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Local constancy for reductions of two-dimensional crystalline representations

par Emiliano TORTI

Résumé. Nous prouvons l'existence du phénomène de constance locale pour les réductions modulo p^n (p impair) des représentations de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ de dimension 2 qui sont cristallines et irréductibles. Ces représentations dépendent de deux paramètres : une trace a_p et un poids k. Par exemple, elles apparaissent comme les représentations attachées aux formes modulaires classiques dont le niveau n'est pas divisible par p. Nous trouvons un résultat de constance locale (explicite) par rapport à a_p, en utilisant la théorie des (φ,Γ)-modules de Fontaine, son raffinement cristallin via les modules de Wach par Berger et leurs propriétés de continuité. Le résultat de constance locale par rapport à k (pour $a_p \neq 0$) découlera d'une étude locale de l'espace analytique rigide de Colmez qui paramétrise les représentations triangulines. Ce travail généralise certains résultats de Berger obtenus dans le cas résiduellement semi-simple.

Abstract. We prove the existence of local constancy phenomena for reductions in a general (odd) prime power setting of two-dimensional irreducible crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. These representations depend on two parameters: a trace a_p and a weight k. They appear for example in the context of classical modular forms of tame level. We find an (explicit) local constancy result with respect to a_p using Fontaine's theory of (φ,Γ)-modules, its crystalline refinement due to Berger via Wach modules and their continuity properties. The local constancy result with respect to k (for $a_p \neq 0$) will follow from a local study of Colmez's rigid analytic space parametrizing trianguline representations. This work extends some results of Berger obtained in the residually semi-simple case.

1. Introduction

Crystalline representations play a central role in the study of p-adic representations of the local absolute Galois group $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (see, for example, some density results due to Berger (see [6, Thm. IV.2.1]), Chenevier (see [18, Thm. A]), Colmez (see [21, §5.1]), and Kisin (see [30, Thm. 0.3])). We are interested in studying irreducible crystalline representations of $G_{\mathbb{Q}_p}$ of dimension two and in particular their reductions modulo prime powers.

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Let p be an odd prime, let $k \geq 2$ be an integer and $a_p \in \mathfrak{m}_{\mathbb{E}}$ where $\mathbb{E}$ is a finite extension of $\mathbb{Q}_p$, $\mathfrak{m}_{\mathbb{E}}$ denotes the maximal ideal of the ring of integers $\mathcal{O}_{\mathbb{E}}$ with residue field $k_{\mathbb{E}}$ and e will denote the ramification index of $\mathbb{E}$ over $\mathbb{Q}_p$. Fix once and for all a choice of a uniformizer, say $\pi_{\mathbb{E}}$. Let $D_{k,a_p} := \mathbb{E}e_1 \oplus \mathbb{E}e_2$ be the filtered φ-module whose structure is given by:

$$\varphi = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix} \text{ and a filtration } \mathrm{Fil}^i(D_{k,a_p}) = \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \\ \mathbb{E}e_1 & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k \end{cases}$$

By a theorem of Colmez and Fontaine (see [23, Thm. A]), there exists a unique crystalline irreducible $\mathbb{E}$-linear representation V_{k,a_p} of dimension two, with Hodge–Tate weights $\{0, k-1\}$ such that $D_{\mathrm{cris}}(V^*_{k,a_p}) = D_{k,a_p}$, where V^*_{k,a_p} denotes the $\mathbb{E}$-linear dual representation of V_{k,a_p}. By a result of Breuil (see [16, Prop. 3.1.1]), up to twist, any irreducible two-dimensional crystalline representation is isomorphic to V_{k,a_p} for some $k \geq 2$ and $a_p \in \mathfrak{m}_{\mathbb{E}}$.

These results give rise to the natural questions of whether it is possible to completely classify V_{k,a_p} in terms of k and a_p, and how V_{k,a_p} varies when the parameters k and a_p vary p-adically.

In general, classifying the representations V_{k,a_p} in characteristic zero turns out to be an hard problem even though some progress have been made in particular cases via the local Langlands correspondence. (e.g. see [31]). Nevertheless, much progress has been made in describing the semi-simple residual reductions of the representations V_{k,a_p} using different approaches. We will briefly recall the state of art in the residual case. Consider the $\mathbb{E}$-linear representation V_{k,a_p} and let T_{k,a_p} be a $G_{\mathbb{Q}_p}$-stable lattice inside V_{k,a_p}, we have an isomorphism $T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E} \cong V_{k,a_p}$ of $G_{\mathbb{Q}_p}$-modules. Denote by $\overline{V}_{k,a_p}$ the semi-simplification of $T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} k_{\mathbb{E}}$; by the Brauer–Nesbitt's theorem, the representation $\overline{V}_{k,a_p}$ does not depend on the chosen $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p}.

The problem of describing the representations $\overline{V}_{k,a_p}$ has been deeply studied by many authors via the p-adic and mod p Langlands correspondence (see for example [16] and [17]), via Fontaine's theory of (φ, Γ)-modules and its crystalline refinement via Wach modules (see for example [10]), via Kisin modules (see [5]) or via deformation theory (see for example [32])). However, the problem of classifying them is still open and only partial results are known (see for example [10, 12, 13] and several preprints of Arsowski such as [1] and [2]) although partial conjectures have been formulated (see [16, Conj. 1.5] and [27, Conj. 1.1]).

In order to try to describe the reductions $\overline{V}_{k,a_p}$, one different approach consists in finding isomorphisms between different residual representations of the form $\overline{V}_{k,a_p}$ when we let k and a_p vary p-adically. This approach has

been developed by Berger et al. with the so-called local constancy results both in the trace and in the weight (see [8, Thm. A and Thm. B] and for the case $a_p = 0$ see [10, Thm. 1.1.1]).

The purpose of this article is to extend Berger's result to a prime power setting. The main difficulty lies in keeping track of the Galois stable lattices involved in the congruences because no semi-simplification process is, a priori, allowed (or defined) for general reductions modulo prime powers. Hence, in proving the existence of such congruences, a dependency on a choice of the Galois stable lattices is expected; but as we will see later, in some cases, the result will be independent of such choice.

The first result of the article is the following local constancy result with respect to the trace, i.e. we fix the weight and we let the trace of the crystalline Frobenius vary p-adically:

Theorem 1.1 (Local constancy with respect to a_p)**.** *Let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$ and $k \geq 2$ be an integer. Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1)+m$, then for every $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p} inside V_{k,a_p} there exists a $G_{\mathbb{Q}_p}$-stable lattice T_{k,a'_p} inside V_{k,a'_p} such that*

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules;}$$

where $\alpha(k-1) = \sum_{n\geq 1} \lfloor \frac{k-1}{p^{n-1}(p-1)} \rfloor$.

This result will be proven in two steps. First we will prove that if a_p and a'_p are sufficiently p-adically close then it is possible to deform p-adically the Wach module attached to the representation T_{k,a_p} into a new Wach module which will correspond to the representation T_{k,a'_p}. Afterwards, we will prove that if the two Wach modules involved are p-adically close (in some sense that will be clarified later) then the corresponding representations T_{k,a_p} and T_{k,a'_p} will be p-adically close as well. We will refer to this feature as continuity property of the Wach modules.

The second result of the article is the following local constancy result with respect to the weight, i.e. we fix the trace of the crystalline Frobenius and we let the weight vary in a neighborhood of the weight space $\mathcal{W}$:

Theorem 1.2 (Local constancy with respect to k)**.** *Let $a_p \in m_{\mathbb{E}} - \{0\}$ for some finite extension $\mathbb{E}$ of $\mathbb{Q}_p$. Let $k \geq 2$ and $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ be fixed. Assume that*

$$k \geq (3v(a_p) + m) \cdot \left(1 - \frac{p}{(p-1)^2}\right)^{-1} + 1.$$

There exists an integer $r = r(k, a_p) \geq 1$ such that if $k' - k \in p^{r+m}(p-1)\mathbb{Z}_{\geq 0}$ then there exist $G_{\mathbb{Q}_p}$-stable lattices $T_{k,a_p} \subset V_{k,a_p}$ and $T_{k',a_p} \subset V_{k',a_p}$ such that

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k',a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

The idea is to prove that the representations V_{k,a_p} and V_{k',a_p} are respectively congruent modulo p^m to two representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ which fit into an analytic family of trianguline representations in the sense of Berger and Colmez (see [9]); as a consequence, the claims will follow from proving that if k and k' are sufficiently close in the weight space $\mathcal{W}$ then the representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ are p-adically close as well (in a sense that will be clarified precisely later in the article). The result constitutes a converse (in a particular crystalline case) to a non-published theorem of Wintenberger, also proven by Berger and Colmez (see [9, Thm. 7.1.1 and Cor. 7.1.2]), concerning the continuity property of the Sen periods and the Hodge–Tate weights.

Specializing the above theorems to the case $m = 1/e$, we get a slightly stronger result than the known local constancy results in the semi-simple residual case (see [8, Thm. A and Thm. B]); indeed our conclusions do not involve any semi-simplification process, so for example, being residually decomposable (i.e. direct sum of two characters modulo p) for some choice of lattice is also a locally constant phenomenon.

The motivation behind the study of local constancy phenomena modulo prime powers is two-fold. From a purely representation theoretical point of view, the interest in understanding reductions modulo prime powers of crystalline representations lies in the result of Berger on limits of crystalline representations (see [6]). To be more precise, Berger's result implies that if V is any p-adic representation of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights in a bounded interval I and if $\{V_i\}_{i\in I}$ is a countable family of crystalline representations with HT weights in I such that $T \equiv T_i \bmod p^i$, where T is a fixed $G_{\mathbb{Q}_p}$-stable lattice in V and T_i is a $G_{\mathbb{Q}_p}$-stable lattice in V_i, then V is also crystalline.

Moreover, we observe that a good source of examples for the crystalline representations of the form V_{k,a_p} comes from restriction at $G_{\mathbb{Q}_p}$ of Galois representations attached to classical modular forms of tame level. To be precise, let f be a classical normalized cuspidal eigenform in $S_k(\Gamma_0(N))$ where N is a positive integer prime with p and denote by ρ_f its attached p-adic Galois representation constructed by Deligne and Shimura. We define by $V_p(f) := \rho_f|_{G_{\mathbb{Q}_p}}$ the restriction of ρ_f at the decomposition group at p. It is well-known (see [33]), that under the mild hypothesis $a_p^2 \neq 4p^{k-1}$ (see [20]), the $G_{\mathbb{Q}_p}$-representation $V_p(f)$ is crystalline and moreover we have that $D_{\mathrm{cris}}(V_p(f)^*) = D_{k,a_p}$ and so $V_p(f) \cong V_{k,a_p}$ where a_p is the p-th coefficient of the q-expansion of f. A straightforward application of the results in this article consists in using the explicit local constancy results in the trace (see Theorem 1.1 and Corollary 4.6) to find upper and lower bounds for

the number of non-isomorphic classes of reductions modulo prime powers of modular crystalline representations of $G_{\mathbb{Q}_p}$ coming from classical modular forms of tame level.

The paper is organized as follows. In Section 2, we will recall the notions of (φ, Γ)-module of Fontaine and of Wach module and their main properties which will be used later in the article. In Section 3, we will recall the continuity property of Wach modules which will play a key role in proving the local constancy result in the trace. In Section 4, we will show how to p-adically deform Wach modules and we will state and prove the explicit local constancy result when the weight k is fixed and we let the trace of the crystalline Frobenius a_p vary. Finally, in Section 5, we are going to state and prove the local constancy result when we fix the trace of the crystalline Frobenius a_p and we let the weight k vary.

