via the Artin map on divisors $B \mapsto [[B, \Sigma_0/K]]$. Furthermore, for any level D, the distance map ρ_D corresponding to the genus $\mathbb{O}_D$ satisfies the identity

$$\rho_D(\mathfrak{E}_{\Lambda,\Lambda'}, \mathfrak{E}_{L,L'}) = \rho_0(\mathfrak{D}_\Lambda, \mathfrak{D}_L)\Big|_{\Sigma_D},$$

for any four lattices Λ, Λ', L and L', as follows from [7, Prop. 6.1] and the discussion thereafter. If $D = \sum_P a_P P$, the field Σ_D can be characterized as the largest subfield of Σ_0 splitting at all places P with an odd coefficient a_P. This is a straightforward generalization of [4, Thm. 1.2].

Assume that the orders in a genus $\mathbb{O}$ are maximal at some place P. Two orders $\mathfrak{R}, \mathfrak{R}' \in \mathbb{O}$ are called P-neighbors if both of the following conditions are satisfied:

- $d_P(\widehat{\mathfrak{R}}_P, \widehat{\mathfrak{R}}'_P) = 1$, where d_P is the local distance, equivalently, the completions $\widehat{\mathfrak{R}}_P$ and $\widehat{\mathfrak{R}}'_P$ correspond to neighbors in the tree $\mathfrak{t}(K_P)$ (cf. Section 5).
- $\widehat{\mathfrak{R}}_Q = \widehat{\mathfrak{R}}'_Q$ for any place $Q \neq P$.

One last fact, which we often quote in the sequel, is the formula

$$\rho(\mathfrak{R}, \mathfrak{R}') = [[P, \Sigma(\mathbb{O})/K]], \tag{4.1}$$

for the distance between P-neighbors. It is immediate from the definition of ρ. The distance between P-variants can be computed by iteration from the preceding formula.

Remark 4.2. In an arbitrary quaternion algebra $\mathfrak{A}$, we have the following general version of Lemma 4.1 (cf. [2, §2]):

> $\mathfrak{R}' \in \mathrm{Spn}(\mathfrak{R})$ *precisely when* $\mathfrak{R}'(U)$ *is conjugate to* $\mathfrak{R}(U)$ *whenever there is a place in the complement of* U *that splits* $\mathfrak{A}$.

5. Eichler orders and trees

In all that follows, by a graph $\mathfrak{h}$, we mean a 5-tuplet (V, E, s, t, r) satisfying the following statements:

- $V = V_\mathfrak{h}$ and $E = E_\mathfrak{h}$ are sets, called the vertex set and the edge set.
- The three last symbols denote functions. They are the source $s : E \to V$, the target $t : E \to V$ and the reverse $r : E \to E$, and satisfy the identities $r(a) \neq a$, $r\big(r(a)\big) = a$ and $s\big(r(a)\big) = t(a)$, for each edge a.

Graphs are the objects in a category $\mathfrak{Graphs}$ whose morphisms are simplicial maps $\gamma : \mathfrak{h} \to \mathfrak{h}'$, i.e., pairs of functions $\gamma_V : V_\mathfrak{h} \to V_{\mathfrak{h}'}$ and $\gamma_E : E_\mathfrak{h} \to E_{\mathfrak{h}'}$ commuting with the preceding functions. Group actions are defined analogously. An action without inversions, of a group Γ on a graph $\mathfrak{h}$, is an action by simplicial maps where no edge is in the same orbit as its reverse. The definition of quotient graphs for these actions is straightforward. This type of quotients is used in Bass–Serre Theory to study the structure of groups acting on trees, as we mention in Section 1. When the action has inversions, the definition is more subtle. We replace the graph by its first barycentric subdivision to obtain a bipartite graph, with old vertices (or actual vertices) and new vertices (or baricenters). The only baricenters drawn in pictures are those having valency one in the quotient. We call them nonvertices[2]. For details on this definition, the reader can see [5, Rem. 1.6] or [6, Rem. 3.1]. We use the term *half-edge* for the edge joining a nonvertex to a vertex. We often speak of vertices with the meaning of actual vertices. We also use other notations that purposely ignore baricenters. As an example, we refer to paths of length 2, as defined below, joining two actual vertices as edges. Likewise, we assume throughout that a simplicial map always takes an actual vertex to an actual vertex, and a baricenter to a baricenter.

To define the notion of path in a graph, we introduce the concept of integral interval. We introduce the graph-theoretical "real line" as a graph $\mathfrak{r}$ satisfying $V_{\mathfrak{r}} = \{m_j | j \in \mathbb{Z}\}$, $E_{\mathfrak{r}} = \{b_j, r(b_j) | j \in \mathbb{Z}\}$, $s(b_j) = m_j$ and $t(b_j) = m_{j+1}$. An integral interval is defined as a connected subgraph of the real line. For these intervals we use the notations $\mathfrak{i}_{k,k'}$, $\mathfrak{i}_{-\infty,k}$, $\mathfrak{i}_{k,\infty}$ and $\mathfrak{i}_{-\infty,\infty} = \mathfrak{r}$ in a way that correlates naturally with the standard notation for closed intervals in the real line. The length of a finite interval is defined by $l(\mathfrak{i}_{k,k'}) = k' - k$. Vertices in the real line have a natural order. A shift is an order-preserving simplicial isomorphism between intervals. A path in a graph $\mathfrak{h}$ is a class of simplicial maps γ from integral intervals to $\mathfrak{h}$, under the smallest equivalence relation $\equiv$ satisfying $\gamma \circ \sigma \equiv \gamma$ whenever σ is a shift. An order-reversing simplicial isomorphism between integral intervals is called a flip. The reverse of the path represented by a simplicial map γ is the path corresponding to $\gamma \circ \phi$ where ϕ is a flip. The length of a path is, by definition, the length of the interval, which is invariant under shifts. This convention is naturally extended to infinite path, i.e., those defined from the intervals $\mathfrak{i}_{-\infty,k}$, $\mathfrak{i}_{k,\infty}$ or $\mathfrak{i}_{-\infty,\infty}$. A path $\gamma : \mathfrak{i}_{k,\infty} \to \mathfrak{h}$ is called a ray, and a path $\gamma : \mathfrak{i}_{-\infty,\infty} \to \mathfrak{h}$ is called a maximal path. A line is the graph-theoretical image of a path. Note that such image is shift-invariant. We also identify a line with a pair of mutually reverse paths. Maximal lines are defined analogously. When dealing with quotient graphs with nonvertices, baricenter can be added to the real line to define intervals and paths of half-integral length. We skip the details.

It is often convenient to highlight the vertices $v = \gamma_V(m_k)$ and $v' = \gamma_V(m_{k'})$ of the path with a representative $\gamma : \mathfrak{i}_{k,k'} \to \mathfrak{h}$. They are called initial and final vertices of the path, and also endpoints of the corresponding

[2]In previous work we used the expression "virtual endpoint", but the word "endpoint" is reserved for paths by most authors.

line. We often say a path from v to v', or between v and v', in this context. A tree is a connected graph with a unique path between every pair of vertices.

Bruhat–Tits trees. A vertex in the Bruhat-tits tree $\mathfrak{t}(K_P)$ for $\mathrm{SL}_2(K_P)$ [28, §II.1] can be described, for our purposes, by giving any of the following three equivalent pieces of data:

- A ball $B = B_a^{[\nu_P(u)]}$ with center $a \in K_P$ and radius $|u|_P$, for $u \in K_P$.
- As the homothety class of the $\widehat{\mathcal{O}}_P$-lattice $\widehat{\Lambda}_{a,u,P} = \left\langle \binom{a}{1}, \binom{u}{0} \right\rangle$.
- As the endomorphism ring $\widehat{\mathfrak{D}}_{a,u,P} = \mathrm{End}_{\widehat{\mathcal{O}}_P}(\widehat{\Lambda}_{a,u,P})$.

The natural graph distance in this tree coincide with the local distance for maximal orders described in Section 4. In particular, neighbors are precisely the maximal orders at distance 1. More generally, there is a canonical bijection between local Eichler orders $\widehat{\mathfrak{E}}_P$ of level k and lines of length k in $\mathfrak{t}(K_P)$. Furthermore, if we write $\widehat{\mathfrak{D}}_{P,v}$ for the maximal order corresponding to a vertex v and $\widehat{\mathfrak{E}}_{P,\gamma}$ for the Eichler order corresponding to a line represented by a map $\gamma : \mathfrak{i}_{0,k} \to \mathfrak{t}(K_P)$, the maximal orders containing $\widehat{\mathfrak{E}}_{P,\gamma}$ are precisely those of the form $\widehat{\mathfrak{D}}_{P,\gamma_V(m_i)}$, for $0 \leq i \leq k$, and in fact $\widehat{\mathfrak{E}}_{P,\gamma} = \widehat{\mathfrak{D}}_{P,\gamma_V(m_0)} \cap \widehat{\mathfrak{D}}_{P,\gamma_V(m_k)}$. This can be used to prove that the maximal orders in the expression of a local Eichler order as an intersection are unique.

