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Oscillations in the Goldbach conjecture

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Résumé. Soit $R(n) = \sum_{a+b=n} \Lambda(a)\Lambda(b)$, où $\Lambda(\,\cdot\,)$ est la fonction de von Mangoldt. La fonction $R(n)$ est souvent étudiée en relation avec la conjecture de Goldbach. Sous l'hypothèse de Riemann (RH), on sait que $\sum_{n\le x} R(n) = x^2/2 - 4x^{3/2}G(x) + O(x^{1+\epsilon})$, où $G(x) = \Re\sum_{\gamma>0} \frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)}$ et la somme est prise sur les ordonnées des zéros non triviaux de la fonction zêta de Riemann dans le demi-plan supérieur. Nous prouvons (sous l'hypothèse de Riemann) que chacune des inégalités $G(x) < -0.02297$ et $G(x) > 0.02103$ est vérifiée infiniment souvent, et établissons une amélioration de cette dernière borne sous une hypothèse d'indépendance linéaire pour les zéros de la fonction zêta. Nous montrons également que les bornes obtenues sont très proches de l'optimal.

Abstract. Let $R(n) = \sum_{a+b=n} \Lambda(a)\Lambda(b)$, where $\Lambda(\,\cdot\,)$ is the von Mangoldt function. The function $R(n)$ is often studied in connection with Goldbach's conjecture. On the Riemann hypothesis (RH) it is known that $\sum_{n\le x} R(n) = x^2/2 - 4x^{3/2}G(x) + O(x^{1+\epsilon})$, where $G(x) = \Re\sum_{\gamma>0} \frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)}$ and the sum is over the ordinates of the nontrivial zeros of the Riemann zeta function in the upper half-plane. We prove (on RH) that each of the inequalities $G(x) < -0.02297$ and $G(x) > 0.02103$ holds infinitely often, and establish an improvement on the latter bound under an assumption of linearly independence for zeros of the zeta function. We also show that the bounds we obtain are very close to optimal.

1. Introduction

Let $\Lambda(n)$ denote the von Mangoldt function, and define $R(n)$ by

$$R(n) = \sum_{a+b=n} \Lambda(a)\Lambda(b), \tag{1.1}$$

where the sum is over positive integers a and b that sum to n. This function arises naturally in the study of Goldbach's problem: clearly $R(n) > 0$ precisely when n is the sum of two positive prime powers. The use of the von Mangoldt function makes the problem more amenable to analysis, and

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Goldbach's conjecture would follow if it could be shown that $R(n)$ were sufficiently large at even integers $n > 2$. It is natural then to study the average value of $R(n)$. It is known that

$$\sum_{n\le x} R(n) = \frac{1}{2}x^2 + O\left(x^2\exp(-C(\log x)^{3/5}(\log\log x)^{-1/5})\right), \tag{1.2}$$

unconditionally, for some positive constant C. In a series of articles in 1991, Fujii obtained improvements on the error term in (1.2) that are conditional on RH. (In fact, Fujii [5, p. 173] cited the weaker unconditional error term $O(x^2(\log x)^{-A})$, for any positive constant A, which follows from the Prime Number Theorem; (1.2) follows by the same reasoning but using the Vinogradov–Korobov zero-free region for the zeta function.) In the first of this series, he established [5] that

$$\sum_{n\le x} R(n) = \frac{1}{2}x^2 + O(x^{3/2}),$$

and in the second paper [6] he refined the error term, proving that[1]

$$\sum_{n\le x} R(n) = \frac{1}{2}x^2 - 4x^{3/2}\Re\sum_{\gamma>0}\frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + O\left((x\log x)^{4/3}\right). \tag{1.3}$$

Similar statements, with a slightly larger power on the $\log x$ term, were noted by Goldston [8] and by Granville [10, 11] (where a main result [10, Theorem 1A] states that RH is equivalent to an estimate regarding the average number of solutions in the Goldbach problem). Reductions in the error term in (1.3) were made by Bhowmik and Schlage-Puchta [3] and then by Languasco and Zaccagnini [15], who established $O(x\log^3 x)$. See also Goldston and Yang [9] for a proof of this result. This is fairly close to optimal, since Bhowmik and Schlage-Puchta also proved that the error term here is $\Omega(x\log\log x)$. Analogous results for forms of (1.1), where n is written as the sum of k prime powers, have been proved by Languasco and Zaccagnini [15] and by Bhowmik, Ramaré, and Schlage-Puchta [2]. For a recent survey of results in this area, see [1].

