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Conjecture A and μ-invariant for Selmer groups of supersingular elliptic curves

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Résumé. Soit p un nombre premier impair et soit E une courbe elliptique sur un corps de nombres F ayant bonne réduction en toute place au-dessus de p. Dans cet article de synthèse, nous donnons un aperçu de certains des résultats importants sur le groupe de Selmer fin et les groupes de Selmer signés dans les tours cyclotomiques aussi que sur les groupes de Selmer signés dans les $\mathbb{Z}_p^2$-extensions d'un corps quadratique imaginaire, où p est complètement décomposé. Nous discutons uniquement des aspects algébriques en utilisant des outils de la théorie d'Iwasawa. Nous donnons un survol de certains des résultats récents impliquant l'annulation de l'invariant μ sous l'hypothèse de la conjecture A. En outre, nous esquissons une analogie entre le groupe de Selmer classique dans le cas de bonne réduction ordinaire et le groupe de Selmer signé de Kobayashi dans le cas supersingulier. Nous mettons l'accent sur les propriétés des groupes de Selmer signés dans le cas où E a bonne réduction supersingulière, qui sont complètement analogues à celles des groupes de Selmer classiques quand E a bonne réduction ordinaire. Cet article ne contient pas de démonstrations, cependant pour le lecteur intéressé, nous donnons des références pour les résultats exposés dans le texte.

Abstract. Let p be an odd prime and let E be an elliptic curve defined over a number field F with good reduction at the primes above p. In this survey article, we give an overview of some of the important results proven for the fine Selmer group and the signed Selmer groups over cyclotomic towers as well as the signed Selmer groups over $\mathbb{Z}_p^2$-extensions of an imaginary quadratic field where p splits completely. We only discuss the algebraic aspects of these objects through Iwasawa theory. We also attempt to give some of the recent results implying the vanishing of the μ-invariant under the hypothesis of Conjecture A. Moreover, we draw an analogy between the classical Selmer group in the ordinary reduction case and that of the signed Selmer groups of Kobayashi in the supersingular reduction case. We highlight properties of signed Selmer groups, when E has good supersingular reduction, which are

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completely analogous to the classical Selmer group, when E has good ordinary reduction. In this survey paper we do not present any proofs, however, we have tried to give references of the discussed results for the interested reader.

Introduction

Classical Iwasawa theory began from the work of Iwasawa who explored the growth of ideal class groups in towers of number fields. Iwasawa theory for elliptic curves, which shares many of its fundamental ideas with classical Iwasawa theory, deals with the arithmetic of elliptic curves over various infinite extensions. Among the main objects of study in Iwasawa theory are Selmer groups of the Galois representations attached to the elliptic curves at a prime p.

Over the cyclotomic $\mathbb{Z}_p$-extension of a number field, when the Galois representation (or equivalently, the elliptic curve) has good ordinary reduction over a prime p, then the Pontryagin dual of the Selmer group is conjectured to be a finitely generated torsion module over the Iwasawa algebra of a p-adic Lie group of dimension 1 (cf. Conjecture 6.1). This conjecture has been proved by Kato for elliptic curves over $\mathbb{Q}$ for the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\text{cyc}}$ of $\mathbb{Q}$.

Mazur's conjecture can be generalized for arbitrary p-adic Lie extensions provided the elliptic curve has good ordinary reduction at p. Given a compact p-valued p-adic Lie group, the corresponding Iwasawa algebra admits a nice structure theory for finitely generated torsion modules (up to pseudo-isomorphism) (cf. [17]). Therefore, the dual Selmer group enjoys important algebraic properties, and in the ordinary reduction case, it has been studied extensively.

When the elliptic curve has good supersingular reduction at primes over p, however, the Pontryagin dual of the Selmer group, even though still finitely generated, is not a torsion module over the corresponding Iwasawa algebra. In fact, it is shown to have positive rank [20, Theorem 2.6] for the cyclotomic $\mathbb{Z}_p$-extension and it is believed to be true more generally.

Nevertheless, this problem can be partially addressed if we focus our attention to certain subgroups of the Selmer groups. The signed Selmer groups, which were first introduced by Kobayashi (in [37]), are subgroups of Selmer groups and the emerging picture seems to be that the signed Selmer groups enjoy many of the properties in the supersingular case that are enjoyed by the Selmer groups in the ordinary case. This survey intends to give evidence for this claim.

Over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\text{cyc}}/\mathbb{Q}$, the dual of the signed Selmer groups (often called plus/minus Selmer groups) are finitely generated torsion modules over the Iwasawa algebra of $\text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q})$. In [19], Coates and

Sujatha considered the notion of the *fine* Selmer group, which is a subgroup of the signed Selmer groups. Therefore, the Pontryagin dual of fine Selmer is a finitely generated module over the corresponding Iwasawa algebra. They conjectured that the μ-invariant of the dual fine Selmer is zero; this is known as Conjecture A to Iwasawa theorists.

Thus, we are left with four finitely generated torsion modules, viz. dual fine Selmer, dual plus and minus Selmer, dual of the torsion submodule of the Selmer. This article attempts to bring in light the connections between μ-invariants of these torsion modules.

This article is organized as follows. In Section 1, we define the Selmer group, the fine Selmer group, and the signed Selmer groups over cyclotomic towers as well as the signed Selmer groups over a $\mathbb{Z}_p^2$-extension of an imaginary quadratic field where p splits completely.

In Section 2, we give some of the definitions that we will use in the rest of this article, such as the Iwasawa algebra, pseudo-nullity, and the Euler characteristic. We discuss the structure theorem and use it to define the μ-invariant and the characteristic polynomial of a finitely generated module over a given Iwasawa algebra. Furthermore, we recall Conjecture A and discuss how it is related to Iwasawa μ-invariant conjecture over cyclotomic extensions. We finish Section 2 by describing what is known as Iwasawa main conjecture which beautifully relates an algebraically defined object to an analytically defined one, both coming from the arithmetic of number fields.

Then, in Section 3, we recall some of the most important results proved so far for the signed Selmer groups over the cyclotomic $\mathbb{Z}_p$-extension and $\mathbb{Z}_p^2$-extension. We note an exact sequence, due to Kobayashi [37], connecting the fine Selmer group with that of the signed Selmer groups. This exact sequence can be thought of as a tool to transport information from the fine Selmer group (like Conjecture A or its dual being torsion) to the signed Selmer groups. Moreover, we talk about how the Euler characteristic varies as we climb up the Iwasawa theoretic tower from the cyclotomic $\mathbb{Z}_p$-extension to a $\mathbb{Z}_p^2$-extension; this is a result of Lei and Sujatha [41] (see Theorem 3.18).

In Section 4, over the cyclotomic $\mathbb{Z}_p$-extension, when the elliptic curve has supersingular reduction at the primes above p, we explain how Conjecture A implies that the μ-invariant attached to the torsion part of the dual Selmer group is zero. To show this, we use important results proved by Billot [10] and then generalized by Wingberg [66].

Our goal in Section 5 is to give another view of Kobayashi's signed Selmer groups for the cyclotomic $\mathbb{Z}_p$-extension. Lei, Loeffler and Zerbes [40, 42] have redefined Kobayashi's signed Selmer groups using Fontaine's p-adic Hodge theory and Fontaine's ring $\widetilde{\mathbb{E}}$ (see [42, Section 2.1]). The Fontaine's

ring $\widetilde{\mathbb{E}}$ is the tilt of the perfectoid field $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ in the terminology of Scholze. In Section 5, we start by recalling the notion of perfectoid spaces of Scholze [60] which gives a geometric understanding of Fontaine's p-adic Hodge theory and Fontaine–Winterberger theorem [24]. Then, we use the language of perfectoids to give the definition of signed Selmer groups. Since the titling construction of Scholze works for any general perfectoid field, this poses a natural open question of whether it is possible to define signed Selmer groups for more general p-adic Lie extensions using Scholze's construction of perfectoid fields. Note that, at present, all the constructions: $\mathbb{E}_L$, $\mathbb{A}_L$, (φ, Γ)-modules and their relation with Iwasawa cohomology discussed in Section 5 are difficult to generalize for extensions other than the cyclotomic one (see for example [8] for an obstruction of the existence of a lift of Frobenius and the Galois action from $\mathbb{E}_L$ to $\mathbb{A}_L$). The reader can consult [9, 38, 58] for progress in the Lubin–Tate extensions. For a result on general perfectoid fields in Iwasawa theory see [55] which is a generalization of [15].

Finally, in Section 6, we turn our attention to elliptic curves with good ordinary reduction at p and we list some properties of the dual Selmer group that are analogous to the case of signed Selmer groups in the supersingular reduction case. To finish, we pose two questions that remain open for elliptic curves with supersingular reduction concerning signed Selmer groups and their μ-invariants, while their analogue holds for Selmer groups in the ordinary reduction case.

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1. Definitions of Selmer, signed Selmer, and fine Selmer groups

Let p be an odd prime. Let F be a number field, F' be a subfield of F, and E/F' be an elliptic curve with good reduction at all the primes above p. Let S be a finite set of primes of F' containing primes above p, the archimedean places and the primes where E has bad reduction.

1.1. Selmer group. Let F_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F, and let Γ denote $\mathrm{Gal}(F_{\mathrm{cyc}}/F)$ which is topologically isomorphic to $\mathbb{Z}_p$. For each integer $n \geq 0$, let F_n be the sub-extension of F_{cyc} such that F_n is a cyclic extension of degree p^n over F. The p-Selmer group over F_n is defined by the sequence

$$0 \longrightarrow \mathrm{Sel}_p(E/F_n) \longrightarrow H^1(F_n, E_{p^\infty}) \longrightarrow \prod_w H^1(F_{n,w}, E),$$

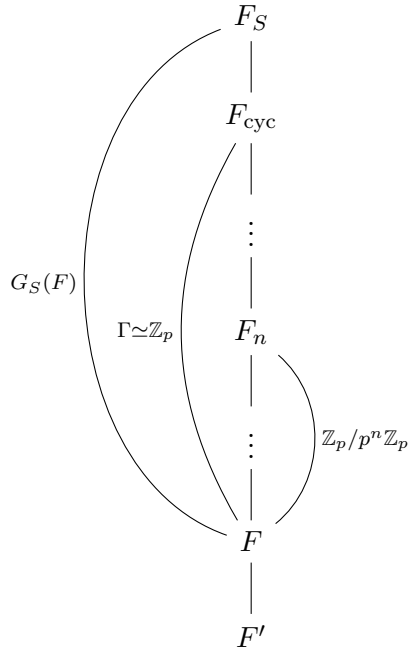
where w runs over all primes of F_n and $F_{n,w}$ is the completion of F_n at prime w. For $v \in S$, we let

$$J_v(E/F_n) := \bigoplus_{w|v} H^1(F_{n,w}, E)(p) \cong \bigoplus_{w|v} \frac{H^1(F_{n,w}, E_{p^\infty})}{E(F_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p},$$

where the isomorphism is due to the Kummer map (cf. [20, Section 1.6]). Then, we have an exact sequence[1]

$$0 \longrightarrow \mathrm{Sel}_p(E/F_n) \longrightarrow H^1(F_S/F_n, E_{p^\infty}) \xrightarrow{\lambda_n} \bigoplus_{v \in S} J_v(E/F_n). \tag{1.1}$$

Here F_S is the maximal extension of F unramified outside S and λ_n is consists of localization maps. Here, the choice of S implies that $F_{\mathrm{cyc}} \subset F_S$.



We define

$$\mathrm{Sel}_p(E/F_{\mathrm{cyc}}) := \varinjlim_n \mathrm{Sel}_p(E/F_n),$$

and we obtain the following exact sequence by taking the direct limit of the exact sequence (1.1) over intermediate field extensions F_n

$$0 \longrightarrow \mathrm{Sel}_p(E/F_{\mathrm{cyc}}) \longrightarrow H^1(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) \xrightarrow{\lambda_{\mathrm{cyc}}} \bigoplus_{v \in S} J_v(E/F_{\mathrm{cyc}}).$$

[1]We refer the reader to [20, Sections 1.7 and 2.2] for more details about the equivalent definitions of the Selmer group.

Definition 1.1. Suppose K is a p-adic Lie extension of F'. The p-Selmer group over K is

$$\mathrm{Sel}_p(E/K) := \ker\left(H^1(K, E_{p^\infty}) \longrightarrow \prod_w H^1(K_w, E)\right),$$

where w runs over all primes of K and K_w is the union of the completions at w of all finite extensions of F' contained in K.

It is not hard to see that

$$\mathrm{Sel}_p(E/K) = \varinjlim_L \mathrm{Sel}_p(E/L)$$

where L runs over all finite extensions of F' contained in K.

1.2. Signed Selmer groups over cyclotomic extension. We write S_p^{ss} to be the set of primes of F' lying above p where E has supersingular reduction. Let F'_v be the completion of F' at a prime $v \in S_p^{\mathrm{ss}}$ with the residue field k'_v. Let $\widetilde{E}(k'_v)$ be the k'_v-points of the reduction of E at place v. We assume the following:

(i) $S_p^{\mathrm{ss}} \neq \emptyset$;
(ii) For all $v \in S^{\mathrm{ss}}$ we have that F'_v, the completion of F' at v, is $\mathbb{Q}_p$;
(iii) $a_v = 1 + p - \#\widetilde{E}(k'_v) = 0$;
(iv) v is unramified in F.

