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Résumé. Des bornes explicites pour la fonction zêta ζ loin de la droite réelle sont nécessaires pour des applications, notamment aux intégrales de ζ sur des lignes verticales ou bien sur d'autres chemins. Ici, nous bornons des normes L^2 ponderées de la fonction zêta loin de la droite réelle.

Nous suivons deux approches, chacune donnant le meilleur ré-sultat dans un certain rang. La première est inspirée par le théo-rème de la valeur moyenne pour les polynômes de Dirichlet. La deuxième, supérieure pour T grand, est basée sur des résultats classiques, en commençant par une approximation de ζ via la formule d'Euler–Maclaurin.

Ces bornes donnent toutes les deux des termes principaux d'or-dre correct pour $0 < \sigma \leq 1$. Elles sont assez fortes pour être d'utilité pratique dans le calcul numérique rigoureux d'intégrales impropres.

Nous présentons aussi des bornes pour la norme L^2 de ζ dans $[1, T]$ pour $0 \leq \sigma \leq 1$.

Abstract. Explicit bounds on the tails of the zeta function ζ are needed for applications, notably for integrals involving ζ on vertical lines or other paths going to infinity. Here we bound weighted L^2 norms of tails of ζ.

Two approaches are followed, each giving the better result on a different range. The first one is inspired by the proof of the standard mean value theorem for Dirichlet polynomials. The second approach, superior for large T, is based on classical lines, starting with an approximation to ζ via Euler–Maclaurin.

Both bounds give main terms of the correct order for $0 < \sigma \leq 1$ and are strong enough to be of practical use for the rigorous computation of improper integrals.

We also present bounds for the L^2 norm of ζ in $[1, T]$ for $0 \leq \sigma \leq 1$.

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Mots-clefs. Riemann zeta function, L^2 norm, mean square bounds, explicit bounds, mean value theorem.

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1. Introduction

1.1. Motivation. Say we want to compute a line integral from $\sigma - i\infty$ to $\sigma + i\infty$ involving the zeta function. Such integrals arise often in work in number theory as inverse Mellin transforms. For example, during his work on [14], the second author had to estimate the double sum

$$D_{\alpha_1,\alpha_2}(y) = \sum_{d\leq y}\sum_{l\leq y/d} \frac{\log\left(\frac{y}{dl}\right)}{d^{\alpha_1}l^{\alpha_2}},$$

and others of the same kind. Now, it is not hard to show that

$$D_{\alpha_1,\alpha_2}(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s+\alpha_1)\zeta(s+\alpha_2)}{s^2} y^s \mathrm{d}s$$

for $\sigma > 1$. Let $0 < \alpha_1, \alpha_2 < 1$. Shifting the line of integration to the left, we obtain main terms coming from the poles at $s = 1 - \alpha_2$ and $s = 1 - \alpha_2$, and, as a remainder term, the integral

$$\frac{1}{2\pi i}\int_{R_\beta} \frac{\zeta(s+\alpha_1)\zeta(s+\alpha_2)}{s^2} y^s \mathrm{d}s,$$

where R_β is some contour to the left of the poles going from $\beta - i\infty$ to $\beta + i\infty$.

It is possible to do rigorous numerical integration on bounded contours in the complex plane, using, for instance, the ARB package [18]. It then remains to bound the integral

$$\int_{\beta+iT}^{\beta+i\infty} \frac{|\zeta(s+\alpha_1)||\zeta(s+\alpha_2)|}{|s|^2} \mathrm{d}s,$$

the integral from $\beta - i\infty$ to $\beta - iT$ having the same absolute value. By the Cauchy–Schwarz inequality, the problem reduces to that of giving explicit bounds for the integral

$$\int_{\beta+iT}^{\beta+i\infty} \frac{|\zeta(s+\alpha_1)|^2}{|s|^2} \mathrm{d}s. \tag{1.1}$$

Finding such bounds is the main subject of this paper.

1.2. Methods and results. Convexity bounds on ζ have been known explicitly for more than 100 years [4]. Since they are of the form $\zeta(\sigma+it) = O(t^{\frac{1-\sigma}{2}}\log t)$ for $0 \leq \sigma \leq 1$, they imply that (1.1) converges for $0 < \sigma \leq 1$. There are also explicit subconvexity bounds (that is, bounds stronger than convexity) for $\sigma = \frac{1}{2}$ ([22, 7, 30, 15]) and for $\frac{1}{2} \leq \sigma \leq 1$ [8].

Here, we produce better results in the L^2 norm than can be obtained from such L^∞ bounds. Non-explicit bounds on the L^2-norm of $\zeta(\sigma+it)$ are well known ([10], [12], [20, Vol. 2, 806–819, 905–906], [23]; see the introduction to [16] for an exposition).

Our main result collects in a simplified form the results in Theorems 3.1 and 4.6.

Theorem 1.1. *Let $0 < \sigma \leq 1$. Then, the integral $\int_T^\infty \left|\frac{\zeta(\sigma+it)}{\sigma+it}\right|^2 \mathrm{d}t$ is bounded as follows*

(1) *if $\sigma = 1$, by*

$$\frac{\pi^2}{6} \cdot \frac{1}{T} + 28.3 \cdot \frac{\log T}{T^2} \qquad \textit{for} \quad T \geq 200;$$

(2) *if $\frac{1}{2} < \sigma < 1$, by*

$$\begin{aligned} &\frac{3\pi\zeta(2\sigma)}{5} \cdot \frac{1}{T} + \left(18.98 - \frac{0.61}{\sigma - \frac{1}{2}}\right) \cdot \frac{1}{T^{2\sigma}} &&\textit{for} \quad T \geq 200, \\ &\zeta(2\sigma) \cdot \frac{1}{T} + \frac{12.95}{(\sigma - \frac{1}{2})(1-\sigma)} \cdot \frac{1}{T^{2\sigma}} &&\textit{for} \quad T \geq 4; \end{aligned}$$

(3) *if $\sigma = \frac{1}{2}$, by*

$$\begin{aligned} &\frac{3\pi}{5} \cdot \frac{\log T}{T} + 7.72 \cdot \frac{1}{T} &&\textit{for} \quad T \geq 200, \\ &\frac{\log T}{T} + 9.2 \cdot \frac{\sqrt{\log T}}{T} &&\textit{for} \quad T \geq 10^{40}; \end{aligned}$$

(4) *if $0 < \sigma < \frac{1}{2}$, by*

$$\begin{aligned} &\left(\frac{0.5}{\sigma} + \frac{0.95}{\frac{1}{2} - \sigma} + 5.62\right) \cdot \frac{1}{T^{2\sigma}} - 2.55\zeta(2\sigma) \cdot \frac{1}{T} &&\textit{for} \quad T \geq 200, \\ &\frac{\zeta(2-2\sigma)}{2\sigma(2\pi)^{1-2\sigma}} \cdot \frac{1}{T^{2\sigma}} + \frac{20.7}{\sigma^2(\frac{1}{2} - \sigma)} \cdot \frac{1}{T} &&\textit{for} \quad T \geq 4. \end{aligned}$$

In each pair of bounds above, the second one is stronger for large T and fixed σ. The first bounds in cases (2), (3), (4) are obtained by a method explained in Section 3, based on the fact that the Mellin transform is an isometry. The second set of bounds and the single bound in case (1) use a different approach, explained in Section 4; it is based on the following explicit bounds on the L^2 norm of the restriction of $\zeta(\sigma + it)$ to a segment.

Theorem 1.2. *Let* $0 \leq \sigma \leq 1$ *and* $T \geq 4$. *Then, the integral* $\int_1^T |\zeta(\sigma+it)|^2 \mathrm{d}t$ *is bounded from above by*

$$\begin{aligned}
&\frac{\pi^2}{6} \cdot T + 18.49 \cdot \sqrt{T} && \text{if } \sigma = 1,\\
&\zeta(2\sigma) \cdot T + \frac{5.22}{(\sigma - \frac{1}{2})(1-\sigma)^2} \cdot \max\{T^{2-2\sigma} \log T, \sqrt{T}\} && \text{if } \frac{1}{2} < \sigma < 1,\\
&T \log T + 2.0 \cdot T\sqrt{\log T} + 23.06 \cdot T && \text{if } \sigma = \frac{1}{2},\\
&\frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)} \cdot T^{2-2\sigma} + \frac{10.35}{\sigma^2(\frac{1}{2}-\sigma)} \cdot T && \text{if } 0 < \sigma < \frac{1}{2},\\
&\frac{\pi}{24} \cdot T^2 + 9.37 \cdot T \log T && \text{if } \sigma = 0.
\end{aligned}$$

The error terms above are not optimal: bounds with the correct coefficient for the second-order term (and a non-explicit lower-order term) are known; for $\sigma = \frac{1}{2}$, see Ingham [16], Titchmarsh [36], Atkinson [3], and Balasubramanian [5] (vd. Heath-Brown [13] for an L^2 estimate of the lower-order term, and Good [9] for a lower bound on its order). For $\frac{1}{2} < \sigma < \frac{3}{4}$, an estimate was given by Matsumoto [24], later extended by Matsumoto and Meurman [26] to $\frac{1}{2} < \sigma < 1$. We will be more precise in Theorems 4.3 and 4.5.

It would seem feasible to improve on Theorem 1.2 by starting from Atkinson's formula for $\sigma = \frac{1}{2}$, or Matsumoto–Meurman's for $\frac{1}{2} < \sigma < 1$, estimating all terms while foregoing cancellation. One could then deduce a bound for $0 < \sigma < \frac{1}{2}$ by the functional equation, as in Theorem 4.5 here. For $\sigma = \frac{1}{2}$, yet another possibility would be to attempt to make the work of Titchmarsh or Balasubramanian explicit.

An exposition of these alternative procedures (in their current non-explicit versions) can be found in [25, §1]. They are based on the approximate functional equation, or the Riemann–Siegel formula, which is closely related. Shortly after the appearance of the original version of the present paper, Simonič provided an explicit bound in [35, Cor. 5] for the case $\sigma = \frac{1}{2}$, improving on Theorems 1.2 and 4.3 for T large enough.

For the sake of rigor, we have used interval arithmetic throughout, implemented by ARB [18], which we used via Sage.

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2. Classical foundations revisited

2.1. O and O^* notation. When we write $f(x) = O(g(x))$ as $x \to a$ ($a = \pm\infty$ is allowed) for a real or complex valued function f and a real valued function g, we mean that there is a constant C such that $|f(x)| \leq Cg(x)$ in a neighborhood of a. We write $f(x) = O^*(h(x))$ to mean that $|f(x)| \leq h(x)$ (either for all x or in an explicitly stated neighborhood of a).

2.2. Bernoulli polynomials. We define the Bernoulli polynomials $B_k : \mathbb{R} \to \mathbb{R}$ inductively: $B_0(x) = 1$ and for $k \geq 1$, $B_k(x)$ is determined by $B_k'(x) = kB_{k-1}(x)$ and $\int_0^1 B_k(x) = 0$. The k-th Bernoulli number b_k is the constant term of $B_k(x)$. In particular, $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$.

Lemma 2.1 ([29, Cor. B.4, Exer. B.5(e)])**.** *For $k \geq 1$, $\max_{x\in[0,1]} |B_{2k}(x)| = |b_{2k}|$ and $\max_{x\in[0,1]} |B_{2k+1}(x)| < \frac{2(2k+1)!}{(2\pi)^{2k+1}}$. In general, for every $k \geq 2$,*

$$\max_{x\in[0,1]} |B_k(x)| \leq \frac{2\zeta(k)k!}{(2\pi)^k}. \tag{2.1}$$

2.3. Euler–Maclaurin summation formula. Bernoulli polynomials appear naturally in the Euler–Maclaurin summation formula.

Theorem 2.2 (Euler–Maclaurin)**.** *Let K be a positive integer. Let $X < Y$ be two real numbers such that the function $f : [X, Y] \to \mathbb{C}$ has continuous derivatives up to the K-th order on the interval $[X, Y]$. Then*

$$\sum_{X<n\leq Y} f(n) = \int_X^Y f(x)\mathrm{d}x + S(K) - \frac{(-1)^K}{K!} \int_X^Y B_K(\{x\})f^{(K)}(x)\mathrm{d}x, \tag{2.2}$$

where

$$S(K) = \sum_{k=1}^{K} \frac{(-1)^k}{k!} \left(B_k(\{Y\})f^{(k-1)}(Y) - B_k(\{X\})f^{(k-1)}(X)\right), \tag{2.3}$$

and $B_k : [0, 1] \to \mathbb{R}$ is the k-th Bernoulli polynomial.

The reader may refer to [29, App. B] for a proof of Theorem 2.2.

Corollary 2.3. *Let $X \geq 1$ be an arbitrary real number. Let K be a positive integer. For every $s = \sigma + it \in \mathbb{C}$ such that $\sigma > 1 - K$ and $s \neq 1$, we have*

$$\zeta(s) = \sum_{n\leq X} \frac{1}{n^s} + \frac{X^{1-s}}{s-1} + \left(\{X\} - \frac{1}{2}\right)\frac{1}{X^s} + \sum_{k=2}^{K} \frac{a_k(s)B_k(\{X\})}{k!X^{s+k-1}} - \frac{a_{K+1}(s)}{K!} \int_X^\infty \frac{B_K(\{X\})}{x^{s+K}}\mathrm{d}x,$$

where $a_k(s) = s(s + 1) \dots (s + k - 2)$ for $k \geq 2$.

For $\sigma > 1$, Corollary 2.3 is a direct application of Theorem 2.2 upon defining $f : [X, Y] \to \mathbb{C}$, as $x \mapsto x^{-s}$, $\Re(s) > 1$, and letting $Y \to \infty$. We extend the statement to $\sigma > 1 - K$ by analytic continuation.

We consider Theorem 2.2 into a broader class of functions than C^K. The following formulation (from [14, §3.1]) improves slightly on a constant value: it replaces the factor $\frac{1}{12}$, coming from a direct application of Theorem 2.2 with $K=2$, by a factor of $\frac{1}{16}$.

Lemma 2.4 (Improved Euler–Maclaurin summation formula of second order)**.** *Let $f:[0,\infty)\to\mathbb{C}$ be a continuous, piecewise C^1 function such that f, f', f'' are in $L^1([0,\infty))$. Then*

$$\sum_{n=1}^{\infty} f(n)=\int_0^\infty f(x)\mathrm{d}x-\frac{f(0)}{2}-\lim_{t\to 0^+}\frac{f'(t)}{16}+O^*\left(\frac{1}{16}\|f''\|_1\right). \tag{2.4}$$

Here and elsewhere (for instance in Proposition 3.2), we mean f'' and $\|f''\|_1$ in the sense of distributions or measures, so that $\|f''\|_1$ stands for the total variation of the function f' on the interval $[0,\infty)$. If f is in C^2, this is the same as the usual meaning.

Proof. As f has bounded total variation, $f(x)$ converges to a real number R as $x\to\infty$. If R were non-zero, then f could not be in L^1; thus $\lim_{x\to\infty} f(x)=0$. By the same reasoning, since f' is differentiable and f', f'' are in L^1, we have $\lim_{x\to\infty} f'(x)=0$.

Suppose first that f' is continuous at the positive integers. Let $F(x)$ be a differentiable function with $F'(x)=x-\frac{1}{2}$. Then $\int_0^1 F'(x)\mathrm{d}x=0$, $F(0)=F(1)$, and so, by integration by parts,

$$\begin{aligned}\int_{n-1}^{n} f(x)\mathrm{d}x&=\frac{f(n)}{2}-\frac{f(n-1)}{2}-\int_{n-1}^{n} f'(x)F'(\{x\})\mathrm{d}x\\&=\frac{f(n)}{2}-\frac{f(n-1)}{2}-(f'(n)-f'(n-1))F(0)+\int_{n-1}^{n} f''(x)F(\{x\})\mathrm{d}x,\end{aligned}$$

where we write $f'(0)$ for $\lim_{t\to 0^+} f'(t)$. Therefore, $\int_0^n f(x)dx$ equals

$$\sum_{k=1}^{n} f(k)-\frac{f(n)}{2}+\frac{f(0)}{2}-f'(n)F(0)+f'(0)F(0)+\int_0^n f''(x)F(\{x\})\mathrm{d}x.$$

By using the fact that $\lim_{n\to\infty} f(n)=\lim_{n\to\infty} f'(n)=0$, we obtain finally that

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)=\int_0^\infty f(x)\mathrm{d}x-\frac{f(0)}{2}-f'(0)F(0)&\\+O^*\left(\int_0^\infty |f''(x)||F(\{x\})|\mathrm{d}x\right).&\end{aligned} \tag{2.5}$$

It remains to choose F with $F'(x)=x-\frac{1}{2}$ such that $\max_{x\in[0,1]}|F(x)|$ is minimal. We take $F(x)=\frac{1}{2}\left(x^2-x+\frac{1}{8}\right)$, in which case $\max_{x\in[0,1]}|F(x)|=\frac{1}{16}$. We obtain (2.4).