In this article, we study the oscillations in the sum on the right side of (1.3). To this end, define $G(x)$ by

$$G(x) = \Re\sum_{\gamma>0}\frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)}, \tag{1.4}$$

where the sum is over the ordinates of the zeros of the Riemann zeta function in the upper half-plane. We assume RH, so each such zero has real part 1/2. Any multiple zeros that may occur appear with the appropriate

[1] We note that the sum over zeros, written in the form $2\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$, appears here even without assuming RH.

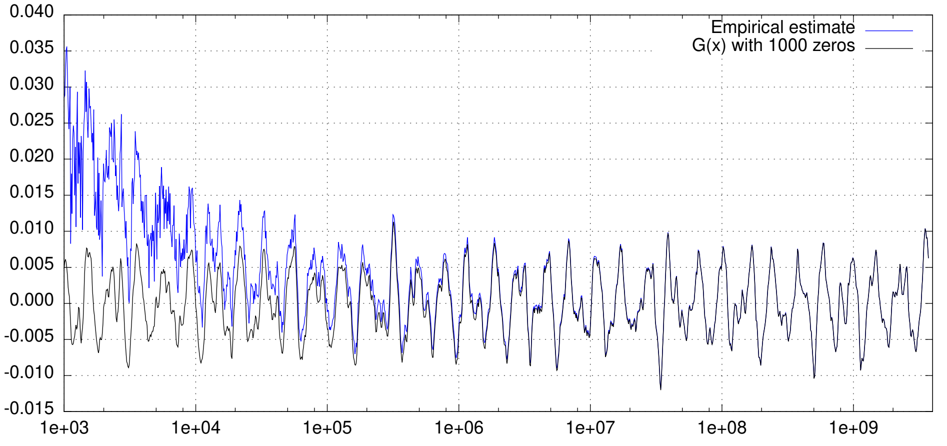

FIGURE 1.1. $G(x)$ using 1000 zeros of $\zeta(s)$ (lower curve), and the empirical estimate $(\frac{x^2}{2} - \sum_{n\le x} R(n))/4x^{3/2}$.

multiplicity in (1.4) and similar such sums throughout this article. A plot of $G(x)$ using the first 1000 zeros of the zeta function on the critical line appears in Figure 1.1 (courtesy of T. Oliveira e Silva), along with an estimate for this function obtained by computing the sum of $R(n)$ for $n \le x$, then subtracting off the main term $x^2/2$ and scaling by $-1/4x^{3/2}$.

In his third paper of 1991 on this topic [7], Fujii proved that if the ordinates of the first 70 zeros of the Riemann zeta function on the critical line are linearly independent over the rationals, then each of the inequalities

$$G(x) < -0.012, \quad G(x) > 0.012 \tag{1.5}$$

would hold for an unbounded sequence of positive real numbers x. He noted that this conclusion could also be established without the linear independence hypothesis, if one instead employed a method of Odlyzko and te Riele to solve certain inhomogeneous simultaneous approximation problems involving these 70 real numbers. In 1985 Odlyzko and te Riele [20] famously employed this method to disprove the Mertens conjecture regarding the size of oscillations in the function $M(x) = \sum_{n\le x} \mu(n)$, where $\mu(\cdot)$ represents the Möbius function. Recently, Hurst [13] used the same method, along with additional techniques, to obtain the presently best known result in this problem.

Odlyzko and te Riele established large oscillations in the positive direction by determining a real number y and integers $m_1, \ldots, m_{70}$ with the property that

$$\left|\gamma_{k_j} y - \psi_{k_j} - 2m_j\pi\right| < \epsilon,$$

for $1 \leq j \leq 70$, for a small positive number ϵ. Here $-\psi_{k_j}$ represents the argument of the residue of $1/\zeta(s)$ at $s = 1/2 + i\gamma_{k_j}$, and $1 \leq k_1 < k_2 < \cdots < k_{70} \leq 400$ denotes a particular sequence of positive integers corresponding to the zeros which produced the most beneficial contributions in the method employed there. Likewise, to establish large oscillations in the negative direction, they determined z, $n_1, \ldots, n_{70}$ so that

$$\left|\gamma_{k_j} z - \psi_{k_j} - (2n_j + 1)\pi\right| < \epsilon',$$

for $1 \leq j \leq 70$, for a small positive number ϵ'. In [7], Fujii required analogous results for the same problems, but with each ψ_{k_j} eliminated, $k_j = j$ for each j, and $\epsilon = \epsilon' = 0.1$. (The first case is then a simpler homogeneous approximation problem.) It is not clear however if the required computations were in fact performed in [7]: it is stated that the argument there implies the bounds (1.5) "in principle".