Let $S_{p,F}^{\mathrm{ss}}$ denote the set of primes of F above S^{ss}. Note that every prime $v \in S_{p,F}^{\mathrm{ss}}$ is totally ramified in the extension F_{cyc}/F and therefore there exists a unique prime in F_n for each prime $v \in S_{p,F}^{\mathrm{ss}}$. Let $F_{n,v}$ be the completion of F_n at the unique prime over $v \in S_{p,F}^{\mathrm{ss}}$. For every $v \in S_{p,F}^{\mathrm{ss}}$, following Kobayashi [37] and Kitajima–Otsuki [36], we define

$$E^+(F_{n,v}) = \left\{ P \in \widehat{E}(F_{n,v}) \;\middle|\; \begin{array}{l} \mathrm{Trace}_{n/m+1}\, P \in \widehat{E}(F_{m,v}) \\ \text{for all even } m, 0 \le m \le n-1 \end{array} \right\}, \tag{1.2}$$

$$E^-(F_{n,v}) = \left\{ P \in \widehat{E}(F_{n,v}) \;\middle|\; \begin{array}{l} \mathrm{Trace}_{n/m+1}\, P \in \widehat{E}(F_{m,v}) \\ \text{for all odd } m, -1 \le m \le n-1 \end{array} \right\}. \tag{1.3}$$

Here $\mathrm{Trace}_{n/m+1}$ is the trace map from $\widehat{E}(F_{n,v})$ to $\widehat{E}(F_{m+1,v})$. For all $n \ge 0$, we define

$$E_{p^{n+1}} = \ker\left(E(\overline{\mathbb{Q}}) \xrightarrow{p^{n+1}} E(\overline{\mathbb{Q}})\right),$$
$$E_{p^\infty} = \bigcup_{n \ge 0} E_{p^{n+1}}.$$

Identifying $E^{\pm}(F_{n,w})\otimes\mathbb{Q}_p/\mathbb{Z}_p$ with a subgroup of $H^1(F_{n,w},E_{p^\infty})$ via the Kummer map, we may define the local terms

$$J_v^{\pm}(E/F_n)=\bigoplus_{w|v}\frac{H^1(F_{n,w},E_{p^\infty})}{E^{\pm}(F_{n,w})\otimes\mathbb{Q}_p/\mathbb{Z}_p}. \tag{1.4}$$

Definition 1.2. The plus and minus Selmer groups over F_n are defined by

$$\mathrm{Sel}_p^{\pm}(E/F_n)=\ker\left(\mathrm{Sel}_p(E/F_n)\longrightarrow\bigoplus_{v\in S_{p,F}^{\mathrm{ss}}}J_v^{\pm}(E/F_n)\right).$$

The plus and minus Selmer groups over F_{cyc} are obtained by taking direct limits

$$\mathrm{Sel}_p^{\pm}(E/F_{\mathrm{cyc}})=\varinjlim_n\mathrm{Sel}_p^{\pm}(E/F_n).$$

1.3. Signed Selmer groups over a $\mathbb{Z}_p^2$-extension. Suppose F is an imaginary quadratic field where p splits completely. Let F_∞ denote the compositum of all $\mathbb{Z}_p$-extensions of F. By Leopoldt's conjecture[2] we know that

$$G=\mathrm{Gal}(F_\infty/F)\simeq\mathbb{Z}_p^2,$$

which implies that F_∞ over F_{cyc} is a $\mathbb{Z}_p$-extension. Let v be a place of F and w be a place of F_∞ above v. If $v\mid p$, then $F_v\simeq\mathbb{Q}_p$ and $F_{\infty,w}$ is an abelian pro-p extension over F_v and $\mathrm{Gal}(F_{\infty,w}/F_v)\simeq\mathbb{Z}_p^2$. Under this setting, it is possible to define the plus and minus norm groups $E^{\pm}(F_{\infty,w})\subset\widehat{E}(F_{\infty,w})$ via trace maps as in (1.2) and (1.3) (cf. [41, Section 5.2] which is a generalization of a construction by Kim [35]). Similar to the above construction, we may define the local terms

$$J_v^{\pm}(E/F_\infty)=\bigoplus_{w|v}\frac{H^1(F_{\infty,w},E_{p^\infty})}{E^{\pm}(F_{\infty,w})\otimes\mathbb{Q}_p/\mathbb{Z}_p}. \tag{1.5}$$

Definition 1.3. Retain the settings described above. The plus and minus Selmer groups over the $\mathbb{Z}_p^2$-extension F_∞ are defined by

$$\mathrm{Sel}_p^{\pm}(E/F_\infty)=\ker\left(\mathrm{Sel}_p(E/F_\infty)\longrightarrow\bigoplus_{v\in S_{p,F}^{\mathrm{ss}}}J_v^{\pm}(E/F_\infty)\right). \tag{1.6}$$

Here the (classical) Selmer group $\mathrm{Sel}_p(E/F_\infty)$ is defined by taking inductive limit of Selmer groups over all finite extensions contained in the p-adic Lie extension F_∞.

[2]Good references for Leopoldt's conjecture are Sections 5.1 and 13.1 of [65]. In particular, for our claim about the number of independent $\mathbb{Z}_p$-extensions above a number field, please look at Theorem 13.4 and Corollary 5.32.

1.4. Fine Selmer group. The fine Selmer group of an elliptic curve over a number field is a subgroup of the Selmer group. The fine Selmer group is obtained when we impose extra stronger conditions at the primes above p on the sequence defining the Selmer groups.

Definition 1.4. The *fine Selmer group* of E/F' over a number field L/F' is

$$\mathrm{Sel}_p^0(E/L) := \ker\left(H^1(L, E_{p^\infty}) \longrightarrow \prod_\nu H^1(L_\nu, E_{p^\infty})\right),$$

where ν runs over all places of L and L_ν is the completion of L at ν. Let

$$\mathrm{Sel}_p^0(E/F_{\mathrm{cyc}}) := \varinjlim_n \mathrm{Sel}_p^0(E/F_n).$$

When K is a p-adic Lie extension of F', then

$$\mathrm{Sel}_p^0(E/K) = \varinjlim_L \mathrm{Sel}_p^0(E/L)$$

where L runs over all finite extensions of F' contained in K.

Remark 1.5. We can turn the direct product in the above Definition 1.4 into a direct sum in the following way. Let S be a finite set of primes of F' containing primes above p, the archimedean places and the primes where E has bad reduction. Let $G_S(L) = \mathrm{Gal}(L_S/L)$ where L_S denote the maximal extension of L unramified outside of S. Then, the above definition becomes (cf. [19] for more details):

$$\mathrm{Sel}_p^0(E/L) := \ker\left(H^1(G_S(L), E_{p^\infty}) \longrightarrow \bigoplus_{\nu\in S} K_\nu(L)\right),$$

where

$$K_v(L) = \bigoplus_{\omega|\nu} H^1(L_\omega, E_{p^\infty}).$$

2. Preliminaries of Iwasawa theory

2.1. Iwasawa Algebra.

Definition 2.1. Let G be a compact p-adic Lie group. We define the *Iwasawa algebra* of G, denoted by $\Lambda(G)$, to be the completed group algebra of G over $\mathbb{Z}_p$. That is

$$\Lambda(G) = \mathbb{Z}_p[\![G]\!] := \varprojlim_{N\subset G} \mathbb{Z}_p[G/N],$$

where N runs over open normal subgroups of G.

Remark 2.2. If there exists $n \geq 1$ such that the $G \simeq \mathbb{Z}_p^n$ (as topological groups), then $\Lambda(G)$ is isomorphic to the ring of formal power series $\mathbb{Z}_p[\![T_1, \ldots, T_n]\!]$, in indeterminate variables $T_1, \ldots, T_n$.

When $G = \mathrm{Gal}(F_{\mathrm{cyc}}/F) \simeq \mathbb{Z}_p$ (resp. $G \simeq \mathbb{Z}_p^2$) then $\Lambda(G) \simeq \mathbb{Z}_p[\![T]\!]$ (resp. $\Lambda(G) \simeq \mathbb{Z}_p[\![T_1, T_2]\!]$). In this case, $\Lambda(G)$ is a commutative, regular local ring of Krull dimension 2 (resp. 3). More generally, when our Galois group G is a non-commutative, torsion-free, compact p-adic Lie group of dimension d, then $\Lambda(G)$ is a non-commutative, left and right Noetherian local domain, and Auslander regular ring with global dimension $d+1$ (cf. [64]). This allows $\Lambda(G)$ to admit a nice dimension theory for its finitely generated modules[3].

Definition 2.3. Let M be a finitely generated $\Lambda(G)$-module. Then, M is said to be *pseudo-null* if

$$\dim_{\Lambda(G)}(M) \leq \dim_{\Lambda(G)}(\Lambda(G)) - 2.$$

Furthermore, M is $\Lambda(G)$-torsion if

$$\dim_{\Lambda(G)}(M) \leq \dim_{\Lambda(G)}(\Lambda(G)) - 1.$$

Therefore, a pseudo-null $\Lambda(G)$-module is $\Lambda(G)$-torsion.

Remark 2.4. Equivalently, whenever G is a torsion-free pro-p group, a finitely generated $\Lambda(G)$-module M is torsion if

$$\mathrm{Ext}^0_{\Lambda(G)}(M, \Lambda(G)) = 0,$$

and it is pseudo-null if

$$\mathrm{Ext}^i_{\Lambda(G)}(M, \Lambda(G)) = 0$$

for $i = 0, 1$.

Remark 2.5. We note that if $\Lambda(G)$ is commutative, a $\Lambda(G)$-module M is pseudo-null exactly when the annihilator ideal $\mathrm{Ann}_{\Lambda(G)}(M)$ has height at least 2. If

$$\Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F) \simeq \mathbb{Z}_p,$$

and hence $\Lambda(\Gamma) \simeq \mathbb{Z}_p[\![T]\!]$, then we can show that a finitely generated $\Lambda(\Gamma)$-module is pseudo-null if and only if it is finite.

Remark 2.6. For a general compact torsion-free p-adic Lie group G, a finite $\Lambda(G)$-module is always pseudo-null but the converse is not true.

Definition 2.7. Suppose G is a compact p-adic Lie group and M is a $\Lambda(G)$-module. The *Pontryagin dual* of M, denoted by $\widehat{M}$, is defined to be

$$\widehat{M} := \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$$

where Hom denotes the group of continuous homomorphisms.

[3]For information on the ring-theoretic properties of Iwasawa algebras we suggest Ardakov's survey paper [4].

2.2. Euler characteristic. Suppose $G = \mathrm{Gal}(K/F)$, where K is a p-adic Lie extension of a number field F. Suppose G has finite dimension d as a p-adic Lie group. Then, provided that G has no non-trivial p-torsion, G has p-cohomological dimension equal to d (cf. [62, Corollary 1]).

Definition 2.8. Suppose G is a p-adic Lie group with p-cohomological dimension equal to d. Given a discrete primary G-module M and if $H^i(G, M)$ is finite for $i = 0, \ldots, d$, the *G-Euler characteristic*[4] of M is defined to be

$$\chi(G, M) := \prod_{i=0}^{d} \Big(\#H^i(G, M)\Big)^{(-1)^i}.$$

2.3. Structure theorem. Suppose K/F is a $\mathbb{Z}_p$-extension, and therefore

$$G = \mathrm{Gal}(K/F) \simeq \mathbb{Z}_p.$$

Moreover, suppose M is a finitely generated $\Lambda(G)$-module. Then, the structure of M as a $\Lambda(G)$-module is well understood up to pseudo-isomorphism.

We say that a $\Lambda(G)$-homomorphism $f : M \to M'$ between two $\Lambda(G)$-modules M and M' is a pseudo-isomorphism if f has finite kernel and cokernel. We say M and M' are pseudo-isomorphic and we denote this by $M \sim M'$. The following *structure theorem* is due to Iwasawa.

Theorem 2.9. *Let M be a finitely generated $\Lambda(G)$-module. Then*

$$M \sim \Lambda(G)^r \oplus \left(\bigoplus_{i=0}^{s} \Lambda(G)/p^{n_i}\Lambda(G)\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda(G)/f_j(T)^{m_j}\Lambda(G)\right),$$

where r, s, t, n_i, and m_j are non-negative integers and are unique up to reordering. Moreover, $f_j(T) \in \mathbb{Z}_p[T]$ is an irreducible monic distinguished polynomial[5] *for each $j \in \{1, \ldots, t\}$.*

Note that in Theorem 2.9, r is the rank of M as a $\Lambda(G)$-module and M is torsion if and only if $r = 0$. This enables us to define some important structural invariants for finitely generated $\Lambda(G)$-modules.