Grids and classifying graphs. Define the support of a divisor $D = \sum_{P \in |X|} \alpha_P P$ by the formula $\mathrm{Supp}(D) = \{P \in |X| \,|\, \alpha_P > 0\}$. Since contention is a local property, the set of maximal orders containing a given global Eichler order $\mathfrak{E}$, with completions $\widehat{\mathfrak{E}}_P = \widehat{\mathfrak{E}}_{P,\gamma_P}$, can be interpreted as the set of vertices in the product $\mathbb{S}(\mathfrak{E}) = \prod_{P \in \mathrm{Supp}(D)} \mathrm{Im}(\gamma_P)$. This product can be visualized as a finite grid. A corner of the grid is a vertex whose P-coordinates are all endpoints. Write $\mathfrak{D}_v$ for the global order corresponding to a vertex v. Then, next result is immediate:

Lemma 5.1. *In the preceding notations, $\mathfrak{E} = \mathfrak{D}_{v_1} \cap \mathfrak{D}_{v_2}$ for every pair (v_1, v_2) of opposite corners in the grid.*

The divisor valued distance between orders is easily represented as a coordinate-wise distance between the corresponding vertices. To compare different orders, we fix an effective divisor $D = \sum_P \alpha_P P$, and a finite set of places $T \supseteq \mathrm{Supp}(D)$. Call $\mathrm{Eich}(D,T)$ the set of Eichler orders of level D satisfying $\widehat{\mathfrak{E}}_Q = \mathbb{M}_2(\widehat{\mathcal{O}}_Q)$ for $Q \notin T$. Then, the grid corresponding to an Eichler order in this set can be seen naturally as a sub-complex of the product $\prod_{P \in T} \mathfrak{t}(K_P)$. We often emphasize the fact that these grids are contained in this explicit product of Bruhat–Tits trees by calling them concrete D-grids.

Set $D = D' + \alpha_P P$, where $P \notin \mathrm{Supp}(D')$. If the Eichler order $\mathfrak{E}$ has level D, the parallelotope $\mathbb{S}(\mathfrak{E})$ is regarded as a concrete D-grid, for a suitable T. If $\alpha_P > 0$, there are precisely two Eichler orders $\mathfrak{E}_1$ and $\mathfrak{E}_2$ of level D' for which $\mathbb{S}(\mathfrak{E})$ is the only D-grid containing simultaneously $\mathbb{S}(\mathfrak{E}_1)$ and $\mathbb{S}(\mathfrak{E}_2)$. We refer to the latter grids as the P-faces of $\mathbb{S}(\mathfrak{E})$.

Fix now a place $Q \in |X|$, write $U = X \smallsetminus \{Q\}$, and consider a genus $\mathbb{O}$ such that $\widehat{\mathfrak{R}}_Q$ is maximal for some (and hence every) order $\mathfrak{R} \in \mathbb{O}$. Fix such order $\mathfrak{R}$, and define $\Psi = \Psi(\mathfrak{R}, Q) = \{\mathfrak{R}' \in \mathbb{O} | \mathfrak{R}'(U) = \mathfrak{R}(U)\}$, the set of Q-variants of $\mathfrak{R}$. Let $\mathcal{N} = \mathcal{N}(\mathfrak{R}, Q)$ be the normalizer of $\mathfrak{R}(U)$ in $\mathrm{GL}_2(K)$. Any element $\mathfrak{R}' \in \Psi$ is fully determined by its completion $\widehat{\mathfrak{R}}'_Q$ at Q. The local order $\widehat{\mathfrak{R}}'_Q$ corresponds to a vertex of the Bruhat–Tits tree $\mathfrak{t}_Q = \mathfrak{t}(K_Q)$, whence the vertices of the quotient graph $\mathfrak{c}_Q(\mathfrak{R}) = \mathcal{N} \backslash \mathfrak{t}_Q$, which we call the classifying graph, are in bijection with the $\mathcal{N}$-orbits of orders in Ψ. It follows from Formula (4.1) that all the orders in Ψ belong to the same spinor genus precisely when $|[Q, \Sigma(\mathbb{O})/K]|$ is the identity. Otherwise, these orders belong to two different spinor genera and the quotient graph is bipartite. Since $\mathfrak{R}'''(U)$ is conjugate to $\mathfrak{R}(U)$, for any order $\mathfrak{R}'''$ in the spinor genus of $\mathfrak{R}$ (cf. Lemma 4.1), next result follows:

Proposition 5.2. *Under the preceding hypotheses and notations, every conjugacy class in the spinor genus of* $\mathfrak{R}$ *is represented in* Ψ*. In particular, every conjugacy class in this spinor genus is represented by a vertex in* $\mathfrak{c}_Q(\mathfrak{R})$*.*

Note that the group $G_T = \mathrm{GL}_2\big(\mathcal{O}_X(X \smallsetminus T)\big)$ acts naturally in the product $\prod_{P \in T} \mathfrak{t}(K_P)$. Orbits are called D-grid classes. This action preserves P-faces, so we can define the P-faces of a D-grid class, which are also grid classes. We use this convention throughout.

Proposition 5.3. *For any fix divisor* $D > 0$*, and for any finite set* T *containing* $\mathrm{Supp}(D)$*, there is a natural bijection between the set of* G_T*-orbits in the set* $\mathrm{Eich}(D, T)$ *and the set of* D*-grid classes in the corresponding product of Bruhat–Tits trees. We can choose the set* T *in a way that every conjugacy class of Eichler orders of level* D *contains a representative in* $\mathrm{Eich}(D, T)$*.*

Proof. Everything is straightforward except for the last statement. Fix an order $\mathfrak{E}_0 \in \mathbb{O}_D$. Assume that the Frobenius maps of the places $Q_1, \dots, Q_N \in T \smallsetminus \mathrm{Supp}(D)$ generate the Galois group $\mathrm{Gal}(\Sigma_D/K)$. Then, for any order $\mathfrak{E} \in \mathbb{O}_D$ satisfying $\mathfrak{E}_P = \mathfrak{E}_{0,P}$ for $P \notin \{Q_1, \dots, Q_N\}$, we have $\rho_D(\mathfrak{E}_0, \mathfrak{E}) = \prod_{i=1}^N [[Q_i, \Sigma_D/K]]^{\beta_i}$, where $\beta_i = d_{Q_i}(\widehat{\mathfrak{E}}_{0,Q_i}, \widehat{\mathfrak{E}}_{Q_i})$, as Formula (4.1) shows. In particular, we can find an order $\mathfrak{E}$ in any given spinor genus by choosing a suitable family of local maximal orders $\widehat{\mathfrak{E}}_{Q_i}$. Now the result follows from Proposition 5.2, for any place $Q \neq Q_1, \dots, Q_N$ in $T \smallsetminus \mathrm{Supp}(D)$. ☐

We have one quotient graph for every spinor genera, or pair of such, so we can define the full classifying graph as the coproduct $\mathfrak{c}_Q(\mathbb{O}) = \bigcup_{\mathfrak{R}} \mathfrak{c}_Q(\mathfrak{R})$. Here, we choose one representative $\mathfrak{R}$ in each spinor genus or pair, generalizing thus the definition in [5]. Next result is apparent at this point:

Proposition 5.4. *Consider a divisor $D > 0$ and a point $Q \notin \mathrm{Supp}(D)$. There exists a canonical bijection between the vertices of the classifying graph $\mathfrak{c}_Q(\mathbb{O}_D)$ and the set of D-grid classes, and another between the geometric edges (i.e., pairs of the form $\{a, r(a)\}$) and the $(D+Q)$-grid classes, such that the Q-faces of the grid corresponding to such an edge $\{a, r(a)\}$ correspond to the endpoints $s(a)$ and $t(a)$.*

Remark 5.5. Classifying graphs can have multiple edges, or even loops or half edges, the latter two only when $|[Q, \Sigma(\mathbb{O})/K]| = \mathrm{Id}_{\Sigma(\mathbb{O})}$. See [5] for some examples. In particular, two different $(D + Q)$-grid classes, as above, can have the same, or even repeated, Q-faces.

6. Some preliminary results

In the sequel, we use the following convention:

$$\psi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \delta_f = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \quad \nu = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right). \tag{6.1}$$

Our next objective is to prove Lemma 6.7, which is the main result of this section. This requires some preparation, which we have divided into six lemmas. The first of them is a straightforward computation (cf. Section 2).

Lemma 6.1. *The orders having the idempotent η as a global section are those of the form $\mathfrak{R} = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L}_1 \\ \mathfrak{L}_2 & \mathcal{O}_X \end{pmatrix}$, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are bundles in K satisfying $\mathfrak{L}_1\mathfrak{L}_2 \subseteq \mathcal{O}_X$. In particular, the maximal orders having η as a global section have the form $\mathfrak{D} = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L} \\ \mathfrak{L}^{-1} & \mathcal{O}_X \end{pmatrix}$.*

In particular, $\mathfrak{D}_B = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L}^B \\ \mathfrak{L}^{-B} & \mathcal{O}_X \end{pmatrix}$ is a maximal order for any divisor B. Next result is an immediate consequence, as all nontrivial idempotents in $\mathbb{M}_2(K_Q)$ are conjugates. See also [4, Cor. 4.3].