Finally, suppose that f' not continuous at all the positive integers. Since there are countably many points in which f is not differentiable, there are only countably many $x \in \mathbb{R}$ such that f is not differentiable at $n+x$ for at least one $n \in \mathbb{Z}_{>0}$. Thus, there is a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \varepsilon_k = 0$ such that the functions $f_k : x \mapsto f(x+\varepsilon_k)$ are differentiable at all positive integers. Then, by the above, (2.4) holds for all of these functions. Since $f' \in L^1([0,\infty))$, dominated convergence gives us that $\sum_{n=1}^{\infty} f_k(n) \to \sum_{n=1}^{\infty} f(n)$ as $k \to \infty$. It is clear that $\lim_{k\to\infty} \lim_{t\to 0^+} f_k'(t) = \lim_{k\to\infty} \lim_{t\to 0^+} f'(t+\varepsilon_k) = \lim_{t\to 0^+} f'(t)$, because the last limit exists. Obviously, $\int_0^{\infty} f_k(x)\mathrm{d}x \to \int_0^{\infty} f(x)\mathrm{d}x$ as $k \to \infty$ and $\|f_k''\|_1 \le \|f''\|_1$ for all k. We let $k \to \infty$ and obtain that f satisfies (2.4). $\square$

2.4. The Mellin transform. Let $f : [0,\infty) \to \mathbb{C}$. Its Mellin transform is defined as $\mathcal{M}f(s) = \int_0^{\infty} f(x)x^{s-1}\mathrm{d}x$ for all s such that the integral converges absolutely. It is a Fourier transform up to changing variables, so a version of Plancherel's identity holds:

$$\int_0^{\infty} |f(x)|^2 x^{2\sigma-1}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}f(\sigma+it)|^2\mathrm{d}t, \tag{2.6}$$

provided that $f(x)x^{\sigma-\frac{1}{2}}$ is in $L^2([0,\infty))$ and $f(x)x^{\sigma-1}$ is in $L^1([0,\infty))$.

For f continuous and piecewise C^1, by integration by parts,

$$\mathcal{M}f'(s) = -(s-1)\mathcal{M}f(s-1). \tag{2.7}$$

In particular, we have that $\mathcal{M}\mathbb{1}_{(0,a]}(s) = \frac{a^s}{s}$, where $\mathbb{1}_S$ denotes the indicator function of a set S. Considering now $f(x) = \sum_{n=1}^{\infty} a_n \mathbb{1}_{(0,1/n]}(x)$, where $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is a Dirichlet series converging in the half-plane $\{s \in \mathbb{C} \,|\, \Re(s) > \sigma_c\}$, we observe that

$$\mathcal{M}f(s) = \sum_{n=1}^{\infty} \int_0^{\infty} a_n \mathbb{1}_{(0,1/n]}(x) x^{s-1}\mathrm{d}x = \sum_{n=1}^{\infty} a_n \int_0^{1/n} x^{s-1}\mathrm{d}x = \frac{A(s)}{s},$$

in the set $\{s \in \mathbb{C} \,|\, \Re(s) > \max\{0, \sigma_c\}\}$. As the above holds for every Dirichlet series, we have, for the function $J(x) = \sum_{n=1}^{\infty} \mathbb{1}_{(0,1/n]}(x) = \lfloor 1/x \rfloor$, the equality

$$\mathcal{M}J(s) = \frac{\zeta(s)}{s}, \tag{2.8}$$

which is valid for the set $\{s \in \mathbb{C} \,|\, \Re(s) > 1\}$. Moreover, for a general function f, the function $\widetilde{f} : x \mapsto f(nx)$ has Mellin transform $\mathcal{M}\widetilde{f}(s) = \frac{\mathcal{M}f(s)}{n^s}$ for all s in the domain of definition of $\mathcal{M}f$. Thus, for every well-defined function $F(x) = \sum_{n=1}^{\infty} f(nx)$, by considering

$$h(x) = \left\lfloor \frac{1}{x} \right\rfloor - F(x) = \sum_{n=1}^{\infty} \mathbb{1}_{(0,1/n]}(x) - \sum_{n=1}^{\infty} f(nx),$$

we obtain

$$\mathcal{M}F(s) = \mathcal{M}f(s)\zeta(s), \tag{2.9}$$

$$\mathcal{M}h(s) = \left(\frac{1}{s} - \mathcal{M}f(s)\right)\zeta(s), \tag{2.10}$$

for all s in the domain of definition of $\mathcal{M}f$ such that $\Re(s) > 1$.

The following can be readily proved by induction.

Lemma 2.5. *For every $a \in \mathbb{R}$, $j \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{C}$ such that $\Re(s) > 0$, we have*

$$\mathcal{M}\left((a-x)^j \mathbb{1}_{(0,a]}(x)\right)(s) = \frac{j!a^{s+j}}{s(s+1)\dots(s+j)}. \tag{2.11}$$

2.5. The Gamma function. The Gamma function Γ is defined for all $s \in \mathbb{C}$ such that $\Re(s) > 0$ as $\Gamma : s \mapsto \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t$. This function can be extended meromorphically to $\mathbb{C}$, with poles on the set $\{0, -1, -2, -3, \dots\}$ and vanishing nowhere. Where well-defined, it satisfies the relationship $\Gamma(s+1) = s\Gamma(s)$. This function is closely related to the ζ function, by means of the functional equation, valid for all $s \in \mathbb{C}\backslash\{0, 1\}$,

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{2.12}$$

Theorem 2.6 (Stirling's formula, explicit form)**.** *Let $0 < \theta < \pi$. Let $s \in \mathbb{C}\backslash(-\infty, 0]$ such that $|\arg(s)| \leq \pi - \theta$, where $\arg(s)$ is the principal argument of s. Then*

$$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}\exp\left(O^*\left(\frac{F}{|s|}\right)\right),$$

where $F = F_\theta = \frac{1}{12\sin^2\left(\frac{\theta}{2}\right)}$.

Proof. Since $\Gamma(s)$ has neither zeroes nor poles in the simply connected domain $\mathbb{C}\backslash(-\infty, 0]$, $\log\Gamma(s)$ is a well-defined analytic function on $\mathbb{C}\backslash(-\infty, 0]$. By [2, Thm. 1.4.2] with $m = 1$,

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s - \frac{1}{2}\right)\log s - s + \mu(s), \tag{2.13}$$

where log is the principal branch of the logarithm defined on $\mathbb{C}\backslash(-\infty, 0]$ and $\mu(s) = \frac{1}{12s} - \frac{1}{2}\int_0^\infty \frac{B_2(\{x\})}{(s+x)^2}\mathrm{d}x$. Moreover, as explained in [34, §2.4.4], μ can be expressed as a Gudermann series so that, for all $s \in \mathbb{C}\backslash(-\infty, 0]$,

$$|\mu(s)| \leq \frac{1}{12\cos^2\left(\frac{1}{2}\arg(s)\right)|s|}. \tag{2.14}$$

Now, if $|\arg(s)| \leq \pi - \theta$ then $\cos\left(\frac{1}{2}\arg(s)\right) = \cos\left(\frac{1}{2}|\arg(s)|\right) \geq \cos\left(\frac{\pi-\theta}{2}\right) = \sin\left(\frac{\theta}{2}\right)$. Thus, upon exponentiating both sides of (2.13) and implementing the final bound for (2.14), we derive the result. $\square$

Corollary 2.7 (Rapid decay of Γ in non-negative vertical strips)**.** *Let $T \geq 1$ and $\sigma \geq 0$. Then, for every complex number $s = \sigma + it$ such that $|t| \geq T$,*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\exp\left(O^*\left(\frac{G_\sigma}{T}\right)\right),$$

where $G_\sigma = \frac{\sigma^3}{3} + \frac{\sigma^2}{2}\left|\sigma - \frac{1}{2}\right| + \frac{1}{6}$.

Proof. As s is such that $|\arg(s)| \leq \frac{\pi}{2}$, we use Theorem 2.6 with $\theta = \frac{\pi}{2}$, and obtain

$$\Re(\log\Gamma(s)) = \frac{\log(2\pi)}{2} + \left(\sigma - \frac{1}{2}\right)\frac{\log(\sigma^2+t^2)}{2} - t\arg(\sigma+it) - \sigma + O^*\left(\frac{1}{6T}\right).$$

As $\log(1+x) \leq x$ for all $x \geq 0$, we have that $\log(\sigma^2+t^2) = 2\log|t| + O^*(\frac{\sigma^2}{T^2})$. Furthermore, observe that $\arg(\sigma+it) = \arctan\left(\frac{t}{\sigma}\right)$, and that

$$\begin{aligned}\arctan\left(\frac{t}{\sigma}\right) &= \int_0^{\frac{t}{\sigma}}\frac{\mathrm{d}x}{1+x^2} = \pm\frac{\pi}{2} - \int_0^{\frac{\sigma}{t}}(1+O^*(x^2))\mathrm{d}x\\ &= \pm\frac{\pi}{2} - \frac{\sigma}{t} + O^*\left(\frac{\sigma^3}{3t^3}\right),\end{aligned}$$

where the sign $\pm$ corresponds to the sign of t. Putting everything together, we obtain that $\Re(\log\Gamma(s))$ equals

$$\frac{1}{2}\log(2\pi) + \left(\sigma - \frac{1}{2}\right)\log|t| - \frac{\pi|t|}{2} + O^*\left(\left(\frac{\sigma^3}{3} + \frac{\sigma^2}{2}\left|\sigma - \frac{1}{2}\right|\right)\frac{1}{T^2} + \frac{1}{6T}\right).$$

As $\frac{1}{T^2} \leq \frac{1}{T}$, the above error term can thus be compressed to $O^*(\frac{G_\sigma}{T})$. By exponentiating the above equation, we obtain the result. □

2.6. Bounds on some sums.

Lemma 2.8. *For any $X \geq 1$ we have*

$$\log X + \gamma - \frac{c}{X} \leq \sum_{n\leq X}\frac{1}{n} \leq \log X + \gamma + \frac{1}{2X}, \tag{2.15}$$

where $c = 2(\log 2 + \gamma - 1)$ and $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant.

The constant c in the lower bound was pointed out in [33, Lem. 2.1].

Proof. By applying Theorem 2.2 with $K = 2$ to the function $x \mapsto x^{-1}$, we obtain

$$\sum_{n\leq X}\frac{1}{n} = \log X + \frac{7}{12} - \int_1^\infty\frac{B_2(\{x\})}{x^3}\mathrm{d}x + R(X), \tag{2.16}$$

where $R(X) = -\frac{B_1(\{X\})}{X} - \frac{B_2(\{X\})}{2X^2} + \int_X^\infty \frac{B_2(\{x\})}{x^3}\mathrm{d}x$. By (2.1), B_1 and B_2 are bounded functions on $[0,1]$; hence, $R(X) = O(\frac{1}{X})$ and the integral in (2.16) is convergent. We conclude that γ, defined as $\lim_{X\to\infty}\sum_{n\leq X}\frac{1}{n} - \log X$, equals $\frac{7}{12} - \int_1^\infty \frac{B_2(\{x\})}{x^3}\mathrm{d}x$. Therefore, $\sum_{n\leq X}\frac{1}{n} = \log X + \gamma + R(X)$.

Since $B_1(t) = t - \frac{1}{2}$, $B_2(t) = t^2 - t + \frac{1}{6}$ and $\max|B_2(\{x\})| = \frac{1}{6}$,

$$R(X) = \frac{1}{2X} - \frac{\{X\}}{X}\left(1 - \frac{1-\{X\}}{2X}\right) - \frac{1}{12X^2} + O^*\left(\frac{1}{12X^2}\right).$$

Since $1 - (1-\{X\})/(2X) \geq 0$, the upper bound in (2.15) follows immediately. We also obtain that $R(X) \geq -1/(2X) - 1/(6X^2)$, and so the lower bound in (2.15) holds for $X \geq 5$; we check it for $1 \leq X \leq 5$ by hand. □

Lemma 2.9. *Let $\alpha \in \mathbb{R}^+ \setminus \{1\}$ and $X > 0$. Then*

$$\zeta(\alpha) - \frac{1}{(\alpha-1)X^{\alpha-1}} - \frac{1}{2X^\alpha} \leq \sum_{n\leq X}\frac{1}{n^\alpha} \leq \zeta(\alpha) - \frac{1}{(\alpha-1)X^{\alpha-1}} + \frac{1}{X^\alpha}.$$

Proof. By definition of $\zeta(s)$ for $\Re(s) > 1$, and by analytic continuation for $\Re(s) > 0$,

$$\zeta(s) - \frac{1}{(s-1)X^{s-1}} - \sum_{n\leq X}\frac{1}{n^s} = \sum_{n=1}^\infty \left(\int_{n-1}^n \frac{\mathrm{d}x}{(X+x)^s} - \frac{1}{(\lfloor X\rfloor + n)^s}\right) \tag{2.17}$$

for $s \neq 1$. Set $s = \alpha$. Since $t \mapsto t^{-\alpha}$ is decreasing, the right side of (2.17) is at least

$$\sum_{n=1}^\infty \left(\frac{1}{(X+n)^\alpha} - \frac{1}{(X+n-1)^\alpha}\right) = -\frac{1}{X^\alpha}.$$

By the convexity of $t \mapsto t^{-\alpha}$ and $(\lfloor X\rfloor + n)^{-\alpha} \geq (X+n)^{-\alpha}$,

$$\int_{n-1}^n \frac{\mathrm{d}x}{(X+x)^\alpha} - \frac{1}{(\lfloor X\rfloor + n)^s} \leq \frac{1}{2}\left(\frac{1}{(X+n-1)^\alpha} + \frac{1}{(X+n)^\alpha}\right) - \frac{1}{(X+n)^\alpha}.$$

Telescoping again, we see that the right side of (2.17) is at most $1/(2X^\alpha)$. □

The non-explicit form of the lemma below is classical: see for instance [37, Thm. 4.11] (from [11, Lem. 2]), though the implied constant there depends on σ as $\sigma \to 0^+$.

Lemma 2.10. *Let $s = \sigma + it$. Suppose that $X \geq 1$, $s \neq 1$, $0 < \sigma \leq 1$ and $|t| \leq X$. Then*

$$\zeta(s) = \sum_{n\leq X}\frac{1}{n^s} + \frac{X^{1-s}}{s-1} + O^*\left(\frac{D}{X^\sigma}\right),$$

where $D = 2/3$. If we assume $X \geq C$ for some integer $C > 1$, we may use

$$D = \frac{1}{2} + \sqrt{1+\frac{1}{C^2}}\left(\frac{1}{12} + \frac{\zeta(3)\sqrt{1+\frac{4}{C^2}}}{4\pi^3\left(1-\frac{1}{2\pi}\sqrt{1+(1+\frac{1}{C})^2}\right)}\right) + \frac{\zeta(C+2)}{\pi(\sqrt{2}\pi)^{C+1}}\sqrt{1+\left(1+\frac{1}{C}\right)^2}. \tag{2.18}$$

In the statement above, we can actually take $D = 0.66189$.

Proof. We apply Corollary 2.3: then we obtain $D \leq \frac{1}{2} + D_1 + D_2$, where

$$D_1 = \sum_{k=2}^{K} \frac{|a_k(s)| \max |B_k(\{X\})|}{k! X^{k-1}}, \qquad D_2 = \frac{|a_{K+1}(s)| \max |B_K(\{X\})|}{(\sigma+K-1)K! X^{K-1}}.$$

Our K has to depend on X, since the factor $|a_{K+1}(s)|$ in the numerator of D_2 has absolute value at least X^K, and so a factor of X remains once we divide by the factor of X^{K-1} in the numerator; K will have to be large enough for $\max |B_K(\{X\})|/(\sigma + K - 1)K!$ to be smaller than a (small) constant times $1/X$.

For $X \geq C \geq 2$, we choose $K = \lfloor X \rfloor + 2 \geq C + 2$. By the bound in Lemma 2.1 we have

$$D_1 \leq \sum_{k=2}^{K} \frac{2\zeta(k)|a_k(s)|}{(2\pi)^k X^{k-1}}, \qquad D_2 \leq \frac{2\zeta(K)|s(s+1)\dots(s+K-1)|}{(\sigma+K-1)(2\pi)^K X^{K-1}}. \tag{2.19}$$

We have $|s+k|/X \leq \sqrt{1+(k+1)^2/X^2}$ for all $k \geq 0$. Then, for $2 \leq k \leq K-1$, the ratio of two consecutive addends in the sum above is

$$\left(\frac{2\zeta(k+1)|a_{k+1}(s)|}{(2\pi)^{k+1}X^k}\right)\left(\frac{2\zeta(k)|a_k(s)|}{(2\pi)^k X^{k-1}}\right)^{-1} = \frac{\zeta(k+1)|s+k-1|}{2\pi\zeta(k)X}$$
$$\leq \frac{|s+X|}{2\pi X} \leq \frac{1}{2\pi}\sqrt{1+\left(1+\frac{1}{C}\right)^2} = Q_C,$$

since $\zeta(k)$ is decreasing for $k > 1$. Observe that $Q_C < 1$ for all $C \geq 1$. Therefore

$$\begin{aligned} D_1 &\leq \frac{2|s|\zeta(2)}{(2\pi)^2 X} + \frac{2|s(s+1)|\zeta(3)}{(2\pi)^3 X^2} \sum_{k=0}^{\infty} Q_C^k \\ &\leq \sqrt{1+\frac{1}{C^2}}\left(\frac{1}{12} + \frac{\zeta(3)}{4\pi^3(1-Q_C)}\sqrt{1+\frac{4}{C^2}}\right). \end{aligned} \tag{2.20}$$

By definition $K \geq 3$, so in (2.19) we can bound the terms $|s+k|$ with $k \in \{K-2, K-3\}$ separately. Hence

$$\begin{aligned} D_2 &\leq \frac{\zeta(K)}{\pi} \frac{|s+K-1|}{\sigma+K-1} \frac{|s+K-2|}{2\pi X} \frac{|s+K-3|}{2\pi X} \prod_{k=0}^{K-4} \frac{|s+k|}{2\pi X} \\ &\leq \frac{\zeta(K)}{4\pi^3} \sqrt{1+\left(\frac{X}{\lfloor X\rfloor+1}\right)^2} \sqrt{1+\left(\frac{X+1}{X}\right)^2} \sqrt{2} \left(\frac{1}{2\pi}\sqrt{1+\left(\frac{X-1}{X}\right)^2}\right)^{K-3} \\ &\leq \frac{\zeta(C+2)}{\pi(\sqrt{2}\pi)^{C+1}} \sqrt{1+\left(1+\frac{1}{C}\right)^2}, \end{aligned} \tag{2.21}$$

since the first and last square-roots in the second line are bounded by $\sqrt{2}$. From (2.20) and (2.21), we obtain (2.18).