Definition 2.10. Let M be a finitely generated $\Lambda(G)$-module. Then the *characteristic polynomial* of M, denoted by $\mathrm{Charpoly}_G(M)$, is defined by

$$\mathrm{Charpoly}_G(M) := p^{\mu_G(M)} \prod_{j=1}^{t} f_j(T)^{m_j},$$

where $\mu_G(M) := \sum_{i=0}^{s} p^{n_i}$ and s, t, n_i, and m_j are as in Theorem 2.9. Moreover, the *characteristic ideal* of M, denoted by $\mathrm{Char}_G(M)$, is the principle $\Lambda(G)$-ideal generated by $\mathrm{Charpoly}_G(M)$

$$\mathrm{Char}_G(M) := (\mathrm{Charpoly}_G(M)) \subset \Lambda(G)$$

[4] see [46, Definition 3.3.12].

[5] A monic polynomial in $\mathbb{Z}_p[T]$ is distinguished if p divides all of its coefficients except for the leading one.

It turns out that the above theorem can be naturally extended[6] to any extension K/F with $G = \mathrm{Gal}(K/F) \simeq \mathbb{Z}_p^n$ where $n \geq 1$. Suppose G is a pro-p, p-adic Lie group that is topologically isomorphic to $\mathbb{Z}_p^n$ for some $n \geq 1$, and as a result, $\Lambda(G) \simeq \mathbb{Z}_p[\![T_1, \ldots, T_n]\!]$ for indeterminate variables $T_1, \ldots, T_n$. Suppose M is a finitely generated $\Lambda(G)$-module. Recall from the Definition 2.3 that we say M is a pseudo-null if

$$\dim_{\Lambda(G)}(M) \leq \dim_{\Lambda(G)}(\Lambda(G)) - 2 = (n+1) - 2 = n - 1$$

Moreover, we call a $\Lambda(G)$-homomorphism $f : M \to M'$ between two finitely generated $\Lambda(G)$-modules a *pseudo-isomorphism* if the kernel and cokernel of f are pseudo-null $\Lambda(G)$-modules.

Suppose M is a finitely generated torsion $\Lambda(G)$-module. Let us denote the p-primary torsion $\Lambda(G)$-submodule of M by $M(p)$. Then, we have the following theorem [11, Chapter VII, Section 4.4, Theorems 4 and 5]:

Theorem 2.11. *Let G be a pro-p p-adic Lie group that is isomorphic to $\mathbb{Z}_p^n$ for some $n \geq 1$ and let M be a finitely generated torsion module over $\Lambda(G)$ with no elements of order p. Then, there exist a pseudo-isomorphism*

$$M(p) \sim \bigoplus_{i=0}^{s} \Lambda(G)/p^{n_i}\Lambda(G)$$

where $n_1, \ldots, n_s$ are unique up to reordering.

The *μ-invariant* of M, denoted by $\mu_G(M)$, is defined to be

$$\mu_G(M) = \sum_{i=0}^{s} n_i,$$

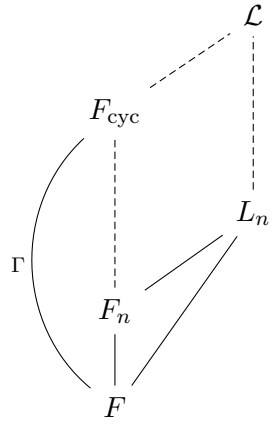
where $n_1, \ldots, n_s$ are as in Theorem 2.11 described above.

2.4. Introduction to Conjecture A. Let F be a number field and let p be an odd prime. Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension of F and let $\Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Then, let F_n/F be the sub-extension of F_{cyc} such that

$$\mathrm{Gal}(F_n/F) = \Gamma/\Gamma^{p^n} = \mathbb{Z}/p^n\mathbb{Z}.$$

[6]When G is a non-commutative torsion-free, compact p-adic Lie group, there is a structure theory for finitely generated torsion modules over the Iwasawa algebra $\Lambda(G)$ [18].

Let L_n/F_n be the maximal unramified abelian p-extension of F_n (so L_n is the p-Hilbert class field extension of F_n).



Note that by class field theory $\mathrm{Gal}(L_n/F_n)$ is isomorphic to the p-Sylow subgroup of the ideal class group of F_n. Let $\mathcal{L} := \bigcup_n L_n$, and then

$$X(F_{\mathrm{cyc}}) := \mathrm{Gal}(\mathcal{L}/F_{\mathrm{cyc}})$$

can be given a (natural) Γ-action, and therefore it can be turned into a $\Lambda(\Gamma)$-module. Iwasawa proved that $X(F_{\mathrm{cyc}})$ is a finitely generated torsion $\Lambda(\Gamma)$-module (cf. [31]). Moreover, in his celebrated work [32], Iwasawa conjectured the following.

For any number field F, $X(F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.

By the structure theorem, this is equivalent to say that $\mu_\Gamma(X(F_{\mathrm{cyc}})) = 0$. We have the following important theorem of Ferrero–Washington [21].

Theorem 2.12. *Iwasawa μ-invariant conjecture holds for all abelian number fields.*

Let $\mathfrak{X}^0(E/F_{\mathrm{cyc}})$ denote the Pontryagin dual of $\mathrm{Sel}_p^0(E/F_{\mathrm{cyc}})$. Then, we have the following conjecture of Coates and Sujatha [19].

Conjecture A. For any number field F, $\mathfrak{X}^0(E/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.

We note that this is equivalent to $\mathfrak{X}^0(E/F_{\mathrm{cyc}})$ being a torsion $\Lambda(\Gamma)$-module and having μ-invariant equal to zero (for the definition of μ-invariant see Section 2.1).

Remark 2.13. See Section 4.2 for several examples of elliptic curves where it is shown that Conjecture A is satisfied.

The following theorem of Coates–Sujatha [19, Theorem 3.4] relates Conjecture A and the Iwasawa μ-invariant conjecture for cyclotomic field extensions of algebraic number fields.

Theorem 2.14. *Suppose E/F is an elliptic curve over a number field F. Suppose p is an odd prime such that the field extension $F(E_{p^\infty})/F$ is a pro-p extension. Then Conjecture A holds for E over F_{cyc} if and only if the Iwasawa μ-invariant conjecture holds for F_{cyc}.*

Using Theorem 2.14, Coates and Sujatha gave the following conditions for an elliptic curve E/F to satisfy Conjecture A [19, Corollary 3.5].

Theorem 2.15. *Suppose p is an odd prime. If there exists a finite extension L/F of F such that*

(1) *$L \subset F(E_{p^\infty})$;*
(2) *$F(E_{p^\infty})$ is pro-p, and*
(3) *the Iwasawa μ-invariant conjecture hold for L_{cyc},*

then Conjecture A holds for E over F_{cyc}. That is, $\mathfrak{X}^0(E/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.

Moreover, Coates and Sujatha combined the above theorem and Theorem 2.12 to prove [19, Corollary 3.6]:

Theorem 2.16. *Suppose p is an odd prime and let $F/\mathbb{Q}$ be a number field such that $\mathrm{Gal}(F/\mathbb{Q})$ is abelian. If $E_{p^\infty}(F) \neq 0$ then Conjecture A holds for E over F_{cyc}.*

Remark 2.17. In [63], the authors investigate the behavior of Conjecture A under the change of base field, short exact sequences of representations, and isogenies. Moreover, they give various equivalent formulations of Conjecture A and they discuss the relation between Conjecture A and the Iwasawa μ-invariant conjecture in more details.[7]

2.5. Iwasawa main conjecture. In this section, we give a brief account of the classical Iwasawa main conjecture. Iwasawa main conjecture describes a striking relation between two objects attached to number fields, a p-adic analytic object, in the form of the values of the p-adic L-functions and certain algebraically defined Iwasawa module. Therefore, Iwasawa main conjecture gives us a bridge between algebraic treatments of arithmetic objects and that of the analytic side. To give the statement of Iwasawa main conjecture, we need to introduce more notations. We follow Chapter 15 of Washington's book [65].

Let p be an odd prime. Let $F = \mathbb{Q}(\mu_p)$ and consider the cyclotomic extension $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p) = F_{\mathrm{cyc}}/F$, where μ_{p^n} denotes a p^n-th primitive root of unity and $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n\geq 0} \mathbb{Q}(\mu_{p^n})$. Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. Furthermore,

(1) Let A_n be the p-Sylow part of the ideal class group of $\mathbb{Q}(\mu_{p^{n+1}})$ for $n \geq 0$ and $A_\infty := \varinjlim_n A_n$.

[7] We thank the referee for suggesting Remark 2.17 to us.

(2) Let $\mathcal{L}$ be the maximal unramified abelian p-extension of $\mathbb{Q}(\mu_{p^\infty})$ and $X(F_{\text{cyc}}) = \text{Gal}(\mathcal{L}/\mathbb{Q}(\mu_{p^\infty}))$.
(3) Let ε_i denote the i^{th} idempotent element of $\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ where i is odd.
(4) Let $L_p(s, \omega^j)$ be the p-adic L function for ω^j where ω is the Teichmüller character at p and j is even (cf. [65, Chapter 5-7] for a construction of the p-adic L function).
(5) Let $f(T, \omega^j) \in \mathbb{Z}_p[\![T]\!] \cong \mathbb{Z}_p[\![\Gamma]\!]$ be such that $f((1+p)^s - 1, \omega^j) = L_p(s, \omega^j)$ (cf. [65, Chapter 13, Section 6]).

Then, as we discussed, $X(F_{\text{cyc}})$ is a $\Lambda(\Gamma)$-module and Iwasawa proved that $X(F_{\text{cyc}})$ is finitely generated and torsion. Since $F = \mathbb{Q}(\mu_p)$ is an abelian extension of $\mathbb{Q}$. Theorem 2.12 implies that $\mu_\Gamma(X(F_{\text{cyc}})) = 0$ and by the structure theorem (see Theorem 2.9), we see that

$$X(F_{\text{cyc}}) \sim \bigoplus_{j=1}^{t} \Lambda(\Gamma)/f_j(T)^{m_j}\Lambda(\Gamma)$$

and

$$\text{Charpoly}_\Gamma(X(F_{\text{cyc}})) = \prod_{j=1}^{t} f_j(T)^{m_j}$$

where $f_j \in \Lambda(\Gamma) \cong \mathbb{Z}_p[\![T]\!]$ is an irreducible distinguished polynomial for each $j \in \{1, \ldots, t\}$, as described in Theorem 2.9. The (classical) Iwasawa main conjecture was proved by Mazur and Wiles in [45] and it states that (cf. [65, Chapter 15, Section 4, Theorem 15.14]):

Theorem 2.18. *Retain the notations* (1) *to* (5) *described above. Let i be an odd integer such that $i \neq 1 \pmod{p-1}$. Then*

$$\text{Charpoly}_\Gamma(\varepsilon_i X(F_{\text{cyc}})) = f(T, \omega^{1-i})u(T)$$

where $u(T) \in \Lambda(\Gamma)^\times$ is a unit of $\Lambda(\Gamma)$.

The Iwasawa main conjecture was generalized (and proved in many cases) to more general number fields and elliptic curves.

3. Properties of fine Selmer and signed Selmer groups

Let p be an odd prime and let F be a number field. Suppose E/F is an elliptic curve with good reduction at all the primes above p. Let $\mathfrak{X}(E/F_{\text{cyc}})$ denote the Pontryagin dual of $\text{Sel}_p(E/F_{\text{cyc}})$. Then $\mathfrak{X}(E/F_{\text{cyc}})$ is finitely generated as a $\Lambda(G)$-module where $G = \text{Gal}(F_{\text{cyc}}/F)$. If E has good ordinary reduction over all the primes above p then $\mathfrak{X}(E/F_{\text{cyc}})$ is conjectured to be torsion as a $\Lambda(\Gamma)$-module (see Conjecture 6.1 for more details). Similarly, assume now that F is a quadratic imaginary field where p splits and F_∞ is the compositum of all $\mathbb{Z}_p$-extensions of F which by Leopoldt's conjecture we know that it is a $\mathbb{Z}_p^2$-extension. Let $\mathfrak{X}(E/F_\infty)$ denote the Pontryagin dual of

$\mathrm{Sel}_p(E/F_\infty)$. Then, $\mathfrak{X}(E/F_\infty)$ is finitely generated as a $\Lambda(G)$-module and if E has good ordinary reduction over all the primes above p, it is conjectured to be torsion as a $\Lambda(G)$-module (a generalization of Conjecture 6.1). For simplicity, let $\mathfrak{X}(E/F_\square)$ denote the Pontryagin dual of $\mathrm{Sel}_p(E/F_\square)$, where $\square \in \{\mathrm{cyc}, \infty\}$. Unlike the ordinary case, $\mathfrak{X}(E/F_\square)$ is believed (shown for the cyclotomic $\mathbb{Z}_p$-extension, see [20, Theorem 2.5]) to have positive rank, and therefore it is no longer a torsion Iwasawa module. However, the Pontryagin dual of the signed Selmer groups $\mathrm{Sel}_p^\pm(E/F_\square)$, which we denote by $\mathfrak{X}^\pm(E/F_\square)$, is conjectured to be torsion.