In this article, we analyze the oscillations in $G(x)$, and prove two main results. First, we use the method of Odlyzko and te Riele to establish a lower bound on the oscillations exhibited by this function, improving (1.5). We also establish an improved bound under an assumption of linear independence for the zeros of the zeta function. Second, we establish an upper bound on these oscillations, which shows that our results are close to optimal. We prove the following theorem.

Theorem 1.1. *With $G(x)$ as in (1.4), on the Riemann hypothesis each of the following inequalities holds for an unbounded sequence of positive real numbers x:*

$$G(x) < -0.022978, \quad G(x) > 0.021030. \tag{1.6}$$

Moreover, for all $x > 0$,

$$|G(x)| < \sum_{\substack{\zeta(1/2+i\gamma)=0 \\ \gamma>0}} \frac{1}{\sqrt{(\gamma^2 + 1/4)(\gamma^2 + 9/4)}} < 0.023059. \tag{1.7}$$

In addition, if the ordinates of the first 10^6 zeros of the Riemann zeta function in the upper half-plane are linearly independent over $\mathbb{Q}$, then the inequality

$$G(x) > 0.022978 \tag{1.8}$$

holds for an unbounded sequence of positive real numbers x.

Thus, by (1.6) the oscillations in $G(x)$ are close to best possible, since they cannot exceed the bound given in (1.7). Moreover, the limit in (1.7) is itself presumably close to best possible, owing to (1.8). It may be possible to weaken the hypotheses leading to (1.8), for example, by assuming that the first 10^6 zeros of $\zeta(s)$ contain no nontrivial linear relations in which all coefficients are integers bounded, say by 10^{100}. Such an adaptation is possible, following the authors' work on weak independence (see, e.g., [19] and the references therein) but we have not pursued this here.

This paper is organized in the following way. Section 2 establishes the upper bound (1.7) of Theorem 1.1. Section 3 obtains lower bounds for positive values of $G(x)$, conditioned on the existence of solutions to particular simultaneous approximation problems involving a number of zeros of the Riemann zeta function, and establishes (1.8). It also establishes the first inequality in (1.6). Last, Section 4 describes the calculations required to establish the second bound in (1.6) on the positive oscillations in this function without assuming any linear independence conditions, in order to complete the proof of Theorem 1.1.

We remark that Hardy and Littlewood [12] conjectured that $R(n) \sim nS(n)$ for even integers n, where

$$S(n) = \prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right),$$

and that several authors encountered $G(x)$ when estimating the average value of $R(n)-nS(n)$. For example, Fujii [6] in fact established (1.3) in the form

$$\sum_{n\le x}(R(n)-nS(n)) = -4x^{3/2}G(x) + O\left((x\log x)^{4/3}\right).$$

It is readily seen that the two forms are equivalent, since from Montgomery and Vaughan [18, Lemma 1] we have that

$$\sum_{n\le x} nS(n) = \frac{1}{2}x^2 + O(x\log x).$$

Additional estimates involving $S(n)$ and related functions and their application in problems in additive number theory can be found in [18].