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2. Wach modules and crystalline representations

Let p be an odd prime and let $\mathbb{E} \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$. We denote by $\mathcal{O}_{\mathbb{E}}$ the ring of integers of $\mathbb{E}$, by $\pi_{\mathbb{E}}$ a uniformizer, by $k_{\mathbb{E}}$ the residue field and denote by e the ramification index of $\mathbb{E}$ over $\mathbb{Q}_p$. Let Γ be a group isomorphic to $\mathbb{Z}_p^\times$ via a map $\chi : \Gamma \to \mathbb{Z}_p^\times$. Fix once and for all a topological generator of Γ (which is procyclic as $p \neq 2$), say γ. For the sake of completeness, we will briefly recall the construction of some of the rings of Fontaine necessary for introducing the (φ, Γ)-modules and the theorem characterizing their relation with certain Galois representations of $G_{\mathbb{Q}_p}$.

Let $\{\epsilon^{(n)}\}_{n\geq 1} \subset \bar{\mathbb{Q}}_p$ be a system of roots of unity such that:

(1) $\epsilon^{(1)} \neq 1$,
(2) $\epsilon^{(n)} \in \mu_{p^n} \subset \overline{\mathbb{Q}}_p$,
(3) $(\epsilon^{(n+1)})^p = \epsilon^{(n)}$.

One can think of $\epsilon := (\epsilon^{(1)}, \epsilon^{(2)}, \dots)$ as an element of Fontaine's ring $\mathcal{E} = \varprojlim \mathbb{C}_p$ (with projective limit maps given by the Frobenius maps $z \mapsto z^p$) where $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$. It is well known that $\mathcal{E}$ is an algebraically closed field of characteristic p. Now, consider the fields $\mathbb{Q}_p^{(n)} := \mathbb{Q}_p(\epsilon^{(n)})$ and define $\mathbb{Q}_p^{(\infty)} = \cup_{n\geq 1}\mathbb{Q}_p^{(n)}$. Denote by $H_{\mathbb{Q}_p}$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{(\infty)})$.

The p-adic cyclotomic character ω gives the exact sequence:

$$1 \longrightarrow H_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}_p} \xrightarrow{\omega} \Gamma \cong \mathbb{Z}_p^{\times} \longrightarrow 1.$$

Consider the field $\mathbb{F} = \mathbb{F}_p((\epsilon-1))$ inside $\mathcal{E}$. Let $\mathcal{A}_{\mathbb{Q}_p}$ be the p-adic completion of $\mathbb{Z}_p[\![x]\!][\frac{1}{x}]$; it is a complete discrete valuation ring whose residue field can be identified with $\mathbb{F}$ (one can identify x with a suitable Teichmuller lift of $\epsilon-1$). Let $\mathcal{A}$ be the p-adic completion of the strict henselization $\mathcal{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ of $\mathcal{A}_{\mathbb{Q}_p}$ inside $\widetilde{\mathcal{A}} := W(\mathcal{E})$. Note that $\mathcal{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ can be identified with the ring of integers of the maximal unramified extension of the field $\mathcal{A}_{\mathbb{Q}_p}[\frac{1}{p}]$ inside $\widetilde{\mathcal{A}}[\frac{1}{p}]$.

The Galois group $G_{\mathbb{Q}_p}$ acts on $\mathcal{E}$ by acting on $\mathbb{C}_p$ and by functoriality on the projective limit. By functoriality of the Witt vectors, the group $G_{\mathbb{Q}_p}$ also acts on $\widetilde{\mathcal{A}} = W(\mathcal{E})$ and we have that $\mathcal{A}$ is $G_{\mathbb{Q}_p}$-stable. It is also true that $\mathcal{A}^{H_{\mathbb{Q}_p}} = \mathcal{A}_{\mathbb{Q}_p}$.

Now, we define $\mathcal{A}_{\mathbb{E}}$ as $\mathcal{A}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$, by a result of Dee (see [25, Prop. 2.2.2]) we have that $\mathcal{A}_{\mathbb{E}}$ is isomorphic to the $\pi_{\mathbb{E}}$-adic completion of $\mathcal{O}_{\mathbb{E}}[\![x]\!][\frac{1}{x}]$. One can think of $\mathcal{A}_{\mathbb{E}}$ inside the ring $\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$. The ring $\mathcal{A}_{\mathbb{E}}$ inherits a $G_{\mathbb{Q}_p}$-action via the natural action of $G_{\mathbb{Q}_p}$ on $\mathcal{A}$ and trivial action on $\mathcal{O}_{\mathbb{E}}$. It follows that $(\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})})^{H_{\mathbb{Q}_p}} = \mathcal{A}_{\mathbb{E}}$ and so $\mathcal{A}_{\mathbb{E}}$ has a structure of Γ-module.

Hence, the ring $\mathcal{A}_{\mathbb{E}}$ has a natural $\mathcal{O}_{\mathbb{E}}$-linear action of Γ and a $\mathcal{O}_{\mathbb{E}}$-linear Frobenius endomorphism φ given by the following expressions:

$$\varphi(f(x)) = f((1+x)^p - 1) \qquad \text{for all } f(x) \in \mathcal{A}_{\mathbb{E}},$$
$$\eta(f(x)) = f((1+x)^{\chi(\eta)} - 1) \quad \text{for all } f(x) \in \mathcal{A}_{\mathbb{E}}, \text{ for all } \eta \in \Gamma.$$

Finally, we can recall the following:

Definition 2.1. An étale (φ, Γ)-module D over $\mathcal{O}_{\mathbb{E}}$ is an $\mathcal{A}_{\mathbb{E}}$-module of finite type endowed with a semilinear Froebenius map φ such that $\varphi(D)$ generates D as $\mathcal{A}_{\mathbb{E}}$-module (this is the étale property) and a semilinear continuous action of Γ which commutes with φ. The category of étale (φ, Γ)-module over $\mathcal{O}_{\mathbb{E}}$ will be denoted by $\mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$.

Denote by $\mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$ the category of $\mathcal{O}_{\mathbb{E}}$-linear representations of $G_{\mathbb{Q}_p}$, i.e. the category of $\mathcal{O}_{\mathbb{E}}$-modules of finite type with a continuous $\mathcal{O}_{\mathbb{E}}$-linear action of $G_{\mathbb{Q}_p}$.

By a theorem of Fontaine (see [26, A.3.4]) and its generalization by Dee (see [25, 2.2]) we have the following:

Theorem 2.2. *There exists a natural isomorphism*

$$\mathfrak{D} : \mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p}) \to \mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$$

given by $\mathfrak{D}(T) = (\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} \otimes_{\mathcal{O}_{\mathbb{E}}} T)^{H_{\mathbb{Q}_p}}$, *where* $\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$.

A quasi-inverse functor, which is a natural isomorphism as well, is given by:

$$\mathfrak{T} : \mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \to \mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$$

given by $\mathfrak{T}(D) = (\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} \otimes_{\mathcal{A}_{\mathbb{E}}} D)^{\varphi=1}$.

Remark 2.3. Note that the equivalence of categories given by the above theorem preserves the objects killed by a fixed power of a chosen uniformizer. This essentially follows from the exactness of the functor $\mathfrak{D}$ (see [25, Prop. 2.1.9]) and so $(\pi_{\mathbb{E}}^n) \cdot \mathfrak{D}(T) = \mathfrak{D}((\pi_{\mathbb{E}}^n) \cdot T)$ in the category $\mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$. Same goes for the quasi-inverse functor $\mathfrak{T}$ (see [25, Prop. 2.1.24]).

Remark 2.4. Let $\mathbf{Mod}^{\text{ét,tors}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ and $\mathbf{Mod}^{\text{ét,free}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ be respectively the categories of torsion and free (as $\mathcal{A}_{\mathbb{E}}$-modules) étale (φ,Γ)-modules. There is a notion of Tate dual for such étale (φ,Γ)-modules. Let $\mathcal{B}_{\mathbb{E}}$ be $\mathcal{A}_{\mathbb{E}}[\frac{1}{p}]$ and define the Tate dual as follows (see [22, §I.2]):

$$\text{if } D \in \mathbf{Mod}^{\text{ét,tors}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \text{ then } D^* := \operatorname{Hom}_{\mathcal{A}_{\mathbb{E}}}\Big(D, \mathcal{B}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{\mathrm{d}x}{1+x}\Big)$$

$$\text{if } D \in \mathbf{Mod}^{\text{ét,free}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \text{ then } D^* := \operatorname{Hom}_{\mathcal{A}_{\mathbb{E}}}\Big(D, \mathcal{A}_{\mathbb{E}}\frac{\mathrm{d}x}{1+x}\Big)$$

where $\mathcal{B}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{\mathrm{d}x}{1+x}$ is the inductive limit of $p^{-n}\mathcal{A}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{\mathrm{d}x}{1+x}$ in the category $\mathbf{Mod}^{\text{ét,tors}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ and the structure of (φ,Γ)-module is given by:

$$\gamma\Big(\frac{\mathrm{d}x}{1+x}\Big) = \chi(\gamma)\frac{\mathrm{d}x}{1+x} \text{ and } \varphi\Big(\frac{\mathrm{d}x}{1+x}\Big) = \frac{\mathrm{d}x}{1+x}.$$

Note the important fact that these two notions of dual are compatible in the following sense: if $D \in \mathbf{Mod}^{\text{ét,free}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ then for all $n \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ we have that $D^*/p^nD^* \cong (D/p^nD)^*$ (see [22, Prop. I.2.5]).

The étale (φ,Γ)-modules corresponding to crystalline representations can be described more explicitly via the theory of Wach modules developed by Berger and Wach. We will briefly recall the definition of Wach modules and their main properties. First, we define $\mathcal{A}^+_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}[\![x]\!]$ inside $\mathcal{A}_{\mathbb{E}}$. It inherits naturally the actions of φ and Γ by restriction from $\mathcal{A}_{\mathbb{E}}$. Note also that $\mathcal{A}_{\mathbb{E}}$ is obtained by taking $\pi_{\mathbb{E}}$-adic completion of the localization $\mathcal{O}_{\mathbb{E}}[\![x]\!][1/x]$ of $\mathcal{A}^+_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}[\![x]\!]$ at the multiplicative set $\{1, x, x^2, \ldots\}$. Moreover, since $\mathcal{A}_{\mathbb{E}}$ is Noetherian, we have that $\mathcal{A}_{\mathbb{E}}$ is flat as $\mathcal{A}^+_{\mathbb{E}}$-module since localization and completion preserve such property. Following Berger (see [8]), we have the following

Definition 2.5. A Wach module of height $h \geq 1$ is a free $\mathcal{A}^+_{\mathbb{E}}$-module of finite rank endowed with commutative $\mathcal{A}^+_{\mathbb{E}}$-semilinear actions of a Frobenius

map φ and of the group Γ such that:

(1) $D(N) := \mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}_{\mathbb{E}}^+} N \in \mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$,
(2) Γ acts trivially on N/xN,
(3) $N/\varphi^*(N)$ is killed by Q^h,

where $\varphi^*(N)$ denotes the $\mathcal{A}_{\mathbb{E}}^+$-module generated by $\varphi(N)$, and $Q = \frac{(1+x)^p-1}{x} \in \mathcal{A}_{\mathbb{E}}^+$.

We recall that the Wach modules are the right linear algebra objects to specialize Fontaine's equivalence to crystalline representations. Indeed, we have the following (see [8, Prop. 1.1]):

Proposition 2.6. *Let N be a Wach module of height h. The $\mathbb{E}$-linear representation $\mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathfrak{T}(\mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}_{\mathbb{E}}^+} N)$ of $G_{\mathbb{Q}_p}$ is crystalline with Hodge–Tate weights in the interval $[-h;0]$; and*

$$D_{\text{cris}}(\mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathfrak{T}(\mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}_{\mathbb{E}}^+} N)) \cong \mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} N/xN \quad \text{as } \varphi\text{-modules.}$$

Moreover, all crystalline representations with Hodge–Tate weights in $[-h;0]$ arise in this way.