3.1. Cyclotomic case. Let E be an elliptic curve over $\mathbb{Q}$ with good supersingular reduction at p and suppose $a_p = 0$. (The signed Selmer groups are defined by Sprung [29] when a_p is nonzero). In this survey paper we mainly focus on $a_p = 0$ case. Let us denote $\mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q})$ by Γ. Then, Kobayashi proved that ([37, Theorem 1.2]):

Theorem 3.1. *The dual Selmer group $\mathfrak{X}^\pm(E/\mathbb{Q}_{\mathrm{cyc}})$ is a finitely generated torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module.*

Moreover, Kobayashi used Kato's Euler system to prove the following theorem [37, Theorem 1.3]:

Theorem 3.2. *Suppose $E/\mathbb{Q}$ does not have complex multiplication. Then for almost all primes p, we have that the ideal generated by Pollack's p-adic L-function $\mathcal{L}_p^\pm(E,X)$ as $\Lambda(\Gamma)$-modules (see* [54] *for more details) is contained in the characteristic ideal of the Pontryagin dual of $\mathrm{Sel}_p^\pm(E/\mathbb{Q}_{\mathrm{cyc}})$. That is,*

$$(\mathcal{L}_p^\pm(E,X)) \subseteq \mathrm{Char}_\Gamma(\mathfrak{X}^\pm(E/\mathbb{Q}_{\mathrm{cyc}})).$$

The plus/minus-Iwasawa main conjectures predict that these containments are an equality[8].

More generally, let us adapt the notations of Section 1.2. Here, F' is a number field and E/F' is an elliptic curve with good reduction at all the primes above p, where p is an odd prime. Suppose F/F' is a finite extension of F'. Then, we have the exact sequence

$$0 \longrightarrow \mathrm{Sel}_p^\pm(E/F_{\mathrm{cyc}}) \longrightarrow H^1(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) \xrightarrow{\lambda_{\mathrm{cyc}}^\pm} \bigoplus_{\nu\in S} J_v^\pm(E/F_{\mathrm{cyc}}). \tag{3.1}$$

Lei and Sujatha proved the following statement ([41, Proposition 4.4]).

Theorem 3.3. *$\mathfrak{X}^\pm(E/F_{\mathrm{cyc}})$ is torsion as a $\Lambda(G)$-module if and only if $\lambda_{\mathrm{cyc}}^\pm$ is surjective in* (3.1) *and $H^2(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) = 0$.*

[8]Kobayashi showed that, in this supersingular setting, the plus/minus-Iwasawa main conjectures are equivalent to Kato's and Perrin-Riou's main conjectures (cf. [37, Theorem 7.4]).

Remark 3.4. $H^2(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) = 0$ is called the *weak Leopoldt's conjecture* for E_{p^∞} over F_{cyc}. This is equivalent to say that $\mathfrak{X}^0(E/F_{\mathrm{cyc}})$, the Pontryagin dual of $\mathrm{Sel}_p^0(E/F_{\mathrm{cyc}})$, is a finitely generated torsion $\Lambda(\Gamma)$-module [19, Lemma 3.1]. More generally, the weak Leopoldt's conjecture for E_{p^∞} on a number field F predicts that

$$H^2(F_S/F, E_{p^\infty}) = 0.$$

Remark 3.5. Note that if $\mathrm{Sel}_p(E/F)$ is finite then $\mathrm{Sel}_p^{\pm}(E/F)$ is finite and the dual signed Selmer groups $\mathfrak{X}^{\pm}(E/F_{\mathrm{cyc}})$ are $\Lambda(\Gamma)$-torsion (cf. Remark 4.5 of [41]). This provides examples for $\mathfrak{X}^{\pm}(E/F_{\mathrm{cyc}})$ to be torsion. By the above theorem, it also provides examples for the weak Leopoldt's conjecture for E_{p^∞} over F_{cyc}.

For an elliptic curve E defined over $\mathbb{Q}$, we have:

Theorem 3.6 ([37, Theorem 5.1 and Corollary 7.2])**.** *$\mathfrak{X}^0(E/\mathbb{Q}_{\mathrm{cyc}})$ is a finitely generated torsion $\Lambda(\Gamma)$-module.*

In the following, we give an exact sequence due to Kobayashi [37] which connects the signed Selmer groups with that of the fine Selmer group.

Let $F_n = \mathbb{Q}(\zeta_{p^{n+1}})$. Let k_n denote $\mathbb{Q}_p(\zeta_{p^{n+1}})$. Let $T_p(E)$ be the p-adic Tate module of the elliptic curve E and $V_p(E) := T_p(E) \otimes \mathbb{Q}_p$. Suppose S_n is the set of places of F_n over S, $G_{n,S} := \mathrm{Gal}(F_{n,S}/F_n)$ where $F_{n,S}$ is the maximal unramified extension of F_n outside S_n. We let

$$H^i_{/S}(T_p(E)) := \varprojlim_n H^i(G_{n,S}, T_p(E)), \quad H^i_{\mathrm{Iw},\nu}(T_p(E)) := \varprojlim_n H^i(F_{n,\nu}, T_p(E)).$$

Moreover, we define $H^1_{\mathrm{Iw},\pm}(T_p(E))$ as the inverse limit of the exact annihilator of subgroups

$$E^{\pm}(k_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subseteq H^1(k_n, V_p(E)/T_p(E))$$

with respect to the Tate pairing (cf. [37])

$$H^1(k_n, V_p(E)/T_p(E)) \times H^1(k_n, T_p(E)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Additionally, we have the following exact sequence relating the fine Selmer group with the signed Selmer groups ([37, Theorem 7.3]):

$$\begin{aligned} (3.2)\quad 0 \longrightarrow H^1_{/S}(T_p(E)) \xrightarrow{\alpha} H^1_{\mathrm{Iw}}(T_p(E))/H^1_{\mathrm{Iw},\pm}(T_p(E)) \\ \longrightarrow \mathfrak{X}^{\pm}(E/\mathbb{Q}_{\mathrm{cyc}}) \longrightarrow \mathfrak{X}^0(E/\mathbb{Q}_{\mathrm{cyc}}) \longrightarrow 0. \end{aligned}$$

We note that if $T_p(E)/pT_p(E)$ is irreducible as a two dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{F}_p$, then $H^1_{/S}(T_p(E))$ is a free $\Lambda(\Gamma)$-module of rank 1 ([33, Theorem 12.4]).

Remark 3.7. Kobayashi uses the Coleman maps and Kato's Euler system to show that the map α in (3.2) is a non-zero map and the cokernel of α is a $\Lambda(\Gamma)$-torsion module ([37, Theorem 6.3]). Then Theorem 3.6 and the exact sequence (3.2) give a proof of the fact that $\mathfrak{X}^{\pm}(E/\mathbb{Q}_{\mathrm{cyc}})$ is torsion.

The following theorem tells us about the triviality of the finite $\Lambda(\Gamma)$-submodules of the dual signed Selmer groups of an elliptic curve $E/\mathbb{Q}$ over the cyclotomic $\mathbb{Z}_p$-extension ([34, Theorem 1.1]).

Theorem 3.8. *Let p be an odd prime and suppose F is a finite extension of $\mathbb{Q}$ in which p is unramified. Let $E/\mathbb{Q}$ have good supersingular reduction at p with $a_p = 0$.*

1. *Suppose $\mathfrak{X}^{-}(E/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. Then, $\mathfrak{X}^{-}(E/F_{\mathrm{cyc}})$ has no nontrivial finite $\Lambda(\Gamma)$-submodules.*
2. *Suppose $\mathfrak{X}^{+}(E/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion, and that p splits completely in F. Then $\mathfrak{X}^{+}(E/F_{\mathrm{cyc}})$ has no nontrivial finite $\Lambda(\Gamma)$-submodules.*

Remark 3.9. More generally, suppose E/F is an elliptic curve satisfying the following conditions (cf. [41, Theorem 4.8]):

1. E has supersingular reduction at all the primes of F above p.
2. The map $\lambda^{\pm}_{\mathrm{cyc}}$ from (3.1) is surjective.
3. $H^2(F_S/F, E_{p^\infty}) = 0$.
4. $H^1(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}})) = 0$.

Then, $\mathfrak{X}^{\pm}(E/F_{\mathrm{cyc}})$ has no nontrivial finite $\Lambda(\Gamma)$-submodules.

Remark 3.10. See Theorem 4.2 for a similar result of Billot on elliptic curves with complex multiplication.

Recall the Definition 2.8 of the Euler characteristic. Let $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ and thus Γ is topologically isomorphic to $\mathbb{Z}_p$. Hence Γ has p-cohomological dimension equal to 1. Therefore, if $\chi(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}}))$ is defined, i.e. $H^i(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}}))$ is finite for $i = 0$ and 1, then

$$\chi(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}})) = \frac{\#H^0(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}}))}{\#H^1(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}}))}.$$

The following theorem is due to Kim ([34, Theorem 1.2]).

Theorem 3.11. *Let E/F be an elliptic curve over a number field F and let $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Suppose E has supersingular reduction at every prime above p. Furthermore, suppose that $\mathrm{Sel}_p(E/F)$ is finite. Then, $\chi(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}}))$ is well-defined, and up to a p-adic unit, is equal to*

$$\chi(\Gamma, \mathrm{Sel}^{\pm}_p(E/F_{\mathrm{cyc}})) = \#\,\mathrm{Sel}_p(E/F) \times \prod_{\nu} c_\nu$$

where ν runs over all primes of F, c_ν is the Tamagawa number of E at prime ν and $c_\nu = [E(F_\nu) : E_0(F_\nu)]$. Here, $E_0(F_\nu)$ denotes the subgroup of points of $E(F_\nu)$ with non-singular reduction.

Finally, we mention a result analogous to the above result.

Remark 3.12. Ahmed and Lim in [2] extend the above result to compute the Euler characteristics of the signed Selmer groups of elliptic curves with *mixed reduction type* (not necessarily supersingular reduction at all the primes above p) over the cyclotomic $\mathbb{Z}_p$-extension (cf. [2, Theorem 1.1]).

3.2. $\mathbb{Z}_p^2$-extension case. We work under the setting of Section 1.3 and follow the results of Lei and Sujatha [41]. Let $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}}) \cong \mathbb{Z}_p$ and $G = \mathrm{Gal}(F_\infty/F)$. In particular, note that:

$$G/H = \mathrm{Gal}(F_{\mathrm{cyc}}/F) = \Gamma \cong \mathbb{Z}_p.$$

Recall the left exact sequence (3.1). Let us consider the following commutative diagram (cf. [41, diagram 5.2]):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}_p^{\pm}(E/F_\infty)^H & \longrightarrow & H^1(F_S/F_\infty, E_{p^\infty})^H & \xrightarrow{\lambda_\infty^{H,\pm}} & \bigoplus_{\nu\in S} J_\nu^{\pm}(E/F_\infty)^H \\ & & \uparrow{\scriptstyle \alpha_\infty^{\pm}} & & \uparrow{\scriptstyle \beta_\infty^{\pm}} & & \uparrow{\scriptstyle \gamma_\infty^{\pm}=\bigoplus \gamma_{\nu,\infty}^{\pm}} \\ 0 & \longrightarrow & \mathrm{Sel}_p^{\pm}(E/F_{\mathrm{cyc}}) & \longrightarrow & H^1(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) & \xrightarrow{\lambda_{\mathrm{cyc}}^{\pm}} & \bigoplus_{\nu\in S} J_\nu^{\pm}(E/F_{\mathrm{cyc}}). \end{array} \tag{3.3}$$

Then, Lei and Sujatha proved the following results (cf. [41, Section 5.3]).

Theorem 3.13. *The following statements hold true for the maps appearing in* (3.3)*:*

1. *If $\lambda_{\mathrm{cyc}}^{\pm}$ is surjective, then $\alpha_\infty^{\pm}$ is injective and* $\mathrm{coker}(\alpha_\infty^{\pm}) \cong \ker(\gamma_\infty^{\pm})$.
2. *The map $\beta_\infty^{\pm}$ is an isomorphism.*
3. *The map $\gamma_{\nu,\infty}^{\pm}$ is an isomorphism for all $\nu \in S_{p,F}^{\mathrm{ss}}$ and for all $\nu \in S$ with $\nu \nmid p$.*
4. *The map $\gamma_{\nu,\infty}^{\pm}$ is surjective and* $\ker(\gamma_{\nu,\infty}^{\pm})$ *is $\Lambda(\Gamma)$-cotorsion*[9] *for all $\nu \in S \backslash S_{p,F}^{\mathrm{ss}}$.*

Remark 3.14. In particular, note that if E has good supersingular reduction at all the primes above p, then all the vertical maps in (3.3) become isomorphism maps.

Recall that Theorem 3.3 implies that if $\mathfrak{X}^{\pm}(E/F_{\mathrm{cyc}})$ is torsion, then $\lambda_{\mathrm{cyc}}^{\pm}$ is surjective. Furthermore, Lei and Sujatha proved (cf. [41, Proposition 5.11]):

Theorem 3.15. *Suppose $\mathfrak{X}^{\pm}(E/F_{\mathrm{cyc}})$ is torsion. Then the map $\lambda_\infty^{H,\pm}$ in* (3.3) *is surjective.*

[9] A $\Lambda(G)$-module M is cotorsion if its Pontryagin dual is a torsion $\Lambda(G)$-module.

