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On Short Sums Involving Fourier Coefficients of Maass Forms

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Résumé. Nous étudions les sommes des valeurs propres des opérateurs de Hecke des formes paraboliques de Hecke–Maass pour le groupe $\mathrm{SL}(n, \mathbb{Z})$ avec $n \geq 3$ quelconque, sur des intervalles courts d'une certaine longueur, en admettant l'hypothèse de Lindelöf généralisée et une estimation pour l'exposant en direction de la conjecture de Ramanujan–Petersson, un peu plus forte que celle qui est actuellement connue. En particulier, dans cette situation, nous donnons une évaluation asymptotique du deuxième moment des sommes en question.

Abstract. We study sums of Hecke eigenvalues of Hecke–Maass cusp forms for the group $\mathrm{SL}(n, \mathbb{Z})$, with general $n \geq 3$, over short intervals of certain length under the assumption of the generalised Lindelöf hypothesis and a slightly stronger upper bound concerning the exponent towards the Ramanujan–Petersson conjecture than is currently known. In particular, in this case we evaluate the second moment of the sums in question asymptotically.

1. Introduction

Let f be a non-trivial Maass cusp form of type $\nu \in \mathbb{C}^{n-1}$ for the full modular group $\mathrm{SL}(n, \mathbb{Z})$ with Fourier coefficients $A(m_1, \ldots, m_{n-1}) = A_f(m_1, \ldots, m_{n-1})$, where $n \geq 3$ is a fixed positive integer throughout the article. The Fourier–Whittaker expansion of f is given by

$$
f(z) = \sum_{\gamma \in U(n-1, \mathbb{Z}) \backslash \mathrm{SL}(n-1, \mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}}
$$
$$
\cdot W_{\mathrm{Jacquet}} \left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu; \psi_{\left(1, \ldots, 1, \frac{m_{n-1}}{|m_{n-1}|}\right)} \right),
$$

where

$$M = \begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix},$$

$U(n-1, \mathbb{Z})$ is the group of $(n-1) \times (n-1)$ integral upper triangular matrices with ones on the diagonal, $\psi_{(1,\ldots,1,m_{n-1}/|m_{n-1}|)}$ is a certain character of $U(n-1, \mathbb{Z})$, and W_{Jacquet} is the Jacquet–Whittaker function of type ν for the character $\psi_{(1,\ldots,1,m_{n-1}/|m_{n-1}|)}$. For more details we refer to Goldfeld's book [6]. We further assume that the Maass cusp form f is an eigenfunction for the full Hecke ring and normalised so that $A(1,\ldots,1) = 1$.

In this case it is known that the eigenvalue of f under the mth Hecke operator T_m (see Section 4) is given by $A(m, 1, \ldots, 1)$. The coefficients $A(m, 1, \ldots, 1)$ are fascinating number theoretic objects and they have been studied extensively as are the Fourier coefficients of holomorphic cusp forms and Maass cusp forms in the classical situation $n = 2$.

Obtaining estimates for the sum of Hecke eigenvalues of cusp forms is a classical problem with a long history. For normalised Fourier coefficients of holomorphic cusp forms, denoted by $a(m)$, the trivial bound for the long sum

$$\sum_{m \leq x} a(m)$$

is $\ll_\varepsilon x^{1+\varepsilon}$ for every $\varepsilon > 0$. First to improve this was Hecke [9] who showed essentially square-root cancellation and shortly after this was sharpened by Walfisz [29]. Then Rankin [24] showed that one has an estimate of the form

$$\sum_{m \leq x} a(m) \ll x^{2/5},$$

which was the sharpest result for a long time. The currently best known upper bound is $\ll_\varepsilon x^{1/3}(\log x)^{-\delta+\varepsilon}$ for $\delta = (8 - 3\sqrt{6})/10$ proved by Rankin himself [25]. For classical Maass cusp form coefficients $t(m)$ it follows from Rankin's argument that

$$\sum_{m \leq x} t(m) \ll_\varepsilon x^{1/3+\vartheta/3+\varepsilon},$$

where $\vartheta \geq 0$ is the exponent towards the Ramanujan–Petersson conjecture for classical Maass cusp forms [8]. Towards this it is known that $\vartheta \leq 7/64$. Currently the best known unconditional result for classical Maass cusp form coefficients is $\ll_\varepsilon x^{1027/2827+\varepsilon}$, which is due to Lü [20]. It is a folklore

conjecture that the correct upper bound is $\ll_\varepsilon x^{1/4+\varepsilon}$ for both holomorphic cusp forms and classical Maass cusp forms.