Now we aim to obtain the numerical value of D in the statement: (2.18) is already enough for $C \geq 2$ (since it implies that $D \leq 0.62175$), and so we may assume that $1 \leq X < 2$. We apply Corollary 2.3 for two choices of I such that $X \in I$: for $I = [1, 15/14)$ we use $K = 3$, and for $I = [15/14, 2)$ we use $K = 4$. In either case, we write I_0 for the minimum of the interval. Instead of (2.20) and (2.21), we get

$$D_1 \leq \sum_{k=2}^{K} \frac{\max_{X\in I} |B_k(\{X\})|}{k!} \prod_{k'=0}^{k-2} \sqrt{1+\left(\frac{k'+1}{I_0}\right)^2},$$

$$D_2 \leq \frac{\max_X |B_K(\{X\})|}{K!} D_3,$$

where

$$D_3 = \max_{X\in I} \sqrt{\left(1+\left(\frac{X}{K-1}\right)^2\right) \prod_{k'=0}^{K-2} \left(1+\left(\frac{k'+1}{X}\right)^2\right)},$$

as $|s+K-1|/(\sigma+K-1) \leq \sqrt{1+X^2/(\sigma+K-1)^2} \leq \sqrt{1+X^2/(K-1)^2}$. The maximum in D_3 is achieved at I_0: inside the square root, $\left(1+(\frac{X}{K-1})^2\right) \cdot \left(1+(\frac{K-1}{X})^2\right)$ is maximized for X small since $X < K-1$, and the other factors are decreasing in X. For either choice of I, the numerical result follows. $\square$

2.7. Further results. The following is an explicit *mean value estimate.*

Proposition 2.11. *For any $X, T > 0$ and any sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$,*

$$\int_0^T \left|\sum_{n\leq X} a_n n^{it}\right|^2 \mathrm{d}t = \left(T+\frac{E}{2}\right) \sum_{n\leq X} |a_n|^2 + O^*\left(E \sum_{n\leq X} n|a_n|^2\right),$$

where E can be chosen to be equal to $2\pi\sqrt{1+2/3\sqrt{6/5}} \le 8.26495$.

Proof. We use the main theorem in [31], which improves on [28, Cor. 2] (the theorem states $C = \frac{4}{3}$, which yields $E = \frac{8}{3}\pi$, but it is proved with a lower C that yields our E). We apply it then as in [28, Cor. 3], with a numerical improvement given by $\log^{-1}(\frac{n+1}{n}) < n + \frac{1}{2}$, proved directly by calculus. See also [6, Satz 4.4.3] for an older explicit result that used $15n$ instead of $\frac{8}{3}\pi(n+\frac{1}{2})$. $\square$

If $\{a_n\}_{n=1}^{\infty}$ is a real sequence then the error term factor may be improved to $\frac{E}{2}$. As pointed out in [32, Lem. 6.5], a term cancels out, allowing us to gain a factor of 2 inside the error term.

Lemma 2.12. *For any* $1 < \sigma < 2$ *we have* $\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{1}{\sigma-1} + \zeta(2) - 1$. *The lower bound holds also for* $0 < \sigma < 1$.

See also [32, Lem. 5.4] for a better upper bound than the above for σ close to 1.

Proof. The Laurent expansion of ζ is

$$f(\sigma) = \zeta(\sigma) - \frac{1}{\sigma - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (\sigma - 1)^n \tag{2.22}$$

where the γ_n are the Stieltjes constants. For the upper bound, it suffices to prove that $f'(\sigma)$ is positive for $\sigma \in (1,2)$, so that $f(\sigma) < f(2)$: one can use

$$f'(\sigma) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\gamma_{n+1}}{n!}(\sigma-1)^n > -\gamma_1 - \sum_{n=1}^{\infty} \frac{|\gamma_{n+1}|}{n!},$$

compute the first 10 constants directly and then use the bound $|\gamma_n| \le \frac{n!}{2^{n+1}}$ (for $n \ge 1$) given by Lavrik in [21, Lem. 4], so that $\sum_{n=10}^{\infty} \frac{|\gamma_{n+1}|}{n!} \le \frac{1}{2}\sum_{n=11}^{\infty} \frac{n}{2^n} < 10^{-2}$. The lower bound is even simpler to obtain: in order to prove that $f(\sigma) > 0$ for $0 < \sigma < 2$ and $\sigma \ne 1$, we compute directly $\gamma_0 = \gamma$ and then we bound the absolute value of the rest of the series in (2.22) by using again Lavrik's estimations. $\square$

Lemma 2.13. *Let* $A, B \ge 0$. *Then, for any* $\rho > 0$,

$$(A+B)^2 \le (1+\rho)A^2 + \left(1 + \frac{1}{\rho}\right)B^2,$$
$$(A-B)^2 \ge (1-\rho)A^2 + \left(1 - \frac{1}{\rho}\right)B^2.$$

Note that the inequalities are tight when $\rho = \frac{B}{A}$.

Proof. Expand the square. By the arithmetic-geometric mean inequality, $2|AB| = 2(\sqrt{\rho}|A|) \cdot \frac{|B|}{\sqrt{\rho}} \le \rho A^2 + \frac{B^2}{\rho}$. $\square$

3. First approach: as in a mean value theorem

We will first bound (Proposition 3.2) the L^2 norm of the function $t \mapsto ((\sigma+it)^{-1} - G(\sigma+it))\,\zeta(\sigma+it)$, where G is the Mellin transform of a function $g : [0,\infty) \to \mathbb{R}$. Then we will choose g so that $G(\sigma+it)$ is close to 0 for $|t| \geq T$, while keeping the aforementioned L^2 bound small.

We will first give a general treatment for g arbitrary (Section 3.1). It will turn out to be easy to choose a g that is optimal within our general statement (Section 3.2). However, that optimality will turn out to be an artifact of the form of our general statement. We will be able to do better (at least for $\sigma \geq 1/2$) by choosing a different g, whose transform G we can compute explicitly (Section 3.3). Our final estimates are as follows.

Theorem 3.1. *Let $0 < \sigma \leq 1$ and $T \geq T_0 = 200$. Then the integral*

$$\frac{1}{2\pi i}\left(\int_{\sigma-i\infty}^{\sigma-iT} + \int_{\sigma+iT}^{\sigma+i\infty}\right)\left|\frac{\zeta(s)}{s}\right|^2 \mathrm{d}s$$

is bounded by

$$\begin{cases} \dfrac{3\zeta(2\sigma)}{5T} + \left(\dfrac{c_{111}}{\sigma} + \dfrac{c_{112}}{2\sigma+1} + \dfrac{c_{113}}{\sigma+1} - \dfrac{c_{114}}{2\sigma-1}\right)\dfrac{1}{T^{2\sigma}} + \dfrac{c_{12*}}{T^{2\sigma+1}}, & \text{if } \sigma > \dfrac{1}{2}, \\ \dfrac{3\log T}{5T} + \dfrac{c_{21*}}{T} + \dfrac{c_{22*}}{T^2}, & \text{if } \sigma = \dfrac{1}{2}, \\ \left(\dfrac{c_{311}}{\sigma} + \dfrac{c_{312}}{2\sigma+1} + \dfrac{c_{313}}{\sigma+1} + \dfrac{c_{314}}{1-2\sigma}\right)\dfrac{1}{T^{2\sigma}} + \dfrac{c_{30*}\zeta(2\sigma)}{T} + \dfrac{c_{32*}}{T^{2\sigma+1}}, & \text{if } \sigma < \dfrac{1}{2}, \end{cases}$$

where

$$\begin{aligned} &c_{11i} = \kappa^\sigma \kappa_{11i} \ (i=1,2,3,4), && \kappa = 27.8821, && \kappa_{12*} = 0.60031, \\ &c_{12*} = \kappa^\sigma \kappa_{12*}, && \kappa_{111} = 0.15659, && c_{21*} = 2.4476, \\ &c_{31i} = c_{11i} \ (i=1,2,3), && \kappa_{112} = 0.15655, && c_{22*} = 1.58493, \\ &c_{314} = \kappa^\sigma \kappa_{314}, && \kappa_{113} = 0.00979, && \kappa_{314} = 0.11361, \\ &c_{32*} = c_{12*}, && \kappa_{114} = 0.07407, && c_{30*} = 0.39113. \end{aligned}$$

We have chosen $T_0 = 200$ for simplicity. In actual fact, $T_0 = 192$ is the least T for which we are able to reach $\frac{3}{5}$ as a main term coefficient for $\sigma = \frac{1}{2}$.

3.1. Basic estimate.

Let us first give a bound valid for a function g that satisfies a number of general conditions. The proof is in parts close to, and in fact inspired by, proofs of classical mean value theorems, such as [27, Thm. 6.1] (see in particular the exposition in [17, Thm. 9.1]).

There are differences all the same. First, in a mean value theorem, we typically work with a finite sum $\sum_{n\leq X} a_n n^{it}$, and obtain a bound that contains a term proportional to X, whereas here we work directly with ζ and thus with an infinite sum.

Secondly, the proof in [17, Thm. 9.1] (or [27, Thm. 6.1]) majorizes the characteristic function of a vertical interval by a continuous function of compact support, and then uses the decay in the inverse Mellin transform to bound the contribution of off-diagonal terms. On the vertical line, we choose to work with a function of the form $1-G(s)s$, where G is the Mellin transform of a function g satisfying certain properties. As a consequence, off-diagonal terms vanish, outside an initial interval $[0,\delta]$ that makes a small contribution.

Proposition 3.2. *Let $g:[0,\infty)\to\mathbb{R}$ be a continuous, piecewise C^1 function such that g and g' have bounded total variation. Assume that*

(a) $\int_0^\infty g(t)\mathrm{d}t = 1$,
(b) $0\le g(t)\le 1$ *for all* t,
(c) $g(t)=1$ *for* $0\le t\le 1-\delta$ *and* $g(t)=0$ *for* $t\ge 1+\delta$, *where* $0<\delta\le\frac{1}{2}$,
(d) $g(1+t)=1-g(1-t)$ *for* $0\le t\le\delta$.

Let

$$I(\sigma)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left|\frac{1}{s}-G(s)\right|^2|\zeta(s)|^2\mathrm{d}s, \tag{3.1}$$

where G is the Mellin transform of g. Then, for any $\sigma>0$,

$$I(\sigma)\le c(\sigma,\alpha)\cdot\delta^{2\sigma}+2\beta\delta\cdot\begin{cases}\zeta(2\sigma)-\frac{\delta^{2\sigma-1}}{2\sigma-1}+\frac{\delta^{2\sigma}}{1-\delta^2} & \text{if } \sigma\ne\frac{1}{2},\\ \log\left(\frac{1}{\delta}\right)+\gamma+\frac{\delta}{2(1-\delta^2)} & \text{if } \sigma=\frac{1}{2},\end{cases} \tag{3.2}$$

where $\alpha=\frac{\delta}{16}\int_0^\infty|g''(t)|\mathrm{d}t$, $\beta=\frac{1}{\delta}\int_1^{1+\delta}|g(y)|^2\mathrm{d}y$ and $c(\sigma,\alpha)=\frac{1}{8\sigma}+\frac{\alpha}{2\sigma+1}+\frac{\alpha^2}{2\sigma+2}$.

Proof. Since g is bounded, $G(s)$ is well-defined when $\Re(s)>0$. For $\Re(s)>1$, we know from (2.9) that $G(s)\zeta(s)$ is the Mellin transform of the function $x\mapsto\sum_{n=1}^\infty g(nx)$ (well-defined by (c)) and from (2.8) that $\frac{\zeta(s)}{s}$ is the Mellin transform of $x\mapsto\sum_{n=1}^\infty \mathbb{1}_{(0,1/n]}(x)$.

Let

$$h(x)=\sum_{n=1}^\infty\left(\mathbb{1}_{[0,1/n]}(x)-g(nx)\right)=\left\lfloor\frac{1}{x}\right\rfloor-\sum_{n=1}^\infty g(nx). \tag{3.3}$$

Then

$$\mathcal{M}h(s)=\left(\frac{1}{s}-G(s)\right)\zeta(s), \tag{3.4}$$

for $\Re(s)>1$. On one hand, by (3.6), h is bounded, and thus $\mathcal{M}h(s)$ is well-defined for $\Re(s)>0$. On the other hand, by condition (a), $G(1)=1$

and thus the right side of (3.4) is holomorphic for $\Re(s) > 0$. Hence, by analytic continuation, (3.4) holds for $\Re(s) > 0$ and therefore, by (2.6),

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left|\frac{1}{s}-G(s)\right|^2|\zeta(s)|^2\mathrm{d}s=\int_0^\infty|h(x)|^2x^{2\sigma-1}\mathrm{d}x, \tag{3.5}$$

for any $s\in\mathbb{C}$ with $\Re(s)>0$, provided that the integral on the right side converges. Bounding the integral on the right will suffice to derive the result.

Let us first find an upper bound for the value of $|h(x)|$ to use for small values of x (namely, $x\leq\delta$). Using Lemma 2.4 and recalling that $g(0)=1$, $g(1)=0$, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}g(nx)&=\frac{1}{x}\int_0^\infty g(t)\mathrm{d}t-\frac{1}{2}+O^*\left(\frac{1}{16}\int_{0+}^\infty|g''(tx)|x^2\mathrm{d}t\right)\\&=\frac{1}{x}-\frac{1}{2}+O^*\left(\frac{x}{16}\int_{0+}^\infty|g''(t)|\mathrm{d}t\right).\end{aligned}$$

By putting the above equality inside (3.3), we obtain for any $x\geq0$ that

$$\begin{aligned}|h(x)|&=\left|\left\lfloor\frac{1}{x}\right\rfloor-\frac{1}{x}+\frac{1}{2}+O^*\left(\frac{x}{16}\int_{0+}^\infty|g''(t)|\mathrm{d}t\right)\right|\\&\leq\frac{1}{2}+\frac{x}{16}\int_{0+}^\infty|g''(t)|\mathrm{d}t,\end{aligned} \tag{3.6}$$

since $\left|\lfloor t\rfloor-t+\frac{1}{2}\right|\leq\frac{1}{2}$ for all $t\in\mathbb{R}$.

For $x>\delta$, we bound h in another way; by its definition and condition (c)

$$\begin{aligned}h(x)&=\sum_{nx\leq1}(1-g(nx))-\sum_{1<nx\leq1+\delta}g(nx)\\&=\sum_{1-\delta\leq nx\leq1}(1-g(nx))-\sum_{1<nx\leq1+\delta}g(nx).\end{aligned} \tag{3.7}$$

When $x>2\delta$, there is at most one integer n such that $nx\in[1-\delta,1+\delta]$, since $\frac{1+\delta}{x}-\frac{1-\delta}{x}=\frac{2\delta}{x}<1$. For the same reason, when $\delta<x\leq2\delta$, there can be at most one integer n (call it $n_{0,x}$) such that $nx\in[1-\delta,1]$ and at most one integer n (call it $n_{1,x}$) such that $nx\in[1,1+\delta]$. Since $0\leq g(t)\leq1$ for all t, we know that $1-g(nx)\geq0$ and $-g(nx)\leq0$, and so the last two sums in (3.7) have opposite sign. Hence $|h(x)|\leq\max\{|1-g(n_{0,x}x)|,|g(n_{1,x})|\}$.

It follows that

$$\text{(3.8)} \quad \int_0^\infty |h(x)|^2 x^{2\sigma-1}\mathrm{d}x \le \int_0^\delta |h(x)|^2 x^{2\sigma-1}\mathrm{d}x + \sum_{n\le\frac{1}{\delta}} \int_{\max\left\{\frac{1-\delta}{n},\delta\right\}}^{\frac{1}{n}} |1-g(nx)|^2 x^{2\sigma-1}\mathrm{d}x + \sum_{n\le\frac{1+\delta}{\delta}} \int_{\max\left\{\frac{1}{n},\delta\right\}}^{\frac{1+\delta}{n}} |g(nx)|^2 x^{2\sigma-1}\mathrm{d}x.$$

Setting $y = nx$ and changing the order of summation, we get

$$\sum_{n\le\frac{1}{\delta}} \int_{\max\left\{\frac{1-\delta}{n},\delta\right\}}^{\frac{1}{n}} |1-g(nx)|^2 x^{2\sigma-1}\mathrm{d}x = \int_{1-\delta}^1 \left(\sum_{n\le\frac{y}{\delta}} \frac{1}{n^{2\sigma}}\right) |1-g(y)|^2 y^{2\sigma-1}\mathrm{d}y,$$

and, similarly,

$$\sum_{n\le\frac{1+\delta}{\delta}} \int_{\max\left\{\frac{1}{n},\delta\right\}}^{\frac{1+\delta}{n}} |g(nx)|^2 x^{2\sigma-1}\mathrm{d}x = \int_1^{1+\delta} \left(\sum_{n\le\frac{y}{\delta}} \frac{1}{n^{2\sigma}}\right) |g(y)|^2 y^{2\sigma-1}\mathrm{d}y.$$

Using (3.6) in the first integral on the right hand side of (3.8), we obtain

$$\text{(3.9)} \qquad \int_0^\delta |h(x)|^2 x^{2\sigma-1}\mathrm{d}x \le \delta^{2\sigma}\cdot\left(\frac{1}{8\sigma} + \frac{\alpha}{2\sigma+1} + \frac{\alpha^2}{2(\sigma+1)}\right),$$

where $\alpha = \alpha_{g,\delta} = \frac{\delta}{16}\int_{0^+}^\infty |g''(t)|\mathrm{d}t$.