2. An upper bound for $|G(x)|$

Taking the real part of the sum in (1.4) produces

$$G(x) = -\sum_{\gamma>0}\frac{\cos(\gamma\log x)}{\gamma^2+\frac{1}{4}} + \sum_{\gamma>0}\frac{3\cos(\gamma\log x)+2\gamma\sin(\gamma\log x)}{(\gamma^2+\frac{1}{4})(\gamma^2+\frac{9}{4})} \tag{2.1}$$

$$= \sum_{\gamma>0}\frac{(\frac{3}{4}-\gamma^2)\cos(\gamma\log x)+2\gamma\sin(\gamma\log x)}{(\gamma^2+\frac{1}{4})(\gamma^2+\frac{9}{4})}. \tag{2.2}$$

With a little calculus one can show that the maximal value of the numerator in (2.2) is $\sqrt{(\gamma^2+1/4)(\gamma^2+9/4)}$, occurring when

$$\tan(\gamma\log x) = \frac{2\gamma}{\frac{3}{4}-\gamma^2},$$

and that the minimal value is $-\sqrt{(\gamma^2+1/4)(\gamma^2+9/4)}$, so

$$|G(x)| \le \sum_{\gamma>0} h(\gamma), \quad h(\gamma) := \frac{1}{\sqrt{(\gamma^2+1/4)(\gamma^2+9/4)}}. \tag{2.3}$$

A simple expansion shows that

$$h(\gamma) = \frac{1}{\gamma^2+\frac{1}{4}} - \frac{1}{\gamma^4} + \frac{2}{\gamma^6} - \frac{61}{16\gamma^8} + O\left(\gamma^{-10}\right),$$

and from Davenport [4, Chapter 12] we have that

$$\sum_{\gamma>0} \frac{1}{\gamma^2+\frac{1}{4}} = \sum_{\rho} \Re\left(\rho^{-1}\right) = 1 + \frac{\gamma_0}{2} - \frac{\log 4\pi}{2} = 0.02309\ldots, \tag{2.4}$$

where $\rho = 1/2 + i\gamma$ and $\gamma_0 = 0.577\ldots$ represents the Euler–Mascheroni constant. We can now show easily that

$$h(\gamma) < \frac{1}{\gamma^2+\frac{1}{4}} - \frac{1}{\gamma^4} + \frac{2}{\gamma^6}. \tag{2.5}$$

For, first rearrange the right side of (2.5) to observe that it is nonnegative, and then square both sides of (2.5). Therefore, to obtain an upper bound on $|G(x)|$, we require an upper bound on $\sum_{\gamma>0}\gamma^{-6}$. (We also need a lower bound on the sum over γ^{-4} from (2.5), but clearly any finite sum will work.) For this, we employ the result of Lehman [16, Lemma 3] stating that

$$\sum_{\gamma>T} \gamma^{-n} < \frac{\log T}{T^{n-1}} \tag{2.6}$$

provided $T \ge 2\pi e = 17.079\ldots$ and $n \ge 2$. Using (2.3), (2.4), (2.5), and (2.6), we therefore conclude that

$$|G(x)| < 1 + \frac{\gamma_0}{2} - \frac{\log 4\pi}{2} - \sum_{0<\gamma\le T_1} \frac{1}{\gamma^4} + 2\sum_{0<\gamma\le T_2} \frac{1}{\gamma^6} + \frac{\log T_2}{T_2^5},$$

where we may choose any values for $T_1 > 0$ and $T_2 \ge 2\pi e$. Choosing the first 1000 zeros for each sum, that is, taking $T_1 = T_2 = 1420.41$, we find that $|G(x)| < 0.023058681$, which establishes (1.7).

3. Lower bounds on oscillations

We may determine a lower bound on positive values attained by $G(x)$, conditioned on the existence of solutions to certain simultaneous approximation problems involving a number of nontrivial zeros of the Riemann zeta function. A similar procedure produces an unconditional bound on negative values achieved by $G(x)$.

Consider first the question of large oscillations in the positive direction. Given a positive integer N and a positive real number ϵ, suppose there exists a real number y and integers $m_1, \ldots, m_N$ so that

$$|\gamma_k y - (2m_k + 1)\pi| \le \epsilon \tag{3.1}$$

for $1 \le k \le N$. Then certainly

$$\cos(\gamma_k y) < -1 + \frac{\epsilon^2}{2} \tag{3.2}$$

for each k. Let $T > 2\pi e$ be a real number selected so that the number of nontrivial zeros of the Riemann zeta function with ordinate $\gamma < T$ is exactly N. From (2.1), we have

$$\begin{aligned} G(x) &= -\sum_{\gamma \le T} \frac{\cos(\gamma \log x)}{\gamma^2 + \frac{1}{4}} + \sum_{\gamma \le T} \frac{3\cos(\gamma \log x) + 2\gamma \sin(\gamma \log x)}{(\gamma^2 + \frac{1}{4})(\gamma^2 + \frac{9}{4})} \\ &\quad + \sum_{\gamma > T} \frac{(\frac{3}{4} - \gamma^2)\cos(\gamma \log x) + 2\gamma \sin(\gamma \log x)}{(\gamma^2 + \frac{1}{4})(\gamma^2 + \frac{9}{4})} \\ &=: G_1(x,T) + G_2(x,T) + G_3(x,T). \end{aligned} \tag{3.3}$$