Concerning the higher rank analogue, Goldfeld and Sengupta [7] have recently shown that for the Fourier coefficients of a $\mathrm{GL}(n)$ Maass cusp form, the upper bound

$$\sum_{m \leq x} A(m,1,\ldots,1) \ll_\varepsilon x^{(n^3-1)/(n^3+n^2+n+1)+\varepsilon}$$

holds for any $n \geq 3$. This has been since slightly improved further by Meher and Murty [22]. Again the trivial bound for the sum is $\ll_\varepsilon x^{1+\varepsilon}$. The conjectural upper bound in this case is $\ll_\varepsilon x^{(n-1)/2n+\varepsilon}$.

It is natural to study analogous problems for shorter summation ranges $[x, x+\Delta]$ with $\Delta = o(x)$. Intuitively studying short sums makes sense as one might suspect that shorter intervals capture the erratic nature of the Fourier coefficients better than longer intervals. Furthermore, when Δ is small compared to x, studying short sums is analogous to studying classical error terms in analytic number theory, such as the error term in Dirichlet's divisor problem, in short intervals. The shape of the main term in the truncated $\mathrm{GL}(n)$ Voronoi summation formula and the square-root-cancellation heuristics suggest that the correct upper bound is

$$(1.1) \qquad \sum_{x \leq m \leq x+\Delta} A(m,1,\ldots,1) \ll_\varepsilon \min\left(\Delta^{1/2}x^\varepsilon, x^{(n-1)/2n+\varepsilon}\right),$$

where $\Delta \ll x$, for every $\varepsilon > 0$.

Pointwise upper bounds for short exponential sums involving Fourier coefficients of cusp forms (of which the plain sum of Fourier coefficients corresponds to the case of the trivial twist) have been obtained first by Jutila [16] and later by Ernvall-Hytönen and Karppinen [5] in the $\mathrm{GL}(2)$-setting for holomorphic cusp forms. Recently analogues of many results of [5] have been obtained for sums involving Fourier coefficients of classical Maass cusp forms [14].

In the present article we evaluate the mean square of sums of Hecke eigenvalues asymptotically over certain short intervals in the general $\mathrm{GL}(n)$-situation assuming the generalised Lindelöf hypothesis for the Godement–Jacquet L-function attached to the underlying cusp form in the t-aspect and a weak version of the Ramanujan–Petersson conjecture. Previously an analogous result has been established for the error term in Dirichlet's divisor problem for the k-fold divisor function d_k, which is defined as

$$\Delta_k(x) := \sum_{m \leq x} d_k(m) - \mathrm{Res}_{s=1}\left(\zeta^k(s)\frac{x^s}{s}\right),$$

by Lester [19] under the Lindelöf hypothesis for the Riemann ζ-function and we follow his strategy. The key difference in our situation is that only

a bound of the form $A(m, 1, \ldots, 1) \ll_\varepsilon m^{\vartheta+\varepsilon}$, for some fixed $\vartheta \geq 0$, is known for the Hecke eigenvalues. It is important to keep track of ϑ because unconditionally we only know that $\vartheta \leq 1/2 - 1/(n^2 + 1)$. Indeed, our main theorems are conditional on the assumption $\vartheta < 1/2 - 1/n$.

While analytic number theory of automorphic forms has seen many advances in the classical GL(2)-setting, the results are more sporadic in the case $n \geq 3$. There are not many statements which are currently known to hold for an individual (contrast to on average over a family of) cusp form on GL(n) for arbitrary n. The best known results of this type are the approximations to the Ramanujan–Petersson conjecture discussed below. The main results in the present article add further examples of such properties assuming hypothesis which are expected to be true.

This article is organised as follows. In Section 2 we introduce the statements of the main theorems. In Section 4 we collect some facts and results needed in the proofs. The penultimate section contains the proof of Theorem 1 and Theorem 2 is proved in Section 6.

2. The main results

The average behaviour of short rationally additively twisted exponential sums weighted by Fourier coefficients of holomorphic cusp forms has been studied e.g. by Jutila [15], Ernvall-Hytönen [2, 3], and Vesalainen [28]. In the higher rank setting, the mean square of long rationally additively twisted sums involving Fourier coefficients of GL(3) Maass cusp forms has been considered in [13].