As for the remaining terms, we just use the bounds

$$\sum_{n\le x} \frac{1}{n^{2\sigma}} \le \begin{cases} \zeta(2\sigma) + \frac{x^{1-2\sigma}}{1-2\sigma} + x^{-2\sigma} & \text{if } \sigma \ne \frac{1}{2}, \\ \log x + \gamma + \frac{1}{2x} & \text{if } \sigma = \frac{1}{2}, \end{cases}$$

which we obtain from Lemmas 2.8 and 2.9, valid for $x \ge 1$ (for $\delta \le \frac{1}{2}$ and $y \ge 1-\delta$ we certainly have $\frac{y}{\delta} \ge 1$). Thus, the second and third terms on the right side of (3.8) add up to at most

$$\text{(3.10)} \quad \zeta(2\sigma)\left(\int_{1-\delta}^1 |1-g(y)|^2 y^{2\sigma-1}\mathrm{d}y + \int_1^{1+\delta} |g(y)|^2 y^{2\sigma-1}\mathrm{d}y\right) + \frac{\delta^{2\sigma-1}}{1-2\sigma}\left(\int_{1-\delta}^1 |1-g(y)|^2\mathrm{d}y + \int_1^{1+\delta} |g(y)|^2\mathrm{d}y\right) + \delta^{2\sigma}\left(\int_{1-\delta}^1 |1-g(y)|^2 \frac{\mathrm{d}y}{y} + \int_1^{1+\delta} |g(y)|^2 \frac{\mathrm{d}y}{y}\right),$$

if $0 < \sigma \leq 1$ with $\sigma \neq \frac{1}{2}$, and

$$(3.11)\quad \int_{1-\delta}^{1} |1-g(y)|^2 \log\left(\frac{y}{\delta}\right) \mathrm{d}y + \int_{1}^{1+\delta} |g(y)|^2 \log\left(\frac{y}{\delta}\right) \mathrm{d}y \\ + \gamma\left(\int_{1-\delta}^{1} |1-g(y)|^2 \mathrm{d}y + \int_{1}^{1+\delta} |g(y)|^2 \mathrm{d}y\right) \\ + \frac{\delta}{2}\left(\int_{1-\delta}^{1} |1-g(y)|^2 \frac{\mathrm{d}y}{y} + \int_{1}^{1+\delta} |g(y)|^2 \frac{\mathrm{d}y}{y}\right),$$

if $\sigma = \frac{1}{2}$.

When $\frac{1}{2} \leq \sigma \leq 1$, as the functions $f(y) = y^{2\sigma-1}$ and $f(y) = \log\left(\frac{y}{\delta}\right)$ are concave, we have by condition (d) that

$$(3.12)\quad \int_{1-\delta}^{1} |1-g(y)|^2 f(y) \mathrm{d}y + \int_{1}^{1+\delta} |g(y)|^2 f(y) \mathrm{d}y \\ = \int_{1}^{1+\delta} |g(y)|^2 (f(2-y) + f(y)) \mathrm{d}y \leq 2f(1) \int_{1}^{1+\delta} |g(y)|^2 \mathrm{d}y.$$

In the first line of (3.10), if $\sigma < \frac{1}{2}$, as $\zeta(2\sigma) < 0$ and $f(y) = y^{2\sigma-1}$ is convex, we employ the following *lower* bound

$$(3.13)\quad \int_{1-\delta}^{1} |1-g(y)|^2 f(y) \mathrm{d}y + \int_{1}^{1+\delta} |g(y)|^2 f(y) \mathrm{d}y \geq 2f(1) \int_{1}^{1+\delta} |g(y)|^2 \mathrm{d}y.$$

To estimate the integrals in (3.10), (3.11) that have $\frac{\mathrm{d}y}{y}$ in the integrand, we just use the fact that $y \mapsto y^{-1}$ is convex, so that for all $0 \leq t \leq \delta$, $(1-t)^{-1} + (1+t)^{-1} \leq (1-\delta)^{-1} + (1+\delta)^{-1} = \frac{2}{1-\delta^2}$.

Consider now $\beta = \beta_{g,\delta} = \frac{1}{\delta} \int_1^{1+\delta} |g(y)|^2 \mathrm{d}y$. Putting together (3.9), the cases (3.10) and (3.11) and the estimates (3.12) and (3.13), we finally obtain the following upper bounds for $\int_0^\infty |h(x)|^2 x^{2\sigma-1} \mathrm{d}x$:

$$(3.14)\quad 2\beta\left(\delta\zeta(2\sigma) - \frac{\delta^{2\sigma}}{2\sigma-1} + \frac{\delta^{2\sigma+1}}{1-\delta^2}\right) + \delta^{2\sigma}\left(\frac{1}{8\sigma} + \frac{\alpha}{2\sigma+1} + \frac{\alpha^2}{2(\sigma+1)}\right),$$

if $0 < \sigma \leq 1$ with $\sigma \neq \frac{1}{2}$, and, if $\sigma = \frac{1}{2}$,

$$(3.15)\quad 2\beta\left(\delta \log\left(\frac{1}{\delta}\right) + \gamma\delta + \frac{\delta^2}{2(1-\delta^2)}\right) + \delta\left(\frac{1}{4} + \frac{\alpha}{2} + \frac{\alpha^2}{3}\right). \qquad \square$$

Note that for $0 < \sigma < \frac{1}{2}$ the leading term in (3.14) is of order $\delta^{2\sigma}$, as then $\zeta(2\sigma) < 0$. The bound for $\sigma = \frac{1}{2}$ is what results from (3.14) if we let $\sigma \to \frac{1}{2}^-$ or $\sigma \to \frac{1}{2}^+$.

Remarks. Note that $\frac{\zeta(s)}{s}$ is the Mellin transform of $x \mapsto \lfloor 1/x \rfloor$. What we are doing is substract an approximation $f(x)$ to $\lfloor 1/x \rfloor$ such that the difference

$h(x) = \lfloor 1/x \rfloor - f(x)$ has a well-defined Mellin transform throughout $\Re(s) > 0$. Then the Mellin transform acts as an isometry throughout that region, and so, for $\Re(s) > 0$,

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left| \frac{\zeta(s)}{s} - F(s) \right|^2 \mathrm{d}s \tag{3.16}$$

equals the L^2 norm of $h(x)x^{\sigma - \frac{1}{2}}$ on $[0, \infty)$.

In the proof of Proposition 3.2, we take $f(x) = \sum_{n=1}^{\infty} g(nx)$ with g continuous. Then $F(s) = G(s)\zeta(s)$. We need G close to $\frac{1}{s}$ for $|\Im(s)| \leq T$ and close to 0 for $|\Im(s)| > T$, but not too close or else g would have slow decay, and f would approximate $\lfloor 1/x \rfloor$ poorly. This tension between two sources of error can be seen as reflecting the uncertainty principle.

Our requirement that g be compactly supported is somewhat restrictive, but greatly simplifies the proof of Proposition 3.2: for $x \geq 2\delta$, the sum $f(x) = \sum_{n=1}^{\infty} g(nx)$ contains only one term, and so does its square.

3.2. An "optimal" choice of g. What we want is to bound the integral $\frac{1}{2\pi i}\left(\int_{\sigma - i\infty}^{\sigma - iT} + \int_{\sigma + iT}^{\sigma + i\infty}\right)\left|\frac{\zeta(s)}{s}\right|^2 \mathrm{d}s$, which is at most

$$\frac{I(\sigma)}{\inf_{|\Im(s)| \geq T} |1 - G(s)s|^2} \tag{3.17}$$

where $I(\sigma)$ and $G(s)$ are as in Proposition 3.2 and $\Re(s) = \sigma$.

Proposition 3.2 gives us a bound on $I(\sigma)$, while

$$\begin{aligned} \inf_{|\Im(s)| \geq T} |1 - G(s)s| &\geq 1 - \sup_{|\Im(s)| \geq T} \frac{|G(s)s(s+1)|}{T} \\ &\geq 1 - \frac{1}{T} \int_0^{\infty} |g''(x)| x^{\sigma + 1} \mathrm{d}x, \end{aligned} \tag{3.18}$$

where the second inequality comes from applying (2.7) twice. (Proceeding in this way seems natural, since we already estimated a quantity in terms of g'' in Proposition 3.2. It will later turn out later that we are losing enough in this step to make the result we would obtain in this section worse than the one we will get in Section 3.3.)

From the conditions on g in Proposition 3.2 we have $g'' = 0$ outside $[1 - \delta, 1 + \delta]$ and $g''(1 + x) = -g''(1 - x)$ for $x \in [0, \delta]$. Since $x \mapsto x^{\sigma+1}$ is convex in x for $\sigma \geq 0$ and $(1 + \delta)^{\sigma+1} + (1 - \delta)^{\sigma+1}$ is increasing in $\sigma \geq 0$, we see that $(1 + x)^{\sigma+1} + (1 - x)^{\sigma+1} \leq (1 + \delta)^2 + (1 - \delta)^2 = 2 + 2\delta^2$, and so

$$1 - \frac{1}{T} \int_0^{\infty} |g''(x)| x^{\sigma+1} \mathrm{d}x \geq 1 - \frac{1 + \delta^2}{T} |g''|_1. \tag{3.19}$$

We focus only on the main terms in the bound of $I(\sigma)$ given in Proposition 3.2. Introduce an auxiliary function $\eta : [0, \infty) \to \mathbb{R}$ defined so that

$g(1+x) = \frac{1}{2}\eta\left(\frac{x}{\delta}\right), g(1-x) = 1-\frac{1}{2}\eta\left(\frac{x}{\delta}\right)$. We then have $\beta = \frac{1}{4}|\eta|_2^2$, $|g''|_1 = \frac{1}{\delta}|\eta''|_1$, and so $\alpha = \frac{1}{16}|\eta''|_1$. The main terms for δ small are

$$\frac{|\eta|_2^2\zeta(2\sigma)\delta}{2\left(1-\frac{1}{\delta T}|\eta''|_1\right)^2} \quad \text{for } \frac{1}{2}<\sigma\leq 1, \qquad \frac{|\eta|_2^2\delta\log\left(\frac{1}{\delta}\right)}{2\left(1-\frac{1}{\delta T}|\eta''|_1\right)^2} \quad \text{for } \sigma=\frac{1}{2},$$

$$\frac{\left(\frac{|\eta|_2^2}{2(1-2\sigma)}+\frac{1}{8\sigma}+\frac{|\eta''|_1}{16(2\sigma+1)}+\frac{|\eta''|_1^2}{512(\sigma+1)}\right)\delta^{2\sigma}}{\left(1-\frac{1}{\delta T}|\eta''|_1\right)^2} \quad \text{for } 0<\sigma<\frac{1}{2}.$$

The term $\frac{1}{8\sigma}$ in the case $0<\sigma<\frac{1}{2}$ is not unexpected, as the integral of Theorem 3.1 diverges at $\sigma=0$. We will choose δ so as to minimize the main terms above. For $\frac{1}{2}<\sigma\leq 1$, the minimum of $\frac{x}{(1-ax^{-1})^2}$ is at $x=3a$. Therefore we let $\delta = 3|\eta''|_1T^{-1}$ so that the main term becomes

$$\frac{3\zeta(2\sigma)}{2T\left(1-\frac{1}{3}\right)^2}|\eta''|_1|\eta|_2^2.$$

For $\sigma=\frac{1}{2}$ we let $\delta=3|\eta''|_1T^{-1}$, out of simplicity. Then $\log(\frac{1}{\delta}) = \log T + \log(\frac{2}{3|\eta''|_1})$, the term with $\log T$, which will be the main term in T, contributing

$$\frac{3\log T}{2T\left(1-\frac{1}{3}\right)^2}|\eta''|_1|\eta|_2^2.$$

For $0<\sigma<\frac{1}{2}$, the minimum of $\frac{x^{2\sigma}}{(1-ax^{-1})^2}$ is reached at $x=(1+\frac{1}{\sigma})a$, so that we can choose $\delta=(1+\frac{1}{\sigma})|\eta''|_1T^{-1}$. The main term in this case is at most

$$\frac{\left(1+\frac{1}{\sigma}\right)^{2\sigma}}{T^{2\sigma}\left(1-\frac{1}{1+\frac{1}{\sigma}}\right)^2}\left(\frac{|\eta''|_1^{2\sigma}|\eta|_2^2}{2(1-2\sigma)}+\frac{|\eta''|_1^{2\sigma}}{8\sigma}+\frac{|\eta''|_1^{2\sigma+1}}{16(2\sigma+1)}+\frac{|\eta''|_1^{2\sigma+2}}{512(\sigma+1)}\right). \tag{3.20}$$

In all cases, we conclude that we have to select η so that the factor $|\eta''|_1|\eta|_2^2$ (or, for $0<\sigma<\frac{1}{2}$, the first term in (3.20)) is minimal.

Lemma 3.3. *Let $\eta:[0,\infty)\to\mathbb{R}$ be a decreasing continuous function, continuously differentiable outside a finite number of points, such that $\eta(0)=1$ and $\eta(x)=0$ for all $x\geq 1$. Then there exist $x_0\in(0,1]$ and a function $\eta_{x_0}:[0,\infty)\to\mathbb{R}$ of the form*

$$\eta_{x_0}(x) = \begin{cases} 1-\frac{x}{x_0} & \text{for } 0\leq x<x_0,\\ 0 & \text{for } x\geq x_0,\end{cases}$$

such that $|\eta''_{x_0}|_1\leq|\eta''|_1$ and $|\eta_{x_0}|_2\leq|\eta|_2$.

Proof. If $|\eta''|_1 = \infty$, we just take η_{x_0} with $x_0 > 0$ sufficiently small so that $|\eta_{x_0}|_2 \le |\eta|_2$. Otherwise, suppose that η' is of bounded variation; then one-sided limits of η' always exist.

Since $|\eta'(0^+)| < \infty$, it is clear that there is a $x_1 > 0$ such that $\eta_{x_1}(t) \le \eta(t)$ for all t in some interval $[0,\delta]$, $\delta > 0$. Since η is decreasing and $\eta(x) = 0$, η is non-negative on $[0,1]$. Hence, for $x_2 = \min(x_1, 1/\delta)$, we know that $\eta_{x_2}(x) \le \eta(x)$ for all $x \in [0,1]$. Let x_0 be the largest element of $[0,1]$ such that $\eta_{x_0}(x) \le \eta(x)$ for all $x \in [0,1]$; clearly, $x_0 \ge x_2 > 0$. We readily see that $|\eta_{x_0}|_2 \le |\eta|_2$, so it is sufficient to prove that $|\eta''_{x_0}|_1 \le |\eta''|_1$.

Clearly, $\eta'_{x_0}(0^+) \le \eta'(0^+)$. Suppose first that $\eta'_{x_0}(0^+) = \eta'(0^+)$. By construction, we have $|\eta''_{x_0}|_1 = \frac{1}{x_0} = |\eta'_{x_0}(0^+)|$; furthermore, the total variation $|\eta''|_1$ of η' is at least $|\eta'(0^+) - \eta'(2)|$, which is equal to $|\eta'(0^+)|$, since $\eta'(2) = 0$. Therefore $|\eta''_{x_0}|_1 = |\eta'_{x_0}(0^+)| = |\eta(0^+)| \le |\eta''|_1$. Now suppose instead that $\eta'_{x_0}(0^+) < \eta'(0^+)$. Let $c \in (0, x_0]$ such that $\eta_{x_0}(c) = \eta(c)$, which must exist by definition of x_0: since η is continuous and $\eta_{x_0}(x) \le \eta(x)$ for all $x \in (0,c)$, there must be some $c' \in (0,c)$ with $\eta'(c') \le \eta'_{x_0}(c') = -\frac{1}{x_0}$, and as before we have $|\eta''|_1 \ge |\eta'(c') - \eta'(2)| \ge \frac{1}{x_0}$, concluding the proof. $\square$

Thanks to Lemma 3.3, we can assume that $\eta(x)$ is simply the function given by $\eta(x) = 1 - x$ for $0 \le x \le 1$, and by $\eta(x) = 0$ for $x \ge 1$; the other functions η_0 described in the statement of Lemma 3.3 are just dilations of this one, and can thus be covered by the fact that we can choose δ as we wish.

Corollary 3.4 (to Proposition 3.2)**.** *Let* $0 < \sigma \le 1$, $T > \max\left\{3, 1 + \frac{1}{\sigma}\right\}$. *Then*

$$\frac{1}{2\pi i}\left(\int_{\sigma-i\infty}^{\sigma-iT} + \int_{\sigma+iT}^{\sigma+i\infty}\right)\left|\frac{\zeta(s)}{s}\right|^2 \mathrm{d}s \le \rho_{\sigma,T} \cdot \begin{cases} \frac{\zeta(2\sigma)}{2T} + \frac{c_0(3) - c_1(3)}{T^{2\sigma}} & \text{if } \sigma > \frac{1}{2}, \\ \frac{\log T}{2T} + \frac{c_0(3) - c_2}{T} & \text{if } \sigma = \frac{1}{2}, \\ \frac{c_0(1+\frac{1}{\sigma}) - c_1(1+\frac{1}{\sigma})}{T^{2\sigma}} + \frac{c_3}{T} & \text{if } \sigma < \frac{1}{2}, \end{cases}$$

where

$$c_0(\kappa) = \kappa^{2\sigma}\left(c' + \frac{\kappa}{6T(1 - \frac{\kappa^2}{T^2})}\right), \qquad c_1(\kappa) = \frac{\kappa^{2\sigma}}{6(2\sigma - 1)},$$

$$c_2 = \frac{\log 3 - \gamma}{2}, \qquad c_3 = \frac{(\sigma+1)\zeta(2\sigma)}{6\sigma},$$

$$c' = \frac{1}{8\sigma} + \frac{1}{16(2\sigma+1)} + \frac{1}{512(\sigma+1)}, \quad \rho_{\sigma,T} = \begin{cases} \frac{9}{4\left(1 - \frac{9}{2T^2}\right)^2} & \text{if } \sigma \ge \frac{1}{2}, \\ \frac{(1+\sigma)^2}{\left(1 - \frac{(1+\sigma)^2}{\sigma T^2}\right)^2} & \text{if } \sigma < \frac{1}{2}. \end{cases}$$

Notice that $c_0(\kappa)$ and c_3 go to ∞ when $\sigma \to 0$. Observe also that the numerical optimization in Section 3.3, on which Theorem 3.1 is based, yields results that are asymptotically stronger than the ones above only for $\sigma \geq \frac{1}{2}$: the main coefficient of Corollary 3.4 turns out to be better when $\sigma > 0$ is close to 0, starting from around $\sigma = 0.044$, although not yet reaching the asymptotically correct value proved later in Theorem 4.6.