Remark 3.16. We have a similar statement as Theorem 3.3 for the map

$$\lambda_\infty^\pm \ : \ H^1(F_S/F_\infty, E_{p^\infty}) \longrightarrow \bigoplus_{\nu\in S} J_\nu^\pm(E/F_\infty).$$

Namely, $\mathfrak{X}^\pm(E/F_\infty)$ is $\Lambda(G)$-torsion exactly when $\lambda_\infty^\pm$ is surjective and

$$H^2(F_S/F_\infty, E_{p^\infty}) = 0$$

(cf. [41, proof of Corollary 5.12]).

Remark 3.17. Using Kim's control theorems (cf. [35, Theorem 1.1]), Lei and Sujatha proved that (cf. [41, Corollary 5.12 and Proportion 5.14]):

(i) If we assume $\mathrm{Sel}_p(E/F)$ is finite, then $H^1(H, \mathrm{Sel}_p^\pm(E/F_\infty)) = 0$.
(ii) If we assume $\mathrm{Sel}_p(E/F)$ is finite and E has supersingular reduction at every prime above p, then $H^i(G, \mathrm{Sel}_p^\pm(E/F_\infty)) = 0$ for $i \geq 1$.

The following theorem [41, Theorem 5.15] extends Theorem 3.11 to G-Euler characteristic of $\mathrm{Sel}_p^\pm(E/F_\infty)$.

Theorem 3.18. *Let E/F be an elliptic curve with supersingular reduction at every prime above p. Furthermore, suppose $\mathrm{Sel}_p(E/F)$ is finite. Then, $\chi(G, \mathrm{Sel}_p^\pm(E/F_\infty))$ is well-defined, and*

$$\chi(G, \mathrm{Sel}_p^\pm(E/F_\infty)) = \chi(\Gamma, \mathrm{Sel}_p^\pm(E/F_{\mathrm{cyc}})).$$

4. Conjecture A and μ-invariant of the torsion part of dual Selmer group

In this section, we would like to show that Conjecture A implies that the torsion part of the p-Selmer group has μ-invariant zero. To see this, we need to relate the fine Selmer group and the torsion part of the Selmer group.

4.1. Results of Billot and Wingberg. Suppose the following statements hold:

(i) E/F is an elliptic curve with complex multiplication by an imaginary quadratic field K and $K \subset F$.
(ii) $E_p \subset E(F)$.
(iii) p is an odd prime number inert in K.
(iv) E has good reduction on all the places of F over p.

Let $F_\infty = F(E_{p^\infty})$. Then, $\mathrm{Gal}(F_\infty/F)$ is topologically isomorphic to $\mathbb{Z}_p^2$ [10, Section 3]. Let $G = \mathrm{Gal}(F_\infty/F)$ and let $\Lambda(G)$ be the Iwasawa algebra of G. Furthermore, we assume:

(v) The Leopoldt's conjecture[10] holds for all the intermediate fields in the $\mathbb{Z}_p^2$-extension F_∞ of F.

[10] See [65, Sections 5.5 and 13.1] for more information on Leopoldt's conjecture.

In the case when E has good supersingular reduction for some of the primes above p, we expect the Pontryagin dual of the Selmer group $\mathfrak{X}(E/F_\infty)$ to have a positive rank. On the other hand, assuming conditions (i) to (iv), Billot proved that the dual of $\mathrm{Sel}^0(E/F_\infty)$ is a torsion $\Lambda(G)$-module (cf. [10, Proposition 3.8]). Let us denote the $\Lambda(G)$-torsion submodule of $\mathfrak{X}(E/F_\infty)$ by $T_{\Lambda(G)}(\mathfrak{X}(E/F_\infty))$. We have the following results by Billot.

Theorem 4.1 ([10, Theorem 3.23])**.** *Given the above assumptions* (i) *to* (v)*, the $\Lambda(G)$-torsion submodule of the Pontryagin dual of the Selmer group $T_{\Lambda(G)}(\mathfrak{X}(E/F_\infty))$ is pseudo-isomorphic to the Pontryagin dual of the fine Selmer group $\mathfrak{X}^0(E/F_\infty)$.*

Theorem 4.2 ([10, Proposition 3.26])**.** *Suppose the assumptions* (i) *to* (v) *hold true. Further assume*

(1) *$\mathfrak{X}(E/F_\infty)$ has no non-trivial pseudo-null $\Lambda(G)$-submodule.*
(2) *Let K_∞ be a $\mathbb{Z}_p$-extension of F contained in F_∞ and let $H = \mathrm{Gal}(K_\infty/F)$. Suppose*

$$\mathrm{rank}_{\Lambda(G)}(\mathfrak{X}(E/F_\infty)) = \mathrm{rank}_{\Lambda(H)}(\mathfrak{X}(E/K_\infty)).$$

Then, $\mathfrak{X}(E/K_\infty)$ has no non-trivial pseudo-null $\Lambda(H)$-submodule.

Now, suppose F is a number field and K_∞/F is a $\mathbb{Z}_p$-extension of F. The Tate–Shafarevich group $Ш(E/F)$ of an elliptic curve E/F is defined by the exact sequence

$$0 \longrightarrow Ш(E/F) \longrightarrow H^1(F, E) \longrightarrow \bigoplus_\omega H^1(F_\omega, E),$$

where ω runs over all archimedean and non-archimedean places of F. We let $Ш(E/F)_{p^i}$ to be the p^i-th torsion subgroup of $Ш(E/F)$ and

$$Ш(E/F)_{p^\infty} = \varinjlim_i Ш(E/F)_{p^i}.$$

Let F_n/F be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}/F. Here, we denote $Ш(E/F_n)_{p^\infty}$ by $Ш_n$. The following theorem by Wingberg ([66, Corollary 2.5]) is a generalization of Billot's result.

Theorem 4.3. *Suppose E/F is an elliptic curve with supersingular reduction for every prime of F over p. Assume that $Ш_n$ is finite for all n and the Pontryagin dual of $E(F_{\mathrm{cyc}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is $\Lambda(\Gamma)$-torsion. Then,*

$$T_{\Lambda(\Gamma)}(\mathfrak{X}(E/F_{\mathrm{cyc}})) \sim \mathfrak{X}^0(E/F_{\mathrm{cyc}}).$$

Remark 4.4. Assuming Conjecture A, we get that $T_{\Lambda(\Gamma)}(\mathfrak{X}(E/F_{\mathrm{cyc}}))$ has μ-invariant equal to zero.

Remark 4.5. Wingberg shows that the dual fine Selmer group can be identified with the adjoint of M, where $M = T_{\Lambda(\Gamma)}(\mathfrak{X}(E/F_{\mathrm{cyc}}))$ is the torsion submodule of the dual Selmer group. The adjoint of M is defined as

$\mathrm{Ext}^1_{\Lambda(G)}(M, \Lambda(G))$ (cf. [66, p. 473]). This adjoint is pseudo-isomorphic to M, as M is a torsion $\Lambda(G)$-module (cf. [49, Chapter 1, Section 2.2]).

Remark 4.6. In [43], Ahmed Matar gives a Galois theoretic proof of Wingberg's result (Theorem 4.3) with slightly different hypotheses. See [43, Theorem 1.1].

4.2. Numerical examples. In this section, we give some evidences for Conjecture A. We use the results of C. Wuthrich [67].

Theorem 4.7 ([67, Proposition 8.1])**.** *Suppose p is an odd prime and $E/\mathbb{Q}$ is an elliptic curve which admits an isogeny over $\mathbb{Q}$ of degree p. Then, the μ-invariant of the Pontryagin dual of the fine Selmer group $\mathfrak{X}^0(E/\mathbb{Q}_{\mathrm{cyc}})$ is zero.*

Recall that, by Theorem 3.6, $\mathfrak{X}^0(E/\mathbb{Q}_{\mathrm{cyc}})$ is a finitely generated $\Lambda(\Gamma)$-module. Therefore, if $E/\mathbb{Q}$ satisfies the hypothesis of Theorem 4.7, then Conjecture A is satisfied for $E/\mathbb{Q}$.

Here we summarize several numerical examples of elliptic curves $E/\mathbb{Q}$ with a description of their fine Selmer groups for specific primes p discussed in [67, Sections 9, 10, and 11].

Cremona label of $E/\mathbb{Q}$	Mordell–Weil rank of E	Fine Selmer groups $\mathrm{Sel}_p^0(E/\mathbb{Q}_{\mathrm{cyc}})$
11A1	0	Trivial for all odd primes $p \neq 5$ Finite (but non-trivial) for $p = 5$
11A2	0	Trivial for all odd primes
11A3	0	Trivial for all odd primes
182D1	0	Conjecture A holds for $p = 5$
37A1	1	Finite for all odd primes $p < 1000$
53A1	1	Finite for $p = 3$
5692A1	2	Conjecture A holds for $p = 3$

Remark 4.8. Note that the Pontryagin dual of a finite (resp. trivial) group is again finite (resp. trivial). Therefore, when the fine Selmer $\mathrm{Sel}_p^0(E/\mathbb{Q}_{\mathrm{cyc}})$ is finite, its Pontryagin dual $\mathfrak{X}^0(E/\mathbb{Q}_{\mathrm{cyc}})$ is finite (and so pseudo-null). In particular, Conjecture A holds here.

Remark 4.9. The first three elliptic curves (11A1, 11A2, and 11A3) in the above table represent all the elliptic curves $E/\mathbb{Q}$ of conductor 11 up to isogeny.

5. Signed Selmer groups and p-adic Hodge Theory

In this section, we revisit signed Selmer groups using Fontaine's p-adic Hodge Theory. These definitions of signed Selmer groups are due to Lei,

Loeffler and Zerbes (see [39, 40, 42]). They defined the "local conditions" (cf. (5.2)) on these Selmer groups using Fontaine's rings $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ (cf. Section 5.3) over the extension $L(\mu_{p^\infty})$ where L is a finite extension of $\mathbb{Q}_p$. The Fontaine's ring $\widetilde{\mathbb{E}}$ is the tilt of the perfectoid field $\widehat{L(\mu_{p^\infty})}$ in the terminology of Scholze (cf. Theorem 5.4). This tilting construction works in general for any perfectoid field. Therefore, a natural question is to ask whether the local conditions on the signed Selmer groups can be generalized using the tilt of any general perfectoid field and thereby extend the definition of signed Selmer groups for any general p-adic Lie extension (other than the cyclotomic extension, for which they are already defined by Kobayashi for $a_p = 0$ and Sprung for a_p nonzero). We record this as an open question (see Question 6.10). This is the motivation behind this section and therefore we start by defining the Fontaine rings using the language of perfectoid spaces of Scholze [60]. At the moment, there are certain obstructions mentioned in the introduction in generalizing signed Selmer groups to any p-adic Lie extension.

5.1. Perfectoid and Tilts. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. Here, we will write Γ_L for the Galois group $\mathrm{Gal}(L(\mu_{p^\infty})/L)$ which we identify with $\mathbb{Z}_p^*$ via the cyclotomic character χ_L. Fix π a uniformizer of L normalized by $|\pi| = q^{-1}$ where $q = |k_L|$, the cardinality of the residue field k_L of L. Here, we will follow the exposition of [57]. Up to notation, this is very close to the presentation of Fontaine–Wintenberger's theory for cyclotomic extension (see [23]). Although we will cite results in [57] because we want to use the perfectoid language, the constructions that we will review below are originally due to Fontaine [22].

Definition 5.1. Any intermediate field $L \subset K \subset \mathbb{C}_p$ is called perfectoid, if it satisfies the following conditions:

(1) K is complete,
(2) The value group $|K^*|$ is dense in $\mathbb{R}_{>0}^*$, and
(3) The p-power map

$$\begin{aligned} \mathcal{O}_K/p\mathcal{O}_K &\longrightarrow \mathcal{O}_K/p\mathcal{O}_K \\ x &\longmapsto x^p \end{aligned}$$

is surjective.