In the present article we study sums of Hecke eigenvalues of Hecke–Maass cusp forms for the group SL($n, \mathbb{Z}$) over short intervals under certain assumptions. However, the method used in the proofs is slightly different compared to the works mentioned above. Those methods are not applicable in our situation essentially for two reasons; first one being that trigonometric polynomials in the truncated GL(n)-Voronoi summation formula are more complicated than in the lower rank setting and the other one is that the error term in the Voronoi formula gives a larger contribution than the expected main term. Instead, we follow Lester's work [19] where a similar problem for the error term of the Dirichlet divisor problem for the k-fold divisor function is treated by combining Jutila's method [15] with the one of Selberg [26]. Selberg's method has been also applied to other problems concerning automorphic forms, see e.g. [23]. The assumptions concerning the truth of the generalised Lindelöf hypothesis for the relevant Godement–Jacquet L-function in the t-aspect and a weak form of the Ramanujan–Petersson conjecture are needed to guarantee that the expected error term in the truncated Voronoi summation formula is small enough on average.

In both of the main results, let the underlying non-trivial Hecke–Maass cusp form have Hecke eigenvalues $A(m, 1, \ldots, 1)$. Our first main result computes the variance of short sums of these coefficients. These types of averages appear for example when studying the value distribution of short sums involving Hecke eigenvalues.

Theorem 1. *Let f be a non-trivial Hecke–Maass cusp form for the group $\mathrm{SL}(n, \mathbb{Z})$ normalised so that $A(1, \ldots, 1) = 1$. Assume the generalised Lindelöf hypothesis for $L(s, f)$ in the t-aspect and that the exponent towards the Ramanujan–Petersson conjecture for the Hecke–Maass form in question satisfies $0 \leq \vartheta < 1/2 - 1/n$. Furthermore, suppose that $2 \leq L \ll_\varepsilon X^{1/(n(n-1+2n\vartheta))-\varepsilon}$ for some small fixed $\varepsilon > 0$ and that $L = L(X) \longrightarrow \infty$ as $X \longrightarrow \infty$. Then we have*

$$\frac{1}{X} \int_X^{2X} \left| \sum_{x \leq m \leq x + x^{(n-1)/n}/L} A(m, 1, \ldots, 1) \right|^2 \mathrm{d}x \sim C_f \cdot \frac{X^{(n-1)/n}}{L}.$$

Here C_f is a constant given by

$$(2.1) \qquad C_f := \frac{2^{(n-1)/n} - 1}{2n - 1} \cdot r_f \cdot H_f(1),$$

where r_f is the residue of the Rankin–Selberg L-function $L(s, f \times \widetilde{f})$ attached to the underlying Hecke–Maass cusp form f at $s = 1$, and $\widetilde{f}$ is the dual Maass form of the form f. The residue is given by

$$r_f = \frac{4\pi^{n^2/2}}{n \cdot w(f)} \|f\|^2.$$

For the proof of this, see Appendix A in [18]. The Petersson norm of f is given by

$$\|f\|^2 := \int_{\mathrm{SL}(n, \mathbb{Z}) \backslash \mathbb{H}^n} |f(z)|^2 \, \mathrm{d}^* z,$$

where $\mathrm{d}^* z$ is the $\mathrm{SL}(n, \mathbb{R})$-invariant measure on the generalised upper-half plane $\mathbb{H}^n \simeq \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$, see Section 1.5 of [6]. Furthermore,

$$w(f) := \prod_{1 \leq j \leq n} \Gamma\left(\frac{1 + 2\Re(\lambda_j(\nu))}{2}\right) \prod_{1 \leq j < k \leq n} \left|\Gamma\left(\frac{1 + \lambda_j(\nu) + \overline{\lambda_k(\nu)}}{2}\right)\right|^2,$$

where $\lambda_j(\nu)$, $j = 1, \ldots, n$, are the Langlands parameters of the form f. These are complex numbers expressed in terms of the type $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ of f. Finally, the constant $H_f(1)$ is given by

$$H_f(1) := \prod_p P_n(\alpha_p(f), \alpha_p(\widetilde{f}), p^{-1}),$$

where P_n is the polynomial defined in (4.1) below, $\alpha_p(f) := \{\alpha_{1,p}(f), \ldots,$ $\alpha_{n,p}(f)\}$ is the set of Satake parameters of f at prime p. It is also known that $\alpha_p(\tilde{f}) = \overline{\alpha_p(f)} := \{\overline{\alpha_{1,p}(f)}, \ldots, \overline{\alpha_{n,p}(f)}\}$. The fact that $H_f(1)$ is non-zero and finite is shown in [18, Appendix B].

The other main theorem computes the mean square of the sum of Hecke eigenvalues over certain short intervals of fixed length.