Proof. As per the discussion above, we let

$$g(t) = \begin{cases} 1 & \text{for } 0 < t \leq 1 - \delta, \\ \frac{1}{2} - \frac{t-1}{2\delta} & \text{for } 1 - \delta \leq t \leq 1 + \delta, \\ 0 & \text{for } t > 1 + \delta. \end{cases}$$

It is clear that $|g''|_1 = \frac{1}{\delta}$; hence, $\alpha = \frac{1}{16}$ and $\beta = \frac{1}{12}$, for α and β as in the statement of Proposition 3.2. We let $\delta = \frac{3}{T}$ if $\frac{1}{2} \leq \sigma \leq 1$ and $\delta = \frac{1+\sigma^{-1}}{T}$ if $0 < \sigma < \frac{1}{2}$. We bound $\inf_{|\Im(s)| \geq T} |1 - G(s)s|$ from below by (3.18) and (3.19). Then apply Proposition 3.2. $\square$

We will not use Corollary 3.4 in our main results.

3.3. A better choice of g for $\Re(s) \in \left[\frac{1}{2}, 1\right]$. The choice of g in Section 3.2 is optimal only once we commit ourselves to bounding $|1 - G(s)s|$ as in (3.18). Alternatively, we can choose g from a class of functions whose Mellin transforms $G(s)$ we can compute explicitly. We can then optimize g within that class. Consider, for instance, $g : [0, \infty) \to \mathbb{R}$ such that g is given by a polynomial in the interval $[1 - \delta, 1 + \delta]$, where the transition from 1 to 0 occurs. So that the conditions in Proposition 3.2 are fulfilled, we ask for g with $g(x) = 1$ for $x < 1 - \delta$, $g(x) = 0$ for $x > 1 + \delta$, and

$$g(x) = \frac{1}{2} + \sum_{k=0}^{n} a_k \frac{(1 + \delta - x)^k (1 - x)(1 - \delta - x)^k}{\delta^{2k+1}} \quad \text{if } 1 - \delta \leq x \leq 1 + \delta \tag{3.21}$$

for some appropriate parameters n, δ and a sequence $\{a_k\}_{k=0}^{n}$. This choice in turn will allow us to give the Mellin transform of such g explicitly, according to Lemma 2.5.

Lemma 3.5. *Let $g : [0, \infty) \to \mathbb{R}$ be a function of the form (3.21). Suppose that*

(a) *$a_0 = \frac{1}{2}$ and $a_1 = -\frac{1}{4}$,*
(b) *for every $0 \leq k \leq n$ the coefficient a_k has sign $(-1)^k$,*
(c) *for every $0 \leq k < n$ we have $|a_{k+1}| \leq \frac{2k+1}{2k+2} |a_k|$.*

Then g is continuously differentiable on $(0, \infty)$ and $0 \leq g \leq 1$ everywhere.

Proof. Each of the three pieces in which g is split by (3.21) is continuously differentiable, so we just have to check the property for the points $1-\delta$ and $1+\delta$. We have $g(1 \pm \delta) = \frac{1}{2} \mp a_0$ and setting $a_0 = \frac{1}{2}$ makes it so that $g(1-\delta) = 1$, $g(1+\delta) = 0$, implying the continuity of g. Supposing that $a_0 = \frac{1}{2}$, we also obtain $\lim_{x \to \delta^-} g'(1 \pm x) = -\frac{1}{2\delta} - \frac{2a_1}{\delta}$ and having $a_1 = -\frac{1}{4}$ makes it so that this limit becomes 0, thus giving us the continuity of the first derivative for g.

To prove that $0 \leq g \leq 1$ in the interval $[1-\delta, 1+\delta]$, it is sufficient to show that $g'(x) \leq 0$ in that interval. If we substitute $\varepsilon = 1-x$, we have

$$\begin{aligned} g'(x) &= -\frac{1}{2\delta} - \sum_{k=1}^{n} a_k \frac{(\varepsilon^2-\delta^2)^{k-1}}{\delta^{2k+1}}((2k+1)(\varepsilon^2-\delta^2)+2k\delta^2) \\ &= -\sum_{k=1}^{n-1} \frac{(\varepsilon^2-\delta^2)^k}{\delta^{2k+1}}((2k+1)a_k+(2k+2)a_{k+1}) - a_n \frac{(\varepsilon^2-\delta^2)^n}{\delta^{2n+1}}(2n+1). \end{aligned}$$

Since we are working in $[1-\delta, 1+\delta]$ we have $\varepsilon^2 - \delta^2 \leq 0$. To ensure that the product $a_n(\varepsilon^2-\delta^2)^n$ in the last term is not negative, it is sufficient to ask for a_n to have sign $(-1)^n$. We can now proceed backwards by induction on the terms in the sum. Indeed, supposing that $(-1)^{k+1}a_{k+1} \geq 0$, in order to have $(\varepsilon^2-\delta^2)^k((2k+1)a_k+(2k+2)a_{k+1}) \geq 0$ it is enough to ask that $(-1)^k a_k \geq 0$ and $(2k+1)|a_k| \geq (2k+2)|a_{k+1}|$. □

Computing the parameter β in Proposition 3.2 is routine.

Lemma 3.6. *Let $g : [0,\infty) \to \mathbb{R}$ be a function of the form* (3.21) *such that $a_0 = \frac{1}{2}$. Define $\beta = \frac{1}{\delta}\int_1^{1+\delta} |g(x)|^2 dx$. Then*

$$\beta = \sum_{i=0}^{4n+2} \frac{(-1)^i}{i+1} \sum_{l=0}^{i} b_{n,l} b_{n,i-l},$$

where

$$b_{n,j} = \sum_{k=0}^{n} 2^{2k-j} \left(2\binom{k}{j-k-1} + \binom{k}{j-k} \right) a_k$$

for $1 \leq j \leq 2n+1$ and $b_{n,j} = 0$ for $j = 0$ or $j > 2n+1$.

Proof. We substitute $y = 1+\delta-x$ inside the definition of $g(x)$. Then, for $1-\delta \leq x \leq 1+\delta$,

$$\begin{aligned} g(x) &= \frac{1}{2} + \sum_{k=0}^{n} a_k \frac{y^k(y-\delta)(y-2\delta)^k}{\delta^{2k+1}} \\ &= \frac{1}{2} + \sum_{k=0}^{n}\sum_{i=0}^{k} \left((-1)^{k-i}\binom{k}{i} \frac{a_k 2^{k-i}}{\delta^{k+i+1}} y^{k+i+1} - (-1)^{k-i}\binom{k}{i}\frac{a_k 2^{k-i}}{\delta^{k+i}} y^{k+i} \right). \end{aligned}$$

Inside the sums, we substitute $j=k+i+1$ in the first term and $j=k+i$ in the second term, we shift one summation symbol outside, with the new index j, and we uniformize the range of each of the inner sums. We obtain

$$g(x)=\frac{1}{2}+\sum_{j=0}^{2n+1}(-1)^{j+1}\frac{y^j}{\delta^j}\sum_{k=0}^{n}2^{2k-j}\left(2\binom{k}{j-k-1}+\binom{k}{j-k}\right)a_k.$$

For $1\leq j\leq 2n+1$, we just define $b_{n,j}$ to be as in the statement. For $j=0$, we include in the definition of $b_{n,0}$ the term $\frac{1}{2}$ that was outside the sums, so that $b_{n,0}=\frac{1}{2}-\sum_{k=0}^{n}2^{2k}\left(2\binom{k}{-k-1}+\binom{k}{-k}\right)a_k=\frac{1}{2}-a_0=0$. Therefore

$$g(x)=\sum_{j=0}^{2n+1}(-1)^{j+1}b_{n,j}\frac{y^j}{\delta^j}. \tag{3.22}$$

Imposing also $b_{n,j}=0$ for $j>2n+1$, we finally get

$$\int_1^{1+\delta}|g(x)|^2\mathrm{d}x=\sum_{i=0}^{4n+2}\left(\sum_{l=0}^{i}(-1)^i b_{n,l}b_{n,i-l}\right)\frac{\delta^{i+1}}{(i+1)\delta^i},$$

which gives β. $\square$

In order to choose δ and g optimally, we need to detect first what to minimize.

Proposition 3.7. *If $0<\sigma\leq 1$, then $\frac{1}{2\pi i}\left(\int_{\sigma-i\infty}^{\sigma-iT}+\int_{\sigma+iT}^{\sigma+i\infty}\right)\left|\frac{\zeta(s)}{s}\right|^2\mathrm{d}s$ is bounded from above by quantities whose main terms are*

$$\begin{cases}\dfrac{2\zeta(2\sigma)r\sum_{i=0}^{4n+2}\frac{(-1)^i}{i+1}\sum_{l=0}^{i}b_{n,l}b_{n,i-l}}{\left(1-\sum_{j=1}^{2n+1}\frac{2j!|b_{n,j}|}{r^j}\right)^2}\cdot\dfrac{1}{T} & \textit{if } \sigma>\dfrac{1}{2},\\[2ex] \dfrac{2r\sum_{i=0}^{4n+2}\frac{(-1)^i}{i+1}\sum_{l=0}^{i}b_{n,l}b_{n,i-l}}{\left(1-\sum_{j=1}^{2n+1}\frac{2j!|b_{n,j}|}{r^j}\right)^2}\cdot\dfrac{\log T}{T} & \textit{if } \sigma=\dfrac{1}{2},\\[2ex] \dfrac{r^{2\sigma}\left(\frac{1}{8\sigma}+\frac{\alpha}{2\sigma+1}+\frac{\alpha^2}{2\sigma+2}+\frac{2}{1-2\sigma}\sum_{i=0}^{4n+2}\frac{(-1)^i}{i+1}\sum_{l=0}^{i}b_{n,l}b_{n,i-l}\right)}{\left(1-\sum_{j=1}^{2n+1}\frac{2j!|b_{n,j}|}{r^j}\right)^2}\cdot\dfrac{1}{T^{2\sigma}} & \\ & \textit{if } \sigma<\dfrac{1}{2},\end{cases} \tag{3.23}$$

where g is any polynomial as in (3.21), for any choice of $(n,r,\{a_k\}_{k=0}^n)$ such that $0<r\leq\frac{T}{2}$, $n\geq 1$, $\{a_k\}_{k=0}^n$ satisfies the conditions of Lemma 3.5, the $b_{n,j}$ are defined as in Lemma 3.6, α is defined as in Proposition 3.2, and the expression inside the square in the denominator is positive.

Proof. Recall inequality (3.17). By Lemma 3.5, all the conditions are met so that we can derive a bound (depending on δ) for its numerator $I(\sigma)$ as given in Proposition 3.2.

Let us concentrate on its denominator. For $x \in [1-\delta, 1+\delta]$, we write $g(x)$ as in (3.22), where $y = 1+\delta-x$. We proceed similarly for $z = 1-\delta-x$. Observe that, since $g(x) = 0$ for all $x > 1+\delta$ and $g = 1$ in $[0, 1-\delta]$,

$$g = \sum_{j=0}^{2n+1} (-1)^{j+1} b_{n,j} \frac{y^j}{\delta^j} \mathbb{1}_{[0,1+\delta]} - \sum_{j=1}^{2n+1} b_{n,j} \frac{z^j}{\delta^j} \mathbb{1}_{[0,1-\delta]}, \tag{3.24}$$

where the $b_{n,j}$ are as in Lemma 3.6. Now, g is written as linear combination of expressions as in (2.11) with $a = 1 \pm \delta$, and, by Lemma 2.5, its Mellin transform is

$$G(s) = \sum_{j=1}^{2n+1} \frac{j! b_{n,j}((-1)^{j+1}(1+\delta)^{s+j} - (1-\delta)^{s+j})}{\delta^j s(s+1)\dots(s+j)}.$$

Furthermore, we have $|s+1|, \dots, |s+j| > |\Im(s)|^j$, and $\sigma + j \leq j+1$ implies that $|(1+\delta)^{s+j} \pm (1-\delta)^{s+j}| \leq (1+\delta)^{j+1} + (1-\delta)^{j+1}$, since the left hand side is an increasing function of σ. These two facts imply that

$$\inf_{|\Im(s)| \geq T} |1 - G(s)s| \geq 1 - \sum_{j=1}^{2n+1} \frac{j! |b_{n,j}|}{\delta^j T^j} \sum_{i=0}^{\left\lfloor \frac{j+1}{2} \right\rfloor} 2 \binom{j+1}{2i} \delta^{2i}. \tag{3.25}$$

We want δ to be small, so as to keep the upper bound in (3.2) small, but not too small, since we want the expression on the right of (3.25) to be positive.

The terms $\delta^j T^j$ in (3.25) tell us that we cannot afford more than taking $\delta = \frac{r}{T}$, which we choose, for some $0 < r \leq \frac{T}{2}$ large enough (depending only on n) to make the right hand side of (3.25) positive. Therefore, all conditions requested in the above paragraph hold. Let $D_{\min}$ be the square of the expression on the right of (3.25), so that $\inf_{|\Im(s)| \geq T} |1 - G(s)s|^2 \geq D_{\min}$.

Now, the substitution $\delta = \frac{r}{T}$ in the bounds (3.2) makes evident that the obtained main terms, as $T \to \infty$, are of order $\frac{1}{T}$, $\frac{\log T}{T}$, $\frac{1}{T^{2\sigma}}$ for $\frac{1}{2} < \sigma \leq 1$, $\sigma = \frac{1}{2}$, $0 < \sigma < \frac{1}{2}$, respectively. Moreover, thanks to the definitions of α, β, implemented for a function g of the form (3.21), it is the choice of $\{a_k\}_{k=0}^n$ and of r that will determine the optimal constants in front of these main terms.

We derive the result once we put everything together and set aside the summands of order $\frac{1}{T^{2i}}$ that come from the inner sum defining $\sqrt{D_{\min}}$. $\square$

Proof of Theorem 3.1. First we bound $\frac{1}{2\pi i} \left(\int_{\sigma-i\infty}^{\sigma-iT} + \int_{\sigma+iT}^{\sigma+i\infty} \right) \left| \frac{\zeta(s)}{s} \right|^2 ds$ as in Proposition 3.7. As aforementioned, it is the choice of n, a_k $(0 \leq k \leq n)$ and r that suffices to optimize those main terms in each case. For simplicity, we will carry out the optimization process and the corresponding choice of parameters according to (3.23) only for $\sigma \geq \frac{1}{2}$, the same choice being used for the remaining cases.

For $n = 2, 3$, we determine by computer all possibilities for coefficients of g satisfying the conditions in Lemma 3.5 with precision 10^{-n-1}. We then proceed inductively for larger n; given an optimized $g = g_n$ for a certain n, a better $g = g_{n+1}$ with $n+1$ is found as follows: start with the set of coefficients provided by the original g, attaching $a_n = 0$ as a new variable, and compute the first bound in (3.23), for any fixed $\frac{1}{2} < \sigma \leq 1$ (in fact, σ does not participate in our analysis), by adding $\vec{x}$ to the tuple $\vec{a} = (a_2, \ldots, a_n)$ (a_0, a_1 being fixed) for every $\vec{x} \in (\{0, \pm 10^{-n-1}\})^{n-1}$ such the conditions of Lemma 3.5 hold. We thus determine an optimal $\vec{x}$, call it $\vec{x}_*$, and compute the first bound in (3.23) with $\vec{a} + j\vec{x}_*$, $j \geq 1$, as long as we encounter improvements, until we stop and consider the last tuple $\vec{a}_* = \vec{a} + j\vec{x}_*$, that produces an improvement on (3.23) (meaning that $\vec{a} + (j+1)\vec{x}_*$ does not). We repeat the described process starting with $\vec{a}_*$ rather than $\vec{a}$ until we find an optimized set of coefficients $a_2, \ldots, a_n$ for which no increment $\vec{x}$ produces any improvement; this final $(a_2, \ldots, a_n)$ will define g_{n+1}.

By taking $n = 6$, our parameters are

$$\begin{aligned} a_0 &= \frac{1}{2}, \quad a_1 = -\frac{1}{4}, \quad a_3 = -\frac{533639}{10000000}, \quad a_5 = -\frac{1483}{2000000}, \\ & \qquad\quad a_2 = \frac{3}{16}, \quad a_4 = \frac{10139}{1250000}, \quad a_6 = \frac{37}{1000000}, \end{aligned} \tag{3.26}$$

$r = 5.28035$ and $T \geq T_0 = 200$.

Consider $D_{\min}$ and let

$$D_{\max} = \left(1 + \sum_{j=1}^{2n+1} \frac{2j!|b_{n,j}|}{r^j} \left(1 + \binom{j+1}{2} \frac{r^2}{T^2} + \cdots \right)\right)^2,$$

so that, recalling again Proposition 3.7, $\sup_{|\Im(s)| \geq T} |1 - G(s)s|^2 \leq D_{\max}$.