Lemma 6.2. *For any place $Q \in |X|$, the Q-variants of the order $\mathfrak{D}_B$ containing η as a global sections are those of the form $\mathfrak{D}_{B+nQ}$, for $n \in \mathbb{Z}$. The maximal orders in $\mathbb{M}_2(K_Q)$ containing η are the completions $\widehat{\mathfrak{D}}_{nQ,Q} = \begin{pmatrix} \hat{\mathcal{O}}_Q & \pi_Q^n \hat{\mathcal{O}}_Q \\ \pi_Q^{-n} \hat{\mathcal{O}}_Q & \hat{\mathcal{O}}_Q \end{pmatrix}$, whose corresponding vertices are in maximal path in the Bruhat–Tits tree $\mathfrak{t}_Q$. More generally, the maximal orders in $\mathbb{M}_2(K_Q)$ containing any fix idempotent correspond to the vertices in a maximal path.*

Lemma 6.3. *The maximal orders $\mathfrak{D}_B$ and $\mathfrak{D}_{B'}$ are conjugates precisely when B is linearly equivalent to either B' or $-B'$.*

Proof. In the notations of (6.1), we have $\mathfrak{D}_B = \psi\mathfrak{D}_{-B}\psi^{-1}$ and $\mathfrak{D}_B = \delta_f\mathfrak{D}_{B+\mathrm{div}(f)}\delta_f^{-1}$. To prove the converse, we observe that the ring of global sections $\mathfrak{D}_B(X)$ spans the four dimensional matrix algebra precisely when $\mathfrak{L}^B$ and $\mathfrak{L}^{-B}$ have a non-trivial global section simultameously. This is possible only when B is principal. Therefore, we might assume neither B nor B' is principal. Replacing B by $-B$ if needed, we might assume $\mathfrak{L}^{-B}(X) = 0$. For the same reason, we might assume $\mathfrak{L}^{-B'}(X) = 0$. Now, the condition $\mu\mathfrak{D}_B\mu^{-1} = \mathfrak{D}_{B'}$, for $\mu \in \mathbb{M}_2(K)$, implies $\mu\mathfrak{D}_B(X)\mu^{-1} = \mathfrak{D}_{B'}(X)$. It follows that the span V of $\mathfrak{D}_B(X)$ has the same dimension as the span V' of $\mathfrak{D}_{B'}(X)$. In particular $\mathfrak{L}^B(X)$ vanishes if and only if $\mathfrak{L}^{B'}(X)$ vanishes. In either case $V = V'$. If $\mathfrak{L}^B(X) = \mathfrak{L}^{B'}(X) = 0$, V is the ring of diagonal matrices, so either μ or $\psi\mu$ is a diagonal matrix. A straightforward computation finish the proof in this case. In the remaining case, V is the ring of upper triangular matrices, so μ is upper triangular. Setting $\eta' = 1 - \eta$, it is easy to see that $\mu^{-1}\eta = \eta\mu^{-1}\eta$ and $\eta'\mu = \eta'\mu\eta'$. On one hand we have $\eta'\mathfrak{D}_{B'}\eta = \mathfrak{L}^{B'}\nu$, while on the other

$$\eta'\left(\mu\mathfrak{D}_B\mu^{-1}\right)\eta = (\eta'\mu\eta')\,\mathfrak{D}_B\left(\eta\mu^{-1}\eta\right) = \mathfrak{L}^B\left(\eta'\mu\nu\mu^{-1}\eta\right).$$

A straightforward computation shows that the last parenthesis is a scalar multiple of ν. The conclusion follows. □

Denote by $[D] \in \mathrm{Pic}(X)$ the class of a divisor D. The absolute value on divisors is defined by $|\sum_P \alpha_P P| = \sum_P |\alpha_P| P$.

Lemma 6.4. *Let $\mathfrak{D} \cong \mathfrak{D}_B$ and $\mathfrak{D}' \cong \mathfrak{D}_{B'}$ be two maximal orders whose divisor valued distance is D. Assume that $\mathfrak{E} = \mathfrak{D} \cap \mathfrak{D}'$ is split. Then there exist two divisors B_0 and B_0' satisfying the following relations:*

$$|B_0 - B_0'| = D, \quad [B_0] \in \{[B], [-B]\} \quad \textit{and} \quad [B_0'] \in \{[B'], [-B']\}.$$

Proof. Replacing by a conjugate if needed, we might assume that the order $\mathfrak{E}$ contains the element η in (6.1), and therefore so do $\mathfrak{D}$ and $\mathfrak{D}'$. In particular, we can write $\mathfrak{E} = \mathfrak{E}[C, C'] = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L}^{C'} \\ \mathfrak{L}^{-C} & \mathcal{O}_X \end{pmatrix}$, with $\mathfrak{L}^{-C}\mathfrak{L}^{C'} \subseteq \mathcal{O}_X$. In particular, $C - C'$ is an effective divisor, so we can write $\mathfrak{E} = \mathfrak{D}_C \cap \mathfrak{D}_{C'}$. Note, however, that this is not the only way to write this order as an intersection. By Lemma 5.1, the orders $\mathfrak{D}$ and $\mathfrak{D}'$ could be any pair of opposite corners in the grid. It is not hard to see that the maximal orders containing $\mathfrak{E}$ are precisely the orders of the form $\mathfrak{D}_{C''}$ with $C \leq C'' \leq C'$. For this reason, we need a formula for the level of any intersection of the form $\mathfrak{D}_{B_0} \cap \mathfrak{D}_{B_0'}$.

Note that the intersection of two invertible sheaves is given by the formula $\mathfrak{L}^{D_1} \cap \mathfrak{L}^{D_2} = \mathfrak{L}^{\min\{D_1, D_2\}}$, where, as one would expect, the minimum is defined by $\min\{D_1, D_2\} = \frac{1}{2}(D_1 + D_2 - |D_1 - D_2|)$. Analogously, we set $\max\{D_1, D_2\} = -\min\{-D_1, -D_2\}$. With this in mind, it is easy to see that the level of $\mathfrak{D}_{B_0} \cap \mathfrak{D}_{B_0'} = \mathfrak{E}[\max\{B_0, B_0'\}, \min\{B_0, B_0'\}]$ is $\max\{B_0, B_0'\} - \min\{B_0, B_0'\} = |B_0 - B_0'|$. The result is, therefore, a consequence of Lemma 6.3. ☐

Finally, we need to recall a few facts from the description of the classifying graph $\mathfrak{c}_P(\mathbb{O}_0)$ for maximal orders in the case $X = \mathbb{P}^1$.

Lemma 6.5 (cf. [5, Fig. 1 and Fig. 7])**.** *Assume $X = \mathbb{P}^1$, and consider a point P of degree $d = 1$. Then $\mathfrak{c}_P(\mathbb{O}_0)$ is a ray whose vertices correspond to the conjugacy classes $\{c_0, c_1, \dots\}$, with $c_n = [\mathfrak{D}_{nP}]$ for $n \in \mathbb{Z}_{\geq 0}$. When $d = 2$, then $\mathfrak{c}_P(\mathbb{O}_0)$ is the graph depicted in Figure 6.1.*

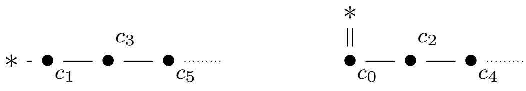

FIGURE 6.1. The two connected components of $\mathfrak{c}_P(\mathbb{O}_0)$ when $X = \mathbb{P}^1$ and $\deg(P) = 2$.

The reader must be warned that the identification $c_n = [\mathfrak{D}_{nP}]$ is valid only for a point $P \in |\mathbb{P}^1|$ of degree 1. For a point P of degree $d > 1$, c_n corresponds to the class $c_n = [\mathfrak{D}_{qP+rP_1}]$, where $n = qd + r$, for an arbitrary point P_1 of degree 1. Note that we have a linear equivalence $P \sim dP_1$.

Lemma 6.6. *Assume $X = \mathbb{P}^1$, and let $P \in X$ be a place of degree d, then, in the notations of previous lemma, the following statements hold:*

(1) *For each residue class $\bar{r} \in \mathbb{Z}/d\mathbb{Z}$, there is a ray $\mathfrak{t}_{\bar{r}}$ in $\mathfrak{c}_P(\mathbb{O}_0)$ whose vertices, in order, are the classes $c_r, c_{r+d}, c_{r+2d}, c_{r+3d}, \dots$.*
(2) *If d is odd, then $\mathfrak{c}_P(\mathbb{O}_0)$ is connected. Otherwise, there are two connected components.*
(3) *When d is even, the connected component containing the class c_r depends only on the parity of r.*

Proof. Statement (1) is a direct application of Lemma 6.2, while statement (2) is in [5, Thm. 1.3]. To prove statement (3), we observe that, for any idele $a \in J_X$, the element a^2 is the determinant of the scalar matrix $a1_{\mathbb{M}_2(\mathbb{A})}$, whence $a^2 \in H(\mathfrak{D})$, for any $\mathfrak{D} \in \mathbb{O}_0$. This shows that Σ_0/K is an extension of exponent two, and therefore, if P_1 is a place of degree 1, then $\rho(\mathfrak{D}_{rP_1}.\mathfrak{D}_{r'P_1}) = |[P_1, \Sigma_0/K]|^{r-r'}$ is trivial when $r - r'$ is even. We conclude that c_r and $c_{r'}$ are in the same connected component in this case. If this were so also when $r - r'$ is odd, there would be just one connected component, so this is not the case. ☐

Lemma 6.7. *Assume $X = \mathbb{P}^1$, and consider a point P of degree $d \geq 2$. Then the genus $\mathbb{O}_P$ contains non-split orders.*