Theorem 2. *Let f be a non-trivial Hecke–Maass cusp form for the group* $\mathrm{SL}(n, \mathbb{Z})$ *normalised so that $A(1, \ldots, 1) = 1$. Suppose that $X^{(n-1)/n+\varepsilon} \ll_\varepsilon$* $\Delta \ll_\varepsilon X^{1-\varepsilon}$ *for some small fixed $\varepsilon > 0$ and that the generalised Lindelöf hypothesis for $L(s, f)$ holds in the t-aspect. Suppose also that the exponent towards the Ramanujan–Petersson conjecture for the Hecke–Maass cusp form in question satisfies $0 \le \vartheta < 1/2 - 1/n$. Then we have*

$$\frac{1}{X} \int_X^{2X} \left| \sum_{x \le m \le x+\Delta} A(m, 1, \ldots, 1) \right|^2 \mathrm{d}x \sim B_f \cdot X^{(n-1)/n},$$

where

$$B_f := \frac{1}{\pi^2} \cdot \frac{2^{(2n-1)/n} - 1}{2n - 1} \sum_{m=1}^{\infty} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}}.$$

The fact that B_f is finite follows from (4.6) below and partial summation. This partly generalises results of Ivić [11], Jutila [15], and Vesalainen [28] to the higher rank setting, and is an analogue to Lester's result [19] in the setting of cusp forms.

Remark 3. *Theorem 2 is not expected to hold in the range $\Delta \ll_{\varepsilon'} X^{(n-1)/n-\varepsilon'}$ as in that case the sum of coefficients over the interval $[x, x + \Delta]$, with $x \asymp X$, is conjectured to be bounded from above by $\Delta^{1/2} X^\varepsilon$ (see (1.1)).*

3. Notation

The symbols $\ll$, $\gg$, $\asymp$, O, and $\sim$ are used for the usual asymptotic notation: for complex valued functions f and g in some set X, the notation $f \ll g$ means that $|f(x)| \le C |g(x)|$ for all $x \in X$ for some implicit constant $C \in \mathbb{R}_+$. When the implied constant depends on some parameters $\alpha, \beta, \ldots$, we use $\ll_{\alpha,\beta,\ldots}$ instead of mere $\ll$. The notation $g \gg f$ means $f \ll g$, and $f \asymp g$ means $f \ll g \ll f$.

All the implicit constants are allowed to depend on the underlying Maass cusp form and on ε, which denotes an arbitrarily small fixed positive number, which may not be the same on each occurrence, unless stated otherwise.

As usual, we write $e(x)$ for $e^{2\pi i x}$. The notation $\prod_p$ means the product over primes. The real and imaginary parts of a complex number s are denoted by $\Re(s)$ and $\Im(s)$, respectively. Finally, sometimes we also write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$.

4. Useful results

We start by recalling standard facts about higher rank Hecke operators and automorphic L-functions. Let f be a non-trivial Maass cusp form of type $(\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ for the group $\mathrm{SL}(n, \mathbb{Z})$ normalised so that $A(1, \ldots, 1) = 1$. By analogue to the classical situation, it follows that for every integer $m \geq 1$ there is a Hecke operator given by

$$
T_m f(z) := \frac{1}{m^{n-1/2}} \sum_{\substack{\prod_{\ell=1}^{n} c_\ell = m \\ 0 \leq c_{i,\ell} < c_\ell \ (1 \leq i < \ell \leq n)}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \cdot z\right)
$$

acting on the space $L^2(\mathrm{SL}(n, \mathbb{Z}) \backslash \mathbb{H}^n)$ of square-integrable automorphic functions (which contains the space of Maass cusp forms for $\mathrm{SL}(n, \mathbb{Z})$). Unlike in the classical situation, these operators are not self-adjoint, but they are normal. If the Maass cusp form f is an eigenfunction of every Hecke operator T_m, it is called a Hecke–Maass cusp form. We remark that if the Fourier coefficient $A(1, \ldots, 1)$ of a Hecke–Maass cusp form is zero, then the form vanishes identically. For more theory of Hecke operators for the group $\mathrm{SL}(n, \mathbb{Z})$, see [6, Section 9.3].