One can show that every element of the value group $|K^*|$ is a p-th power. Let K be a perfectoid field as in the definition above. From the field K, we will construct a new field $K^\flat$ (the tilt of K) of characteristic p. We first

choose an element $\varpi \in \mathfrak{m}_K$ such that $|\varpi| \geq |\pi|$ and define

$$\mathcal{O}_{K^\flat} := \varprojlim \left(\cdots \xrightarrow{(\cdot)^q} \mathcal{O}_K/\varpi\mathcal{O}_K \xrightarrow{(\cdot)^q} \mathcal{O}_K/\varpi\mathcal{O}_K \xrightarrow{(\cdot)^q} \cdots \xrightarrow{(\cdot)^q} \mathcal{O}_K/\varpi\mathcal{O}_K \right)$$
$$= \{(\ldots, \alpha_i, \ldots, \alpha_1, \alpha_0) \in (\mathcal{O}_K/\varpi\mathcal{O}_K)^{\mathbb{N}_0} \ : \ \alpha_{i+1}^q = \alpha_i \ \text{ for all } i \geq 0\}.$$

Let $K^\flat$ be the fraction field of the integral domain $\mathcal{O}_{K^\flat}$. Let k be the residue field of K. The k-algebra $\mathcal{O}_{K^\flat}$ is perfect and one can further show that, for any element $\alpha = (\ldots, \alpha_i, \ldots, \alpha_0) \in \mathcal{O}_{K^\flat}$, if we choose $a_i \in \mathcal{O}_K$ such that $a_i \bmod \varpi\mathcal{O}_K = \alpha_i$, then the limit

$$\alpha^\sharp := \varinjlim_{i \to \infty} a_i^{q^i} \in \mathcal{O}_K$$

exists and is independent of the choice of lifts a_i's. This gives the (untilt) map

$$\begin{aligned} \mathcal{O}_{K^\flat} &\longrightarrow \mathcal{O}_K \\ \alpha &\longrightarrow \alpha^\sharp \end{aligned}$$

which is a well-defined multiplicative map such that $\alpha^\sharp \bmod \varpi\mathcal{O}_K = \alpha_0$. We have a multiplicative bijection

$$\mathcal{O}_{K^\flat} \xrightarrow{\sim} \varprojlim_{(\cdot)^q} \mathcal{O}_K$$

given by

$$\alpha \longmapsto (\ldots, (\alpha^{\frac{1}{q^i}})^\sharp, \ldots, \alpha^\sharp)$$

which shows that $\mathcal{O}_{K^\flat}$ is independent of the choice of the element ϖ.

5.2. Values on Tilt. The map

$$\begin{aligned} |\cdot|_\flat : \mathcal{O}_{K^\flat} &\longrightarrow \mathbb{R}_{\geq 0} \\ \alpha &\longrightarrow |\alpha^\sharp| \end{aligned}$$

is a non-archimedean absolute value such that

(i) $|\mathcal{O}_{K^\flat}|_\flat = |\mathcal{O}_K|$;
(ii) $\mathfrak{m}_{K^\flat} := \{\alpha \in \mathcal{O}_{K^\flat} \ : |\alpha|_{K^\flat} < 1\}$ is the unique maximal ideal in $\mathcal{O}_{K^\flat}$ and we have $\mathcal{O}_{K^\flat}/\mathfrak{m}_{K^\flat} \cong \mathcal{O}_K/\mathfrak{m}_K$;
(iii) Let $\varpi^\flat \in \mathcal{O}_{K^\flat}$ be any element such that $|\varpi^\flat|_\flat = |\varpi|$, then we have an isomorphism $\mathcal{O}_{K^\flat}/\varpi^\flat\mathcal{O}_{K^\flat} \cong \mathcal{O}_K/\varpi\mathcal{O}_K$.

Since $K^\flat$ is the fraction field of the integral domain $\mathcal{O}_{K^\flat}$, the norm $|\cdot|_\flat$ extends to a non-archimedean absolute value on $K^\flat$. One can show that the value group of $K^\flat$ under the norm $|\cdot|_\flat$ is same as the value group of K under $|\cdot|$. Under the norm $|\cdot|_\flat$, $\mathcal{O}_{K^\flat}$ is the ring of integers of $K^\flat$. Further, $K^\flat$ with $|\cdot|_\flat$ is a perfect and complete non-archimedean field of

characteristic p. When $K = \mathbb{C}_p$, Lemma 1.4.10 of [57] shows that its tilt $\mathbb{C}_p^\flat$ is algebraically closed.

An important example of a perfectoid field is the completion $\widehat{L(\mu_{p^\infty})}$ of the field $L(\mu_{p^\infty})$ (cf. [57, Proposition 1.4.12]). We have a natural continuous action (cf. [57, Lemma 1.4.13]) of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ on $\mathbb{C}_p^\flat$ which is continuous with respect to the topology induced by $|\cdot|_\flat$.

The subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L(\mu_{p^\infty}))$ fixes $\widehat{L(\mu_{p^\infty})}^\flat \subseteq \mathbb{C}_p^\flat$ and hence we have a continuous action of $\Gamma_L = \mathrm{Gal}(L(\mu_{p^\infty})/L)$ on the tilt $\widehat{L(\mu_{p^\infty})}^\flat$. Let

$$\varepsilon = (\ldots, \varepsilon_i, \ldots, \varepsilon_1, \varepsilon_0) \in \widehat{L(\mu_{p^\infty})}^\flat$$

be such that $\varepsilon_0 = 1$, $\varepsilon_1 \neq 1$ and let $X = \varepsilon - 1$. Then, we have a well-defined embedding of fields (cf. [57, p. 50])

$$k((X)) \xrightarrow{\theta} \widehat{L(\mu_{p^\infty})}^\flat, \tag{5.1}$$

where k is the common residue field of $L(\mu_{p^\infty})$, $\widehat{L(\mu_{p^\infty})}$, and $\widehat{L(\mu_{p^\infty})}^\flat$ (cf. [57, Proposition 1.3.12(i) and Lemma 1.4.6(iv)]). We denote the image of θ by $\mathbb{E}_L$. Then, $(\mathbb{E}_L, |\cdot|_\flat)$ is a complete non-archimedean discretely valued field with residue field k which is preserved by the Γ_L-action on $\widehat{L(\mu_{p^\infty})}^\flat$.

Remark 5.2. See [1] and [3] for some progress in the generalization of the construction of the "field of norms" $k((X))$ and see [59] for the construction of the field of norms when the residue field is imperfect.

Remark 5.3. If K/F is an infinite Galois extension whose Galois group is a p-adic Lie group in which the inertia subgroup is open, then the completion of K is a perfectoid field. This is essentially due to Sen [61], see also [15, Theorem 2.13]. The fact that the extension K/F is deeply ramified if and only if the completion of K is a perfectoid field can be found (in a much more general setting) in [25, Proposition 6.6.6].[11]

5.3. Perfect hull. Suppose E is any field of characteristic $p > 0$ and let $\overline{E}/E$ be an algebraic closure. Then, we define

$$E^{perf} := \{a \in \overline{E} \mid a^{p^m} \in E \text{ for some } m \geq 0\},$$

which is called the *perfection (or perfect hull)* of E. In particular, E^{perf} is the smallest intermediate field of $\overline{E}/E$ which is perfect. Here are two most crucial theorems whose proofs can be found in [57, Proposition 1.4.17 and Proposition 1.4.27].

[11] We thank the referee for pointing this out to us.

Theorem 5.4 (Scholze)**.** *Let $\mathbb{E}_L$ be the image of the injective map θ in* (5.1)*. Then, we have:*

(1) $\widehat{\mathbb{E}_L^{perf}} = \widehat{L(\mu_{p^\infty})}^\flat$,
(2) $\widehat{\mathbb{E}_L^{sep}} = \mathbb{C}_p^\flat$.

For simplicity, we let $\mathbb{E} := \mathbb{E}_L^{sep}$ and $\widetilde{\mathbb{E}} := \widehat{\mathbb{E}_L^{perf}}$. We can identify $\mathcal{H}_L = \mathrm{Gal}(\overline{L}/L(\mu_{p^\infty}))$ with $\mathrm{Gal}(\mathbb{E}/\mathbb{E}_L)$; this is Fontaine–Wintenberger's theorem [24]. (This now holds for all perfectoid fields by the work of Scholze). By continuity, the action of $\mathrm{Gal}(\overline{L}/L)$ on $\mathbb{E}$ extends uniquely to an action on $\widetilde{\mathbb{E}}$ and we let $\widetilde{\mathbb{E}}_L := \widetilde{\mathbb{E}}^{\mathcal{H}_L}$. Let $\widetilde{\mathbb{A}}$ be the ring of Witt vectors of $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{B}} = \widetilde{\mathbb{A}}[p^{-1}]$.

Recall from (5.1) that $\mathbb{E}_L \cong k((\varepsilon-1))$ and let $\mathbb{E}_L^+ \cong k[\![\varepsilon-1]\!]$ be its valuation ring. Let $\mathbb{A}_L^+$ be a complete regular local ring of dimension 2 such that $\mathbb{A}_L^+/(p) \cong \mathbb{E}_L^+$. Such a lift exists and it is unique (see [14]). Let $\mathbb{A}_L$ be the p-adic completion of $\mathbb{A}_L^+[\frac{1}{\varepsilon-1}]$. Define $\mathbb{B}_L^+ := \mathbb{A}_L^+[\frac{1}{p}]$ and $\mathbb{B}_L := \mathbb{A}_L[\frac{1}{p}]$.

The rings $\mathbb{A}_L$, $\mathbb{A}_L^+$, $\mathbb{B}_L$, and $\mathbb{B}_L^+$ are endowed with an action of Γ, a semi-linear action of the Frobenius φ, and its left inverse ψ. When L is unramified, we have simple formulae for Γ, φ, and ψ-action. Write $X := \varepsilon - 1$ and $\gamma \in \Gamma$, then we have (cf. [5, 51]),

$$\begin{aligned}\gamma(X) &= (1+X)^{\chi_L(\gamma)} - 1,\\ \varphi(X) &= (1+X)^p - 1,\\ \varphi\circ\psi(f(X)) &= \frac{1}{p}\sum_{\zeta^p=1} f\left(\zeta(1+X)-1\right).\end{aligned}$$

Let $\mathbb{B}$ be the p-adic completion of the maximal unramified extension of $\mathbb{B}_{\mathbb{Q}_p} = \mathbb{A}_{\mathbb{Q}_p}[p^{-1}]$ in $\widetilde{\mathbb{B}} = W(\widetilde{\mathbb{E}})[\frac{1}{p}] = \widetilde{\mathbb{A}}[\frac{1}{p}]$ and let $\mathbb{A} = \mathbb{B} \cap \widetilde{\mathbb{A}}$.

These rings are also stable under Frobenius and the absolute Galois group $\mathcal{G}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ (under this identification, $\mathbb{A}_L = \mathbb{A}^{\mathcal{H}_L}$). We write $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\mathbb{Q}_p})$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathbb{Q}_p})$) for the category of finitely generated $\mathbb{Z}_p$-modules (resp. finite dimensional $\mathbb{Q}_p$-vector space) with a continuous action of $\mathcal{G}_{\mathbb{Q}_p}$.

For a p-adic representation $T \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\mathbb{Q}_p})$ (resp. $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathbb{Q}_p})$), Fontaine defines a free finitely generated module over $\mathbb{A}_L$ of rank equal to $\mathrm{rank}_{\mathbb{Z}_p}(T)$ (resp. a finite dimensional vector space over $\mathbb{B}_L$ of dimension equal to $\dim_{\mathbb{Q}_p}(V)$) by $D_L(T) = (\mathbb{A}\otimes_{\mathbb{Z}_p} T)^{\mathcal{H}_L}$ (resp. $D_L(V) = (\mathbb{B}\otimes_{\mathbb{Q}_p} V)^{\mathcal{H}_L}$). The module $D_L(T)$ is a (φ,Γ)-module [23]. These are equipped with a commuting semi-linear action of φ and Γ_L. Further, we can define the left inverse ψ of φ on $\mathbb{A}$ which then extends to a left inverse ψ of φ on $D_L(V)$ and $D_L(T)$ (cf. [13, 14, 28]).

5.4. Iwasawa Cohomology. Let $T \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}_L)$ and let L_∞ be a p-adic Lie extension of L. We define

$$H^i_{\mathrm{Iw}}(L_\infty, T) := \varprojlim_K H^i(K, T),$$

where K varies over the finite extensions of L contained in L_∞ and the inverse limit is taken with respect to the corestriction maps.

If $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, then we write

$$H^i_{\mathrm{Iw}}(L_\infty, V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{\mathrm{Iw}}(L_\infty, T),$$

which is independent of the choice of the lattice $T \subset V$. These are finitely generated $\Lambda(G)$-modules, where $G = \mathrm{Gal}(L_\infty/L)$.

5.5. Fontaine Isomorphism. Suppose $L/\mathbb{Q}_p$ is unramified. In the case when $L_\infty = L(\mu_{p^\infty})$, we have a canonical isomorphism of $\Lambda(\Gamma_L)$-modules:

$$h^1_{\mathrm{Iw},T} : D_L(T)^{\psi=1} \xrightarrow{\cong} H^1_{\mathrm{Iw}}(L(\mu_{p^\infty}), T).$$

Further, if L is an unramified extension of $\mathbb{Q}_p$ and $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is a crystalline representation of $\mathrm{Gal}(\overline{L}/L)$ with non-negative Hodge–Tate weights and no quotient isomorphic to $\mathbb{Q}_p$, then we have

$$h^1_{\mathrm{Iw},T} : N_L(T)^{\psi=1} \xrightarrow{\cong} H^1_{\mathrm{Iw}}\left(L(\mu_{p^\infty}), T\right)$$

where $N_L(T)$ is the Wach module associated to T (see [14]; for the crystalline version see [6]).

5.6. Iwasawa Cohomology and μ-invariant. Recall the structure theorem of finitely generated modules over the Iwasawa algebra (see Theorem 2.9 for more details). By Shapiro's lemma, we know that for $L_\infty = L(\mu_{p^\infty})$, we have

$$H^i_{\mathrm{Iw}}(L_\infty, V) = H^i\left(\mathcal{G}_L, \mathcal{O}_L[\![\Gamma_L]\!] \otimes_{\mathcal{O}_L} V\right),$$

where the right hand side is the usual Galois cohomology with continuous cochains and $\mathcal{G}_L = \mathrm{Gal}(\overline{L}/L)$. We have the following [52, Appendix A].