Proof. Assume first that $d > 2$. We claim that there are two rays in the same connected component that are not of the form $\mathfrak{k}_{\bar{r}}$ and $\mathfrak{k}_{\overline{d-r}}$. When d is odd, we can choose $\mathfrak{k}_{\bar{0}}$ and $\mathfrak{k}_{\bar{1}}$, while for an even integer $d > 2$, we can choose $\mathfrak{k}_{\bar{0}}$ and $\mathfrak{k}_{\bar{2}}$. We conclude that there is, at least, one edge joining a pair of vertices of the form (c_m, c_n), where neither $m+n$ nor $m-n$ is a multiple of d. Now we are in the setting of Lemma 6.4. If the Eichler order of level P corresponding to a preimage of that edge is split, we can find divisors B_0' of degree $\pm n$ and B_0 of degree $\pm m$ such that $|B_0 - B_0'| = P$. This is only possible if we have $B_0 - B_0' = \pm P$, which contradicts the preceding assumptions on their degrees, thus proving the statement in this case. For the case $d = 2$, we use Lemma 6.5. Note that there is a half edge, denoted by a double line in Figure 6.1, connecting the class c_0 to itself, i.e., we can find an Eichler order contained precisely in two maximal orders that are conjugate to $\mathfrak{D}_0$. If it corresponds to a split Eichler order, we must have a pair (B_0, B_0') of degree-0 divisors satisfying $|B_0 - B_0'| = P$, which is not possible. $\square$

Remark 6.8. We could use Lemma 6.2 to prove that, in fact, the line corresponding to the set of P-variants of a given order containing a fixed idempotent global section has one of the following images in $\mathfrak{c}_P(\mathbb{O}_0)$:

- The double line obtained as the union of two rays of the form $\mathfrak{k}_{\bar{r}}$ and $\mathfrak{k}_{\overline{d-r}}$, plus an edge joining its endpoints.
- The ray $\mathfrak{k}_{\bar{0}}$.
- The ray $\mathfrak{k}_{\overline{d/2}}$ with a half line attached to the endpoint.

This tell us, in particular, that the conjugacy class of the non-split Eichler order is unique in the case $d = 2$. The edge joining two vertices in the class c_1 corresponds to a split order.

On valency computations. In order to compute vertex valencies, it is often useful to consider the S-graph as defined in [5]. This is an older notion than the classifying graph we consider above, as it was already studied by Serre in [28]. Let us recall the definition here. Write $U = X \smallsetminus \{Q\}$ for a maximal affine subset, and consider the unit group $\Gamma_{\mathfrak{R}} = K^*\mathfrak{R}(U)^* \subseteq \mathcal{N}$ of $\mathfrak{R}$. The S-graph is defined by $\mathfrak{s}_Q(\mathfrak{R}) = \Gamma_{\mathfrak{R}} \backslash \mathfrak{t}(K_Q)$. This is useful for us because there is a ramified cover of graphs $\phi : \mathfrak{s}_Q(\mathfrak{R}) \to \mathfrak{c}_Q(\mathfrak{R})$, as $\Gamma_{\mathfrak{R}} \leq \mathcal{N}$ is normal. In this context, a ramified cover is, by definition, a simplicial map ϕ that induces a surjective map from the set of neighbors of each vertex v to the set of neighbors of the image $\phi_V(v)$. To compute the valency in $\mathfrak{s}_Q(\mathfrak{R})$ of the vertex v' corresponding to an order $\mathfrak{R}' \in \Psi$ with $\widehat{\mathfrak{R}}'_Q = \mathrm{End}_{\hat{\mathcal{O}}_Q}(\widehat{\Lambda}'_Q)$, we need to observe that matrices in the stabilizer $\mathfrak{R}'(X)^*$ can be viewed as invertible linear maps on the vector space $\widehat{\Lambda}'_Q/\pi_Q\widehat{\Lambda}'_Q \cong \mathbb{F}(Q)^2$, where π_Q is a local uniformizer. As usual, this induces an action by Moebius transformations on the projective space $\mathbb{P}^1(\mathbb{F}(Q))$. The neighbors of v' in the S-graph are in correspondence with the orbits of the latter action (cf. [5, §5]).

In the notations of (6.1), we have $\psi\mathfrak{E}[B,B']\psi^{-1} = \mathfrak{E}[-B',-B]$. When the divisor $B-B'$ is effective and non-zero, this allows us to assume that B has positive degree, so that $\mathfrak{L}^{-B}(X)=\{0\}$, whence the ring of global sections has the following form:

$$\mathfrak{E}[B,B'](X) = \begin{pmatrix} \mathbb{F} & \mathfrak{L}^{B'}(X) \\ 0 & \mathbb{F} \end{pmatrix}. \tag{6.2}$$

Next result is an immediate consequence:

Lemma 6.9. *Let $\mathfrak{E}=\mathfrak{E}[B,B']$, with $B-B'$ an effective non zero divisor as above, i.e., $\mathfrak{E}$ is a non-maximal Eichler order. Assume B has positive degree. Then ∞ is left invariant by the $\mathfrak{E}(X)^*$-action on the projective line $\mathbb{P}^1\big(\mathbb{F}(Q)\big)$. The remaining orbits are in correspondence with $\mathbb{F}^*\backslash(\mathbb{F}(Q)/V)$, where*

$$V = V(B',Q) = \mathfrak{L}^{B'}(X)/\mathfrak{L}^{B'-Q}(X).$$

In particular, if $q=\sharp\mathbb{F}$, the valency of the corresponding vertex $v_{\mathfrak{E}}$ of $\mathfrak{s}_Q(\mathfrak{E})$ is given by

$$\mathrm{val}(v_{\mathfrak{E}}) = 2 + (q-1)^{-1}\Big(q^{\deg(P)-\dim_{\mathbb{F}} V(B',Q)} - 1\Big).$$

Corollary 6.9.1. *In the notations of the lemma, if $\deg P = 1$ the valency is 2 or 3.*

Next result follows from applying, to both B' and $B'-Q$, Riemann–Roch Theorem:

Corollary 6.9.2. *In the notations of the lemma, the valency is 2 whenever $\deg(B') \geq 2g$.*

The same holds for split maximal orders, when $B=B'$ has positive degree, or when it is non-principal of degree 0. When B is principal, the computation is more involved, but the valency is 1 when $\deg(Q)=1$, see [5] for details.

7. Proof of Theorem 3.1 and Theorem 3.2

In all that follows, we let ψ and δ_f be as in Equation (6.1), and use the following straightforward identities:

$$\psi\delta_f\mathfrak{E}[B,D]\,(\psi\delta_f)^{-1} = \mathfrak{E}[\mathrm{div}(f)-D,\mathrm{div}(f)-B] \tag{7.1}$$

and

$$\delta_f\mathfrak{E}[B,D]\delta_f^{-1} = \mathfrak{E}[B-\mathrm{div}(f),D-\mathrm{div}(f)]. \tag{7.2}$$

To simplify the proofs of our main results, we subdivide them in several lemmas that take care of the individual borderline cases.

Lemma 7.1. *If $P_1, P_2, P_3 \in |\mathbb{P}^1|$ denote three diferent points of degree 1, the genus $\mathbb{O}_{P_1+P_2}$ contains only split orders, while the genus $\mathbb{O}_{P_1+P_2+P_3}$ contains a unique conjugacy class of non-split orders.*

Proof. According to Lemma 6.2, the P_1-variants of the maximal order $\mathfrak{D}_0 = \mathbb{M}_2(\mathcal{O}_X)$ containing the idempotent η are the orders of the form $\mathfrak{D}_{rP_1}$, and they are all in a maximal line $\mathfrak{m}$, as shown in Figure 7.1.B. Furthermore, the classifying graph $\mathfrak{c}_{P_1}(\mathfrak{D}_0)$ is a ray, as shown in Figure 7.1.A, and there is a two-to-one ramified cover $\alpha : \mathfrak{m} \to \mathfrak{c}_{P_1}(\mathfrak{D}_0)$. (cf. Lemma 6.5 and Lemma 6.3). Note that any order in a genus $\mathbb{O}_{B+P}$, with B supported away from P, can be uniquely written as the intersection of two P-neighbors in $\mathbb{O}_B$ (cf. Proposition 5.4). The main trick in this proof is taking advantage from this fact to export most of the preceding description to the next genus, adding one place at a time. For instance, the edges of $\mathfrak{c}_{P_1}(\mathfrak{D}_0)$ correspond precisely to conjugacy classes of orders in the genus $\mathbb{O}_{P_1}$. Each has a representative in $\mathfrak{m}$. We conclude that every order in this genus is split. More precisely, all conjugacy classes in the latter genus have a representative in the following set:

$$\{\mathfrak{E}[P_1, 0], \mathfrak{E}[2P_1, P_1], \mathfrak{E}[3P_1, 2P_1], \dots\}. \tag{7.3}$$

To continue the proof, we draw the classifying graph $\mathfrak{c}_{P_2}(\mathbb{O}_{P_1})$. In order to accomplish this, we replace the representatives in (7.3) by a set of P_2-variants. We obtain $\left[\mathfrak{E}[P_1 + nP_2, nP_2]\right] = \left[\mathfrak{E}[(n+1)P_1, nP_1]\right]$ from Equation (7.2). We write b_n to denote this common class. By Corollary 6.9.2, all vertex valencies equal 2, as in Figure 7.1.C. Edges in the latter graph correspond to conjugacy classes in the genus $\mathbb{O}_{P_1+P_2}$. Note that, again, η is a common global section for all the vertices, and therefore also for all the edges of this graph. This proves that every order in the genus $\mathbb{O}_{P_1+P_2}$ splits. A set of representatives for all conjugacy classes is

$$\{\mathfrak{E}[P_1, -P_2], \mathfrak{E}[P_1 + P_2, 0], \mathfrak{E}[P_1 + 2P_2, P_2], \mathfrak{E}[P_1 + 3P_2, 2P_2], \dots\}.$$

It is easy to draw the grid corresponding to each of these orders and deduce which is the corresponding line based on the orders containing each P_2-face. For example, consider the order $\mathfrak{E}[P_1, -P_2]$. The maximal orders containing it are those of the form $\mathfrak{D}_B$ for $-P_2 \leq B \leq P_1$, which gives $B \in \{0, P_1, -P_2, P_1 - P_2\}$. The first two correspond to the vertices c_0 and c_1 in $\mathfrak{c}_{P_1}(\mathfrak{D}_0)$. The same holds for the second pair. We conclude that both P_2-faces of this order correspond to the vertex b_0 in $\mathfrak{c}_{P_2}(\mathbb{O}_{P_1})$, and therefore $\mathfrak{E}[P_1, -P_2]$ corresponds to the half edge in Figure 7.1.C.