3. Continuity of the Wach modules

Berger proved (see [6, §III.4 or Thm. 2]) that the there exists an equivalence of categories between rational Wach modules (over $\mathbb{E}\otimes_{\mathcal{O}_{\mathbb{E}}}\mathcal{A}_{\mathbb{E}}^+$) and $G_{\mathbb{Q}_p}$ crystalline representations. As a consequence, Berger proved that if we fix a $\mathbb{E}$-linear representation V of $G_{\mathbb{Q}_p}$ and denote by D its corresponding étale (φ,Γ)-module via Fontaine's functor there is an inclusion preserving bijection between lattices inside V and Wach modules (over $\mathcal{A}_{\mathbb{E}}^+$) contained in D and which are $\mathcal{A}_{\mathbb{E}}^+$-lattices. Denote by $\mathfrak{N}$ such bijective map that associates to each $G_{\mathbb{Q}_p}$-stable lattice of a crystalline representation its corresponding Wach module.

In this section, we will prove that in some natural sense p-adically close Wach modules will correspond (via $\mathfrak{N}$) to p-adically close $\mathcal{O}_{\mathbb{E}}$-linear representations sitting in crystalline representation and viceversa. We will start by clarifying what we mean by p-adically close Wach modules. Given two Wach modules N_1 and N_2 we say that N_1 and N_2 are congruent modulo some prime power, i.e. $N_1 \equiv N_2 \bmod \pi_{\mathbb{E}}^m$ for some $m \in \mathbb{Z}_{\geq 1}$, if there exists a $\mathcal{A}_{\mathbb{E}}^+$-module isomorphism between $N_1 \otimes_{\mathcal{A}_{\mathbb{E}}^+} \mathcal{A}_{\mathbb{E}}^+/(\pi_{\mathbb{E}}^m)$ and $N_2 \otimes_{\mathcal{A}_{\mathbb{E}}^+} \mathcal{A}_{\mathbb{E}}^+/(\pi_{\mathbb{E}}^m)$ which is (φ,Γ)-equivariant.

Note that, essentially by definition, we have that $N_1 \equiv N_2 \bmod \pi_{\mathbb{E}}^m$ if and only if there exist basis of N_1 and N_2 as $\mathcal{A}_{\mathbb{E}}^+$-modules such that, after defining $P_1 = \text{Mat}(\varphi|_{N_1})$, $P_2 = \text{Mat}(\varphi|_{N_2})$, $G_1 = \text{Mat}(\gamma|_{N_1})$, $G_2 = \text{Mat}(\gamma|_{N_2})$ (note that $P_1, P_2, G_1, G_2 \in \text{M}_{d\times d}(\mathcal{A}_{\mathbb{E}}^+)$ where $d = \text{rank}_{\mathcal{A}_{\mathbb{E}}^+}(N_i)$ for $i = 1, 2$)

it follows that:

$$\begin{cases} P_1 \equiv P_2 \bmod \pi_{\mathbb{E}}^m, \\ G_1 \equiv G_2 \bmod \pi_{\mathbb{E}}^m. \end{cases}$$

We recall that we have defined the arithmetic function α as $\alpha(r-1) = \sum_{n\geq 1} \lfloor \frac{r-1}{p^{n-1}(p-1)} \rfloor$ for any positive integer r. We have the following continuity result (this is [6, Thm. IV.1.1]):

Proposition 3.1. *Let T_1 and T_2 be two Galois stable lattices inside respectively two crystalline $\mathbb{E}$-linear representation V_1 and V_2 of Hodge–Tate weights inside $[-r;0]$, and assume there is an $n \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ with $n \geq \alpha(r)$ such that $T_1 \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^n) \cong T_2 \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^n)$ as $G_{\mathbb{Q}_p}$-modules, then $\mathfrak{N}(T_1) \equiv \mathfrak{N}(T_2) \bmod p^{n-\alpha(r)}$.*

Now, if N is a Wach module denote by $\mathfrak{T}(N) := \mathfrak{T}(D(N))$ the $\mathcal{O}_{\mathbb{E}}$-linear representation of $G_{\mathbb{Q}_p}$ attached to N. We recall that $\mathfrak{T}(N) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$ is a crystalline representation.

We are interested in the following result:

Proposition 3.2. *Let N_1 and N_2 be two Wach modules (over $\mathcal{O}_{\mathbb{E}}$) with the same rank as $\mathcal{A}_{\mathbb{E}}^+$-modules. Assume that $N_1 \equiv N_2 \bmod \pi_{\mathbb{E}}^n$ for some $n \in \mathbb{Z}_{\geq 1}$. Then $\mathfrak{T}(N_1) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi_{\mathbb{E}}^n) \cong \mathfrak{T}(N_2) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi_{\mathbb{E}}^n)$ as $G_{\mathbb{Q}_p}$-modules.*

Proof. Consider the étale (φ,Γ)-modules $D(N_i) = N_i \otimes_{\mathcal{A}_{\mathbb{E}}^+} \mathcal{A}_{\mathbb{E}}$ for $i = 1, 2$. Since $\mathcal{A}_{\mathbb{E}}$ is flat as $\mathcal{A}_{\mathbb{E}}^+$-module (module structure given by inclusion) we have the following chain of isomorphisms of torsion étale (φ,Γ)-modules:

$$\begin{aligned} D(N_1)/\pi_{\mathbb{E}}^n D(N_1) \cong N_1/\pi_{\mathbb{E}}^n N_1 \otimes_{\mathcal{A}_{\mathbb{E}}^+} \mathcal{A}_{\mathbb{E}} \cong N_2/\pi_{\mathbb{E}}^n N_2 \otimes_{\mathcal{A}_{\mathbb{E}}^+} \mathcal{A}_{\mathbb{E}} \\ \cong D(N_2)/\pi_{\mathbb{E}}^n D(N_2). \end{aligned}$$

Now, the claim follows just by applying Fontaine's functor $\mathfrak{T}$, which is exact (see Theorem 2.2 and see Remark 2.3). □

4. Local constancy with respect to the trace

In this section we are going to prove an explicit local constancy result with respect to the trace (i.e. the weight k will be fixed) for reductions modulo prime powers of representations of the type V_{k,a_p}.

4.1. Some linear algebra of Wach modules. As explained in the previous section, a congruence between Wach modules (modulo some prime power) can be translated into a congruence (modulo the same prime power) between systems of matrices representing the (φ,Γ) actions on the Wach modules involved. In this section, we will see how to p-adically deform a Wach module into another one via linear algebra means for the systems of matrices associated to the (φ,Γ)-module structure.

As long as it will be possible, we will keep the same notation as Berger (see [8]). We recall that p is an odd prime and $\mathbb{E}$ is a finite extension of $\mathbb{Q}_p$ with ramification index e. Let v be the normalized p-adic valuation (i.e. $v(p) = 1$). Let $r \geq 1$ be a integer and define $\alpha(r) := \sum_{j=1}^{r} v(1 - \chi(\gamma)^j)$ where we recall that γ is a fixed topological generator of the pro-cyclic group Γ. The constant $\alpha(r)$ has also an explicit description given by $\alpha(r) = \sum_{n\geq 1} \lfloor \frac{r}{p^{n-1}(p-1)} \rfloor$ for a proper choice of γ (see [8]).

We start by recalling two useful results in linear algebra (see [8, Lem. 2.1]):

Lemma 4.1. *If $P_0 \in M_2(\mathcal{O}_\mathbb{E})$ is a matrix with eigenvalues $\lambda \neq \mu$, and if $\delta = \lambda - \mu$, then there exists $Y \in M_2(\mathcal{O}_\mathbb{E})$ such that $Y^{-1} \in \delta^{-1} M_2(\mathcal{O}_\mathbb{E})$ and $Y^{-1} P_0 Y = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$.*

and the following corollary (see [8, Cor. 2.2]):

Corollary 4.2. *If $\alpha \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $\epsilon \in \mathcal{O}_\mathbb{E}$ are such that $v(\epsilon) \geq 2v(\delta) + \alpha$, then there exists $H_0 \in p^\alpha M_2(\mathcal{O}_\mathbb{E})$ such that $\det(\mathrm{Id} + H_0) = 1$ and $\mathrm{Tr}(H_0 P_0) = \epsilon$.*

The corollary above represents the starting point to deform a Wach module into another one. Given the matrix P_0, it gives a p-adically small matrix H_0 such that the product $H_0 P_0$ will have a prescribed p-adically small trace. In practice, this will be applied when P_0 is obtained by the action of φ on $D_{\mathrm{cris}}(V_{k,a_p})$ for some $k \geq 2$ and $a_p \in \mathfrak{m}_\mathbb{E}$.

The idea behind the next results is to show how H_0 gives rise to a deformation of a whole system of matrices (attached to a Wach module) to a p-adically close one preserving the characterizing linear algebra properties of the action on φ and Γ on the Wach module.

We have the following result (it is a little generalization of [8, Prop. 2.3]):

Proposition 4.3. *Let $m \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 2}$. If $G \in \mathrm{Id} + xM_d(\mathcal{O}_\mathbb{E}[\![x]\!])$ and $k \geq 2$ and $H_0 \in p^{\alpha(k-1)+m} M_d(\mathcal{O}_\mathbb{E})$, then there exists $H \in p^m M_d(\mathcal{O}_\mathbb{E}[\![x]\!])$ such that $H(0) = H_0$ and $HG \equiv G\gamma(H) \bmod x^k$.*

Proof. As $G \in \mathrm{Id} + xM_d(\mathcal{O}_\mathbb{E}[\![x]\!])$, we can write $G = \mathrm{Id} + xG_1 + x^2 G_2 + \dots$ where $G_i \in \mathrm{M}_d(\mathcal{O}_\mathbb{E})$ for all $i \in \mathbb{Z}_{\geq 1}$. We prove that for any positive integer r, there exists an $H_r \in p^{\alpha(k-1)-\alpha(r)+m} M_d(\mathcal{O}_\mathbb{E})$ such that if we define $H = H_0 + xH_1 + x^2 H_2 + \dots + x^{k-1} H_{k-1}$ we have that $HG \equiv G\gamma(H) \bmod x^k$.

We start from $r = 1$, then since $\gamma(H) = H_0 + \gamma(x)H_1 + \gamma(x^2)H_2 + \dots + \gamma(x^{k-1})H_{k-1}$ and for all $w \in \mathbb{Z}_{\geq 1}$ we have $\gamma(x^w) = ((1+x)^{\chi(\gamma)} - 1)^w$, we deduce that we can define H_1 such that $(1-\chi(\gamma))H_1 = G_1 H_0 - H_0 G_1$. Since by hypothesis $H_0 \equiv 0 \bmod p^{\alpha(k-1)+m}$ and by definition $\alpha(1) = v_p(1-\chi(\gamma))$, we deduce that $H_1 \in p^{\alpha(k-1)-\alpha(1)+m} M_d(\mathcal{O}_\mathbb{E})$.

Using now the same argument, one can actually see how H_r is uniquely determined by $H_0, H_1, \dots, H_{r-1}$ and moreover we have that $(1-\chi(\gamma)^r)H_r \in p^{\alpha(k-1)-\alpha(r-1)+m} M_d(\mathcal{O}_\mathbb{E})$, note that $\alpha(r) = \alpha(k-1) - \alpha(r-1)$. To be precise,

we prove this by induction on $r \leq k-1$. The first case $r = 1$ is proven above, now assume the case $r-1$, we are going to prove the statement for r.