Fourier coefficients and Satake parameters of a given Hecke–Maass cusp form are closely related by the work of Shintani [27] together with results of Casselman and Shalika [1]. They showed that for any prime number p and $\beta_1, \ldots, \beta_n \in \mathbb{Z}_+ \cup \{0\}$ one has

$$
A_f(p^{\beta_1}, \ldots, p^{\beta_{n-1}}) = S_{\beta_{n-1}, \ldots, \beta_1}(\alpha_{1,p}(f), \ldots, \alpha_{n,p}(f)),
$$

where

$$
\begin{aligned}
&S_{\beta_{n-1}, \ldots, \beta_1}(x_1, \ldots, x_n) \\
&:= \frac{1}{V(x_1, \ldots, x_n)} \det\left[\begin{pmatrix} x_1^{n-1+\beta_{n-1}+\cdots+\beta_1} & \cdots & x_n^{n-1+\beta_{n-1}+\cdots+\beta_1} \\ \vdots & \vdots & \vdots \\ x_1^{2+\beta_{n-1}+\beta_{n-2}} & \cdots & x_n^{2+\beta_{n-1}+\beta_{n-2}} \\ x_1^{1+\beta_{n-1}} & \cdots & x_n^{1+\beta_{n-1}} \\ 1 & \cdots & 1 \end{pmatrix}\right]
\end{aligned}
$$

is a Schur polynomial, and $V(x_1, \ldots, x_n)$ is the Vandermonde determinant given by

$$
V(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n} (x_i - x_j).
$$

Kowalski and Ricotta proved in [18, Proposition B.1] that there exists a polynomial $P_n(\mathbf{x}, \mathbf{y}, T)$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and T

are indeterminates, such that

$$(4.1) \qquad \sum_{k \geq 0} S_{0,\ldots,0,k}(\mathbf{x}) S_{0,\ldots,0,k}(\mathbf{y}) T^k = \frac{P_n(\mathbf{x}, \mathbf{y}, T)}{\prod_{1 \leq j, \ell \leq n}(1 - x_j y_\ell T)}.$$

Next, we define an important notion of a dual Maass cusp form. Let

$$\widetilde{f}(z) := f(w \cdot {}^t(z^{-1}) \cdot w), \qquad \text{where} \quad w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}.$$

Then $\widetilde{f}$ is a Maass cusp form of type $(\nu_{n-1}, \ldots, \nu_1) \in \mathbb{C}^{n-1}$ for the group $SL(n, \mathbb{Z})$ and it is called the dual Maass cusp form of f. It turns out that

$$(4.2) \qquad A_f(m_1, \ldots, m_{n-1}) = A_{\widetilde{f}}(m_{n-1}, \ldots, m_1)$$

for every $m_1, \ldots, m_{n-1} \geq 1$.

Fourier coefficients of a Hecke–Maass cusp form satisfy the multiplicativity relation

$$A(m, 1, \ldots, 1) A(m_1, \ldots, m_{n-1})$$
$$= \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_j | m_j \text{ for } 1 \leq j \leq n-1}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right)$$

for positive integers $m, m_1, \ldots, m_{n-2}$, and a non-zero integer m_{n-1}. Furthermore, the relation

$$(4.3) \qquad A(m_1, \ldots, m_{n-1}) A(m'_1, \ldots, m'_{n-1}) = A(m_1 m'_1, \ldots, m_{n-1} m'_{n-1})$$

holds if $(m_1 \cdots m_{n-1}, m'_1 \cdots m'_{n-1}) = 1$. For the proofs of these facts, see [6, Theorem 9.3.11.]

For an eigenfunction of Hecke operators, one can use Möbius inversion to show that the relation

$$A(m_1, \ldots, m_{n-1}) = \overline{A(m_{n-1}, \ldots, m_1)}$$

holds [6, Theorem 9.3.6, Theorem 9.3.11, Addendum]. In particular, together with the relation (4.2) this yields the equality

$$\overline{A_f(m, 1, \ldots, 1)} = A_{\widetilde{f}}(m, 1, \ldots, 1).$$

Also, it follows that $|A(m, 1, .., 1)| = |A(1, \ldots, 1, m)|$. Associated to the form f is the L-series given by

$$L(s, f) := \sum_{m=1}^{\infty} \frac{A_f(m, 1, \ldots, 1)}{m^s},$$

which converges absolutely for large enough $\Re(s)$. This has an entire continuation to the whole complex plane via the functional equation

$$(4.4) \qquad L(s, f) = \pi^{ns - n/2} \frac{G(1 - s, \widetilde{f})}{G(s, f)} L(1 - s, \widetilde{f}),$$

where

$$G(s, f) := \prod_{j=1}^{n} \Gamma\left(\frac{s - \lambda_j(\nu)}{2}\right) \quad \text{and so} \quad G(s, \widetilde{f}) = \prod_{j=1}^{n} \Gamma\left(\frac{s - \widetilde{\lambda}_j(\nu)}{2}\right).$$

Recall that here $\lambda_j(\nu)$ and $\widetilde{\lambda}_j(\nu)$ are the Langlands parameters of f and $\widetilde{f}$, respectively. The resulting L-function attached to the form f is called the Godement–Jacquet L-function.