From (2.3), we have

$$|G_3(x,T)| \le \sum_{\gamma > T} \frac{1}{\sqrt{(\gamma^2 + 1/4)(\gamma^2 + 9/4)}} < \sum_{\gamma > T} \frac{1}{\gamma^2},$$

and from [16, Lemma 1] we obtain

$$\sum_{\gamma > T} \frac{1}{\gamma^2} = \frac{1}{2\pi} \int_T^\infty \frac{\log(t/2\pi)}{t^2} \, \mathrm{d}t + \vartheta \left(\frac{4}{T^2} \log T + 2 \int_T^\infty \frac{\mathrm{d}t}{t^3} \right), \tag{3.4}$$

where ϑ is a complex number satisfying $|\vartheta| \le 1$. This provides a better estimate than that in (2.6), which we shall need in what follows. While the constants in the error in (3.4) could be improved by the results in [21, 23], the range of T that we are considering here makes any potential gain negligible. Consequently,

$$|G_3(x,T)| < B_3(T) := \frac{1}{2\pi T} \left(\log T + 1 - \log 2\pi + \frac{2\pi}{T}(1 + 4\log T) \right) \tag{3.5}$$

for all $x > 0$. For G_1, we use (3.2) to find

$$G_1(e^y, T) > \left(1 - \frac{\epsilon^2}{2}\right) \sum_{\gamma \le T} \frac{1}{\gamma^2 + \frac{1}{4}}.$$

For G_2, we observe that $3\cos t + 2\gamma \sin t$ is decreasing near $t = \pi$, so

$$G_2(e^y, T) \ge -\sum_{\gamma \le T} \frac{3\cos\epsilon + 2\gamma \sin\epsilon}{(\gamma^2 + \frac{1}{4})(\gamma^2 + \frac{9}{4})}.$$

Therefore,

$$G(e^y) > \left(1 - \frac{\epsilon^2}{2}\right) \sum_{\gamma \le T} \frac{1}{\gamma^2 + \frac{1}{4}} - \sum_{\gamma \le T} \frac{3\cos\epsilon + 2\gamma\sin\epsilon}{(\gamma^2 + \frac{1}{4})(\gamma^2 + \frac{9}{4})} - B_3(T). \tag{3.6}$$

In Table 3.1 we list a few values for the bound (3.6) for a number of choices of N. In each case we assume $\epsilon = 0.01$, and take $T = T^*(N)$, where

$$T^*(N) = \gamma_{N+1} - \frac{\gamma_{N+1} - \gamma_N}{100}. \tag{3.7}$$

Table 3.1. Conditional lower bounds for large positive values of $G(x)$ from (3.6), assuming the simultaneous approximation problem (3.1) has a solution with $\epsilon = 0.01$.

N	Bound	N	Bound
70	0.014756	500	0.020630
100	0.016352	600	0.020902
150	0.017837	700	0.021109
200	0.018692	800	0.021272
250	0.019269	900	0.021404
300	0.019684	1000	0.021515
350	0.020001	2000	0.022079
400	0.020254	10^4	0.022699
450	0.020459	10^5	0.022925

If the ordinates of the first N nontrivial zeros of the zeta function are linearly independent, then by Kronecker's theorem the corresponding bound in Table 3.1 would necessarily hold infinitely often, as would any value computed with an arbitrary choice of $\epsilon > 0$. Selecting $\epsilon = 10^{-6}$ with $N = 10^6$ produces the value $0.02297864\ldots$, which appears in (1.8) in Theorem 1.1.