We need to compute one more graph in the study of the genus $\mathbb{O}_{P_1+P_2+P_3}$. In Figure 7.1.D, we draw $\mathfrak{c}_{P_3}(\mathbb{O}_{P_1+P_2})$. Again, we have representatives for all vertices $d_n = \left[\mathfrak{E}[P_1 + nP_3, nP_3 - P_2]\right] = \left[\mathfrak{E}[nP_2 + P_1, (n-1)P_2]\right]$ in the maximal path corresponding to η, but now the vertex d_0 has valency 3 in the S-graph, as follows from Lemma 6.9.

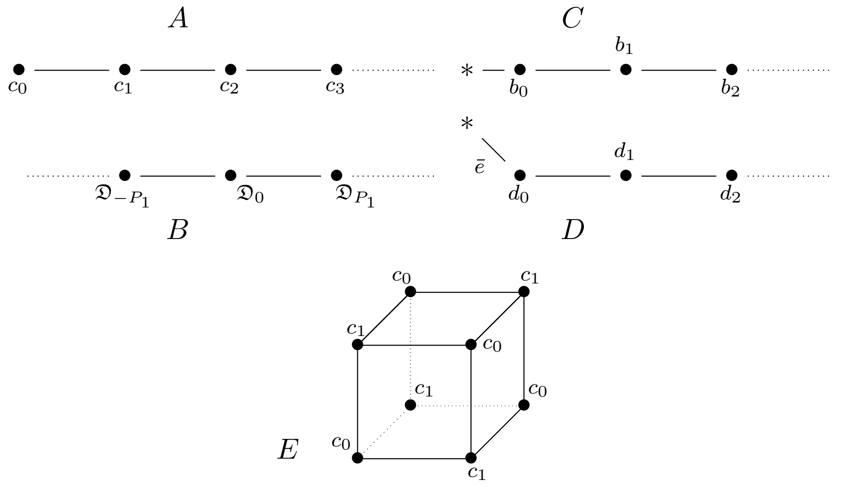

FIGURE 7.1. All graphs used in the proof of Lemma 7.1.

Two of its edges, say e' and e'', are represented by the edges joining $\mathfrak{E}[P_1, -P_2]$ with $\mathfrak{E}[P_1 + P_3, P_3 - P_2]$ and $\mathfrak{E}[P_1 - P_3, -P_3 - P_2]$, respectively, and they coincide in the classifying graph, but the third edge must remain distinct there, as the covering is only two-to-one. This implies that the S-graph, and therefore also the classifying graph, has an edge sticking out of the maximal path, that can only be a half edge in the latter. We claim that the corresponding Eichler order $\mathfrak{E}$ cannot be split. Since the corresponding edge joins the vertex d_0 to itself, the corresponding grid must look like the one in Figure 7.1.E, where we label each vertex with the corresponding class. In fact, we can assume that $\mathfrak{D}_0$ is a vertex of a concrete grid in this class. If it where split, we would have an idempotent global section in $\mathfrak{E}(X) \subseteq \mathfrak{D}_0(X) = \mathbb{M}_2(\mathbb{F})$. By an $\mathbb{F}$-rational change of basis, we can assume it to be η. In that case, each neighbor of $\mathfrak{D}_0$ must be $\mathfrak{D}_{P_i}$ or $\mathfrak{D}_{-P_i}$, for $i = \{1, 2, 3\}$, but no choice of the signs gives us the correct configuration of classes, whence we must conclude that the order is non-split. $\square$

Recall that the S-graph of a maximal order, as defined after Remark 6.8, is combinatorially finite, as defined in Section 1 (cf. Theorem S). We make the latter definition precise by considering as cusps all the images of rays $\gamma : \mathfrak{i}_{0,\infty} \to \mathfrak{g}$, where $\gamma_V(n_i)$ has valency 2 for $i \geq 1$, and the valency of $\gamma_V(n_0)$ is different from 2. These would be typically all cusps in the graph. We make an exception if $\mathfrak{g}$ looks like the classifying graph in Figure 7.1.C, where we assume the initial vertex of the only cusp is $\gamma_V(n_0) = b_0$, or when $\mathfrak{g}$ is a maximal path. In the latter case, we assume there are two cusps and we choose an arbitrary point as the initial vertex of either cusp.

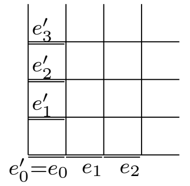

Figure 7.2. Horizontal neighbors are P-neighbors, while vertical neighbors are P'-neighbors.

Example 7.2. The procedure to compute the quotient graphs in the preceding proof can be iterated to describe the classifying graph at P_∞ for every genus of the form $\mathbb{O}_{P_1+\cdots+P_n}$, where $P_1,\dots,P_n,P_\infty \in |\mathbb{P}^1|$ are different points of degree 1. Note that $n \leq q = \sharp\mathbb{F}$. In every step, almost all edges in the cusp of the previous step become vertices in a cusp of the new graph, which is therefore unique. In fact, Equation (7.2), with $\mathrm{div}(f) = n(P'-P)$, tells us that δ_f sends the edge e_n in Figure 7.2 to the edge e'_n. The square between e'_n and e'_{n+1} corresponds to an edge in the next step.

Lemma 7.3. *For any arbitrary smooth projective curve X, and for any closed point $P \in |X|$, there are infinitely many conjugacy classes of non-split orders in the genus $\mathbb{O}_{2P}$.*

Proof. Let $\mathfrak{E}$ denote a fix Eichler order of level $2P$ and let $\mathfrak{D} = \mathfrak{D}_B \supseteq \mathfrak{E}$ be a maximal order. Consider a ray of the classifying graph whose vertices correspond to the classes $a_n = [\mathfrak{D}_{B+nP}]$, as in Lemma 6.2. Note that Corollary 6.9.2 proves that this ray contains a cusp, which, by redefining B, we can assume that look as in Figure 7.3.A. Moreover, by Lemma 6.9, we can further assume that, for $n \geq 1$, every order in the class $a_n = [\mathfrak{D}_{B+nP}]$ has precisely one neighbor in the class a_{n+1} and all the remaining neighbors in the class a_{n-1}. Recall that the Eichler orders $\mathfrak{E}' \in \Psi(\mathfrak{E}, P)$ are in natural bijection with the lines of length 2 in the Bruhat–Tits tree $\mathfrak{t}(K_P)$ (cf. Proposition 5.4). In particular, for every integer $n > 1$ there is an Eichler order contained precisely in one order of the class a_n and in two orders of the class a_{n-1}, located in the corresponding branch as depicted in Figure 7.3.B. We claim that every such order $\mathfrak{E}$, for $n > -\deg(B)$, is non-split. They are pairwise non-conjugate, whence the result follows from our claim. Assume $\mathfrak{E}$ is split. Replacing by a conjugate if needed, we can assume $\mathfrak{E} = \mathfrak{E}[D, D']$, for some pair of divisors satisfying $D - D' = 2P$, or equivalently $D' = D - 2P$. In this case, the maximal orders in the branch must equal $\mathfrak{D}_D$, $\mathfrak{D}_{D-P}$ and $\mathfrak{D}_{D-2P}$, respectively, whence $D - P = D' + P$ must be linearly equivalent to either $B + nP$ or $-(B + nP)$. Permuting D and $-D'$ if needed, we can assume $D - P$ is linearly equivalent to $B + nP$, and therefore it has positive degree.

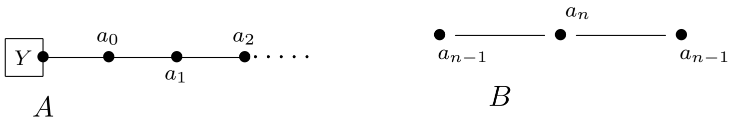

FIGURE 7.3. The graphs used to prove Lemma 7.3. The box marked with a "Y" denotes a connected subgraph.