Given the choice in (3.26), we have

$$\begin{aligned} \alpha &= 0.12496\ldots, & \beta &= \frac{5173290592354408399}{11408158125000000000}, \\ D_{\min} &> 0.79831, & D_{\max} &< 1.22439. \end{aligned}$$

Hence, the coefficient of the leading term $\frac{1}{T}$ in the case of $\frac{1}{2} < \sigma \leq 1$ becomes

$$\frac{2\beta\zeta(2\sigma)r}{D_{\min}}, \quad \text{with} \quad \frac{2\beta r}{D_{\min}} < 0.5999 \leq \frac{3}{5}, \tag{3.27}$$

and the coefficients of the smaller terms $\frac{1}{T^{2\sigma}}, \frac{1}{T^{2\sigma+1}}$ are bounded as follows

$$\begin{aligned}
&\kappa := 27.8821 \in r^2 + [0, 10^{-5}],\\
&c_{111} := \kappa^\sigma \kappa_{111} := \kappa^\sigma \cdot 0.15659 > \frac{r^{2\sigma}}{8D_{\min}},\\
&c_{112} := \kappa^\sigma \kappa_{112} := \kappa^\sigma \cdot 0.15655 > \frac{\alpha r^{2\sigma}}{D_{\min}},\\
&c_{113} := \kappa^\sigma \kappa_{113} := \kappa^\sigma \cdot 0.00979 > \frac{\alpha^2 r^{2\sigma}}{2D_{\min}},\\
&c_{114} := \kappa^\sigma \kappa_{114} := \kappa^\sigma \cdot 0.07407 < \kappa^\sigma \cdot \frac{2\beta\left(1 - \frac{10^{-5}}{\kappa}\right)}{D_{\max}} \leq \frac{2\beta r^{2\sigma}}{D_{\max}},\\
&c_{12*} := \kappa^\sigma \kappa_{12*} := \kappa^\sigma \cdot 0.60031 > \frac{2\beta r^{2\sigma+1}}{\left(1 - \frac{r^2}{T^2}\right) D_{\min}},
\end{aligned}$$

where the numbers c_{ijk} are the ones given in the statement.

In the case of $\sigma = \frac{1}{2}$, the coefficient of the leading term $\frac{\log T}{T}$ is, as in (3.27), bounded by $\frac{3}{5}$, while the lower order terms $\frac{1}{T}, \frac{1}{T^2}$ have their coefficients bounded as follows

$$\begin{aligned}
c_{21*} &:= 2.4476 > \frac{r}{D_{\min}}\left(2\beta\gamma + \frac{1}{4} + \frac{\alpha}{2} + \frac{\alpha^2}{3}\right),\\
c_{22*} &:= 1.58493 > \frac{\beta r^2}{\left(1 - \frac{r^2}{T^2}\right) D_{\min}}.
\end{aligned}$$

Finally, in the case of $0 < \sigma < \frac{1}{2}$, the coefficients are bounded in the same way as in the case of $\frac{1}{2} < \sigma \leq 1$, with the exception of

$$c_{314} := \kappa^\sigma \kappa_{314} := \kappa^\sigma \cdot 0.11361 > \frac{2\beta r^{2\sigma}}{D_{\min}}, \qquad c_{30*} := 0.39113 < \frac{2\beta r}{D_{\max}}. \qquad \square$$

Remarks. The coefficient $\frac{3}{5} = 0.6$ appearing in the case $\frac{1}{2} \leq \sigma \leq 1$ is an artificial threshold that the authors have set, $n = 6$ being the smallest value for which it could be reached for some choice of parameters a_k. These parameters, together with r and T_0, were then determined by our choice of threshold and n through computer calculations, as already described during the proof.

The chosen threshold could have been improved by choosing a larger n than $n = 6$, albeit very slightly; computer investigations up to $n = 9$ did not manage to give less than 0.596. Nevertheless, the correct value in that case, as given in Theorem 1.1 and suggested for example by the asymptotics in Theorems 7.2 and 7.3 in [37], should have been $\frac{1}{\pi} = 0.3183\ldots$.

In Section 4 we obtain such a coefficient. However, for small values of T, the estimations in Theorem 3.1 coming from our work in this section are better, whence its importance.

4. Second approach: Euler–Maclaurin and a standard mean value theorem

Rather than working directly with ζ as in Section 3, we work with its L^2 mean through a finite truncation, as given in Lemma 2.10. We will thus obtain not only bounds of the integral of $t \mapsto \left|\frac{\zeta(\sigma+it)}{\sigma+it}\right|^2$ on the tails but also mean square asymptotic expressions for ζ.

4.1. General bounds. We start by providing bounds for the integral of $|\zeta(s)|^2$ with general extrema. We follow two similar paths, according to whether in Lemma 2.10 the index X of the sum is chosen to be a constant (as in Proposition 4.1) or dependent on t (as in Proposition 4.2): the two choices are advantageous in different situations, as observed in the next subsections.

Proposition 4.1. *Let $\frac{1}{2} \leq \sigma \leq 1$ and T_1, T_2 be real numbers such that $1 \leq T_1 \leq T_2$. Then, for any $\rho > 0$, $\int_{T_1}^{T_2} |\zeta(\sigma+it)|^2 \mathrm{d}t$ is at most*

$$(4.1)\quad (1+\rho)\left(\left(T_2 - T_1 + \frac{E}{2}\right) f_{1,1}^{+}(\sigma, T_2) + E f_{1,2}^{+}(\sigma, T_2)\right) + \left(1+\frac{1}{\rho}\right)\left(T_2^{2-2\sigma}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + \frac{D^2(T_2-T_1)}{T_2^{2\sigma}} + 2DT_2^{1-2\sigma}\log\left(\frac{T_2}{T_1}\right)\right)$$

where

$$f_{1,1}^{+}(\sigma,T) = \begin{cases} \log T + \gamma + \frac{1}{2T} & \text{if } \sigma = \frac{1}{2}, \\ \zeta(2\sigma) - \frac{1}{(2\sigma-1)T^{2\sigma-1}} + \frac{1}{T^{2\sigma}} & \text{if } \frac{1}{2} < \sigma \leq 1, \end{cases}$$

$$f_{1,2}^{+}(\sigma,T) = \begin{cases} \frac{T^{2-2\sigma}}{2(1-\sigma)} + \frac{1}{2} & \text{if } \frac{1}{2} \leq \sigma < 1, \\ \log T + \gamma + \frac{1}{2T} & \text{if } \sigma = 1, \end{cases}$$

and the constants D and E are as in Lemma 2.10, with $C = \lfloor T_2 \rfloor$, and as in Proposition 2.11, respectively. Moreover, for any $-1 < \rho < 0$, $\int_{T_1}^{T_2} |\zeta(\sigma+it)|^2 \mathrm{d}t$ is bounded from below by the expression in (4.1) *where $f_{1,1}^{+}, f_{1,2}^{+}$ are replaced respectively by*

$$f_{1,1}^{-}(\sigma,T) = \begin{cases} \log T + \gamma - \frac{c}{T} & \text{if } \sigma = \frac{1}{2}, \\ \zeta(2\sigma) - \frac{1}{(2\sigma-1)T^{2\sigma-1}} - \frac{1}{2T^{2\sigma}} & \text{if } \frac{1}{2} < \sigma \leq 1, \end{cases}$$

$$f_{1,2}^{-}(\sigma,T) = \begin{cases} \frac{T^{2-2\sigma}}{2(1-\sigma)} + \zeta(2\sigma-1) - \frac{1}{2T^{2\sigma-1}} & \text{if } \frac{1}{2} \leq \sigma < 1, \\ \log T + \gamma - \frac{c}{T} & \text{if } \sigma = 1, \end{cases}$$

where c is as in Lemma 2.8.

Proof. Let $1 \le T_1 \le T_2$. By Lemma 2.10, for any $X \ge T_2$ we have

$$\begin{aligned} &\int_{T_1}^{T_2} |\zeta(\sigma+it)|^2 \mathrm{d}t \le \int_{T_1}^{T_2} (|Z(t)|+|R(t)|)^2 \mathrm{d}t, \\ &Z(t) = \sum_{n\le X} \frac{1}{n^s}, \qquad R(t) = \left|\frac{X^{1-s}}{s-1}\right| + \frac{D}{X^\sigma}, \end{aligned} \tag{4.2}$$

where $s=\sigma+it$. We also obtain a lower bound for the expression above by writing $|Z(t)|-|R(t)| \le |\zeta(\sigma+it)|$. Hence, by Lemma 2.13, for any $\rho>0$,

$$\int_{T_1}^{T_2} (|Z(t)|+|R(t)|)^2 \mathrm{d}t \le (1+\rho)\int_{T_1}^{T_2} |Z(t)|^2 \mathrm{d}t + \left(1+\frac{1}{\rho}\right)\int_{T_1}^{T_2} |R(t)|^2 \mathrm{d}t,$$

and for any $-1<\rho<0$,

$$\int_{T_1}^{T_2} (|Z(t)|-|R(t)|)^2 \mathrm{d}t \ge (1+\rho)\int_{T_1}^{T_2} |Z(t)|^2 \mathrm{d}t + \left(1+\frac{1}{\rho}\right)\int_{T_1}^{T_2} |R(t)|^2 \mathrm{d}t.$$

Applying Proposition 2.11 with $T=T_2-T_1$ and $a_n = \frac{1}{n^{\sigma+iT_1}}$, we see that

$$\int_{T_1}^{T_2} |Z(t)|^2 \mathrm{d}t = \left(T_2-T_1+\frac{E}{2}\right)\sum_{n\le X}|a_n|^2 + O^*\left(E\sum_{n\le X} n|a_n|^2\right), \tag{4.3}$$

If $\sigma=\frac12$, we use Lemma 2.8 for the first term and $\sum_{n\le X} 1 = \lfloor X \rfloor \le X$ for the second. If $\frac12<\sigma<1$ we use Lemma 2.9 for both terms and the inequality $\zeta(2\sigma-1)+\frac{1}{X^{2\sigma-1}} < \zeta(0)+1=\frac12$. If $\sigma=1$ we use Lemma 2.9 for the first and Lemma 2.8 for the second. This analysis gives the following upper bounds for $\int_{T_1}^{T_2}|Z(t)|^2$:

$$\left(T_2-T_1+\frac{E}{2}\right)\left(\log X+\gamma+\frac{1}{2X}\right)+EX$$

if $\sigma=\frac12$,

$$\left(T_2-T_1+\frac{E}{2}\right)\left(\zeta(2\sigma)-\frac{1}{(2\sigma-1)X^{2\sigma-1}}+\frac{1}{X^{2\sigma}}\right)+\frac{EX^{2-2\sigma}}{2(1-\sigma)}+\frac{E}{2}$$

if $\frac12<\sigma<1$, and

$$\left(T_2-T_1+\frac{E}{2}\right)\left(\zeta(2)-\frac{1}{X}+\frac{1}{X^2}\right)+E\left(\log X+\gamma+\frac{1}{2X}\right)$$

if $\sigma=1$. Analogous lower bounds can be deduced respectively, using the same lemmas.

As for the second term in (4.2),

$$\int_{T_1}^{T_2} |R(t)|^2 \mathrm{d}t = \int_{T_1}^{T_2} \left|\frac{X^{1-s}}{s-1}\right|^2 + \frac{D^2}{X^{2\sigma}} + \frac{2D}{X^\sigma}\left|\frac{X^{1-s}}{s-1}\right| \mathrm{d}t. \tag{4.4}$$

Thanks to our condition $\rho > -1$ for the lower bound, and as we want non-trivial lower bounds, with $R(t)$ being smaller in magnitude than $Z(t)$, it suffices to have only an upper bound for (4.4). Hence, in order to bound the expression on the above right side, we observe that

$$\int_{T_1}^{T_2} \left| \frac{X^{1-s}}{s-1} \right|^2 \mathrm{d}t \leq X^{2-2\sigma} \int_{T_1}^{T_2} \frac{\mathrm{d}t}{t^2} = X^{2-2\sigma} \left(\frac{1}{T_1} - \frac{1}{T_2} \right).$$

For the second term we simply have $\int_{T_1}^{T_2} D^2 X^{-2\sigma} \mathrm{d}t = (T_2 - T_1) D^2 X^{-2\sigma}$, while the third one is bounded as

$$\int_{T_1}^{T_2} \frac{2D}{X^\sigma} \left| \frac{X^{1-s}}{s-1} \right| \mathrm{d}t \leq 2DX^{1-2\sigma} \int_{T_1}^{T_2} \frac{\mathrm{d}t}{t} = 2DX^{1-2\sigma} \log\left(\frac{T_2}{T_1} \right).$$

We obtain then

$$\int_{T_1}^{T_2} |R(t)|^2 \mathrm{d}t \leq X^{2-2\sigma} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) + \frac{D^2(T_2 - T_1)}{X^{2\sigma}} + 2DX^{1-2\sigma} \log\left(\frac{T_2}{T_1} \right).$$

Putting everything together, and imposing $X = T_2$ in order to minimize the various terms that arise ($X < T_2$ is not possible, by the conditions in Lemma 2.10), we obtain the result in the statement. $\square$

Proposition 4.2. *Let $\frac{1}{2} \leq \sigma \leq 1$ and T_1, T_2 be real numbers such that $1 \leq T_1 \leq T_2$. Then, for any $\rho > 0$, $\int_{T_1}^{T_2} |\zeta(\sigma + it)|^2 dt$ is at most*

$$(1+\rho)\left(f_{2,1}^{+}(\sigma, T_1, T_2) + f_{2,2}^{+}(\sigma, T_2) \right) + \left(1 + \frac{1}{\rho} \right) f_{2,3}^{+}(\sigma, T_1, T_2), \tag{4.5}$$

where

$$f_{2,1}^{+}(\sigma, T_1, T_2) = \begin{cases} T_2 \log T_2 - T_1 \log T_1 - (1-\gamma)(T_2 - T_1) + \frac{1}{2} \log \frac{T_2}{T_1} & \text{if } \sigma = \frac{1}{2}, \\ \zeta(2\sigma)(T_2 - T_1) & \text{if } \sigma > \frac{1}{2}, \end{cases}$$

$$f_{2,2}^{+}(\sigma, T_2) = \begin{cases} 2T_2 \log T_2 + \left(2\gamma + \frac{E}{2} + 16 \right) T_2 + \left(\frac{E}{4} - 1 \right) \log T_2 \\ \quad + 1 + \left(\frac{E}{4} - 1 \right) \gamma + \left(\frac{E}{4} - 1 \right) \frac{1}{2T_2} & \text{if } \sigma = \frac{1}{2}, \\ \frac{2T_2^{2-2\sigma} \log T_2}{1-\sigma} + \left(\frac{E}{2} + 2\gamma + \frac{4}{1-\sigma} \right) \frac{T_2^{2-2\sigma}}{1-\sigma} + \left(\frac{E}{4} - 1 \right) \zeta(2\sigma) \\ \quad + \left(\frac{1}{1-\sigma} - \frac{\frac{E}{4} - 1}{2\sigma - 1} \right) \frac{1}{T_2^{2\sigma-1}} + \left(\frac{E}{4} - 1 \right) \frac{1}{2T_2^{2\sigma}} & \text{if } \frac{1}{2} < \sigma < 1, \\ 3 \log^2 T_2 + \left(6\gamma + \frac{E}{2} \right) \log T_2 + 3\gamma^2 + \frac{E}{2}\gamma \\ \quad + \left(\frac{E}{4} - 1 \right) \zeta(2) + \frac{3 \log T_2}{T_2} + \frac{3\gamma + \frac{E}{4}}{T_2} + \frac{3}{4T_2^2} & \text{if } \sigma = 1, \end{cases}$$

$$f^+_{2,3}(\sigma,T_1,T_2)=\begin{cases}(1+D)^2\log\left(\frac{T_2}{T_1}\right) & \text{if } \sigma=\frac{1}{2},\\ \frac{(1+D)^2}{2\sigma-1} & \text{if } \frac{1}{2}<\sigma\le 1,\end{cases}$$

and the constants D and E are as in Lemma 2.10 with $C=\lfloor T_2\rfloor$ and as in Proposition 2.11, respectively. Moreover, for any $-1<\rho<0$, $\int_{T_1}^{T_2}|\zeta(\sigma+it)|^2\mathrm{d}t$ is bounded from below by the expression in (4.5) *where $f^+_{2,1}$, $f^+_{2,2}$, $f^+_{2,3}$ are replaced respectively by*

$$f^-_{2,1}(\sigma,T_1,T_2)=\begin{cases}T_2\log T_2-T_1\log T_1-(1-\gamma)(T_2-T_1)-c\log\frac{T_2}{T_1} & \text{if } \sigma=\frac{1}{2},\\ \zeta(2\sigma)(T_2-T_1)-\frac{T_2^{2-2\sigma}-T_1^{2-2\sigma}}{2(1-\sigma)(2\sigma-1)}-\frac{T_2^{1-2\sigma}-T_1^{1-2\sigma}}{2(2\sigma-1)} & \text{if } \sigma>\frac{1}{2},\end{cases}$$

$$f^-_{2,2}(\sigma,T_2)=-f^+_{2,2}(\sigma,T_2),\qquad f^-_{2,3}(\sigma,T_1,T_2)=f^+_{2,3}(\sigma,T_1,T_2),$$

where c is as in Lemma 2.8.