For a bound on the negative values, given N and ϵ we require a real number z and integers $m_1, \ldots, m_N$ so that $|\gamma_k z - 2m_k\pi| \le \epsilon$ for $1 \le k \le N$. Certainly $z = m_1 = \cdots = m_k = 0$ suffices with $\epsilon = 0$, and we obtain an expression similar to (3.6) as a simple lower bound on a negative value achieved by $G(x)$, valid for any positive T:

$$G(1) < -\sum_{\gamma \le T} \frac{1}{\gamma^2 + \frac{1}{4}} + \sum_{\gamma \le T} \frac{3}{(\gamma^2 + \frac{1}{4})(\gamma^2 + \frac{9}{4})} + B_3(T).$$

Using $N = 10^6$ and taking $T = T^*(N)$ produces

$$G(1) < -0.02297865\ldots. \tag{3.8}$$

The statement that values this small occur infinitely often follows from Dirichlet's simultaneous approximation theorem, since for any $\delta > 0$ there exist arbitrarily large q and integers $m_1, \ldots, m_N$ so that

$$|q\gamma_i - 2\pi m_i| < \delta. \tag{3.9}$$

This establishes the first bound in (1.6).

To obtain a bound on the positive values of $G(x)$ without linear independence, in the next section we turn to the method of Odlyzko and te Riele for constructing solutions to simultaneous approximation problems.

4. Computations

We complete the proof of Theorem 1.1 by solving the simultaneous approximation problem (3.1) for particular N and ϵ. For this we employ the method of Odlyzko and te Riele [20], which we briefly describe here. Let $\lfloor x \rceil$ denote the integer nearest the real number x, and let $\mathbf{e}_k$ denote the kth elementary unit column vector in the appropriate real vector space. The construction requires values for four integer parameters: N, b, c, and d. Here, b represents the number of bits of precision used in the computation; c and d are small positive integers whose meanings will be described shortly.

Consider the inhomogeneous problem (3.1), where we require a real number y with the property that $\gamma_k y$ is near π, modulo integer multiples of 2π, for $1 \le k \le N$. We construct the $(N+2)\times(N+2)$ integer matrix M whose column vectors are

$$\left\lfloor 2^{b+1}\pi \right\rceil \mathbf{e}_k,\ 1 \le k \le N,$$

$$\mathbf{e}_{N+1} - \sum_{k=1}^{N} \left\lfloor 2^{b-c}\gamma_k \right\rceil \mathbf{e}_k,$$

$$2^b N^d \mathbf{e}_{N+2} + \left\lfloor 2^b\pi \right\rceil \sum_{k=1}^{N} \mathbf{e}_k.$$

That is, M consists of an $(N+2)\times N$ diagonal matrix with entries $\left\lfloor 2^{b+1}\pi \right\rceil$ on the diagonal, augmented with one column carrying rounded multiples of the γ_k, and another largely filled with a rounded multiple of the inhomogeneous part, π. The penultimate vector carries the lone nonzero value in row position $N+1$, set to 1 so that we can recover a coefficient later in the computation. The last vector has the only nonzero value in the last position, chosen to be much larger than the other entries of the matrix.

We apply the LLL algorithm [17] to M to compute a reduced basis for the lattice spanned by its column vectors. This reduced basis consists of vectors that are relatively short, in fact within a factor (whose value is bounded by an expression that is exponential in the dimension) of the shortest independent vectors in the lattice. Since the last coordinate of every vector in the lattice is an integer multiple of the large integer $2^b N^d$, it is likely that there is only one vector in the reduced basis with a nonzero value in this position, which is very likely to be $\pm 2^b N^d$. If this value is negative we can negate the vector, so suppose it is $(r_1, \ldots, r_N, s, 2^b N^d)^T$. We then have that there exist integers $m_1, \ldots, m_N$ such that

$$r_k = m_k \left\lfloor 2^{b+1}\pi \right\rceil + \left\lfloor 2^b \pi \right\rceil - s \left\lfloor 2^{b-c}\gamma_k \right\rceil$$

for $1 \le k \le N$, and that the r_k are relatively small. If $s < 0$ then we can negate this vector so that our inhomogeneous part is $-\pi$, which serves us just as well, so we assume $s \ge 0$ here. We might then expect

$$\gamma_k s 2^{-c} \approx 2\pi m_k + \pi$$

so we take $y = s/2^c$, and use this in (3.3) and (3.5) to compute the resulting lower bound on positive values reached by $G(x)$:

$$G_1(e^y, T^*(N)) + G_2(e^y, T^*(N)) - B_3(T^*(N)), \tag{4.1}$$

with $T^*(N)$ as in (3.7). For each k we also compute $m_k = \lfloor (\gamma_k y - \pi)/2\pi \rceil$, and then

$$\epsilon = \max_{1 \le k \le N} \{ |\gamma_k y - (2m_k + 1)\pi| \}. \tag{4.2}$$

A large value of ϵ (and consequently a small value in (4.1)) likely indicates that insufficient precision was employed. In that case we repeat this process with a larger value of b.