Lemma 5.5.

(i) $H^i_{\mathrm{Iw}}(L_\infty, V) = 0$ *for* $i \neq 1, 2$.
(ii) $H^2_{\mathrm{Iw}}(L_\infty, V)$ *is a finitely generated* $\mathcal{O}_L$*-module (and therefore has* μ*-invariant zero).*

5.7. p-adic Hodge Theory and Signed Selmer groups. In this section, following [42], we define signed Selmer groups using p-adic Hodge theory. The reader is encouraged to consult [53] where the signed constructions first appear in relation to Perrin-Riou's logarithmic map.

5.7.1. ***Good supersingular elliptic curves.*** Suppose E is an elliptic curve over $\mathbb{Q}$. Let $p \geq 3$ and $a_p = 0$. Let $T := T_p(E)$ be the p-adic Tate module of E and write $V = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ which is a crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$ with Hodge–Tate weights 0, 1. Let $\{v_+, v_-\}$ be a basis of $\mathbb{D}_{cris}(V) = (\mathbb{B}_{cris} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{\mathbb{Q}_p}}$ (see [7] or [23]) such that the matrix of φ on $\mathbb{D}_{cris}(V)$ in this basis is

$$\begin{bmatrix} 0 & -1 \\ p & 0 \end{bmatrix}.$$

One can find such a matrix because φ satisfies $\varphi^2 - \frac{a_p}{p}\varphi + \frac{1}{p} = 0$. Let $q = \varphi(\pi)/\pi$ where $\pi := [\varepsilon] - 1$. Define

$$\log_+(1+\pi) = \prod_{i \geq 0} \frac{\varphi^{2i}(q)}{p},$$

and

$$\log_-(1+\pi) = \prod_{i \geq 0} \frac{\varphi^{2i+1}(q)}{p},$$

where $\varepsilon = (\varepsilon^{(i)})$ was the fixed element in $\widetilde{\mathbb{E}}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$, and $[\varepsilon]$ is the Teichmüller lift of $\varepsilon \in \widetilde{\mathbb{E}}$. Suppose

$$\begin{bmatrix} n_+ \\ n_- \end{bmatrix} = M \begin{bmatrix} v_+ \\ v_- \end{bmatrix},$$

where

$$M = \begin{bmatrix} \log_+(1+\pi) & 0 \\ 0 & \log_-(1+\pi) \end{bmatrix}.$$

Then, n_+ and n_- from a basis of the Wach module $\mathbb{N}(T)$.

By Section 5.5, any element

$$x \in \mathbb{N}_{\mathbb{Q}_p}(T)^{\psi=1} \cong \mathbb{D}_{\mathbb{Q}_p}(T)^{\psi=1} \cong H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$$

can be written as

$$x = x_+ v_+ + x_- v_- = x'_+ n_+ + x'_- n_-,$$

where $x_+ = x'_+ \log_+(1+\pi)$ and $x_- = x'_- \log_-(1+\pi)$. Then, for $i = \pm$, we define

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)^i := \{x \in \mathbb{N}_{\mathbb{Q}_p}(T)^{\psi=1} \mid \varphi(x_i) = -p\psi(x_i)\}.$$

For $n \geq 1$, we define $H^1(\mathbb{Q}_{p,n}, T)^i$ to be the image of $H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)^i$ under the natural projection map

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \longrightarrow H^1(\mathbb{Q}_{p,n}, T),$$

where $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$.

In [40, Section 5.2.1], the authors define two Coleman maps

$$\mathrm{Col}_i = H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) \longrightarrow \Lambda(\Gamma),$$

and by Remark 3.2 of [42], we know that

$$H^1(\mathbb{Q}_p(\mu_{p^\infty}),T)^i = H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}),T) \cap \ker(\mathrm{Col}_i).$$

Let $H^1_f(\mathbb{Q}_{p,n},E_{p^\infty})^\pm$ be the exact annihilator of $H^1(\mathbb{Q}_{p,n},T)^\pm$ under the Pontryagin duality

$$[\sim,\sim] : H^1(\mathbb{Q}_{p,n},T) \times H^1(\mathbb{Q}_{p,n},E_{p^\infty}) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Let S be a finite set of primes of $\mathbb{Q}$ containing p and all the primes where E has bad reduction and the infinite primes. For $v \in S$, define the "new" local conditions

$$J_v^\pm(\mathbb{Q}_n) = \bigoplus_{w_n|v} \frac{H^1(\mathbb{Q}_{w_n,n},E_{p^\infty})}{H^1_f(\mathbb{Q}_{w_n,n},E_{p^\infty})^\pm}, \tag{5.2}$$

for w_n prime of $\mathbb{Q}_n = \mathbb{Q}(\mu_{p^n})$ over ν.

We write $J_v^\pm(\mathbb{Q}_{\mathrm{cyc}}) = \varinjlim_n J_v^\pm(\mathbb{Q}_n)$. We define

$$\mathrm{Sel}_p^\pm(E/\mathbb{Q}_n) := \ker\left(H^1(\mathbb{Q}_S/\mathbb{Q}_n,E_{p^\infty}) \longrightarrow \bigoplus_{v\in S} J_v^\pm(E/\mathbb{Q}_n)\right);$$
$$\mathrm{Sel}_p^\pm(E/\mathbb{Q}_{\mathrm{cyc}}) := \varinjlim_n \mathrm{Sel}^\pm(E/\mathbb{Q}_n).$$

Further, Lei shows that the "local conditions" $H^1_f(\mathbb{Q}_{w_n,n},E_{p^\infty})^\pm$ in the definition of $J_v^\pm(\mathbb{Q}_n)$ exactly coincides with $E^\pm(\mathbb{Q}_{w_n,n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ of Kobayashi (see the paragraph after the proof of Lemma 4.12 of [39]). Therefore, this p-adic Hodge theory definitions of Selmer groups are the usual Kobayashi's $\pm$ Selmer groups.

The reader is encouraged to consult Florian Sprung's article [30] published in the same volume for an outline of the construction of multi-signed Selmer groups in the supersingular setting over a $\mathbb{Z}_p^2$-extension of a quadratic imaginary number field in which p splits. Note that [12] give similar constructions to study Iwasawa theory of motives associated to abelian varieties at non-ordinary primes over the cyclotomic extension of a totally real or a CM field.

6. Analogy with the ordinary reduction case

Here, we list some of the main results proved in the case where we consider elliptic curves with good ordinary reduction over the primes above p and we can find analogous results for elliptic curves with supersingular reduction. The main take away here is that, many of the properties that the Selmer groups enjoy in the ordinary reduction case are observed for the signed Selmer groups when the reduction type is supersingular.

Suppose $p \geq 5$ is an odd prime. We have the following conjecture by Mazur ([44]).[12]

Conjecture 6.1. Suppose E/F has good ordinary reduction at all the primes above p. Then, $\mathfrak{X}(E/F_{\mathrm{cyc}})$ is a finitely generated[13] $\Lambda(\Gamma)$-torsion module.

Remark 6.2. This has been shown in several cases. For example, Mazur proved in [44] that, if $\mathrm{Sel}_p(E/F)$ is finite, then the above conjecture holds. Furthermore, Kato in [33] proved this conjecture for the case where $E/\mathbb{Q}$ is an elliptic curve defined over $\mathbb{Q}$.

Often, the results proved for elliptic curves with good ordinary reduction motivate results in the supersingular reduction case like Theorem 3.8. We have the following (cf. [26]).

Theorem 6.3. *Suppose E/F is an elliptic curve with good ordinary reduction at all the primes above p. Assume $\mathfrak{X}(E/F_{\mathrm{cyc}})$ is a finitely generated torsion $\Lambda(\Gamma)$-module. Furthermore, assume that $E(F)$ has no element of order p. Then, $\mathfrak{X}(E/F_{\mathrm{cyc}})$ has no non-zero finite $\Lambda(\Gamma)$-submodule.*

The following theorem (cf. [50]) is analogous to Theorem 3.3.

Theorem 6.4. *Suppose p is an odd prime and E/F is an elliptic curve with good ordinary reduction at all the primes above p. Then, $\mathfrak{X}(E/F_{\mathrm{cyc}})$ is a finitely generated torsion $\Lambda(\Gamma)$-module if and only if the sequence*

$$0 \longrightarrow \mathrm{Sel}_p(E/F_{\mathrm{cyc}}) \longrightarrow H^1(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) \xrightarrow{\lambda_{\mathrm{cyc}}} \bigoplus_{\nu \in S} J_v(E/F_{\mathrm{cyc}}) \tag{6.1}$$

is short exact and $H^2(F_S/F_{\mathrm{cyc}}, E_{p^\infty}) = 0$.

Let

$$\widetilde{F}_\infty := F(E_{p^\infty}) = \bigcup_{n \geq 0} F(E_{p^n}).$$

$\widetilde{F}_\infty$ is often referred as the trivializing extension of F (with respect to E and the prime p). Let $\mathfrak{X}(E/\widetilde{F}_\infty)$ denote the Pontryagin dual of the Selmer group $\mathrm{Sel}_p(E/\widetilde{F}_\infty)$ and $\widetilde{G} := \mathrm{Gal}(\widetilde{F}_\infty/F)$. We mention the following result of Ochi–Venjakob (cf. [47, Theorem 5.1]).

Theorem 6.5. *Let E/F be an elliptic curve without complex multiplication. Let $p \geq 5$ and suppose E has good ordinary reduction at all the primes above p. Moreover, assume that $\mathfrak{X}(E/\widetilde{F}_\infty)$ is a finitely generated $\Lambda(\widetilde{G})$-torsion module. Then, $\mathfrak{X}(E/\widetilde{F}_\infty)$ has no non-zero pseudo-null $\Lambda(\widetilde{G})$-submodule.*

[12] This conjecture can be extended to other Galois extensions where the Galois group is a general p-adic Lie group.

[13] It is not hard to show that, for any $\mathbb{Z}_p$-extension K_∞/F of F, $\mathfrak{X}(E/K_\infty)$ is a finitely generated module over the Iwasawa algebra of $\mathrm{Gal}(K_\infty/F)$ (cf. [20, Lemma 2.4]).

Remark 6.6. The reader may also look at a similar theorem by B. Perrin-Riou for the CM case ([48, Theorem 2.4]).

Recall the definition 2.8 of Γ-Euler characteristic. The following theorem is analogous to Theorem 3.11 (cf. [20, Theorem 3.3]).

Theorem 6.7. *Suppose E/F has good ordinary reduction at all the primes above p and $\mathrm{Sel}_p(E/F)$ is finite. Then, $\chi(\Gamma, \mathrm{Sel}_p(E/F_{\mathrm{cyc}}))$ is well-defined and is given by*

$$\chi(\Gamma, \mathrm{Sel}_p(E/F_{\mathrm{cyc}})) = \frac{\#Ш(E/F)_{p^\infty}}{\#(E(F)_{p^\infty})^2} \times \prod_{\nu} c_\nu^{(p)} \times \prod_{\nu|p} (d_\nu^{(p)})^2.$$

Here,

(1) *ν runs over all the finite primes of F.*
(2) *c_ν is the Tamagawa number of E at the prime ν and $c_\nu := [E(F_\nu) : E_0(F_\nu)]$, where $E_0(F_\nu)$ denotes the subgroup of points of $E(F_\nu)$ with non-singular reduction.*
(3) *$d_\nu := \#(\widetilde{E}_\nu(f_\nu))$, where f_ν is the residue field of F_ν and $\widetilde{E}_\nu(f_\nu)$ is the reduction of E modulo ν.*
(4) *$c_\nu^{(p)}$ (resp. $d_\nu^{(p)}$) is the largest power of p dividing c_ν (resp. d_ν).*

Remark 6.8. Similarly, one may look at Theorem 1.1 of [16] for an analogue of Theorem 3.18 in the ordinary setting.

Now, suppose E/F is an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_F$. Suppose F is an imaginary quadratic field with class number one. Let p be an odd prime such that it splits in F, $p\mathcal{O}_F = \mathfrak{p}\bar{\mathfrak{p}}$ and suppose E has good reduction at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Since E/F has complex multiplication, then $\widetilde{G} = \mathrm{Gal}(\widetilde{F}_\infty/F)$ is topologically isomorphic to $\mathbb{Z}_p^2$ and $\Lambda(\widetilde{G}) \cong \mathbb{Z}_p[\![T_1, T_2]\!]$. We have (cf. [56, Proposition 3.2]):

Theorem 6.9. *$\mathfrak{X}(E/\widetilde{F}_\infty)$ is a finitely generated torsion $\Lambda(\widetilde{G})$-module and*

$$\mu_{\widetilde{G}}(\mathfrak{X}(E/\widetilde{F}_\infty)) = 0.$$

To conclude, we pose two questions that remain open.

Question 6.10. Can we define the signed Selmer groups for general p-adic Lie extensions of a given number field?