It is straightforward to prove that, expanding the expression $HG \equiv G\gamma(H) \bmod x^k$, the following identity holds:

$$(1-\chi(\gamma)^r)H_r = \sum_{h=0}^{r-1} G_{r-h}\left(\sum_{n=0}^{h}\gamma(x^n)_h H_n\right) - \sum_{i=0}^{r-1} H_i G_{r-i}, \quad \text{for } r \leq k-1$$

where $\gamma(x^n)_h$ is the h-th coefficient (i.e. coefficient of x^h) of the polynomial $\gamma(x^n)$. Note now that the map $\alpha(n)$ is non-decreasing as n grows. Hence, by inductive hypothesis, we deduce that for any i such that $0 \leq i \leq r-1$ we have $H_i \equiv 0 \bmod (p^{\alpha(k-1)-\alpha(r-1)+m})$. Since $v(1-\chi(\gamma)^r)+\alpha(r-1) = \alpha(r)$, we deduce that $H_r \equiv 0 \bmod (p^{\alpha(k)-\alpha(r)+m})$. This concludes the proof. □

The following result completes the linear algebra deformation process of a system of matrices that will represent a (φ,Γ)-action on Wach modules:

Proposition 4.4. *Let $m \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 2}$. Let $G \in \mathrm{Id} + xM_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ and $P \in M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ satisfy $P\varphi(G) = G\gamma(P)$ and $\det(P) = Q^{k-1}$ where $Q = \frac{(1+x)^p-1}{x}$.*

If $H_0 \in p^{\alpha(k-1)+m}M_d(\mathcal{O}_{\mathbb{E}})$, then there exist $G' \in \mathrm{Id} + xM_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ and $H \in p^m M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ such that:

(1) $H(0) = H_0$;
(2) $P'\varphi(G') = G'\gamma(P')$, *where* $P' = (\mathrm{Id}+H)P$;
(3) $P \equiv P' \bmod p^m$;
(4) $G \equiv G' \bmod p^m$.

Proof. After applying the previous proposition for the existence of the matrix H which satisfies $H(0) = H_0$, the existence of G' follows directly from [8, Prop. 2.4]. Hence the claims (1) and (2) hold.

The claim (3) is clear since $H \in p^m M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ implies that $P \equiv P' \bmod p^m$. What it is left to prove is (4), i.e. $G \equiv G' \bmod p^m$. In order to prove this, we need to look at how the matrix G' is defined.

The matrix G' is constructed as an x-adic limit inside $\mathrm{Id} + x\,\mathrm{M}_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$ (note that we are dealing with non-commutative rings). Define $G'_k := G$ and observe that it satisfies by construction $G'_k - P'\varphi(G'_k)\gamma(P')^{-1} = x^k R_k$ for some $R_k \in M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$. Then define G' as the x-adic limit of G'_j, for $j \geq k$, which satisfies $G'_{j+1} = G'_j + x^j S_j$ for some $S_j \in M_d(\mathcal{O}_{\mathbb{E}})$, and $G'_j - P'\varphi(G'_j)\gamma(P')^{-1} = x^j R_j$ where $R_j \in M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!])$.

We will prove by induction that:

(i) $R_j \equiv 0 \bmod p^m$,
(ii) $G'_j \equiv G \bmod p^m$.

First the case $j = k$: since $H \equiv 0 \bmod p^m$ then $P \equiv P' \bmod p^m$ which implies that $R_k \equiv 0 \bmod p^m$, and by construction $G'_k = G$ so the first case

of induction is done. Now assume $j \geq k$ and that the above claims hold for j, we will prove them for $j+1$.

We have that there exists $S_j \in M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$\begin{aligned} G'_{j+1} &- P'\varphi(G'_{j+1})\gamma(P')^{-1} \\ &= G'_j + x^j S_j - P'\varphi(G'_j)\gamma(P')^{-1} - P'x^j Q^j S_j \gamma(P')^{-1} \\ &= x^j(R_j + S_j - Q^{j-k+1}P'S_jQ^{k-1})\gamma(P')^{-1}) \in x^{j+1}M_d(\mathcal{O}_{\mathbb{E}}[\![x]\!]). \end{aligned}$$

Note that we used that φ acts trivially on $\mathrm{M}_d(\mathcal{O}_{\mathbb{E}})$ and that $\varphi(x^j) = x^j Q^j$ for all $j \in \mathbb{Z}_{\geq 1}$. We want to prove that there exists $S_j \in p^m M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$R_j + S_j - Q^{j-k+1}P'S_jQ^{k-1}\gamma(P')^{-1} \in xM_d(\mathcal{O}_{\mathbb{E}}).$$

Evaluating the above expression at $x = 0$, the claim is equivalent to prove that there exists $S_j \in p^m M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$S_j - p^{j-k+1}P'(0)S_j p^{k-1}(\gamma(P')^{-1})(0) = -R_j(0).$$

Now, since $R_j \equiv 0 \bmod p^m$, we have that $R_j(0) \equiv 0 \bmod p^m$. It is clear that the map $S \mapsto S - p^{j-k+1}P'(0)Sp^{k-1}(\gamma(P')^{-1})(0)$ gives a bijection of $M_d(\mathcal{O}_{\mathbb{E}})$. Moreover, it is also clear that it is a bijection on $p^m M_d(\mathcal{O}_{\mathbb{E}})$. As $R_j(0) \equiv 0 \bmod p^m$, we have the existence of $S_j \equiv 0 \bmod p^m$ such that the above relations are satisfied. By inductive hypothesis $R_j \equiv 0 \bmod p^m$, so $R_{j+1} \equiv 0 \bmod p^m$. Since $S_j \equiv 0 \bmod p^m$ implies that $G'_{j+1} = G'_j + x^j S_j \equiv G \bmod p^m$. This concludes the proof. □

4.2. Local constancy modulo prime powers with respect to a_p. Let $k \geq 2$ be a positive integer and let $a_p \in \mathfrak{m}_{\mathbb{E}}$. In this section, we will apply the continuity properties of the Wach modules to prove local constancy results modulo prime powers when we fix the weight k and we let the trace of the crystalline Frobenius a_p vary p-adically.

The main result of this section is the following (this is a generalization of [8, Thm. A]):

Theorem 4.5. *Let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$ and $k \geq 2$ be an integer. Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1) + m$, then for every $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p} inside V_{k,a_p} there exists a $G_{\mathbb{Q}_p}$-stable lattice T_{k,a'_p} inside V_{k,a'_p} such that*

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

Proof. Consider the $G_{\mathbb{Q}_p}$-representation $V^*_{k,a_p} = \mathrm{Hom}_{\mathbb{E}}(V_{k,a_p}, \mathbb{E})$; it is crystalline and it has Hodge–Tate weights 0 and $-(k-1)$.

Let $T^*_{k,a_p} := \mathrm{Hom}_{\mathcal{O}_{\mathbb{E}}}(T_{k,a_p}, \mathcal{O}_{\mathbb{E}})$ be the $G_{\mathbb{Q}_p}$-stable lattice in V^*_{k,a_p}, dual of T_{k,a_p}. By a result of Berger (see [6, Prop. III.4.2 and III.4.4]), it is possible to attach to T^*_{k,a_p} a Wach module N_{k,a_p} of height $k-1$. Fixing a basis of N_{k,a_p}

as $\mathcal{A}_{\mathbb{E}}^+$-module (we recall that in our notation $\mathcal{A}_{\mathbb{E}}^+ = \mathcal{O}_{\mathbb{E}}[\![x]\!]$), the actions of φ and γ on N_{k,a_p} can be respectively represented by the matrices $P \in \mathrm{Mat}_2(\mathcal{A}_{\mathbb{E}}^+)$ and $G \in \mathrm{Id} + x\,\mathrm{Mat}_2(\mathcal{A}_{\mathbb{E}}^+)$. Note that since the actions of φ and Γ commute (in a semi-linear sense) we have that $P\varphi(G) = G\gamma(P)$. We recall also that the matrix $P(0)$ has characteristic polynomial $T^2 - a_pT + p^{k-1}$. Now, let $a_p' \in \mathcal{O}_{\mathbb{E}}$ be as in the hypothesis, i.e. it satisfies $v(a_p - a_p') \geq 2 \cdot v(a_p) + \alpha(k-1) + m$ for some $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$. Applying in sequence the results in Section 4.1, we deduce the existence of two matrices P' and G' which give rise to a Wach module N' and such that $P \equiv P' \bmod p^m$ and $G \equiv G' \bmod p^m$. Since by construction $P' = (\mathrm{Id} + H)P$, we have that evaluating at $x = 0$ we deduce that the characteristic polynomial of $P'(0)$ is $T^2 - a_p'T + p^{k-1}$ (note that $\mathrm{Trace}(H(0)P(0)) = a_p' - a_p$ and $\det(\mathrm{Id} + H(0)) = 1$). By a result of Berger (see [8, Prop. 1.2]), we can deduce that $N' = N_{k,a_p'}$, or equivalently $V_{k,a_p'} = \mathfrak{T}(N') \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$ and $D_{\mathrm{cris}}(V_{k,a_p'}^*) = N'/xN' \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$.

Since $P \equiv P' \bmod p^m$ and $G \equiv G' \bmod p^m$, we have that $N_{k,a_p} \equiv N_{k,a_p'} \bmod p^m$. As a consequence of Remark 2.3, we have that $\mathfrak{D}(N_k, a_p) \equiv \mathfrak{D}(N_{k,a_p'}) \bmod p^m$, i.e.

$$\mathfrak{D}(N_k, a_p) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong \mathfrak{D}(N_{k,a_p'}) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ inside } \mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}).$$

Hence, we define $T_{k,a_p'} := \mathfrak{T}(\mathfrak{D}(N_{k,a_p'})^*)$ which is a $G_{\mathbb{Q}_p}$-stable lattice in $V_{k,a_p'}$ that satisfies (since Fontaine's functor $\mathfrak{T}$ is compatible with duals):

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a_p'} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

Indeed, since $\mathfrak{D}(N_{k,a_p}) \equiv \mathfrak{D}(N_{k,a_p'}) \bmod p^m$, by exactness of Fontaine's functor $\mathfrak{T}$ we can deduce that $T_{k,a_p}^* \equiv \mathfrak{T}(\mathfrak{D}(N_{k,a_p'})) \bmod p^m$. This completes the proof of the theorem. □

4.3. Converse of local constancy with respect to the trace. Via the continuity properties of the Wach modules, it is also possible to find an explicit necessary condition for the existence of local constancy phenomena modulo prime powers. In precise terms, let $k \geq 2$ be an integer and let $a_p, a_p' \in \mathfrak{m}_{\mathbb{E}}$; then we have the following:

Proposition 4.6. *Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ and assume $m \geq \alpha(k-1)$. If $V_{k,a_p} \equiv V_{k,a_p'} \bmod p^m$, then $v(a_p - a_p') \geq m - \alpha(k-1)$.*

Proof. This is a straightforward application of Berger's Proposition 3.1. Indeed, we have that $D_{\mathrm{cris}}(V_{k,a_p}^*) = N_{k,a_p}/xN_{k,a_p} \otimes \mathbb{E}$ and $D_{\mathrm{cris}}(V_{k,a_p'}^*) = N_{k,a_p'}/xN_{k,a_p'} \otimes \mathbb{E}$ for some Wach modules N_{k,a_p} and $N_{k,a_p'}$ corresponding respectively to the $G_{\mathbb{Q}_p}$-stable lattices in V_{k,a_p} and $V_{k,a_p'}$ that are congruent modulo p^m. By Proposition 3.1, we have that $N_{k,a_p} \equiv N_{k,a_p'} \bmod p^{m-\alpha(k-1)}$ and looking at the characteristic polynomials of φ acting on $N_{k,a_p}/xN_{k,a_p}$ and $N_{k,a_p'}/xN_{k,a_p}$ the claim follows. □

5. Local constancy with respect to the weight

In this section, we are going to prove a local constancy result for reductions modulo prime powers once we fix the trace of the crystalline Frobenius a_p and we let the weight k vary.