Proof. We start with the bound in Lemma 2.10. For $s=\sigma+it$ and $X=t$, by the triangle inequality we get

$$\begin{aligned}(4.6)\quad \int_{T_1}^{T_2}|\zeta(\sigma+it)|^2\mathrm{d}t&=\int_{T_1}^{T_2}\left|\sum_{n\le t}\frac{1}{n^{\sigma+it}}\right|^2\mathrm{d}t\\ &+O^*\Bigg(2\int_{T_1}^{T_2}\left|\sum_{n\le t}\frac{1}{n^{\sigma+it}}\right|\left|\frac{t^{1-s}}{s-1}\right|\mathrm{d}t+2D\int_{T_1}^{T_2}\left|\sum_{n\le t}\frac{1}{n^{\sigma+it}}\right|t^{-\sigma}\mathrm{d}t\\ &+\int_{T_1}^{T_2}\left|\frac{t^{1-s}}{s-1}\right|^2\mathrm{d}t+2D\int_{T_1}^{T_2}\left|\frac{t^{1-s}}{s-1}\right|t^{-\sigma}\mathrm{d}t+D^2\int_{T_1}^{T_2}t^{-2\sigma}\mathrm{d}t\Bigg).\end{aligned}$$

The second and third term in (4.6) can be treated using the Cauchy–Schwarz inequality and reduced to the other integrals in the expression. Observe that the integrands $\left|\frac{t^{1-s}}{s-1}\right|t^{-\sigma}$, $\left|\frac{t^{1-s}}{s-1}\right|^2$ are both bounded from above by $t^{-2\sigma}$, so that all of their integrals are bounded by $\frac{1}{2\sigma-1}$ if $\frac{1}{2}<\sigma\le 1$ and by $\log(\frac{T_2}{T_1})$ if $\sigma=\frac{1}{2}$. Using then Lemma 2.13 we get

$$\begin{aligned}(4.7)\quad &\int_{T_1}^{T_2}|\zeta(\sigma+it)|^2\mathrm{d}t\\ &\qquad\le(1+\rho)\int_{T_1}^{T_2}\left|\sum_{n\le t}\frac{1}{n^{\sigma+it}}\right|^2\mathrm{d}t+\left(1+\frac{1}{\rho}\right)f^+_{2,3}(\sigma,T_1,T_2),\end{aligned}$$

and an analogous lower bound for $-1<\rho<0$.

We want now to estimate the first term in (4.7), namely we want bounds for the integral $\int_{T_1}^{T_2}\left|\sum_{n\leq t} a_n e^{i\lambda_n t}\right|^2 \mathrm{d}t$, where in our case $a_n = \frac{1}{n^\sigma} \in \mathbb{R}^+$ and $\lambda_n = -\log n$. First, note that

$$(4.8) \qquad \int_{T_1}^{T_2}\left|\sum_{n\leq t} a_n e^{i\lambda_n t}\right|^2 \mathrm{d}t = \int_{T_1}^{T_2}\sum_{n\leq t} a_n^2 \mathrm{d}t + \int_{T_1}^{T_2}\sum_{\substack{l,r\leq t\\ l\neq r}} a_l a_r e^{i(\lambda_l-\lambda_r)t}\mathrm{d}t.$$

If $\frac{1}{2} < \sigma \leq 1$, the first integral in (4.8) is bounded by Lemma 2.9 as

$$\int_{T_1}^{T_2}\left(\zeta(2\sigma) - \frac{t^{1-2\sigma}}{2\sigma-1} - \frac{1}{2t^{2\sigma}}\right)\mathrm{d}t \\ \leq \int_{T_1}^{T_2}\sum_{n\leq t} a_n^2 dt \leq \int_{T_1}^{T_2}\left(\zeta(2\sigma) - \frac{t^{1-2\sigma}}{2\sigma-1} + \frac{1}{t^{2\sigma}}\right)\mathrm{d}t,$$

so that

$$\zeta(2\sigma)(T_2-T_1) - \frac{T_2^{2-2\sigma}-T_1^{2-2\sigma}}{2(1-\sigma)(2\sigma-1)} - \frac{T_2^{1-2\sigma}-T_1^{1-2\sigma}}{2(2\sigma-1)} \\ \leq \int_{T_1}^{T_2}\sum_{n\leq t} a_n^2 \mathrm{d}t \leq \zeta(2\sigma)(T_2-T_1),$$

where we use that $-\frac{t^{1-2\sigma}}{2\sigma-1} + t^{-2\sigma} \leq 0$ (under the conditions for σ, t), and we can extract an analogous lower bound.

If $\sigma = \frac{1}{2}$, the first integral is bounded from above as

$$(4.9) \qquad \int_{T_1}^{T_2}\sum_{n\leq t} a_n^2 \mathrm{d}t \leq \int_{T_1}^{T_2}\left(\log t + \gamma + \frac{1}{2t}\right)\mathrm{d}t \\ = T_2\log T_2 - T_1\log T_1 - (1-\gamma)(T_2-T_1) + \frac{1}{2}(\log T_2 - \log T_1),$$

by Lemma 2.8, from which we can derive an analogous lower bound.

As for the second integral in (4.8), consider first T_1, T_2 to be integers for simplicity: we make use of the fact that a sum for $l, r \leq t$ is the same as a sum for $l,r\leq\lfloor t\rfloor$ and get

$$\begin{aligned}(4.10)\quad \int_{T_1}^{T_2}\sum_{\substack{l,r\leq t\\ l\neq r}} a_l a_r e^{i(\lambda_l-\lambda_r)t}\mathrm{d}t &= \sum_{j=T_1}^{T_2-1}\sum_{\substack{l,r\leq j\\ l\neq r}} a_l a_r\frac{e^{i(\lambda_l-\lambda_r)(j+1)}-e^{i(\lambda_l-\lambda_r)j}}{i(\lambda_l-\lambda_r)}\\ &= \sum_{\substack{l,r\leq T_2-1\\ l\neq r}}\sum_{j=\max\{T_1,l,r\}}^{T_2-1} a_l a_r\frac{e^{i(\lambda_l-\lambda_r)(j+1)}-e^{i(\lambda_l-\lambda_r)j}}{i(\lambda_l-\lambda_r)}\\ &= \sum_{\substack{l,r\leq T_2-1\\ l\neq r}}\frac{a_l a_r}{\lambda_l-\lambda_r}\frac{e^{i(\lambda_l-\lambda_r)T_2}-e^{i(\lambda_l-\lambda_r)\max\{T_1,l,r\}}}{i}.\end{aligned}$$

For T_1,T_2 general, we have to consider two additional integrals $\int_{T_1}^{\lceil T_1\rceil},\int_{\lfloor T_2\rfloor}^{T_2}$; we obtain however the same bound as in (4.10), with the summation going up to $\lfloor T_2\rfloor$ and with T_1 replaced by $\lfloor T_1\rfloor$.

We can divide the last sum in (4.10) into two sums, one for each of the summands in the numerator of the second fraction. For the first sum we can reason as in Proposition 2.11, using [31] and obtaining

$$(4.11)\qquad \left|\sum_{\substack{l,r\leq T_2-1\\ l\neq r}}\frac{a_l a_r e^{i(\lambda_l-\lambda_r)T_2}}{i(\lambda_l-\lambda_r)}\right|\leq\frac{E}{2}\sum_{n\leq\lfloor T_2\rfloor}\frac{a_n^2}{\min_{n'\neq n}|\lambda_n-\lambda_{n'}|}.$$

As for the second sum, we can bound the summand in absolute value by $\frac{a_l a_r}{|\lambda_l-\lambda_r|}$; then we use classical arguments (see [19, (3.5)–(3.6)]), and $\sum_{\substack{l,r\leq\lfloor T_2\rfloor\\ l\neq r}}\frac{a_l a_r}{|\lambda_l-\lambda_r|}$ is at most

$$(4.12)\quad \sum_{\substack{l,r\leq T_2\\ l\neq r}}\frac{a_l a_r}{|\lambda_l-\lambda_r|}\leq\left(\sum_{r\leq T_2}\frac{1}{r^\sigma}\right)^2-\sum_{r\leq T_2}\frac{1}{r^{2\sigma}}+2\left(\sum_{r\leq T_2}\frac{1}{r^{2\sigma-1}}\right)\left(\sum_{r\leq T_2}\frac{1}{r}\right).$$

Upon putting (4.11) and (4.12) together, we resort to Lemmas 2.8 and 2.9 along with the simplifications $\zeta(\alpha)+\frac{1}{T_2^\alpha}<\frac{1}{2T_2^\alpha}$, $\sum_{r\leq T_2}\frac{1}{r^\alpha}\leq\frac{2T_2^{1-\alpha}}{1-\alpha}$ for $0<\alpha<1$ and $T_2\geq 1$, and the bound $\sum_{r\leq T_2}\frac{1}{r^2}\leq\zeta(2)$. Subsequently, we obtain $f_{2,2}^{\pm}(\sigma,T_2)$ as in the statement. $\square$

4.2. Mean value estimates of $\zeta(s)$ for $\Re(s) \in \left[\frac{1}{2}, 1\right]$.

Theorem 4.3. *Let $T \geq T_0 = 4$. Then*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \leq T\log T + 2.0\cdot T\sqrt{\log T} + 23.05779\cdot T$$
$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \geq T\log T - 2.0\cdot T\sqrt{\log T} - 0.93213\cdot T.$$

Moreover, for $\frac{1}{2} < \sigma < 1$,

$$\int_1^T |\zeta(\sigma+it)|^2 \mathrm{d}t \leq \zeta(2\sigma)T + C^+(\sigma)\cdot\max\{T^{2-2\sigma}\log T, \sqrt{T}\}$$
$$\int_1^T |\zeta(\sigma+it)|^2 \mathrm{d}t \geq \zeta(2\sigma)T - C^-(\sigma)\cdot\max\{T^{2-2\sigma}\log T, \sqrt{T}\}$$

with

$$C^+(\sigma) = \frac{7.20238}{(1-\sigma)^2} + \frac{14.96202}{1-\sigma} + \frac{5.45623}{2\sigma-1} + 0.08136$$
$$C^-(\sigma) = \frac{4.0}{(1-\sigma)^2} + \frac{0.70015}{(1-\sigma)(2\sigma-1)} + \frac{8.3095}{1-\sigma} + \frac{4.12046}{2\sigma-1} + 0.69962.$$

Finally,

$$\int_1^T |\zeta(1+it)|^2 \mathrm{d}t \leq \frac{\pi^2}{6}T + \pi\sqrt{\frac{2}{3}}\sqrt{T} + 22.45519\cdot\log T$$
$$\int_1^T |\zeta(1+it)|^2 \mathrm{d}t \geq \frac{\pi^2}{6}T - \pi\sqrt{\frac{2}{3}}\sqrt{T} + 0.39474\cdot\log T.$$

Proof. We substitute $T_1 = 1$ inside either Proposition 4.1 or Proposition 4.2, according to which one gives us the best result. Our choice of ρ for the upper bound will be the square root of the ratio between the leading terms of the expressions multiplying $1+\frac{1}{\rho}$ and $1+\rho$ respectively, the same choice with a negative sign corresponding to the lower bound. Such choice will be very close to the optimal one highlighted by Lemma 2.13, but simpler and easier to handle.

For $\frac{1}{2} < \sigma < 1$, Proposition 4.2 is the better alternative, as ρ will be qualitatively smaller than in Proposition 4.1 and the second order term will be of smaller order (the error term arising in the alternative case being of order $T^{\frac{3}{2}-\sigma}$). We set $\rho = \frac{1+D}{\sqrt{(2\sigma-1)\zeta(2\sigma)T}}$ (where D is as in the proof of Lemma 2.10, choosing $C = \lfloor T_0 \rfloor$) and by imposing $T \geq T_0$ we merge all lower order terms, observing that the bound on $\zeta(2\sigma)$ given in Lemma 2.12 is being used; the condition $T_0 = 4$ is employed to make sure that we actually get $-\rho > -1$, in order to apply Proposition 4.2 in the lower bound correctly.

When $\sigma \in \{\frac{1}{2}, 1\}$ the better alternative is Proposition 4.1: in the first case, the main terms obtained through Propositions 4.1 and 4.2 are qualitatively the same but worse constants arise from Proposition 4.2, while in the second case the same situation occurs for the error terms. For $\sigma = \frac{1}{2}$ we set $\rho = \frac{1}{\sqrt{\log T}}$ and for $\sigma = 1$ we set $\rho = \frac{1}{\sqrt{\zeta(2)T}}$, and then impose $T \geq T_0$ to simplify the second order terms. $\square$

4.3. Extension of asymptotic formulas. We prove here a proposition that allows us to extend the asymptotic formulas in the previous subsection to the case $\sigma < \frac{1}{2}$, via the functional equation (2.12).

Proposition 4.4. *Let $\mathbb{I} = [a_0, a_1]$ be an interval of the real line ($a_i = \pm\infty$ is allowed). Let $Z : \mathbb{I} \to \mathbb{R}_{\geq 0}$ be an integrable function such that, for every $T_1, T_2 \in \mathbb{I}$ with $T_1 \leq T_2$,*

$$F(T_1,T_2) - r^-(T_1,T_2) \ \leq \int_{T_1}^{T_2} Z(t)\mathrm{d}t \leq \ F(T_1,T_2) + r^+(T_1,T_2), \tag{4.13}$$

where F, r^+ and r^- are non-negative real functions, such that F is differentiable and, for every pair $T_1, T_2 \in \mathbb{I}$, $F(T_2,T_2) = F(T_1,T_1) = 0$.

Let $f : \mathbb{I} \to \mathbb{R}_{\geq 0}$ be a differentiable function with f' integrable satisfying either $f' \geq 0$ or $f' \leq 0$ and such that either $f(a_0) = 0$ or $f(a_1) = 0$. We have the following cases.

(i) *If $f(a_0) = 0$ (so $f' \geq 0$) and $\int_{T_1}^{T_2}\int_{a_0}^{T_2} |f'(u)|Z(t)\mathrm{d}u\mathrm{d}t$ converges for every $T_1, T_2 \in \mathbb{I}$ with $T_1 \leq T_2$, then*

$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t \leq \int_{a_0}^{T_2} \left(-f(u)\frac{\partial F(u,T_2)}{\partial u} + f'(u)r^+(u,T_2)\right)\mathrm{d}u,$$
$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t \geq \int_{a_0}^{T_2} \left(-f(u)\frac{\partial F(u,T_2)}{\partial u} - f'(u)r^-(u,T_2)\right)\mathrm{d}u.$$

(ii) *If $f(a_1) = 0$ (so $f' \leq 0$) and $\int_{T_1}^{T_2}\int_{T_1}^{a_1} |f'(u)|Z(t)\mathrm{d}u\mathrm{d}t$ converges for every $T_1, T_2 \in \mathbb{I}$ with $T_1 \leq T_2$ and $\lim_{u\to a_1} f(u)F(T_1,u) = 0$, then*

$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t \leq \int_{T_1}^{a_1} \left(f(u)\frac{\partial F(T_1,u)}{\partial u} - f'(u)r^+(T_1,u)\right)\mathrm{d}u,$$
$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t \geq \int_{T_1}^{a_1} \left(f(u)\frac{\partial F(T_1,u)}{\partial u} + f'(u)r^-(T_1,u)\right)\mathrm{d}u.$$

Proof. As f' is integrable, so is $|f|$. Suppose first that $f(a_0) = 0$; by the Fundamental Theorem of Calculus, for every $t \in [T_1, T_2]$, $f(t) = \int_{a_0}^{t} f'(u)du$. Then

$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t = \int_{T_1}^{T_2}\int_{a_0}^{t} f'(u)Z(t)\mathrm{d}u\mathrm{d}t = \int_{T_1}^{T_2}\int_{T_1}^{T_2} \mathbb{1}_{[a_0,t]}(u)f'(u)Z(t)\mathrm{d}u\mathrm{d}t.$$

Observe that, under the above conditions, $\mathbb{1}_{[a_0,t]}(u) = \mathbb{1}_{[u,T_2]}(t)\mathbb{1}_{[a_0,T_2]}(u)$. Since the double integral $\int_{T_1}^{T_2}\int_{a_0}^{T_2} |f'(x)|Z(t)\mathrm{d}x\mathrm{d}t$ converges, by Fubini's Theorem, we can exchange the order of integration in the above equation and obtain

$$\int_{T_1}^{T_2} f(t)Z(t)\mathrm{d}t = \int_{a_0}^{T_2} f'(u)\int_{u}^{T_2} Z(t)\mathrm{d}t\mathrm{d}u \le \int_{a_0}^{T_2} f'(u)(F(u,T_2)+r^+(u,T_2))\mathrm{d}u$$
$$= -\int_{a_0}^{T_2} f(u)\frac{\partial F(u,T_2)}{\partial u}\mathrm{d}u + \int_{a_0}^{T_2} f'(u)r^+(u,T_2)\mathrm{d}u,$$

where we have used integration by parts in the last step. We also derive the lower bound

$$-\int_{a_0}^{T_2} f(u)\frac{\partial F(u,T_2)}{\partial u}\mathrm{d}u - \int_{a_0}^{T_2} f'(u)r^-(u,T_2)\mathrm{d}u.$$

Case (ii) is obtained by proceeding in a similar manner as above, keeping in mind that $f(t) = -\int_t^{a_1} f'(u)\mathrm{d}u$ for $t \in [T_1, T_2]$ and $\mathbb{1}_{[t,a_1]}(u) = \mathbb{1}_{[T_1,u]}(t)\mathbb{1}_{[T_1,a_1]}(u)$, and then using Fubini's Theorem and integration by parts. Here, the condition $\lim_{u\to a_1} f(u)F(T_1,u) = 0$ is employed so as to make sure that if $a_1 = \infty$, integration by parts is well-performed. $\square$

The sign condition on f' in Proposition 4.4 is not necessary; under the other conditions, one can derive an analogous result by writing $f' = f'_+ - f'_-$, where $f_\pm = \max\{\pm f', 0\}$. In that case, the point $a \in \mathbb{I}$ such that $f(a) = 0$ need not be an extremum of $\mathbb{I}$, and if $T_1 < a < T_2$ one can derive bounds by applying case (i) to $\int_a^{T_2} f(t)Z(t)\mathrm{d}t$ and case (ii) to $\int_{T_1}^{a} f(t)Z(t)\mathrm{d}t$.