In cases where signed Selmer groups are defined, many of the results that have been proven for the Selmer group of elliptic curves with good ordinary reduction are expected to be true in the supersingular reduction for the signed Selmer groups under suitable hypotheses. For example, we may ask:

Question 6.11. Is it possible, under suitable conditions for an elliptic curve with supersingular reduction over the primes above p, to prove that the μ-invariants of the Pontryagin dual of the signed Selmer groups (also known as plus/minus μ-invariants) vanish over the cyclotomic extension?

Remark 6.12. It is worth mentioning that Greenberg conjectured that, in the ordinary setting for an elliptic curve over the rational numbers, if $E_p(\overline{\mathbb{Q}})$ is irreducible then the μ-invariant of the Pontryagin dual of the Selmer group vanishes over the cyclotomic extension (cf. [27, Conjecture 1.11]). Furthermore, Pollack in [54] notes that when p has supersingular reduction for an elliptic curve over the rational numbers, then E_p is always irreducible. Pollack then extends Greenberg's conjecture to the supersingular case (cf. [54, Conjecture 6.3]). In particular, Pollack conjectures that the (algebraic) plus/minus μ-invariants (coming from the Pontryagin dual of the signed Selmer groups) as well as the analytical plus/minus μ-invariants (coming from plus/minus L-functions) vanish over the cyclotomic extension of the rationals.[14]

Answering these questions should involve some new ideas.

References

[1] V. Abrashkin, "An analogue of the field-of-norms functor and of the Grothendieck conjecture", *J. Algebr. Geom.* **16** (2007), no. 4, p. 671-730.

[2] S. Ahmed & M. F. Lim, "On the Euler characteristics of signed Selmer groups", *Bull. Aust. Math. Soc.* **101** (2020), no. 2, p. 238-246.

[3] F. Andreatta, "Generalized ring of norms and generalized (φ, Γ)-modules", *Ann. Sci. Éc. Norm. Supér.* **39** (2006), no. 4, p. 599-647.

[4] K. Ardakov & K. A. Brown, "Ring-theoretic properties of Iwasawa algebras: a survey", *Doc. Math.* **Extra Vol.** (2006), p. 7-33.

[5] D. Benois, "On Iwasawa theory of crystalline representations", *Duke Math. J.* **104** (2000), no. 2, p. 211-267.

[6] L. Berger, "Bloch and Kato's exponential map: three explicit formulas", *Doc. Math.* **Extra Vol.** (2003), p. 99-129, Kazuya Kato's fiftieth birthday.

[7] ———, "Limites de représentations cristallines", *Compos. Math.* **140** (2004), no. 6, p. 1473-1498.

[8] ———, "Lifting the field of norms", *J. Éc. Polytech., Math.* **1** (2014), p. 29-38.

[9] L. Berger & L. Fourquaux, "Iwasawa theory and F-analytic Lubin-Tate (φ, Γ)-modules", *Doc. Math.* **22** (2017), p. 999-1030.

[10] P. Billot, "Quelques aspects de la descente sur une courbe elliptique dans le cas de réduction supersingulière", *Compos. Math.* **58** (1986), no. 3, p. 341-369.

[11] N. Bourbaki, *Elements of Mathematics. Algebra. II. Chapters 4–7*, Springer, 1990, Translated from the French by P. M. Cohn and J. Howie, vii+461 pages.

[12] K. Büyükboduk & A. Lei, "Integral Iwasawa theory of Galois representations for non-ordinary primes", *Math. Z.* **286** (2017), no. 1-2, p. 361-398.

[13] F. Cherbonnier & P. Colmez, "Représentations p-adiques surconvergentes", *Invent. Math.* **133** (1998), no. 3, p. 581-611.

[14] ———, "Théorie d'Iwasawa des représentations p-adiques d'un corps local", *J. Am. Math. Soc.* **12** (1999), no. 1, p. 241-268.

[14] We wish to thank the referee for pointing this, and Remark 6.8 to us.

[15] J. COATES & R. GREENBERG, "Kummer theory for abelian varieties over local fields", *Invent. Math.* **124** (1996), no. 1-3, p. 129-174.
[16] J. COATES & S. HOWSON, "Euler characteristics and elliptic curves. II", *J. Math. Soc. Japan* **53** (2001), no. 1, p. 175-235.
[17] J. COATES, P. SCHNEIDER & R. SUJATHA, "Links between cyclotomic and GL_2 Iwasawa theory", *Doc. Math.* **Extra Vol.** (2003), p. 187-215, Kazuya Kato's fiftieth birthday.
[18] ———, "Modules over Iwasawa algebras", *J. Inst. Math. Jussieu* **2** (2003), no. 1, p. 73-108.
[19] J. COATES & R. SUJATHA, "Fine Selmer groups of elliptic curves over p-adic Lie extensions", *Math. Ann.* **331** (2005), no. 4, p. 809-839.
[20] ———, *Galois cohomology of elliptic curves*, second ed., Narosa Publishing House, 2010, xii+98 pages.
[21] B. FERRERO & L. C. WASHINGTON, "The Iwasawa Invariant μ_p Vanishes for Abelian Number Fields", *Ann. Math.* **109** (1979), no. 2, p. 377-395.
[22] J.-M. FONTAINE, "Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate", *Ann. Math.* **115** (1982), no. 3, p. 529-577.
[23] ———, "Représentations p-adiques des corps locaux. I", in *The Grothendieck Festschrift, Vol. II*, Progress in Mathematics, vol. 87, Birkhäuser, 1990, p. 249-309.
[24] J.-M. FONTAINE & J.-P. WINTENBERGER, "Extensions algébrique et corps des normes des extensions APF des corps locaux", *C. R. Math. Acad. Sci. Paris* **288** (1979), no. 8, p. A441-A444.
[25] O. GABBER & L. RAMERO, *Almost ring theory*, Lecture Notes in Mathematics, vol. 1800, Springer, 2003, vi+307 pages.
[26] R. GREENBERG, "The structure of Selmer groups", *Proc. Natl. Acad. Sci. USA* **94** (1997), no. 21, p. 11125-11128.
[27] ———, "Iwasawa theory for elliptic curves", in *Arithmetic theory of elliptic curves (Cetraro, 1997)*, Lecture Notes in Mathematics, vol. 1716, Springer, 1999, p. 51-144.
[28] L. HERR, "Sur la cohomologie galoisienne des corps p-adiques", *Bull. Soc. Math. Fr.* **126** (1998), no. 4, p. 563-600.
[29] F. ITO SPRUNG, "Iwasawa theory for elliptic curves at supersingular primes: A pair of main conjectures", *J. Number Theory* **132** (2012), no. 7, p. 1483-1506.
[30] ———, "Chromatic Selmer groups and arithmetic invariants of elliptic curves", *J. Théor. Nombres Bordeaux* **33** (2021), no. 3.2, p. 1103-1114.
[31] K. IWASAWA, "On Γ-extensions of algebraic number fields", *Bull. Am. Math. Soc.* **65** (1959), p. 183-226.
[32] ———, "On $\mathbb{Z}_l$-Extensions of Algebraic Number Fields", *Ann. Math.* **98** (1973), no. 2, p. 246-326.
[33] K. KATO, "p-adic Hodge theory and values of zeta functions of modular forms", in *Cohomologies p-adiques et applications arithmétiques. III*, Astérisque, vol. 295, Société Mathématique de France, 2004, p. 117-290.
[34] B. D. KIM, "The plus/minus Selmer groups for supersingular primes", *J. Aust. Math. Soc.* **95** (2013), no. 2, p. 189-200.
[35] ———, "Signed-Selmer groups over the $\mathbb{Z}_p^2$-extension of an imaginary quadratic field", *Can. J. Math.* **66** (2014), no. 4, p. 826-843.
[36] T. KITAJIMA & R. OTSUKI, "On the plus and the minus Selmer groups for elliptic curves at supersingular primes", *Tokyo J. Math.* **41** (2018), no. 1, p. 273-303.
[37] S.-I. KOBAYASHI, "Iwasawa theory for elliptic curves at supersingular primes", *Invent. Math.* **152** (2003), no. 1, p. 1-36.
[38] B. KUPFERER & O. VENJAKOB, "Herr-complexes in the Lubin-Tate setting", 2020, `https://arxiv.org/abs/2006.07895`.
[39] A. LEI, "Iwasawa theory for modular forms at supersingular primes", *Compos. Math.* **147** (2011), no. 3, p. 803-838.
[40] A. LEI, D. LOEFFLER & S. L. ZERBES, "Coleman maps and the p-adic regulator", *Algebra Number Theory* **5** (2011), no. 8, p. 1095-1131.

[41] A. Lei & R. Sujatha, "On Selmer groups in the supersingular reduction case", *Tokyo J. Math.* **43** (2020), no. 2, p. 455-479.
[42] A. Lei & S. L. Zerbes, "Signed Selmer groups over p-adic Lie extensions", *J. Théor. Nombres Bordeaux* **24** (2012), no. 2, p. 377-403.
[43] A. Matar, "On the Λ-cotorsion subgroup of the Selmer group", *Asian J. Math.* **24** (2020), no. 3, p. 437-456.
[44] B. Mazur, "Rational points of abelian varieties with values in towers of number fields", *Invent. Math.* **18** (1972), p. 183-266.
[45] B. Mazur & A. Wiles, "Class fields of abelian extensions of $\mathbb{Q}$", *Invent. Math.* **76** (1984), no. 2, p. 179-330.
[46] J. Neukirch, A. Schmidt & K. Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer, 2008, xvi+825 pages.
[47] Y. Ochi & O. Venjakob, "On the structure of Selmer groups over p-adic Lie extensions", *J. Algebr. Geom.* **11** (2002), no. 3, p. 547-580.
[48] B. Perrin-Riou, "Groupe de Selmer d'une courbe elliptique à multiplication complexe", *Compos. Math.* **43** (1981), no. 3, p. 387-417.
[49] ———, "Arithmétique des courbes elliptiques et théorie d'Iwasawa", *Mém. Soc. Math. Fr., Nouv. Sér.* (1984), no. 17, p. 130.
[50] ———, "Théorie d'Iwasawa et hauteurs p-adiques (cas des variétés abéliennes)", in *Séminaire de Théorie des Nombres, Paris, 1990–91*, Progress in Mathematics, vol. 108, Birkhäuser, 1993, p. 203-220.
[51] ———, "Théorie d'Iwasawa des représentations p-adiques sur un corps local", *Invent. Math.* **115** (1994), no. 1, p. 81-161, with an appendix by Jean-Marc Fontaine.
[52] ———, *Fonctions L p-adiques des représentations p-adiques*, Astérisque, vol. 229, Société Mathématique de France, 1995, 198 pages.
[53] ———, "Arithmétique des courbes elliptiques à réduction supersingulière en p", *Exp. Math.* **12** (2003), no. 2, p. 155-186.
[54] R. Pollack, "On the p-adic L-function of a modular form at a supersingular prime", *Duke Math. J.* **118** (2003), no. 3, p. 523-558.
[55] G. Ponsinet, "Universal norms and the Fargues–Fontaine curve", `https://arxiv.org/abs/2010.02292`, 2020.
[56] S. Ramdorai, "On the μ-invariant in Iwasawa theory", in *WIN—women in numbers*, Fields Institute Communications, vol. 60, American Mathematical Society, 2011, p. 265-276.
[57] P. Schneider, *Galois representations and (φ, Γ)-modules*, Cambridge Studies in Advanced Mathematics, vol. 164, Cambridge University Press, 2017, vii+148 pages.
[58] P. Schneider & O. Venjakob, "Coates-Wiles homomorphisms and Iwasawa cohomology for Lubin-Tate extensions", in *Elliptic curves, modular forms and Iwasawa theory*, Springer Proceedings in Mathematics & Statistics, vol. 188, Springer, 2016, p. 401-468.
[59] A. J. Scholl, "Higher fields of norms and (φ, Γ)-modules", *Doc. Math.* **Extra Vol.** (2006), p. 685-709.
[60] P. Scholze, "Perfectoid spaces", *Publ. Math., Inst. Hautes Étud. Sci.* **116** (2012), p. 245-313.
[61] S. Sen, "Ramification in p-adic Lie extensions", *Invent. Math.* **17** (1972), p. 44-50.
[62] J.-P. Serre, "Sur la dimension cohomologique des groupes profinis", *Topology* **3** (1965), p. 413-420.
[63] R. Sujatha & M. Witte, "Fine Selmer groups and isogeny invariance", in *Geometry, algebra, number theory, and their information technology applications*, Springer Proceedings in Mathematics & Statistics, vol. 251, Springer, 2018, p. 419-444.
[64] O. Venjakob, "On the structure theory of the Iwasawa algebra of a p-adic Lie group", *J. Eur. Math. Soc.* **4** (2002), no. 3, p. 271-311.
[65] L. C. Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer, 1997, xiv+487 pages.
[66] K. Wingberg, "Duality theorems for abelian varieties over $\mathbf{Z}_p$-extensions", in *Algebraic number theory*, Advanced Studies in Pure Mathematics, vol. 17, Academic Press Inc., 1989, p. 471-492.

[67] C. WUTHRICH, "Iwasawa theory of the fine Selmer group", *J. Algebr. Geom.* **16** (2007), no. 1, p. 83-108.

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