In order to simplify the notation, we will say that two $\mathbb{E}$-linear representations V and V' of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ are congruent modulo some prime power (i.e. $V \equiv V' \bmod \pi_{\mathbb{E}}^n$ for some $n \in \mathbb{Z}_{\geq 1}$) if there exist $G_{\mathbb{Q}_p}$-stable lattices $T \subset V$ and $T' \subset V'$ such that we have an isomorphism

$$T \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi^n) \cong T' \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi^n) \quad \text{of } G_{\mathbb{Q}_p}\text{-modules.}$$

Note that the above definition requires a bit of attention when used as it clearly doesn't define an equivalence relation (in general, it is not transitive).

The main result of this section is the following:

Theorem 5.1. *Let p be an odd prime. Let $a_p \in m_{\mathbb{E}} - \{0\}$ for some finite extension $\mathbb{E}/\mathbb{Q}_p$. Let $k \geq 2$ be an integer and $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ be fixed. Assume that*

$$(*) \qquad k \geq (3v(a_p) + m) \cdot \left(1 - \frac{p}{(p-1)^2}\right)^{-1} + 1.$$

There exists an integer $r = r(k, a_p) \geq 1$ such that if $k' - k \in p^{r+m}(p-1)\mathbb{Z}_{\geq 0}$ then $V_{k,a_p} \equiv V_{k',a_p} \bmod p^m$

Remark 5.2. As it will be clear from the proof below, the condition in the hypothesis is not optimal in the sense that it can be replaced by the weaker condition given by the system:

$$\begin{cases} k \geq 3v(a_p) + \alpha(k-1) + 1 + m, \\ k' \geq 3v(a_p) + \alpha(k'-1) + 1 + m. \end{cases}$$

as in Berger's result (see [8, Thm. B]) when $m = 1/e$. For the sake of simplicity, we just assume the stronger condition $(*)$ which has the advantage that it is explicit in the weight, doesn't depend on the function α and automatically holds for k' if it holds for k assuming $k' \geq k$.

The condition $(*)$ in the theorem can be deduced directly from the above conditions in the system by noticing that $\alpha(k-1) = \sum_{n \geq 1} \left\lfloor \frac{k-1}{p^{n-1}(p-1)} \right\rfloor$ satisfies the inequality $\alpha(k-1) \leq \frac{(k-1)p}{(p-1)^2}$.

Remark 5.3. Note that Theorem 5.1 and Theorem 4.5 can be applied in sequence (i.e. one can first deform the weight and then deform the trace) in order to have a local constancy result in which both the trace and the weight vary. This is possible because Theorem 4.5 is independent of the starting chosen lattice inside V_{k,a_p}. Note that the order in which the theorems can

be applied in sequence cannot be switched, as it is always necessary to keep track of the lattices involved in the congruences.

Remark 5.4. It could be interesting to consider the question of finding explicitly a radius for the local constancy in the weight. Partial results have been obtained by Bhattacharya (see [11]). As already pointed out in the introduction, the above theorem can be seen as a converse (in a special crystalline case) of a non-published theorem of Winterberger, proven by Berger and Colmez as a consequence of a continuity property of the Sen periods (see [9]). Such result provides a connection between the local constancy radius of our theorem and the constant $c(2,\mathbb{Q}_p)$ of Berger and Colmez. To be precise, combining Theorem 5.1 above and [9, Cor. 7.1.2] we get the upper bound on the radius for the local constancy in weight $p^{-(r+m)} \leq p^{-(\lfloor \frac{m}{2} \rfloor - c(2,\mathbb{Q}_p))}$.

In order to prove the theorem, the idea is to make use of Kedlaya's theory of (φ,Γ)-modules of slope zero over the Robba ring (see [29]) and to realize the representations V_{k,a_p} (for k suff. big) as trianguline representations in the sense of Colmez (see [21]). A theorem of Colmez will then ensure us that locally such representations vary in a continuous way, in the sense that they come in analytic families. We will make this precise in the next section. We refer the reader to [7] for a nice summary on the theory of trianguline representations and its applications in arithmetic geometry.

Let $\mathcal{R}_{\mathbb{E}}$ be the Robba ring with coefficients in $\mathbb{E}$ and for any multiplicative character $\delta : \mathbb{Q}_p^\times \to \mathbb{E}^\times$, denote by $\mathcal{R}_{\mathbb{E}}(\delta) := \mathcal{R}_{\mathbb{E}} e_\delta$ the (φ,Γ)-module (in the sense of Kedlaya, see [29]) of rank one obtained by defining the actions $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$ for all $\gamma \in \Gamma$, where χ denotes the chosen fixed isomorphism between Γ and $\mathbb{Z}_p^\times$.

Colmez (see [21, Thm. 0.2]) proved that all (φ,Γ)-modules of rank one arise as $\mathcal{R}_{\mathbb{E}}(\delta)$ for a unique multiplicative character δ; moreover, if δ_1, $\delta_2 : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ are multiplicative characters then $\mathrm{Ext}^1(\mathcal{R}_{\mathbb{E}}(\delta_1), \mathcal{R}_{\mathbb{E}}(\delta_2))$ is an $\mathbb{E}$-vector space of dimension 1 unless $\delta_1\delta_2^{-1}$ is of the form x^{-i} for some integer $i \geq 0$, or $|x|x^i$ for some integer $i \geq 1$; in both cases, the dimension over $\mathbb{E}$ is two and the attached projective space is isomorphic to $\mathbb{P}^1(\mathbb{E})$; here x denotes the identity character of $\mathbb{Q}_p^\times$.

Hence, where the extension is not unique (up to isomorphism), one will need to specify the corresponding parameter in $\mathbb{P}^1(\mathbb{E})$ usually called L-invariant and denoted as $\mathfrak{L}$. The corresponding Galois representation will be denoted by $V(\delta_1,\delta_2,\mathfrak{L})$. For an extensive discussion about $\mathfrak{L}$-invariant, we refer the reader to the original article of Colmez (see [21, §4.5]).

Each trianguline representation $V(\delta_1,\delta_2)$ corresponds (up to considering blow-up in case $\delta_1\delta_2^{-1} = x^{-i}$ or $|x|x^i$; see [21]) to the point $(\delta_1,\delta_2) \in \mathfrak{X} \times \mathfrak{X}$ where $\mathfrak{X}$ is isomorphic (non-canonically, as there are choices involved) to

the $\mathbb{Q}_p$-rigid analytic space $\mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p}$, which parametrizes multiplicative characters $\mathbb{Q}_p^\times$ with values in the multiplicative group of some finite extension of $\mathbb{Q}_p$.

From now on, we denote by $\mathbb{B}^1(a,r)^+_{\mathbb{Q}_p}$ the closed affinoid rigid $\mathbb{Q}_p$-ball centered in a and with radius r. The expression $\mathbb{B}^1(a,r)^-_{\mathbb{Q}_p}$ will instead denote the open rigid $\mathbb{Q}_p$-ball as a $\mathbb{Q}_p$-rigid analytic space. If we are working with an algebraic closure $\overline{\mathbb{Q}}_p$, then the expression $\mathbb{B}^1(a,r)^+$ will simply denote the standard p-adic ball centered in a with radius r.

5.1. Proof of Theorem 5.1. Let k' be an integer satisfying $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$. The claim is to prove that if k' and k are sufficiently p-adically close then the corresponding representations are isomorphic modulo a prescribed prime power.

The assumption $(*)$ on the weight k allow us, applying Theorem 4.5, to deduce that:

$$V_{k,a_p+\frac{p^{k-1}}{a_p}} \equiv V_{k,a_p} \mod p^m,$$
$$V_{k',a_p+\frac{p^{k'-1}}{a_p}} \equiv V_{k',a_p} \mod p^m.$$

Indeed, note that assumption $(*)$ implies that $k-1 > 2v(a_p)$ and hence in this specific case, the Theorem 4.5 can be applied both starting from V_{k,a_p} or starting from $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ (same goes for k') and hence this gives us a strong control over the lattices involved in the congruences. Therefore, this first step reduces the claim to prove that if k' and k are sufficiently p-adically close (in the weight space) then we have the congruence $V_{k,a_p+\frac{p^{k-1}}{a_p}} \equiv V_{k',a_p+\frac{p^{k'-1}}{a_p}} \mod p^m$.

The following proposition of Colmez (see [8, Prop. 3.1] or see [21, §4.5]) allow us to realize the above representations as trianguline representations:

Proposition 5.5. *If $z \in \mathfrak{m}_{\mathbb{E}}$ is a root of $z^2 - a_p z + p^{k-1}$ which satisfies $v(z) < k-1$, then we have that $V(\mu_z, \mu_{\frac{1}{z}}\chi^{1-k}, \infty) = V^*_{k,a_p}$.*

Here $\mu_z : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ is the character which satisfies $\mu_z(p) = z$ and $\mu_z(\mathbb{Z}_p^\times) = 1$ and $\chi : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ is the character which satisfies $\chi(p) = 1$ and $\chi(y) = y$ for all $y \in \mathbb{Z}_p^\times$.

Hence, if $k' - 1 \geq k - 1 > v(a_p)$, we have that the crystalline representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ coincide respectively with the trianguline representations $V(\mu_{a_p}, \delta_{k,a_p}, \infty)$ and $V(\mu_{a_p}, \delta_{k',a_p}, \infty)$, where $\delta_{k,a_p} := \mu_{\frac{1}{a_p}}\chi^{1-k}$ and $\delta_{k',a_p} = \mu_{\frac{1}{a_p}}\chi^{1-k'}$. Since the $\mathfrak{L}$-invariant is going to be ∞ for all the trianguline representations involved, we will drop the notation $V(\cdot,\cdot,\infty)$ writing simply $V(\cdot,\cdot)$.

The trianguline representations $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$ define two $\mathbb{E}$-points, respectively $u_{k,a_p} = (\mu_{a_p}, \delta_{k,a_p})$ and $u_{k',a_p} = (\mu_{a_p}, \delta_{k',a_p})$, on the rigid analytic space $\mathfrak{X} \times \mathfrak{X}$ (see [21]) parametrizing couples of multiplicative characters of $\mathbb{Q}_p^\times$ with values in $\mathbb{L}^\times$ where $\mathbb{L}$ is some finite extension of $\mathbb{Q}_p$.

The following basic lemma represents the first step for constructing an explicit 1-parameter family of trianguline representations inside $\mathfrak{X}^2 = (\mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p})^2$ interpolating $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$ when k and k' will be sufficiently p-adically close (in the weight space):

Lemma 5.6. *Let $\alpha \in 1 + p\mathbb{Z}_p$, then we have that*

$$\begin{aligned} \psi_\alpha : \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p} &\longrightarrow \mathbb{B}^1(1, |\alpha - 1|)^+_{\mathbb{Q}_p} \\ [s] &\longmapsto [\exp_p(s \cdot \log_p(\alpha))] \end{aligned}$$

is an isomorphism in the category of $\mathbb{Q}_p$-rigid analytic spaces. Here $[s]$ denotes the maximal ideal of $\mathbb{Q}_p\langle T\rangle$ corresponding to the element $s \in \overline{\mathbb{Z}}_p$ and the analogue for $\psi_\alpha([s])$.