An elementary application of Stirling's formula says that when s lies in the vertical strips $-\delta \leq \Re(s) \leq 1 + \delta$, for a small fixed $\delta > 0$, and has sufficiently large imaginary part, the multiple Γ-factors can be replaced by a single quotient of two Γ-factors [4]:

$$\frac{G(1 - s, \widetilde{f})}{G(s, f)} = n^{ns - n/2} \frac{\Gamma\left(\frac{1 - ns}{2}\right)}{\Gamma\left(\frac{ns - (n-1)}{2}\right)} \left(1 + O(|s|^{-1})\right).$$

The main theorems of the present article are conditional on the generalised Lindelöf hypothesis for the Godement–Jacquet L-function in the t-aspect. It states that on the critical line $\sigma = 1/2$ an estimate of the form $L(1/2 + it, f) \ll_\varepsilon (1 + |t|)^\varepsilon$ holds for every $\varepsilon > 0$. For more detailed discussion about this conjecture, see [12].

The Rankin–Selberg L-function of two Hecke–Maass cusp forms f and g for the group $\mathrm{SL}(n, \mathbb{Z})$ is given by

$$L(s, f \times g) := \zeta(ns) \sum_{m_1, \ldots, m_{n-1} \geq 1} \frac{A_f(m_1, \ldots, m_{n-1}) A_g(m_1, \ldots, m_{n-1})}{(m_1^{n-1} m_2^{n-2} \cdots m_{n-1})^s},$$

which converges absolutely for large enough $\Re(s)$. This L-series has an analytic continuation to the whole complex plane if $g \neq \widetilde{f}$ and a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if $g = \widetilde{f}$. If we set

$$\Lambda(s, f \times g)$$
$$:= \prod_{i=1}^{n} \prod_{j=1}^{n} \pi^{(-s + \lambda_i(\nu_f) + \overline{\lambda_j(\nu_g)})/2} \Gamma\left(\frac{s - \lambda_i(\nu_f) - \overline{\lambda_j(\nu_g)}}{2}\right) L(s, f \times g),$$

then the functional equation

$$(4.5) \qquad \Lambda(s, f \times g) = \Lambda(1 - s, \widetilde{f} \times \widetilde{g})$$

holds; see [6, Theorem 12.1.4].

If $L(s, f)$ has an Euler product representation

$$L(s, f) = \sum_{m=1}^{\infty} \frac{A(m, 1, \ldots, 1)}{m^s} = \prod_p \prod_{j=1}^{n} (1 - \alpha_{j,p}(f) p^{-s})^{-1},$$

where the complex numbers $\alpha_{j,p}(f)$ are the Satake parameters of the underlying Hecke–Maass cusp form f at prime p, for large enough $\Re(s)$, and similar representation holds for g with Satake parameters $\alpha_{j,p}(g)$, then also the Rankin–Selberg L-function has an Euler product representation given by

$$L(s, f \times g) = \prod_p \prod_{k=1}^{n} \prod_{\ell=1}^{n} (1 - \alpha_{k,p}(f) \alpha_{\ell,p}(g) p^{-s})^{-1}$$

when $\Re(s)$ is large enough.

Analytic properties of $L(s, f \times \widetilde{f})$ imply that

$$(4.6) \qquad \sum_{m_1^{n-1} m_2^{n-2} \cdots m_{n-1} \leq x} |A(m_1, m_2, \ldots, m_{n-1})|^2 \sim r_f \cdot x,$$

where r_f is as before, see [6, Proposition 12.1.6, Remark 12.1.8]. This result can be interpreted as saying that the Fourier coefficients $A(m_1, \ldots, m_{n-1})$ are essentially of constant order of magnitude on average. However, currently known pointwise bounds for the Fourier coefficients are quite far from the expected truth. The Ramanujan–Petersson conjecture predicts that an estimate of the form $A(m, 1, \ldots, 1) \ll_\varepsilon m^\varepsilon$ holds for every $\varepsilon > 0$. There are however approximations towards this conjecture. Let $\vartheta = \vartheta(n) \geq 0$ be the smallest non-negative real number so that the estimate $A(m, 1, \ldots, 1) \ll_\varepsilon m^{\vartheta+\varepsilon}$ holds. Hence the Ramanujan–Petersson conjecture states that the value $\vartheta(n) = 0$ is admissible for every $n \geq 2$. It is easy to see that one has $\vartheta \leq 1/2$ [6, Proposition 12.1.6], but currently it is known that $\vartheta \leq 1/2 - 1/(n^2 + 1)$. This result is due to [21]. For small values of n sharper results are known. We have $\vartheta(2) \leq 7/64$, $\vartheta(3) \leq 5/14$ and $\vartheta(4) \leq 9/22$ [17]. An equivalent estimate holds for the Satake parameters of the underlying form f. Namely, we have $\alpha_{j,p}(f) \ll_\varepsilon p^{\vartheta(n)+\varepsilon}$ for every prime p.