This implies that the integers $\deg(D)$ and $\deg(D-2P)$ have different absolute values, whence the orders corresponding to the divisors D and $D-2P$ must fail to be conjugate. The contradiction concludes the proof. ☐

Example 7.4. Consider the case $X=\mathbb{P}^1$, and let $P,Q\in X$ be two different points of degree 1. Set $U=X\smallsetminus\{P\}$, and fix a representative $\mathfrak{E}\in\mathbb{O}_{2P}$. The conjugacy classes in $\mathbb{O}_{2P}$ are canonically in bijection with either, the vertices of the classifying graph $\mathfrak{c}_Q(\mathfrak{E})$ or the $\mathcal{N}(\mathfrak{D},P)$-orbits of length 2 lines in $\mathfrak{t}(K_P)$, for any maximal order $\mathfrak{D}\supseteq\mathfrak{E}$ (cf. Section 5). In this particular case, such orbits of lines are in bijection with the simplicial maps of the form $\gamma:\mathfrak{i}_{0,2}\to\mathfrak{c}_P(\mathfrak{D})$, up to reverse, that can be lifted to paths in Bruhat–Tits tree. It is easy to see that lines in the same orbit define the same map, so we prove the converse. For this we need to describe all such maps, which can be done by looking at Figure 7.1.A. Note that a simplicial map Γ can be described in this setting by the triplet $E(\gamma)=(\gamma_V(n_0),\gamma_V(n_1),\gamma_V(n_2))$. The simplicial map satisfying $E(\gamma)=(c_{n+1},c_n,c_{n+1})$, for $n\geq 1$, cannot be lifted to a path, as any order in the class c_n has a unique neighbor in the class c_{n+1}. The map for which $E(\gamma)=(c_{n-1},c_n,c_{n-1})$ defines a unique conjugacy class since the stabilizer of a vertex in the class c_{n+1} acts transitively on pairs of neighbors in c_n, as this action corresponds to the action by linear maps on the standard affine part of the projective line over the residue field. A similar, but simpler argument works for the map satisfying $E(\gamma)=(c_{n-1},c_n,c_{n+1})$. The path where $E(\gamma)=(c_1,c_0,c_1)$ needs to be consider separately, but again, the stabilizer of a vertex in c_0 is shown to act 2-transitively (or even 3-transitively) on its neighbors, as the corresponding action is the one by Moebius transformations on the projective line. The latter simplicial map corresponds to the class of the split Eichler order $\mathfrak{E}[P,P]$. We denote its class by e_0. If $E(\gamma)=(c_{n-1},c_n,c_{n+1})$, for $n\geq 1$, we can choose the representative $\mathfrak{E}[(1+n)P,(n-1)P]$. We denote its class by e_n. For the case $E(\gamma)=(c_{n-1},c_n,c_{n-1})$, the corresponding order $\mathfrak{F}_n$ is not split. We denote its class by f_n. This gives us all vertices of the classifying graph. To find the edges we study the grids corresponding to orders of level $2P+Q$ (see Figure 7.4.A). These correspond to simplicial maps of the form $\gamma:\mathfrak{i}_{0,2}\to\mathfrak{c}_P(\mathfrak{E}')$, for an Eichler order $\mathfrak{E}'$ of level Q. The corresponding classifying graph is depicted in Figure 7.1.C. The analysis

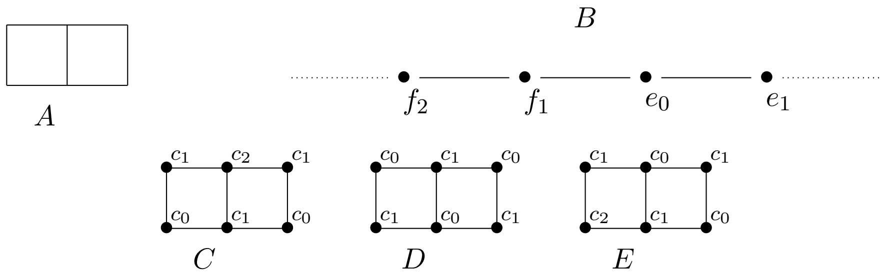

Figure 7.4. A 2×1 grid (A), The graph in Example 7.4. (B), and the classes of the orders in the grids corresponding to each on the central edges (C-E).

is very similar in this case, with every stabilizer of a vertex in b_n acting 2-transitively on the neighbors in the class b_{n-1}. A few differences must be noted however. The map corresponding to the triplet (b_1, b_0, b_1) cannot be lifted in this case, while the presence of a half-edge, force us to consider the triplets (b_0, b_0, b_1) and (b_0, b_0, b_0). These two triplets correspond to the grids in Figure 7.4.E and Figure 7.4.D, respectively, while the grid depicted in Figure 7.4.C corresponds to the triplet (b_0, b_1, b_0). These three grids give us the three central edges of the classifying graph in Figure 7.4.B.

Proof of Theorem 3.2. Consider an effective divisor D on $X = \mathbb{P}^1$. Assume first that $0 \leq D \leq P_1 + P_2$ for two degree-1 places $P_1 \neq P_2$ in $|X|$, and set $\mathfrak{E} \in \mathbb{O}_D$. Equivalently, there exists a paralellotope $\mathbb{S}(\mathfrak{E}')$, for an order $\mathfrak{E}' \in \mathbb{O}_{P_1+P_2}$, which is a concrete 1-times-1 square grid, where horizontal neighbors represent P_1-neighbors and vertical neighbors represent P_2-neighbors, satisfying $\mathbb{S}(\mathfrak{E}) \subseteq \mathbb{S}(\mathfrak{E}')$, or, what is the same, $\mathfrak{E}' \subseteq \mathfrak{E}$. Lemma 7.1 shows that $\mathfrak{E}'$ splits, whence its ring $\mathfrak{E}'(X)$ of global sections contains a non-trivial idempotent. The same must happen for the overorder $\mathfrak{E}$. This concludes the proof in this case. Assume now that D is not bounded by the sum of two degree-1 places. Then $D \geq B$ for some divisor B satisfying one of the following conditions:

(1) $B = P$, for a place P of degree 2 or larger,
(2) B is the sum of three degree-1 places, or
(3) $B = 2P$, for some place P of degree 1.

We conclude by applying Lemma 6.7, Lemma 7.1 or Lemma 7.3, respectively, and following a similar line of reasoning. □

In what follows, we need to quote a version of [28, Ch. II, Thm. 9], a result by Serre. Due to our slightly different setting, we provide a sketch of the proof. See the reference for details.

Lemma 7.5. *Let $\mathfrak{D}$ be a maximal order. The classifying graph $\mathfrak{c}_P(\mathfrak{D}) = \mathcal{N}(\mathfrak{D}, P)\backslash\mathfrak{t}(K_P)$, where $P \in |X|$, is obtained by attaching a finite number of cusps, or infinite half lines, to a certain finite graph Y. The vertices in the cusps correspond to split orders.*

Proof. We know that the vertices in $\mathfrak{c}_P(\mathfrak{D})$ correspond to conjugacy classes of P-variants of $\mathfrak{D}$, that the vertices corresponding to splits orders are precisely the vertices in a finite number of rays and maximal paths, and that every vertex sufficiently far into any of these paths has valency 2 by Corollary 6.9.2, whence almost all of them are located in a finite number of cusps. It remains to show that the graph Y that is left when the cusps are removed is finite. It suffices, therefore, to find a constant bound for the distance from every P-variant $\mathfrak{D}'$ of $\mathfrak{D}$ to the closest split P-variant $\mathfrak{D}''$.

To find this bound, we write every maximal order in the form $\mathfrak{D}' = \mathfrak{D}_\Lambda$, for some bundle Λ in K^2. Recall that $\mathfrak{D}_{\mathfrak{L}\Lambda} = \mathfrak{D}_\Lambda$ for every invertible bundle $\mathfrak{L}$, so we can assume that the bundle is chosen in a way that $2g - 2 < \deg(\Lambda) \leq 2g$, where g is the genus of X. Riemann–Roch Theorem (for higher dimensional bundles) gives us the inequality $\dim \Lambda(X) \geq \deg(\Lambda)+2(1-g) > 0$. This implies that Λ contains a line bundle $\mathfrak{F}$ with non-trivial global sections. Let $W \subseteq K^2$ be the K-span of $\mathfrak{F}$. Replacing $\mathfrak{F}$ by $W \cap \Lambda$ if needed, we can assume that $\mathfrak{F}' = \Lambda/\mathfrak{F}$ is a line bundle in the space K^2/W. Since $\mathfrak{F}$ has nontrivial global section, we have $\deg(\mathfrak{F}) \geq 0$, whence

$$N(\Lambda; \mathfrak{F}) := \deg(\mathfrak{F}) - \deg(\mathfrak{F}') = 2\deg(\mathfrak{F}) - \deg(\Lambda) \geq 2g.$$

If $\mathfrak{F}^v$ and Λ^v are the P-variants of $\mathfrak{F}$ and Λ satisfying $\widehat{\mathfrak{F}}^v_P = \pi_P^{-1}\widehat{\mathfrak{F}}_P$ and $\widehat{\Lambda}^v_P = \widehat{\Lambda}_P + \widehat{\mathfrak{F}}^v_P$, then $N(\Lambda^v; \mathfrak{F}^v) = N(\Lambda; \mathfrak{F}) + \deg(P)$, and the maximal order corresponding to Λ^v is a P-neighbor of $\mathfrak{D}_\Lambda$. By successively taking P-neighbors in this way, we reach, in no more that $4g$ steps, a pair $(\Lambda^V, \mathfrak{F}^V)$ that satisfies $N(\Lambda^V; \mathfrak{F}^V) > 2g - 2$. Now [28, §II.2.2, Prop. 7] proves that Λ^V splits, whence so does the order $\mathfrak{D}_{\Lambda^V}$. □