Proof. First, we will clarify that ψ_α is a well defined map for every $\alpha \in 1 + p\mathbb{Z}_p$. For all $s \in \overline{\mathbb{Z}}_p$, we have that ψ_α converges when evaluated in s since $|s \cdot \log_p(\alpha)| \leq |\alpha - 1| \leq p^{-1}$. Moreover since $\psi_\alpha(s) \in \mathbb{Q}_p(s)$, we have that the map ψ_α is Galois equivariant, i.e. $\psi_\alpha(\sigma(s)) = \sigma(\psi_\alpha(s))$ for every $\sigma \in G_{\mathbb{Q}_p}$. Note also that we can find the explicit expression $\psi_\alpha(s) = \alpha^s = (1 + (\alpha - 1))^s = \sum_{n \geq 0} \binom{s}{n} (\alpha - 1)^n$ which converges for every $s \in \overline{\mathbb{Z}}_p$. This allow us to define ψ_α on the set of $G_{\mathbb{Q}_p}$-orbits of $\overline{\mathbb{Z}}_p$ which can be identified set-theoretically with $\mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$. Proving that ψ_α is a morphism of $\mathbb{Q}_p$-rigid analytic spaces (affinoid spaces in this case) boils down to show that the induced map on the corresponding affinoid algebras is a morphism. Let $\mathcal{O}_{0,1}$ and $\mathcal{O}_{1,|\alpha-1|}$ denote respectively the $\mathbb{Q}_p$-affinoid algebras attached to the $\mathbb{Q}_p$-affinoid spaces $\mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$ and $\mathbb{B}^1(1,|\alpha-1|)^+_{\mathbb{Q}_p}$. We have that the associated map $\psi_\alpha^* : \mathcal{O}_{1,|\alpha-1|} \to \mathcal{O}_{0,1}$ is given by $\psi_\alpha^*(f) = f \circ \psi_\alpha$ for all $f \in \mathcal{O}_{1,|\alpha-1|} \cong \mathbb{Q}_p\langle \frac{T-1}{\alpha-1} \rangle$.

In order to show that ψ_α^* is a morphism of affinoid algebras, since it is given by the pull-back, it is sufficient to show that it is a well-defined map, in the sense that $\psi_\alpha^*(f)$ belongs to $\mathcal{O}_{0,1} \cong \mathbb{Q}_p\langle T\rangle$; or in other words, it is a converging series. Indeed, the problem is that, in general, composition of p-adic analytic functions is not analytic. The convergence property will be deduced by the following convergence criterion (see [28, Thm. 4.3.3]):

Theorem 5.7. *Let $f(X) = \sum a_n X^n$ and $g(X) = \sum b_n X^n$ be formal power series in $\overline{\mathbb{Q}}_p[\![X]\!]$ with $g(0) = 0$, and let $h(X) = f(g(X))$ be their formal composition.*

Suppose that:

(i) $g(x)$ *converges,*
(ii) $f(g(x))$ *converges (i.e. plugging the number to which* $g(x)$ *converges into* $f(X)$ *gives a convergent series),*
(iii) *for every* $n \in \mathbb{Z}_{\geq 0}$*, we have* $|b_n x^n| \leq |g(x)|$ *(i.e. no term of the series converging to* $g(x)$ *is bigger than the sum).*

Then $h(x)$ *also converges, and* $f(g(x)) = h(x)$*.*

Indeed, in our case, it is enough to prove that for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$|c_n s^n| \leq |\psi_\alpha(s)|$$

with $\psi_\alpha(s) = \exp_p(s \cdot \log_p(\alpha)) = \sum_{n\geq 0} c_n s^n$ and $c_n := \frac{(\log_p(\alpha))^n}{n!}$. We have that $|\psi_\alpha(s)| = |\exp_p(s \cdot \log_p(\alpha))| = 1$ for all $s \in \mathbb{B}^1(0,1)^+$, hence since $|s| \leq 1$, it is sufficient to prove that $|c_n| \leq 1$.

Since $\alpha \in 1 + p\mathbb{Z}_p$ and since the p-adic logarithm is an isometry we have that $|\log_p(\alpha)|^n = |\alpha - 1|^n \leq p^{-n}$. We also recall from classical p-adic analysis that $v_p(n!) < \frac{n}{p-1}$ or in other words $|n!| > p^{-\frac{n}{p-1}}$. It follows at once that

$$|c_n| = \frac{|\log_p(\alpha)|^n}{|n!|} < \frac{p^{-n}}{p^{-\frac{n}{p-1}}} < 1.$$

Note that the inverse $\psi_\alpha^{-1} : \mathbb{B}^1(1, |\alpha - 1|)^+_{\mathbb{Q}_p} \longrightarrow \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$ sends $[t]$ to $\left[\frac{\log_p(t)}{\log_p(\alpha)}\right]$ and it is, via the same argument, a well-defined morphism of $\mathbb{Q}_p$-affinoid spaces. This concludes the proof. □

We will make use of the map ψ_α just defined to construct a family of points (i.e. trianguline representations) on $\mathfrak{X}^2$ which will pass through u_{1-k} (i.e. the representation $V(\mu_{a_p}, \mu_{\frac{1}{a_p}}\chi^{1-k})$).

For each $s \in \overline{\mathbb{Z}}_p$, we define a multiplicative character of $\mathbb{Q}_p^\times$ as follows:

$$\begin{aligned} \delta_{k,a_p}^{(s)} : \mathbb{Q}_p^\times &\longrightarrow \mathbb{E}(s)^\times \\ x &\longmapsto \delta_{k,a_p}^{(s)}(x) := \mu_{\frac{1}{a_p}}(x) \cdot \omega(x)^{1-k} \cdot \psi_{\langle x\rangle}(s), \end{aligned}$$

where $\mathbb{E}(s)$ is the finite extension obtained from $\mathbb{E}$ by adding s; and where $x = p^{v_p(x)}\omega(x)\langle x\rangle$ is the unique decomposition given by a fixed isomorphism $\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mu(\mathbb{Q}_p) \times 1 + p\mathbb{Z}_p$.

Note that $\psi_{\langle x\rangle}(s)$ is an element in $\mathbb{Q}_p(s)$ since $\binom{s}{n}(\langle x\rangle - 1)^n \in \mathbb{Q}_p(s)$ for any $n \in \mathbb{Z}_{\geq 1}$.

Finally, we are ready to apply this in the context of rigid analytic spaces, indeed we will define a 1-dimensional p-adic family of points in $\mathfrak{X}^2$ through which we will control the "p-adic" distance between u_{1-k} and $u_{1-k'}$.

We define $\mathcal{Z}$ to be the $\mathbb{Q}_p$-affinoid spaces given by $\{\mu_{a_p}\} \times \{1/a_p\} \times \{\zeta_p^{k-1}\} \times \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$; here $\{\mu_{a_p}\}$ denotes the singleton corresponding to the character μ_{a_p} on $\mathfrak{X}$, the singleton $\{1/a_p\}$ corresponds to the $\mathbb{E}$-point $1/a_p$ in $\mathbb{G}_m^{\mathrm{rig}}$ and the singleton $\{\zeta_p^{k-1}\}$ corresponds to the point ζ_{p-1}^{k-1} in $\mu(\mathbb{Q}_p)$.

By a little abuse of notation, we will still denote a point in $\mathcal{Z}$ by s for the corresponding point $s \in \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$. Now, we define the injective map:

$$\begin{aligned} \Phi : \mathcal{Z} &\longrightarrow \mathfrak{X}^2 \\ s &\longmapsto \Phi(s) := (\mu_{a_p}, \delta_{k,a_p}^{(s)}) \end{aligned}$$

and note that if $k' \in \mathbb{Z}_{\geq 2}$ satisfies $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$ we have, by construction, that $\Phi(1-k) = u_{k,a_p}$ and $\Phi(1-k') = u_{k',a_p}$ since $\delta_{k,a_p}^{(1-k)} = \mu_{\frac{1}{a_p}}\chi^{1-k}$ and $\delta_{k,a_p}^{(1-k')} = \mu_{\frac{1}{a_p}}\chi^{1-k'}$.

Proposition 5.8. *The map $\Phi : \mathcal{Z} \to \mathfrak{X}^2$ is a rigid analytic closed immersion.*

Proof. In order to see that Φ is a morphism of $\mathbb{Q}_p$-rigid analytic spaces it is sufficient to observe that, decomposing $\mathfrak{X}^2$ as $\mathfrak{X} \times \mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p}$, the map Φ is a product of constant morphisms and ψ_{1+p}:

$$\begin{aligned} \Phi : \mathcal{Z} \longrightarrow \mathfrak{X}^2 &= \mathfrak{X} \times \mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p} \\ s \longmapsto (\mu_{a_p}, \delta_{k,a_p}^{(s)}) &= (\mu_{a_p}, \delta_{k,a_p}^{(s)}(\zeta_{p-1}), \delta_{k,a_p}^{(s)}(p), \delta_{k,a_p}^{(s)}(1+p)) \\ &= (\mu_{a_p}, [\zeta_{p-1}^{1-k}], [a_p^{-1}], \psi_{1+p}(s)). \end{aligned}$$

The universal property of fiber products (in the category of rigid analytic spaces) allows us to reduce the claim to prove that the composition of Φ with the projection on the last factor of $\mathfrak{X}^2$, which is exactly ψ_{1+p}, belongs to $\mathrm{Mor}(\mathcal{Z}, \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p})$. This follows at once from Lemma 5.6. Moreover, the image is an affinoid subdomain of $\mathfrak{X}^2$ making Φ a closed immersion. □

Now, the heart of the proof is that the representations attached to points of $\mathfrak{X}^2$ vary locally in a continuous way. In precise terms, this is the following result of Colmez and Chenevier (see [21, Prop. 5.2] and its generalization [18, Prop. 3.9], see also [8, Prop. 3.2]):

Theorem 5.9. *Let $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ be two characters such that $\delta_1\delta_2^{-1} \neq x^{-i}$ for some $i \geq 0$, where x denotes the identity character of $\mathbb{Q}_p^\times$. Let $u = (\delta_1, \delta_2)$ be the corresponding point in $\mathfrak{X}^2$. Then there exists a open neighborhood $\mathfrak{U}$ of u and a finite, free $\mathcal{O}_{\mathfrak{U}}$-module $\mathbb{V}$ of rank 2 with an action of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\bar{\mathbb{Q}}_p, \mathbb{Q}_p)$ such that $\mathbb{V}(\widetilde{u}) = V(\delta_1(\widetilde{u}), \delta_2(\widetilde{u}))$ for every $\widetilde{u} \in \mathfrak{U}$.*

As we are interested in points inside $\mathfrak{U} \subset \mathfrak{X}^2$ which is an open neighborhood of u_{1-k}, we will first prove that if k' is sufficiently p-adically close to k (close as points in the weight space) then also $\Phi(1-k') = u_{1-k'}$ will lie in $\mathfrak{U}$.

Without loss of generality, as $\mathfrak{U}$ is an admissible open we can assume (up to restriction) that $(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$ is an affinoid space. Since Φ is a morphism of rigid analytic spaces, it is in particular continuous for the G-topology, hence $\Phi^{-1}(\mathfrak{U})$ is an admissible open of the affinoid space $\mathcal{Z}$.

In particular, we can deduce that there exists a minimal $r \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that the affinoid subdomain $\mathcal{Z}_r := \{\mu_{a_p}\} \times \{[a_p^{-1}]\} \times \{[\zeta_p^{k-1}]\} \times \mathbb{B}^1(1-k, p^{-r})^+_{\mathbb{Q}_p}$ of $\mathcal{Z}$ is contained in $\Phi^{-1}(\mathfrak{U})$. As usual, we identify the algebra of functions $\mathcal{O}_{\mathcal{Z}_r}$ of the $\mathbb{Q}_p$-affinoid space $\mathcal{Z}_r$ with $\mathbb{E} \otimes \mathbb{Q}_p\langle \frac{T-(1-k)}{p^r} \rangle$. By restricting the morphism Φ to $\mathcal{Z}_r$, we get the morphism of $\mathbb{Q}_p$-affinoid spaces:

$$\Phi : \mathcal{Z}_r \to \mathfrak{U}$$

and as usual, we denote its associated morphism of $\mathbb{Q}_p$-affinoid algebras by $\Phi^* : \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathcal{Z}_r}$.