It follows from (4.6), for a fixed $\delta \in \mathbb{R}_+$, by partial summation that

$$(4.7) \qquad \sum_{m=1}^{\infty} \frac{|A(1, \ldots, 1, m)|^2}{m^{1+\delta}} \ll_\delta 1 \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{|A(1, \ldots, 1, m)|}{m^{1+\delta}} \ll_\delta 1.$$

In the course of the proof of Proposition 9 we will come across certain complex line integrals involving Γ-functions. More precisely, these integrals are of the form

$$\Omega_{\nu,k}(y; \delta, Y) := \frac{1}{2\pi i} \int_{-\delta-iY}^{-\delta+iY} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns+1}{2} + \nu - \frac{n}{2}\right)} (s + \Lambda)^{-k} y^s \, \mathrm{d}s,$$

where the integration is along a straight line segment, and where ν and k are non-negative integers, and y and Y are positive real numbers. The parameter Λ is a large positive real number, which will depend on n and the underlying Hecke–Maass cusp form. The parameter δ will be a sufficiently small positive real constant. All the implicit constants in the following are going to depend on n, δ and Λ. It is proved in [13] that the following lemma holds.

Lemma 4 ([13, Lemma 8]). *Let ν and k be non-negative integers, and let $y, Y \in [1, \infty[$ with $y < (nY/2)^n$. Then*

$$\Omega_{\nu,k}(y; \delta, Y) = \left(\frac{n}{2}\right)^{k-1} y^{1/2+(1-\nu-k)/n} \, J_{\nu+k-n/2}(2y^{1/n}) + O(1)$$

$$+ O\left(Y^{n/2-\nu-k+n\delta}\right) + O\left(Y^{n/2-\nu-k} \frac{1}{\log \frac{n^n Y^n}{2^n y}}\right).$$

Using the asymptotics of J-Bessel functions for $y \gg 1$, we get the following corollary.

Corollary 5. *Let $y, Y \in [1, \infty[$ with $y < (nY/2)^n$. Then*

$$\Omega_{0,1}(y; \delta, Y) = \frac{1}{\sqrt{\pi}} y^{(n-1)/2n} \cos\left(2y^{1/n} + \frac{(n-3)}{4}\pi\right) + O(y^{1/2-1/(2n)-1/n})$$

$$+ O(Y^{n/2-1+n\delta}) + O\left(Y^{n/2-1} \frac{1}{\log \frac{n^n Y^n}{2^n y}}\right).$$

We also need asymptotics for the sum of the coefficients $|A(m, 1, \ldots, 1)|^2$. The proof of the following theorem combines methods from [18] and [22].

Theorem 6. *Let f be a non-trivial Hecke–Maass cusp form for the group $\mathrm{SL}(n, \mathbb{Z})$ normalised so that $A(1, \ldots, 1) = 1$. Then*

$$\sum_{m \leq x} |A(m, 1, \ldots, 1)|^2 \sim r_f \cdot H_f(1) \cdot x,$$

where r_f and $H_f(1)$ are as given above.

Proof. We start by defining a Dirichlet series

$$D_f(s) := \sum_{m=1}^{\infty} \frac{|A(m, 1, \ldots, 1)|^2}{m^s},$$

which is absolutely convergent for $\Re(s) > 1$ due to (4.6) and defines a holomorphic function in this half-plane. Since f is a Hecke eigenform, the coefficients $A(m, 1, \ldots, 1)$ are multiplicative by using (4.3). Therefore we have

$$D_f(s) = \prod_p \sum_{k \geq 0} \frac{|A(p^k, 1, \ldots, 1)|^2}{p^{ks}} =: \prod_p D_{f,p}(s)$$

for $\Re(s) > 1$. Furthermore, by applying (4.1) with $\mathbf{x} = \alpha_p(f)$, $\mathbf{y} = \alpha_p(\widetilde{f})$ and $T = p^{-s}$, we have

$$D_{f,p}(s) = \frac{P_n(\alpha_p(f), \alpha_p(\widetilde{f}), p^{-s})}{\prod_{1 \leq j,k \leq n}(1 - \alpha_{j,p}(f)\alpha_{k,p}(\widetilde{f})p^{-s})}$$

for any prime p, where P_n is the polynomial given by (4.1). Hence, by using the estimates $\alpha_p(f)$, $\alpha_p(\widetilde{f}) \ll_\varepsilon p^{\vartheta+\varepsilon}$, the quotient

(4.8) $$\frac{D_f(s)}{L(s, f \times \widetilde{f})} = \prod_p P_n(\alpha_p(f), \alpha_p(\widetilde{f}), p^{-s}) =: H_f(s)$$

defines a bounded holomorphic function on the half-plane $\Re(s) > 1/2 + \vartheta$.