Lemma 7.6. *Consider an arbitrary smooth curve X, and assume $P_1, \dots, P_n$ are different closed points in X. Then the genus $\mathbb{O}_{P_1+\cdots+P_n}$ contains only a finite number of conjugacy classes of non-split orders.*

Proof. The graph $\mathfrak{c}_P(\mathbb{O}_0)$ has finitely many connected components, and each one is a graph like the one described in Lemma 7.5. This proves the case $n = 0$. Next assume that the statement holds in the case $n = t$, i.e., all but finitely many vertices in $\mathfrak{c}_{P_{t+1}}(\mathbb{O}_{P_1+\cdots+P_t})$ correspond to conjugacy classes of split orders. Consider a cusp like the one depicted in Figure 7.3.A. The induction hypothesis tells us that, for m large enough, the vertex a_m corresponds to a conjugacy class of split orders. One representative is an order of the form $\mathfrak{E} = \mathfrak{E}[B, B']$, by definition of split order. Certainly $B - B' = \sum_{i=1}^t P_i$. Furthermore, two neighbors are the orders

$\mathfrak{E}' = \mathfrak{E}[B + P_{t+1}, B' + P_{t+1}]$ and $\mathfrak{E}'' = \mathfrak{E}[B - P_{t+1}, B' - P_{t+1}]$. Additionally, we can assume without loss of generality that $\deg(B') > 0$, for large m. In that case, $\mathfrak{E}'$ and $\mathfrak{E}''$ cannot be conjugates, as the former is contained in $\mathfrak{D}_{B+P_{t+1}}$, while the second is contained only in orders of the form $\mathfrak{D}_{B''}$ with $B' - P_{t+1} \leq B'' \leq B - P_{t+1}$, none of which is conjugate to $\mathfrak{D}_{B+P_{t+1}}$ by Lemma 6.3. We conclude that one is a representative of the class corresponding to a_{m+1}, while the other is a representative of the class corresponding to a_{m-1}. A straightforward computation shows that either edge corresponds to a split order, and the result follows. □

Proof of Theorem 3.1. When the effective divisor D is multiplicity-free, the result is a direct applications of the preceding lemma. Assume this is not the case. Then $2P \leq D$ for some place $P \in |X|$. We conclude from the combinatorics of the tree that every order in $\mathbb{O}_{2P}$ contains some order in $\mathbb{O}_D$, while each order in $\mathbb{O}_D$ is contained into finitely many orders of the genus $\mathbb{O}_{2P}$. As splitting is equivalent to having an idempotent global section, every order containing a split suborder is split. Lemma 7.3 tell us that there is an infinite set of pairwise non-conjugate non-split orders in $\mathbb{O}_{2P}$, so the same holds for $\mathbb{O}_D$. The result follows. □

8. Computing fundamental domains for congruence subgroups of $\mathbf{GL_2}(A)$

In all of this section, we set $A = \mathbb{F}[t]$, $K = \mathbb{F}(t)$, and we let $K_\infty = \mathbb{F}((t^{-1}))$ be the completion of K at P_∞. We set $\widehat{\mathcal{O}}_\infty = \mathbb{F}[[t^{-1}]]$, the ring of integers of K_∞, and we let $\nu = \nu_{P_\infty} = -\deg$ denote the valuation map on K_∞. In particular, we assume $\deg(0) = -\nu(0) = -\infty$. The same conventions apply to the absolute value $x \mapsto |x| = |x|_{P_\infty}$. We identify the Bruhat–Tits tree for $\mathrm{SL}_2(K_\infty)$ with the Ball-tree $\mathfrak{g}$, as in Section 3. In this tree, two balls are neighbors if one is a maximal proper sub-ball of the other. See [1, §4] for details. By an end of a graph $\mathfrak{h}$, we mean an equivalence class of rays $\rho : \mathfrak{i}_{0,\infty} \to \mathfrak{h}$, where two rays ρ and ρ' are equivalently precisely when $\rho_E(a_n) = \rho'_E(a_{n+t})$ for a fixed integer t and every big enough positive integer n. There is a natural bijection between the ends of the Ball tree, or its subgraphs, and the elements of $\mathbb{P}^1(K_\infty)$. We say that a subgraph $\mathfrak{h}'$ contains an end $a \in \mathbb{P}^1(K_\infty)$ if there is at least one ray $\rho : \mathfrak{i}_{0,\infty} \to \mathfrak{h}'$ in the corresponding equivalence class. We write $a \in \mathfrak{h}'$ in this case. As it is the case for any tree, $\mathfrak{g}$ contains a unique path $\gamma_{a,b}$ between any pair (a, b) of vertices or ends. The smallest subtree containing any set S of ends and vertices, like $\mathfrak{s}$ in Theorem 3.3, is the graph-theoretical union $\bigcup_{(a,b)\in S\times S} \mathrm{Im}(\gamma_{a,b})$.

Recall that, to define quotient graphs in full generality, it is convenient to work with the barycentric subdivision. We extend this convention in the sequel to fundamental domains. To define a fundamental domain, we

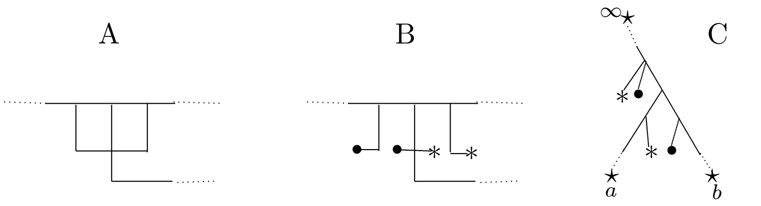

FIGURE 8.1. Quotient graph (A), a surgery (B) and a corresponding choice of a fundamental domain (C). Bullets and asterisks denote two corresponding pairs of nonvertices. Stars denote ends.

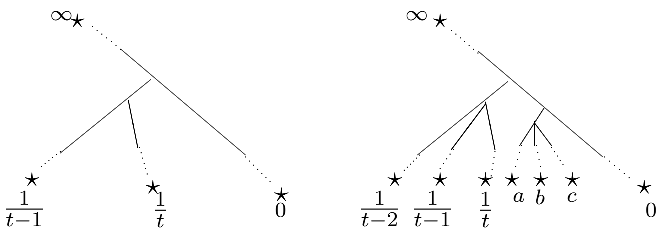

FIGURE 8.2. The global orders in Example 8.1. Here, $a = \frac{1}{t(t-1)}$, $b = \frac{1}{t(t-2)}$ and $c = \frac{1}{(t-1)(t-2)}$.

perform a finite number of surgeries to transform the quotient graph $\mathfrak{q}$ into a tree. See Figure 8.1. A surgery consists on replacing an edge by a pair of half edges, provided that the resulting graph is still connected. Once we get a tree $\mathfrak{q}'$, we fix a vertex v, choose a preimage $\widetilde{v}$ in the Bruhat–Tits tree, and successively lift the path from v to any vertex or nonvertex in $\mathfrak{q}'$ in a consistent fashion. The union of the images of such liftings is the fundamental domain under consideration. See Figure 8.1.C. Note that the quotient graph can be recovered from the fundamental domain and the pairs of corresponding nonvertices. This is done by gluing the latter in an obvious manner.

Example 8.1. Assume $A = \mathbb{F}[t]$. In Figure 8.2 we can see the minimal subgraph $\mathfrak{s}$ containing 0, ∞ and each M^{-1} with M dividing N, for $N = t(t-1)$ or $N = t(t-1)(t-2)$. In the latter case we assume $\mathrm{char}(\mathbb{F}) > 2$.

Proof of Theorem 3.3. Denote by $P_i \in |\mathbb{P}^1|$ the point corresponding to λ_i, or equivalently, assume $\mathrm{div}(t-\lambda_i) = P_i - P_\infty$. Repetitive use of Example 7.2 shows that the classifying graph $\mathfrak{c}_P(\mathbb{O}_{P_1+\cdots+P_n})$ has a unique cusp. Consider the natural cover $\phi : \mathfrak{s}_P(\mathbb{O}_{P_1+\cdots+P_n}) \twoheadrightarrow \mathfrak{c}_P(\mathbb{O}_{P_1+\cdots+P_n})$ defined in Section 5. Note that $\Gamma_0(N) = \Gamma_{\mathfrak{E}}$ for a suitable choice of $\mathfrak{E}$. The cover ϕ is at most m to one, where $m = \sharp\,(\mathcal{N}/\Gamma_{\mathfrak{E}})$.

Assume a matrix ρ satisfies $\rho\mathfrak{D}\rho^{-1} = \mathfrak{D}$ for some maximal A-order. Then, as A is principal, we might assume $\mathfrak{D} = \mathbb{M}_2(A)$. If $\Lambda = A^2 \subseteq K^2$, then $\rho\Lambda = f\Lambda$ for some scalar $f \in K$. It follows that $\rho \in K^*\mathfrak{D}^*$. We conclude that $\Gamma_{\mathfrak{E}}$ is the point-wise stabilizer, in $\mathrm{GL}_2(K)$, of the grid $\mathbb{S}(\mathfrak{E})$. Since $\mathcal{N}$ can only act on the grid by switching the endpoints of each local line, it follows that $\sharp(\mathcal{N}/\Gamma_{\mathfrak{E}}) \leq 2^n$.