Now, observe that if we fix a $\bar{u} \in \mathfrak{U}$ with field of definition $\mathbb{L}_{\bar{u}}$, it induces a $\mathbb{Q}_p$-Banach spaces morphism given by the evaluation $\mathrm{ev}_{\bar{u}} : \mathcal{O}_{\mathfrak{U}} \to \mathbb{L}_{\bar{u}}$. Consider now the finite, free $\mathcal{O}_{\mathfrak{U}}$-module $\mathbb{V}$ of rank 2 considered by Colmez. The ring homomorphism $\mathrm{ev}_{\bar{u}}$ induces on $\mathbb{L}_{\bar{u}}$ a structure of $\mathcal{O}_{\mathfrak{U}}$-module, hence we define $\mathbb{V}(\bar{u}) := \mathbb{V} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathbb{L}_{\bar{u}}$. By Chenevier's and Colmez's Theorem 5.9, we have that $\mathbb{V}(\bar{u}) = V(\delta_1(\bar{u}), \delta_2(\bar{u}))$ where $\bar{u} := (\delta_1(\bar{u}), \delta_2(\bar{u})) \in \mathfrak{X}^2$. In particular, we note that when $\bar{u} = u_{1-k}$, then $\mathbb{L}_{u_{1-k}} = \mathbb{E}$ and $\mathbb{V}(u_{1-k}) = V(\mu_{a_p}, \delta_{k,a_p})$. Clearly the analogue statement holds for $k' \in \mathcal{Z}_r$ such that $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$.

The idea is now to pull back the analytic family of representations given by Colmez in order to create a new analytic family parametrized by points in $\mathcal{Z}_r$ which has the advantage that will depend only on one parameter. The notion of analytic family of representations parametrized by an affinoid space is used in the sense of Berger and Colmez (see [9]) but one could also have approached the problem from the point of view of analytic family of (φ, Γ)-modules over variations of the Robba rings in the sense of Bellaïche (see [3]) considering the existence of a fully faithful functor which connects the two categories (see [3, §3]).

Coming back to the closed immersion $\Phi : \mathcal{Z}_r \to \mathfrak{U}$ of $\mathbb{Q}_p$-affinoid spaces, we have that the induced map $\Phi^* : \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathcal{Z}_r}$ is given by the pull-back, i.e. $\Phi^*(f) = f \circ \Phi$ for all $f \in \mathcal{O}_{\mathfrak{U}}$. The ring homomorphism Φ^* gives to $\mathcal{O}_{\mathcal{Z}_r}$ the structure of $\mathcal{O}_{\mathfrak{U}}$-module and so we can define $\mathbb{V}_{\mathcal{Z}_r} := \mathbb{V} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}$, it is a finite, free $\mathcal{O}_{\mathcal{Z}_r}$-module of rank 2 with a continuous $G_{\mathbb{Q}_p}$-action (given by

the action of the Galois group on $\mathbb{V}$). In particular, for all $s \in \mathcal{Z}_r$ we have by definition that $\mathbb{V}_{\mathcal{Z}_r}(s) \cong \mathbb{V}(\Phi(s))$.

Now, in order to deal with reductions we first need to identify an integral analytic family of lattices. First, we recall that there is a notion of integral model for affinoid algebras. By integral model, or simply model, of a $\mathbb{Q}_p$-affinoid algebra A we mean a $\mathbb{Z}_p$-subalgebra $\widetilde{A}$ topologically of finite type (i.e. it is a quotient of $\mathbb{Z}_p\langle x_1, \ldots, x_n\rangle$ for some integer $n \geq 1$ where each x_i is a power bounded element of A) and such that $\widetilde{A}[\frac{1}{p}] = A$. We denote by A° the subalgebra of power bounded elements. It is a model of A. We recall that under the assumption that the affinoid algebra A is reduced, the model A° is the biggest one and it contains any other model. In our case, we have $\mathcal{O}_{\mathfrak{U}}^\circ = \{g \in \mathcal{O}_{\mathfrak{U}} : \ |g|_{\sup} \leq 1\}$. It is an integral model for $\mathcal{O}_{\mathfrak{U}}$.

The local existence of an integral analytic family of representations inside $\mathbb{V}$ follows from the following lemma (which is a small variation of [19, Lem. 3.18]):

Lemma 5.10. *Let G be a profinite group, let A be a $\mathbb{Q}_p$-affinoid algebra and denote by $X = \mathrm{Spm}(A)$ its attached affinoid $\mathbb{Q}_p$-rigid analytic space. Let $\mathbb{V}$ be a finite, free A-module of rank 2 endowed with an A-linear, continuous action of G. For every $x \in X$, there exists an affinoid open neighborhood $U_x = \mathrm{Spm}(A_{U_x}) \subset X$, a model $\mathcal{A}_{U_x}$ of the affinoid algebra A_{U_x} and a finite, free $\mathcal{A}_{U_x}$-module $\mathbb{T}_{U_x}$ with an action of G such that the natural map given by the extension of scalars induces an isomorphism:*

$$\mathbb{T}_{U_x} \otimes_{\mathcal{A}_{U_x}} A_{U_x} \cong \mathbb{V}_{U_x} := \mathbb{V} \otimes_A A_{U_x}$$

of $A_{U_x}[G]$-modules.

Proof. Let $\mathcal{A}$ be an integral model of A. Let $\mathbb{W}$ be a finite and free $\mathcal{A}$-module such that $\mathbb{W} \otimes_{\mathcal{A}} A = \mathbb{V}$. Since $\mathcal{A}$ is open inside A, and since $\mathbb{V}$ is a topological finite direct sum of copies of A we have that $\mathbb{W}$ is open inside $\mathbb{V}$. The action of G can be represented by a continuous map $G \times \mathbb{W} \to \mathbb{V}$. Since $\mathbb{W}$ is open inside $\mathbb{V}$, then the subgroup $H_{\mathbb{W}} \subset G$ stabilizing $\mathbb{W}$ is an open subgroup of G. Since G is profinite, we have that $H_{\mathbb{W}}$ is of finite index. Let $\{h_i\}_i$ be a finite set of representatives for the left $H_{\mathbb{W}}$-cosets in G. Hence, defining $\mathbb{S}$ as $\sum_i h_i\mathbb{W}$ we have that $\mathbb{S}$ is a G-stable, finite $\mathcal{A}$-module such that $\mathbb{S} \otimes_{\mathcal{A}} A = \mathbb{V}$.

In general, it is not true that $\mathbb{S}$ is free as a $\mathcal{A}$-module, so in order to find a free module satisfying all the required properties we will work locally using Raynaud's theory of formal models.

Let V_x be an affinoid open neighborhood of $x \in X$. Denote by A_{V_x} its corresponding affinoid algebra (it is a quotient of A) and define $\mathbb{V}_{V_x} := \mathbb{V} \otimes_A A_{V_x}$. As every affinoid algebra morphism is in particular a contraction, denote by $\widetilde{\mathcal{A}}_{V_x}$ the model for the affinoid algebra A_{V_x} defined by the natural

projection from A to A_{V_x}. Denote by $\mathfrak{V}_{V_x} = \mathrm{Spf}(\widetilde{\mathcal{A}}_{V_x})$ the integral formal scheme (of rigid generic fiber V_x) attached to the model $\widetilde{\mathcal{A}}_{V_x}$ via Raynaud's theory. We can now restrict our integral subfamilies of G-modules to the affinoid open V_x by defining $\mathbb{S}_{V_x} := \mathbb{S} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}_{V_x}$. We have that $\mathbb{S}_{V_x}$ is a finite $\widetilde{\mathcal{A}}_{V_x}$-module with a continuous $\widetilde{\mathcal{A}}_{V_x}$-linear action of G and such that the natural scalar extension map $\mathbb{S}_{V_x} \otimes_{\tilde{\mathcal{A}}_{V_x}} A_{V_x} \cong \mathbb{V}_{V_x}$ is an isomorphism of $A_{V_x}[G]$-modules.

Now that we have set the problem locally, we can make full use of Raynaud's theory of formal integral models. Indeed, let $\mathcal{I}$ be the Fitting ideal (of $\widetilde{\mathcal{A}}_{V_x}$) of the $\widetilde{\mathcal{A}}_{V_x}$-module $\mathbb{S}_{V_x}$. The ideal $\mathcal{I}$ defines a blow-up morphism of formal schemes (see [24, §3.3] for the construction):

$$\mathrm{Bl}_{\mathcal{I}} : \mathfrak{Z}_{V_x} \to \mathfrak{V}_{V_x}$$

A priori, the formal scheme $\mathfrak{Z}_{V_x}$ is not affine but up to considering an affine open covering of $\mathfrak{Z}_{V_x}$ and without loss of generality, we can substitute the admissible open V_x with a sufficiently small affine open neighborhood of x, say U_x, such that the blow-up morphism constructed is actually a morphism between affine formal schemes. Since we are repeating all the above constructions, all the subscripts V_x will now be substituted without loss of generality with U_x affinoid open neighborhood of x contained in V_x. This implies that the formal integral scheme $\mathfrak{Z}_{U_x}$ is the formal spectrum of a model $\mathcal{A}_{U_x}$ (a priori, different from $\widetilde{\mathcal{A}}_{U_x}$) of the $\mathbb{Q}_p$-affinoid algebra A. Indeed, we recall that the blow-up morphism induces an isomorphism on the generic fiber which in this case is the $\mathbb{Q}_p$-rigid analytic affinoid space $U_x = \mathrm{Spm}(A_{U_x})$.

Now, consider the blow-up morphism

$$\mathrm{Bl}_{\mathcal{I}} : \mathfrak{Z}_{U_x} = \mathrm{Spm}(\mathcal{A}_{U_x}) \to \mathfrak{V}_{U_x} = \mathrm{Spm}(\widetilde{\mathcal{A}}_{U_x}).$$

Identifying the finite $\mathcal{A}_{U_x}$-module $\mathbb{S}$ as a coherent sheaf on the formal scheme $\mathfrak{V}_x$, we can pull it back to a coherent sheaf of the affine formal scheme $\mathfrak{Z}_{U_x} = \mathrm{Spf}(\mathcal{A}_{U_x})$ via the blow-up morphism $\mathrm{Bl}_{\mathcal{I}} : \mathfrak{Z}_{U_x} \to \mathfrak{V}_{U_x}$, i.e. formally we define $\mathbb{T}_{U_x} := \mathrm{Bl}^*_{\mathcal{I}}(\mathbb{S}_{U_x})$. By construction of the blow-up via the Fitting ideal I of the $\widetilde{\mathcal{A}}_{U_x}$-module $\mathbb{S}$, the $\mathcal{A}_{U_x}$-module $\mathbb{T}_{U_x}$ will be finite and free and inherits the $\mathcal{A}_{U_x}$-linear action of G compatible with the base change, i.e. we have an isomorphism $\mathbb{T}_{U_x} \otimes_{\mathcal{A}_{U_x}} A_{U_x} \cong \mathbb{V}_{U_x}$ of $A_{U_x}[G]$-modules. This concludes the proof. □

Remark 5.11. The above result has been generalized also to the context of analytic families of Galois representations parametrized by a generic quasi-separated and quasi-compact rigid analytic space (see the discussion in [4, §2]).

Coming back to our case, up to decomposing $\mathfrak{U}$ in irreducible components, we can assume without loss of generality that $\mathfrak{U}$ is irreducible or equivalently

its associated affinoid algebra is reduced. Applying the above result (up to restricting the affinoid open $\mathfrak{U}$) when $G = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $A = \mathcal{O}_{\mathfrak{U}}$, $A^\circ = \mathcal{O}^\circ_{\mathfrak{U}}$ and $V = \mathbb{V}$ allow us to deduce that there exists $\mathbb{T}$ inside $\mathbb{V}$ finite, free $\mathcal{O}^\circ_{\mathfrak{U}}$-module of rank 2, which is $G_{\mathbb{Q}_p}$-stable and such that $\mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} \mathcal{O}_{\mathfrak{U}} \cong \mathbb{V}$ as $G_{\mathbb{Q}_p}$-modules.