4.4. Mean value estimates of $\boldsymbol{\zeta(s)}$ for $\boldsymbol{\Re(s) \in \left[0, \frac{1}{2}\right)}$. Thanks to Proposition 4.4, we are going to give asymptotic formulas for the integral of $|\zeta(\sigma+it)|^2$ in the case $0 \le \sigma < \frac{1}{2}$.

Theorem 4.5. *If* $0 < \sigma < \frac{1}{2}$ *and* $T \ge T_0 = 4$, *then*

$$\int_1^T |\zeta(\sigma+it)|^2\mathrm{d}t \le \frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)}T^{2-2\sigma} + L^+(\sigma)T,$$
$$\int_1^T |\zeta(\sigma+it)|^2\mathrm{d}t \ge \frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)}T^{2-2\sigma} - L^-(\sigma)T,$$

where

$$L^+(\sigma) = \frac{1}{(2\pi)^{1-2\sigma}}\left(\frac{27.86621}{\sigma^2} + \frac{74.41842}{\sigma} + \frac{42.18032}{1-2\sigma} + 35.54594\right),$$
$$L^-(\sigma) = \frac{1}{(2\pi)^{1-2\sigma}}\left(\frac{27.86621}{\sigma^2} + \frac{66.15347}{\sigma} + \frac{40.4816}{1-2\sigma} + 15.45198\right).$$

If $\sigma = 0$ and $T \geq T_0 = 4$, then

$$\int_1^T |\zeta(it)|^2 \mathrm{d}t \leq \frac{\pi}{24} T^2 + 4.02061 \cdot T \log T + 7.41137 \cdot T,$$
$$\int_1^T |\zeta(it)|^2 \mathrm{d}t \geq \frac{\pi}{24} T^2 - 0.0936 \cdot T \log T - 7.41137 \cdot T.$$

Proof. Consider σ such that $0 \leq \sigma < \frac{1}{2}$. By using the functional equation (2.12) of ζ and knowing that $|\zeta(s)| = |\zeta(\overline{s})|, |\Gamma(s)| = |\Gamma(\overline{s})|$, we readily see that

$$\begin{aligned}(4.14)\quad &\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t \\ &\qquad = \frac{1}{(2\pi)^{2-2\sigma}} \int_1^T \left| 2 \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - \sigma + it) \zeta(1 - \sigma + it) \right|^2 \mathrm{d}t\end{aligned}$$

Let $s = \sigma + it$ with $t \geq 1$. For every complex number z we have the identity $|\sin(z)|^2 = \cosh^2(\Im(z)) - \cos^2(\Re(z))$ (combine 4.5.7 and 4.5.54 in [1]). Hence

$$(4.15)\quad \begin{aligned}\left|\sin\left(\frac{\pi s}{2}\right)\right|^2 &= \frac{e^{\pi t}}{4}\left(1 + \frac{1}{e^{\pi t}}\left(2 + \frac{1}{e^{\pi t}} - 4\cos^2\left(\frac{\pi \sigma}{2}\right)\right)\right) \\ &= \frac{e^{\pi t}}{4}\left(1 + O^*\left(\frac{2}{e^{\pi t}}\right)\right),\end{aligned}$$

since $\frac{1}{2} < \cos^2\left(\frac{\pi\sigma}{2}\right) \leq 1$ for the choice of σ. Moreover, using Corollary 2.7, $|\Gamma(1 - \sigma + it)| = \sqrt{2\pi} t^{\frac{1}{2} - \sigma} e^{-\frac{\pi t}{2}} \exp\left(O^*\left(\frac{G_{1-\sigma}}{t}\right)\right)$, where $G_{1-\sigma} = \frac{(1-\sigma)^3}{3} + \frac{(1-\sigma)^2}{2}\left(\frac{1}{2} - \sigma\right) + \frac{1}{6} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4}$. We then verify that $\exp\left(O^*\left(\frac{G_{1-\sigma}}{t}\right)\right) = 1 + O^*\left(\frac{K_1}{t}\right)$, where $K_1 = e^{\frac{3}{4}} - 1$, as $t(e^{\frac{3}{4t}} - 1)$ is decreasing for $t \geq 1$. This observation and (4.15) allow us to derive in (4.14) that

$$\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t = \frac{1}{(2\pi)^{1-2\sigma}} \int_1^T t^{1-2\sigma} |\zeta(1 - \sigma + it)|^2 \left(1 + O^*\left(\frac{K_2}{t}\right)\right) \mathrm{d}t,$$

where K_2 is defined as below

$$\left(1 + \frac{2}{e^{\pi t}}\right)\left(1 + \frac{K_1}{t}\right)^2 \leq 1 + \left(2K_1 + K_1^2 + \frac{2}{e^{\pi}} + \frac{4K_1}{e^{\pi}} + \frac{2K_1^2}{e^{\pi}}\right)\frac{1}{t} = 1 + \frac{K_2}{t},$$

since $\frac{e^{\pi t}}{t}$ is increasing for $t \geq 1$; we could do better, since the worst cases of (4.15) and $G_{1-\sigma}$ happen at different σ, but the advantage would be negligible. We conclude that

$$\begin{aligned}(4.16)\quad \int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t &= \frac{1}{(2\pi)^{1-2\sigma}} \int_1^T t^{1-2\sigma} |\zeta(1 - \sigma + it)|^2 \mathrm{d}t \\ &\qquad + O^*\left(\frac{K_2}{(2\pi)^{1-2\sigma}} \int_1^T \frac{|\zeta(1 - \sigma + it)|^2}{t^{2\sigma}} \mathrm{d}t\right).\end{aligned}$$

To estimate (4.16), we could resort to Proposition 4.4, using the functions $t \mapsto t^{1-2\sigma}$, $t \mapsto t^{-2\sigma}$ and the bounds for $\int_1^T |\zeta(1-\sigma+it)|^2 \mathrm{d}t$ given in Theorem 4.3. This approach, while simpler, produces less accurate second order terms. One can do better by studying $\int_u^T |\zeta(1-\sigma+it)|^2 \mathrm{d}t$ for $1 \leq u \leq T$. We proceed as in Theorem 4.3, with the general bound of Proposition 4.1. Set $\rho = \frac{T^{\sigma-1/2}}{\sqrt{\zeta(2-2\sigma)u}}$: the dependence on u allows us to use Proposition 4.4 non-trivially, yielding sharper estimates, while ρ in Proposition 4.2 depends solely on T. Afterwards, we merge the second order terms according to either $u \geq 1$ or $u \leq T$, recalling Lemma 2.12 and $T \geq T_0$. The final bounds are

$$(4.17) \quad -r^-(u,T) \leq \int_u^T |\zeta(1-\sigma+it)|^2 \mathrm{d}t - \zeta(2-2\sigma)(T-u) \leq r^+(u,T),$$

where

$$(4.18) \quad r^{\pm}(u,T) = \begin{cases} 2\sqrt{\zeta(2-2\sigma)}\left(\frac{T^{\frac{1}{2}+\sigma}}{\sqrt{u}} + \frac{D\sqrt{u}}{T^{\frac{1}{2}-\sigma}} \log\left(\frac{T}{u}\right)\right) + N^{\pm}(\sigma)T^{2\sigma} & \text{if } \sigma > 0, \\ \pi\sqrt{\frac{2T}{3u}} + W^{\pm} \log T & \text{if } \sigma = 0, \end{cases}$$

and

$$N^+(\sigma) = \frac{8.26495}{\sigma} + \frac{4.69872}{1-2\sigma} + 13.23452, \qquad W^+ = 25.26219,$$
$$N^-(\sigma) = \frac{4.13248}{\sigma} + \frac{3.34936}{1-2\sigma} + 2.86508, \qquad W^- = 0.58811.$$

As remarked, the terms in u in the definition of $r^{\pm}(u,T)$ are those that would have otherwise given larger error terms if we had taken $r^{\pm}$ independent of u.

We further verify by (4.17) that the conditions of Proposition 4.4 are met with the increasing function $f(t) = t^{1-2\sigma} - 1$, $Z(t) = |\zeta(1-\sigma+it)|^2$ and $a_0 = 1$ (we cannot use $f(t) = t^{1-2\sigma}$ directly as (4.17) is only valid for $u \geq 1$). We split the integral as

$$\int_1^T t^{1-2\sigma}|\zeta(1-\sigma+it)|^2 \mathrm{d}t = \int_1^T (t^{1-2\sigma}-1)|\zeta(1-\sigma+it)|^2 \mathrm{d}t + \int_1^T |\zeta(1-\sigma+it)|^2 \mathrm{d}t,$$

and the second integral is already bounded by (4.17). For the first, we thus apply Proposition 4.4 (i) using the bound in (4.17) as

$$\begin{aligned}
&\int_1^T (t^{1-2\sigma}-1)|\zeta(1-\sigma+it)|^2 \mathrm{d}t \\
&\quad\leq \int_0^T u^{1-2\sigma}\zeta(2-2\sigma) \\
&\qquad + (1-2\sigma)\left(2\sqrt{\zeta(2-2\sigma)}T^{\frac{1}{2}+\sigma}u^{-\sigma-\frac{1}{2}} + N^+(\sigma)T^{2\sigma}u^{-2\sigma}\right)\mathrm{d}u \\
&\qquad + \int_1^T 2\sqrt{\zeta(2-2\sigma)}DT^{\frac{1}{2}-\sigma}u^{\frac{1}{2}}\log\left(\frac{T}{u}\right) - \zeta(2-2\sigma)\mathrm{d}u \\
&\quad= \frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + 4\sqrt{\zeta(2-2\sigma)}T + N^+(\sigma)T \\
&\qquad + (1-2\sigma)2\sqrt{\zeta(2-2\sigma)}D\frac{T^{1-\sigma}-T^{\sigma-\frac{1}{2}}}{\left(\frac{3}{2}-2\sigma\right)^2} - \zeta(2-2\sigma)(T-1).
\end{aligned}$$

We proceed similarly for the lower bound (a term $-\frac{\zeta(2-2\sigma)}{2-2\sigma}$ emerges in that case from the approximations) and for $\sigma=0$. Using also Lemma 2.12 and $\frac{1-2\sigma}{\left(\frac{3}{2}-2\sigma\right)^2} \leq \frac{1}{2}$, we obtain

$$\begin{aligned}
\int_1^T t^{1-2\sigma}|\zeta(1-\sigma+it)|^2\mathrm{d}t &\leq \frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + S^+(\sigma,T), \\
\int_1^T t^{1-2\sigma}|\zeta(1-\sigma+it)|^2\mathrm{d}t &\geq \frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} - S^-(\sigma,T)
\end{aligned}$$

where

$$S^+(\sigma,T) = \begin{cases} \left(\frac{16.5299}{\sigma} + \frac{17.20098}{1-2\sigma} + 32.73591\right)T & \text{if } 0<\sigma<\frac{1}{2}, \\ W^+T\log T + 2\pi\sqrt{\frac{2}{3}}T & \text{if } \sigma=0, \end{cases}$$

$$S^-(\sigma,T) = \begin{cases} \left(\frac{8.26495}{\sigma} + \frac{15.50227}{1-2\sigma} + 12.64195\right)T & \text{if } 0<\sigma<\frac{1}{2}, \\ W^-T\log T + 2\pi\sqrt{\frac{2}{3}}T & \text{if } \sigma=0. \end{cases}$$

Finally, for the error term of (4.16), the conditions of Proposition 4.4 are not met with $f(t)=t^{-2\sigma}$ and $0<\sigma<\frac{1}{2}$. Instead, we apply the weaker bound $t^{-2\sigma}<1$, sufficient to have an error term of order T, and use Theorem 4.3 with $1-\sigma$ instead of σ. □

4.5. Square mean of $\frac{\zeta(s)}{s}$ on tails: asymptotically sharp bounds. We will use the bounds for $\zeta(s)$ given in the previous sections and the machinery of Proposition 4.4 to retrieve upper bounds for $\frac{\zeta(s)}{s}$.

Theorem 4.6. *Let $T \geq T_0 = 4$; let L^+, N^+ be as in Theorem 4.5, and let D be defined as in Lemma 2.10 with $C = \lfloor T_0 \rfloor$. Then $\int_T^\infty \frac{|\zeta(s)|^2}{|s|^2} \mathrm{d}t$ is bounded from above by*

$$\begin{cases} \dfrac{\zeta(2-2\sigma)}{2\sigma(2\pi)^{1-2\sigma}} \cdot \dfrac{1}{T^{2\sigma}} + 2L^+(\sigma) \cdot \dfrac{1}{T} & \text{if } 0 < \sigma < \dfrac{1}{2}, \\ \dfrac{\log T}{T} + 4.0 \cdot \dfrac{\sqrt{\log T}}{T} + 49.81422 \cdot \dfrac{1}{T} & \text{if } \sigma = \dfrac{1}{2}, \\ \zeta(2\sigma) \cdot \dfrac{1}{T} + \left(2N^+(1-\sigma) + (D+4)\sqrt{\zeta(2\sigma)}\right) \cdot \dfrac{1}{T^{2\sigma}} & \text{if } \dfrac{1}{2} < \sigma < 1, \\ \dfrac{\pi^2}{6} \cdot \dfrac{1}{T} + 25.26219 \cdot \dfrac{\log T}{T^2} + 16.05123 \cdot \dfrac{1}{T^2} & \text{if } \sigma = 1. \end{cases}$$

Proof. We apply case (ii) of Proposition 4.4 by taking $T_1 = T \geq 1$, $T_2 = \infty$, $f(t) = \frac{1}{\sigma^2+t^2}$, and $a_1 = \infty$. Using the bounds $u^2 < \sigma^2 + u^2$ and $(-(\sigma^2 + u^2)^{-1})' < 2u^{-3}$, we get

$$\int_T^\infty \frac{|\zeta(s)|^2}{|s|^2} \mathrm{d}t < \int_T^\infty \left(\frac{1}{u^2} \frac{\partial F(T,u)}{\partial u} + \frac{2}{u^3} r^+(T,u) \right) \mathrm{d}u,$$

for appropriate choices of F and r^+, which are taken as follows.

For $0 < \sigma < \frac{1}{2}$, we use Theorem 4.5 and the observation that the integral of $|\zeta(\sigma+it)|^2$ in $[T,u]$ is bounded by the integral in $[1,u]$. For $\sigma = \frac{1}{2}$, we use Theorem 4.3 and the same observation. For $\frac{1}{2} < \sigma \leq 1$, we use the upper bound in (4.17) replacing σ by $1-\sigma$. □

When $\sigma = 0$, note by Proposition 4.4 that the main term of $\int_T^\infty \frac{|\zeta(s)|^2}{|s|^2} \mathrm{d}t$ is $\frac{\pi^2}{12} \int_T^\infty \frac{u}{\sigma^2+u^2} \mathrm{d}u = \frac{\pi^2}{12} \cdot \frac{1}{2} \log(\sigma^2+u^2)\Big|_T^\infty = \infty$, so that the integral is divergent.

5. Numerical considerations

In case (1) of Theorem 1.1, we only show the bound from Theorem 4.6, since it is always stronger than the one from Theorem 3.1. In case (3), we chose $T = 10^{40}$ because the threshold where the second bound is better than the first sits in $(10^{39}, 10^{40}]$.

In case (2), $T = 639$ is the lowest integer at which for some $\sigma \in \left(\frac{1}{2}, 1\right)$ the second bound is stronger than the first. In Table 5.1 we give thresholds T for all $\sigma \in \frac{1}{20}\mathbb{N} \cap \left(\frac{1}{2}, 1\right)$. We also present thresholds for $\sigma \in \frac{1}{100}\mathbb{N} \cap \left(\frac{1}{2}, \frac{2}{5}\right)$ for the bounds of Theorems 3.1 and 4.6: for the same σ, these sharper bounds yield a lower T than the ones from Theorem 1.1.

In case (4), $T = 2223$ is the lowest integer at which for some $\sigma \in \left(0, \frac{1}{2}\right)$ the second bound is stronger than the first. Table 5.2 gives thresholds between the bounds of Theorem 1.1 or between those in Theorems 3.1 and 4.6.

In the tables, the significant digits of the higher entries of T have been reduced for simplicity. To obtain the reported approximations, the threshold has been rounded up.

Lastly: the loss of precision in Theorems 1.1 and 1.2 with respect to Theorems 3.1, 4.6, 4.3 and 4.5 may be significant, especially for $\sigma \notin \{0, \frac{1}{2}, 1\}$. In Section 1, we favored simplicity in the statements, provided that they showed the correct asymptotics for the main terms and the correct order of the error terms for $T \to \infty$ and σ tending to $0, \frac{1}{2}, 1$. Readers wanting sharper bounds are advised to rely on the stronger estimates of Section 3 and Section 4.

TABLE 5.1. T for $\sigma > \frac{1}{2}$.

σ	Thm. 1.1	σ	Thms. 3.1-4.6
0.55	$\approx 6.82{\cdot}10^{17}$	0.51	$\approx 4.92{\cdot}10^{43}$
0.6	978617582	0.52	$\approx 2.81{\cdot}10^{19}$
0.65	1197629	0.53	$\approx 1.24{\cdot}10^{11}$
0.7	45124	0.54	3571099
0.75	6802	0.55	1869
0.8	2095	0.56	(< 200)
0.85	1004	0.57	(< 200)
0.9	678	0.58	(< 200)
0.95	694	0.59	298

TABLE 5.2. T for $\sigma < \frac{1}{2}$.

σ	Thm. 1.1	Thms. 3.1-4.6
0.05	2833	1049
0.1	2233	1149
0.15	2953	1961
0.2	5921	4886
0.25	20025	19970
0.3	157195	182209
0.35	6443047	7902814
0.4	$\approx 1.71{\cdot}10^{10}$	$\approx 1.61{\cdot}10^{10}$
0.45	$\approx 1.07{\cdot}10^{21}$	$\approx 1.45{\cdot}10^{20}$

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