It suffices, therefore, to prove that the restriction of ϕ to the tree $\mathfrak{s}$ is an injection. Consequently, the result follows from next statement, which we prove following the techniques in [19]:

Lemma 8.2. *The vertices in $\mathfrak{s}$ are in different $\Gamma_0(N)$-orbits.*

Proof. Note that, if $N = 1$, or equivalently if $\mathfrak{E}$ is maximal, the lemma is essentially Nagao's result, which follows from the description of the graph depicted in Figure 7.1.A. We assume throughout that this is not the case. We use $B_x^{|s|}$ for the ball of radius $|\pi|^s$ centered at $x \in K_\infty$, where $\pi = t^{-1}$ is a uniformizing parameter at P_∞. Set $B_0 = B_0^{|0|}$, the ball corresponding to the local maximal order $\widehat{\mathfrak{D}}_{0,1,P_\infty}$, in the notations used to describe $\mathfrak{t}(K_P)$ in Section 5. Let $B_1 = B_{x_1}^{|r_1|}$ and $B_2 = B_{x_2}^{|r_2|}$ be two vertices in $\mathfrak{s}$, where the center x_1 is either 0 or the multiplicative inverse of a proper monic divisor of N, and the same holds for x_2. Assume that there exists a matrix $g = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_0(N)$ satisfying $g \cdot B_1 = B_2$. Set $h_1 = \left(\begin{smallmatrix} x_1 & \pi^{r_1} \\ 1 & 0 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} x_2 & \pi^{r_2} \\ 1 & 0 \end{smallmatrix}\right)$, so that we have both $B_1 = h_1 \cdot B_0$ and $B_2 = h_2 \cdot B_0$. Then, for some $\lambda \in K_\infty^*$, we must have $h_2^{-1} g h_1 \in \lambda\,\mathrm{GL}_2(\widehat{\mathcal{O}}_\infty)$, since $K_\infty^*\,\mathrm{GL}_2(\widehat{\mathcal{O}}_\infty)$ is the stabilizer of B_0. By taking determinants, we get $2\nu(\lambda) = r_1 - r_2$. Hence, $r_1 - r_2$ is an even integer and $\widetilde{g} = \pi^{\frac{r_2-r_1}{2}} h_2^{-1} g h_1 \in \mathrm{GL}_2(\widehat{\mathcal{O}}_\infty)$. After a simple computation we have

$$\widetilde{g} = \begin{pmatrix} \pi^{\frac{r_2-r_1}{2}}(d + Ncx_1) & \pi^{\frac{r_2+r_1}{2}} Nc \\ \pi^{\frac{-r_1-r_2}{2}}(ax_1 - dx_2 + b - Ncx_1x_2) & \pi^{\frac{r_1-r_2}{2}}(a - Ncx_2) \end{pmatrix}. \tag{8.1}$$

We conclude that $\pi^{\frac{r_1-r_2}{2}}(a - Ncx_2), \pi^{\frac{r_2-r_1}{2}}(d + Ncx_1) \in \widehat{\mathcal{O}}_\infty$. On the other hand, the polynomials $a - Ncx_2$ and $d + Ncx_1$ either vanish or have non-positive valuations. This leaves us three alternatives:

(i) $r := r_1 = r_2$, together with $\nu(a - Ncx_2) = \nu(d + Ncx_1) = 0$,
(ii) $a = Ncx_2$ or
(iii) $d = -Ncx_1$.

The last two alternatives imply $\det(g) \notin \mathbb{F}^*$, so (i) must hold. The result follows if $x_1 = x_2$, as this implies $B_1 = B_2$. The same holds if $r \leq 0$, as $\nu(x_1), \nu(x_2) > 0$, since $B_1 = B_0^{|r|} = B_2$ in this case. We assume in the sequel that $x_1 \neq x_2$ and $r > 0$. From (8.1) and (i) we deduce the following facts:

(a) $a - Ncx_2 = a_0 \in \mathbb{F}^*$,
(b) $d + Ncx_1 = d_0 \in \mathbb{F}^*$,
(c) $Nc \in \pi^{-r}\widehat{\mathcal{O}}_\infty$, or equivalently $\deg(Nc) \le r$, and
(d) $a_0x_1 - d_0x_2 + b + Ncx_1x_2 = ax_1 - dx_2 + b - Ncx_1x_2 \in \pi^r\widehat{\mathcal{O}}_\infty$.

Note that x_1 and x_2 do not vanish simultaneously by the previous assumption. If we suppose that either $\nu(Ncx_1x_2) > 0$ or $x_1x_2 = 0$, then the dominant term on the left hand side of identity (d) is $b \in \mathbb{F}[t]$, unless it vanishes. As $r > 0$ we must conclude the latter. It follows that $g = \left(\begin{smallmatrix} a & 0 \\ Nc & d \end{smallmatrix}\right)$, in particular $a, d \in \mathbb{F}^*$. This can only mean $Ncx_2, Ncx_1 \in \mathbb{F}$, and then $c = 0$, as at least one element $x_i \in \{x_1, x_2\}$ is the inverse of a proper monic divisor of N. Such x_i can be written in the form $x_i = \pi^\nu + \varepsilon$, where $\nu := \nu(x_i)$ and $\nu(\varepsilon) > \nu$. In particular, there are two possibilities, either $r \le \nu$, in which case $0 \in B_i$, or $r > \nu$, and then every center of the ball has the same form. In the first case B_i is invariant by any diagonal matrix, so $B_2 = g \cdot B_1$ implies $B_1 = B_2$. In the second case $B_2 = g \cdot B_1$ means $B^{|r|}_{x_2} = B^{|r|}_{ax_1/d}$, so neither ball contains 0, and $a = d$, as both x_1^{-1} and x_2^{-1} are monic. Again we conclude $B_1 = B_2$.

Finally, assume that both $x_1, x_2 \neq 0$ and $\nu(Ncx_1x_2) \le 0$. We can assume $r > \max\{\nu(x_1), \nu(x_2)\}$ or we could redefine x_1 or x_2 as 0 and return to the preceding case. Let

$$\epsilon = b + Ncx_1x_2 \in -a_0x_1 + d_0x_2 + \pi^r\widehat{\mathcal{O}}_\infty \subseteq \pi\widehat{\mathcal{O}}_\infty. \tag{8.2}$$

By a simple computation, we get $\det(g) = a_0d_0 - \xi \in \mathbb{F}^*$, where $\xi = Nc(a_0x_1 - d_0x_2 + \epsilon) \in \mathbb{F}$. If $\xi = 0$, we have that $c = 0$ or

$$Nc + bx_1^{-1}x_2^{-1} = \epsilon(x_1x_2)^{-1} = d_0x_1^{-1} - a_0x_2^{-1}. \tag{8.3}$$

In the former case $b \in \pi\widehat{\mathcal{O}}_\infty$ by (8.2), so that $b = 0$ and we argue as in the previous paragraph. In the latter case, Equation (8.3) implies that x_1^{-1} divides x_2^{-1} and inversely, as either divides N, whence $B_1 = B_2$.

Assume now that $\xi \neq 0$, so by applying, successively, (c), the definition of ξ, the definition of ϵ, and (d), we prove the following chain of inequalities:

$$r \ge -\nu(Nc) = \nu(a_0x_1 - d_0x_2 + \epsilon) = \nu(a_0x_1 - d_0x_2 + b + Ncx_1x_2) \ge r.$$

From here we conclude the following identity:

$$\nu(a_0x_1 - d_0x_2 + \epsilon) = -\nu(Nc) = r. \tag{8.4}$$

In this case we have

$$|\pi^r| = |a_0x_1 - d_0x_2 + \epsilon| = |x_1x_2||a_0x_2^{-1} - d_0x_1^{-1} + \epsilon(x_1x_2)^{-1}| \ge |x_1x_2|,$$

as the second factor in the third expression is a polynomial by definition of ϵ. On the other hand, the hypothesis $\nu(Ncx_1x_2) \le 0$ implies $|x_1x_2| = |Ncx_1x_2||Nc|^{-1} \ge |\pi^r|$. Thus, $r = \nu(x_1x_2)$, which together with (8.4) shows that $\sigma = a_0x_2^{-1} - d_0x_1^{-1} + b(x_1x_2)^{-1} + Nc$ is a non zero constant polynomial.

But σ is divisible by $\gcd(x_1^{-1}, x_2^{-1})$, and therefore $\gcd(x_1^{-1}, x_2^{-1}) = 1$. If $\epsilon \neq 0$ we conclude that $b(x_1x_2)^{-1} + Nc$ is a multiple of $(x_1x_2)^{-1}$. By the strong triangular inequality, $|\sigma| = 1$ implies

$$|a_0x_1^{-1} - d_0x_2^{-1}| = |b(x_1x_2)^{-1} + Nc| \geq |x_1x_2|^{-1}.$$

The preceding inequality is impossible by a degree argument. To finish the proof we consider $\epsilon = 0$, in which case $|a_0x_1^{-1} - d_0x_2^{-1}| = 1$ by (8.4). As the polynomials are monic, this is only possible when $a_0 = d_0$ and $|x_1 - x_2| \leq |\pi^r|$. We conclude that $B_1 = B_2$. □

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