After defining $\mathbb{V}_{\mathcal{Z}_r}$ as $\mathbb{V}_{\mathcal{O}_{\mathfrak{U}}} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}$, consider the model $\mathcal{O}^\circ_{\mathcal{Z}_r} := \{g \in \mathcal{O}_{\mathcal{Z}_r} : |g|_{\sup} \leq 1\}$ inside $\mathcal{O}_{\mathfrak{U}}$. Since every morphism of affinoid algebras is in particular a contraction, we have that via the restriction of Φ to the power bounded elements inside $\mathcal{O}_{\mathfrak{U}}$ we obtain the $\mathbb{Z}_p$-algebra morphism $\Phi : \mathcal{O}^\circ_{\mathfrak{U}} \twoheadrightarrow \mathcal{O}^\circ_{\mathcal{Z}_r}$. This allows to define $\mathbb{T}_{\mathcal{Z}_r} := \mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} \mathcal{O}^\circ_{\mathcal{Z}_r}$.

The properties of $\mathbb{T}_{\mathcal{Z}_r}$ are summarized in the following:

Lemma 5.12. *The $\mathcal{O}^\circ_{\mathcal{Z}_r}$-module $\mathbb{T}_{\mathcal{Z}_r}$ is finite, free of rank 2 submodule of $\mathbb{V}_{\mathcal{Z}_r}$. It has a natural action of the Galois group $G_{\mathbb{Q}_p}$; in particular we have that:*

$$\mathbb{T}_{\mathcal{Z}_r} \otimes_{\mathcal{O}^\circ_{\mathcal{Z}_r}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{V}_{\mathcal{Z}_r}$$

is an isomorphism of $G_{\mathbb{Q}_p}$-modules.

Proof. The only claim that is not clear is the isomorphism of $G_{\mathbb{Q}_p}$-modules. By definition we have an isomorphism $\mathbb{T} \otimes_{\mathcal{O}^0_{\mathfrak{U}}} \mathcal{O}_{\mathfrak{U}} \cong \mathbb{V}$ of $G_{\mathbb{Q}_p}$-modules, hence tensorizing by $\mathcal{O}_{\mathcal{Z}_r}$ (considered as $\mathcal{O}_{\mathfrak{U}}$-module via Φ^*) we get the isomorphism $(\mathbb{T} \otimes_{\mathcal{O}^0_{\mathfrak{U}}} \mathcal{O}_{\mathfrak{U}}) \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{V}_{\mathcal{Z}_r}$ of $G_{\mathbb{Q}_p}$-modules.

We have the following chain of isomorphism of $G_{\mathbb{Q}_p}$-modules:

$$\begin{aligned}&(\mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} \mathcal{O}_{\mathfrak{U}}) \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} (\mathcal{O}_{\mathfrak{U}} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}) \cong \mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}\\ &\cong \mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} (\mathcal{O}^\circ_{\mathcal{Z}_r} \otimes_{\mathcal{O}^\circ_{\mathcal{Z}_r}} \mathcal{O}_{\mathcal{Z}_r}) \cong (\mathbb{T} \otimes_{\mathcal{O}^\circ_{\mathfrak{U}}} \mathcal{O}^\circ_{\mathcal{Z}_r}) \otimes_{\mathcal{O}^\circ_{\mathcal{Z}_r}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{T}_{\mathcal{Z}_r} \otimes_{\mathcal{O}^\circ_{\mathcal{Z}_r}} \mathcal{O}_{\mathcal{Z}_r},\end{aligned}$$

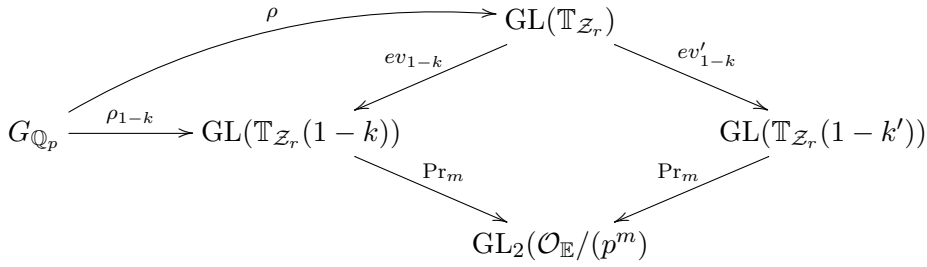
where the isomorphisms are given by the associative property of tensor product for general modules (see [15, Prop. 3.8, Chap. 3]). Hence, the claim follows. □

In terms of Berger's and Colmez's notion of analytic families of p-adic representations (see [9]), the two above results essentially grant that the Galois properties of analytic integral subfamilies are preserved via pull-back. Note that now for all $s \in \mathcal{Z}_r$, if we denote by $\mathbb{L}_{u_s}$ the field of definition of the point $\Phi(s) := u_s$ then $\mathbb{T}_{\mathcal{Z}_r}(s) \otimes \mathbb{L}_{u_s} \cong \mathbb{V}_{\mathcal{Z}_r}(s)$ as $G_{\mathbb{Q}_p}$-modules. Hence, we deduce that $\mathbb{T}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{T}_{\mathcal{Z}_r}(1-k')$ are two $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_{\mathbb{E}}$-lattices inside respectively $\mathbb{V}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{V}_{\mathcal{Z}_r}(1-k')$.

Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(\mathbb{T}_{\mathcal{Z}_r}) \cong \mathrm{GL}_2(\mathcal{O}^\circ_{\mathcal{Z}_r})$ the Galois representations attached to $\mathbb{T}_{\mathcal{Z}_r}$. For every $s \in \mathcal{Z}_r$, we denote by $\rho_s : G_{\mathbb{Q}_p} \to \mathrm{GL}(\mathbb{T}_{\mathcal{Z}_r}(s))$ the specialization of ρ at s. This representation correspond to a $G_{\mathbb{Q}_p}$-stable lattice inside the trianguline representation $V(\delta_1(s), \delta_2(s)) \cong \mathbb{V}_{\mathcal{Z}_r}(s) \cong \mathbb{V}(\Phi(s))$. In particular, we have that ρ_{1-k} and $\rho_{1-k'}$ correspond to $G_{\mathbb{Q}_p}$-stable lattices inside respectively the representations $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$.

Moreover, as we have already seen, the representations ρ_s can be obtained from ρ via composition by the evaluation map at s, i.e. we have $\rho_s = \mathrm{ev}_s \circ \rho$ where we keep ev_s as our notation for the induced map on GL_2 from ev_s.

Now, for a fixed $m \in \mathbb{Z}_{\geq 1}$, we can consider the diagram:



where Pr_m denotes the induced homomorphism on GL_2 from the natural projection $\mathcal{O}_{\mathbb{E}} \twoheadrightarrow \mathcal{O}_{\mathbb{E}}/(p^m)$.

It is clear that the above diamond in the diagram commutes if and only if for all $f \in \mathcal{O}^o_{\mathcal{Z}_r}$ we have $f(1-k) - f(1-k') \in (p^m)$ inside $\mathcal{O}_{\mathbb{E}}$.

Hence, we reduced the claim of Theorem 5.1 to prove that there exists a positive integer $n = n(k, a_p, m) \geq 1$ such that if $k' - k \in p^n(p-1)\mathbb{Z}_{\geq 0}$ then $|\mathrm{ev}_{1-k'}(f) - \mathrm{ev}_{1-k}(f)| = |f(1-k') - f(1-k)| \leq p^{-m}$.

This follows from the general following (this is just a slight variation of [14, Prop. 7.2.1.1]):

Lemma 5.13. *Let $r \geq 0$ be an integer. For every $g \in \overline{\mathbb{Q}}_p\langle \frac{T}{p^r}\rangle := \{\sum_n a_n T^n : a_n p^{rn} \to 0$ as $n \to \infty\}$ and for any $x, y \in \mathbb{B}^1(0, p^{-r})^+$ we have*

$$|g(x) - g(y)| \leq p^r |g|_r |x - y|.$$

Here $|\cdot|_r$ denotes the norm on $\overline{\mathbb{Q}}_p\langle \frac{T}{p^r}\rangle$ given by $|\sum a_n T^n| := \max |a_n p^{rn}|$ and $|\cdot|$ denotes the usual norm on $\overline{\mathbb{Q}}_p\langle T\rangle$.

Proof. Consider the map $\alpha : \overline{\mathbb{Q}}_p\langle \frac{T}{p^r}\rangle \to \overline{\mathbb{Q}}_p\langle T\rangle$ sending T to $p^r T$; it is an isometric isomorphism with respect to the norms $|\cdot|_r$ and $|\cdot|$ respectively. Denote by α^* the induced map on maximal spectra, i.e. $\alpha^* : \mathbb{B}^1(0,1)^+ \to \mathbb{B}^1(0, p^{-r})^+$ sending z to $p^r z$; it is bijective. Let $g \in \overline{\mathbb{Q}}_p\langle \frac{T}{p^r}\rangle$ such that $\alpha(g) = f$.

It is a classical result in the theory of Tate's algebras (see [14, Prop. 7.2.1.1]) that for any $f \in \overline{\mathbb{Q}}_p\langle T\rangle$ and for any $\widetilde{x}, \widetilde{y} \in \mathbb{B}^1(0,1)^+$:

$$|f(\widetilde{x}) - f(\widetilde{y})| \leq |f||\widetilde{x} - \widetilde{y}|.$$

Since α is an isometry and defining $\alpha^*(\widetilde{x}) = x$ and $\alpha^*(\widetilde{y}) = y$, this is equivalent to say:

$$\begin{aligned}|\alpha(g)(\widetilde{x}) - \alpha(g)(\widetilde{y})| &\leq |f| \cdot |\widetilde{x} - \widetilde{y}| \\ &\iff |g(\alpha^*(\widetilde{x})) - g(\alpha^*(\widetilde{y}))| \leq |\alpha(g)| \cdot |\widetilde{x} - \widetilde{y}| \\ &\iff |g(x) - g(y)| \leq |g|_r \cdot |\widetilde{x} - \widetilde{y}| = |g|_r \cdot \Big|\frac{x}{p^r} - \frac{y}{p^r}\Big| = p^r |g|_r |x - y|.\end{aligned}$$

Note that we used the fact that $\alpha(g)(\widetilde{x}) = g(\alpha^*(\widetilde{x}))$ (and the same for y) which is a standard property of affinoid maps (see [14, Lem. 7.1.4.2]). □

Finally, we can complete the proof of Theorem 5.1. Indeed, we have that the model $\mathcal{O}^o_{\mathcal{Z}_r}$ is isomorphc to $\mathbb{E}\langle\frac{T}{p^r}\rangle^\circ := \{g \in \mathbb{E}\langle\frac{T}{p^r}\rangle : |g|_{\sup} = |g|_r \leq 1\}$. Hence, for all $g \in \mathbb{E}\langle\frac{T}{p^r}\rangle^\circ$ we have: $|g(x)-g(y)| \leq p^r|x-y|$ for all $x, y \in \overline{\mathbb{Z}}_p$ representing the corresponding maximal ideals in $\mathcal{Z}_r$. Note that we are considering fixed an embedding of $\mathbb{E}$ in $\overline{\mathbb{Q}}_p$.

For any fixed positive integer m such that the hypothesis of the theorem holds, there exists a positive integer n, namely $n = m + r$, such that the representations $\mathbb{T}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{T}_{\mathcal{Z}_r}(1-k')$ are congruent modulo p^m. By the definition of $\mathbb{T}_{\mathcal{Z}_r}$ we deduce that the same is true for $\mathbb{T}(u_{1-k})$ and $\mathbb{T}(u_{1-k'})$. This completes the proof of Theorem 5.1.

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