We remark that the other vectors in the reduced basis (those with zero in the last component) in a similar way produce candidate values z in the corresponding homogeneous problem, so $\gamma_k z \approx 2\pi m_k$ for $1 \le k \le n$, with integers $m_1, \ldots, m_n$. These numbers produce bounds on negative values achieved by $G(x)$, though not as good as the bound established by (3.8).

Odlyzko and te Riele used $c = 10$ and $d = 4$ in their computations. Both values worked sufficiently well in our application, too, so we did not alter these in our principal runs. Those authors also reported selecting b between $6.6N$ and $13.3N$ (that is, using between $2N$ and $4N$ decimal digits of precision). The larger end of this range sufficed in our application only for N up to about 200, where we produced $\epsilon = 0.0116$. For larger dimensions we needed to select b as large as $26.9N$.

All computations were performed in SageMath [22], using resources at NCI Australia and at the Center for Communications Research. High-precision values for zeros of the Riemann zeta function were computed using the `mpmath` Python library [14], available within SageMath.

Table 4.1 records the bounds we obtained on $G(x)$ in this way, using different values for N and b. The last line in this table records the parameters and results of the computation that establishes the second part of (1.6) in Theorem 1.1. This calculation required 24.4 days of core time on an Intel Xeon Platinum 8175M processor running at 2.5 GHz. Figure 4.1

exhibits the value of $2^{10}y$ obtained for this case, using base 36 for economy of space. The statement that $G(x)$ achieves this bound infinitely often follows by applying Dirichlet's theorem as in (3.9), using $y+q$ for a sequence of qualifying values q.

Finally, the error incurred by the estimate (3.6) for $G(x)$ decays like $1/N$, and the LLL algorithm requires $O(N^6b^3)$ bit operations ($O(N^{5+\epsilon}b^{2+\epsilon})$ with fast multiplication techniques). If we assume that b is linear in N, then we conclude that a new computation designed to reduce the error in our lower bound for $G(x)$ in Theorem 1.1 by a factor of β would require $O(\beta^9)$ (respectively $O(\beta^{7+\epsilon})$) time. However, the LLL algorithm often performs better in practice than its worst-case bound, and indeed our running times in this work grow approximately as N^6 using our choices for b. Thus, we expect that a computation to halve the error in our lower bound for $G(x)$ by doubling N and adjusting b appropriately would require around four years of core time on similar computers.

TABLE 4.1. Guaranteed oscillations in $G(x)$, along with the error ϵ from (4.2), obtained by solving the simultaneous approximation problem (3.1) using the first N zeros of the Riemann zeta function, and using b bits of precision. The displayed values for the bound are truncated at the last displayed digit; those for ϵ are rounded up at the last displayed digit.

N	b	Bound	ϵ
70	930	0.0147720	0.00089
100	1330	0.0163683	0.00125
150	2000	0.0178520	0.00394
200	2660	0.0187115	0.01160
250	3750	0.0192853	0.00932
300	4800	0.0197039	0.01097
350	6300	0.0200147	0.00792
400	7600	0.0202690	0.00990
450	9200	0.0204694	0.01577
500	11250	0.0206430	0.01239
600	15000	0.0209272	0.02106
650	17500	0.0210305	0.01396

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Figure 4.1. $2^{10}y$ (in base 36) for the last line in Table 4.1.

5okbyhk5v96hatbw4kv6843kloupw8dmjhvsau9ncv1ljmhwpqzfdrq3a4040rp5fzp7jpfshge8lyv3hi3x1tedopk
w6lnr24dooyqnbuaeu2f0woo8tvc0p7r7ns6k0mxjvavrwwcp4o5snebtu478rno0twp89oz6pp38cs3o35hei7taom
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e4hicrxy6xrek5qzadlg6djfewxg591a14betd1qnzl57z9

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