Hence, writing $H_f(s)$ as a Dirichlet series

$$H_f(s) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

in this half-plane, we have

(4.9) $$\sum_{m \leq x} c(m) \ll_\varepsilon x^{1/2+\vartheta+\varepsilon}$$

for every $\varepsilon > 0$.

For simplicity, write

$$a_{f \times \tilde{f}}(m) := \sum_{m_1^{n-1} m_2^{n-2} \cdots m_{n-1} = m} |A(m_1, \ldots, m_{n-1})|^2$$

for the Dirichlet series coefficients of $L(s, f \times \widetilde{f})$. By the properties of Dirichlet convolution together with (4.8) we have

$$|A(m, 1, \ldots, 1)|^2 = \sum_{d|m} c(d) a_{f \times \tilde{f}}\left(\frac{m}{d}\right).$$

Therefore

$$\sum_{m \leq x} |A(m, 1, \ldots, 1)|^2 = \sum_{m \leq x} \sum_{d|m} c(d) a_{f \times \tilde{f}}\left(\frac{m}{d}\right)$$

$$= \sum_{d \leq x} \sum_{\ell \leq \frac{x}{d}} c(d) a_{f \times \tilde{f}}(\ell)$$

$$\sim r_f \cdot x \sum_{d \leq x} \frac{c(d)}{d}$$

by using (4.6). Combining this with the observation, which follows from (4.9), the fact that $\vartheta \leq 1/2 - 1/(n^2 + 1)$, and partial summation,

$$\sum_{d > x} \frac{c(d)}{d} \ll_\varepsilon \int_x^\infty \frac{t^{1/2+\vartheta+\varepsilon}}{t^2} \mathrm{d}t$$

$$\ll_\varepsilon x^{-1/2+\vartheta+\varepsilon}$$

$$\ll_\varepsilon x^{-1/(n^2+1)+\varepsilon}$$

it follows that

$$\sum_{m \leq x} |A(m, 1, \ldots, 1)|^2 \sim r_f \cdot x \sum_{d \leq x} \frac{c(d)}{d}$$

$$= r_f \cdot x \sum_{d=1}^\infty \frac{c(d)}{d} + O\left(x \sum_{d > x} \frac{c(d)}{d} \right)$$

$$\sim r_f \cdot H_f(1) \cdot x.$$

Note that $H_f(1)$ is clearly finite and non-zero. This completes the proof. $\square$

As a consequence of this, we can evaluate the sum

$$\sum_{m \leq X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \sin^2\left(\frac{\pi m^{1/n}}{L} \right),$$

where $0 < \theta \leq 1$ is fixed, asymptotically for a certain range of L.

Lemma 7. *Let L be such that $L \ll X^{\theta/n-\varepsilon}$ for some small fixed $\varepsilon > 0$ and that $L = L(X) \longrightarrow \infty$ as $X \longrightarrow \infty$. Then we have*

$$\sum_{m \leq X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \sin^2\left(\frac{\pi m^{1/n}}{L} \right) \sim \frac{r_f \cdot H_f(1) \cdot n}{L} \cdot \frac{\pi^2}{2}.$$

Proof. By using partial summation we have

$$(4.10) \quad \sum_{m \leq X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \sin^2\left(\frac{\pi m^{1/n}}{L} \right)$$

$$= \left(1 + \frac{1}{n} \right) \int_1^{X^\theta} \left(\sum_{m \leq x} |A(m, 1, \ldots, 1)|^2 \right) \frac{\sin^2\left(\frac{\pi x^{1/n}}{L} \right)}{x^{(2n+1)/n}} \mathrm{d}x$$

$$- \frac{2\pi}{n} \cdot \frac{1}{L} \int_1^{X^\theta} \left(\sum_{m \leq x} |A(m, 1, \ldots, 1)|^2 \right) \frac{\sin\left(\frac{\pi x^{1/n}}{L} \right) \cos\left(\frac{\pi x^{1/n}}{L} \right)}{x^2} \mathrm{d}x$$

$$+ O\left(\frac{1}{X^{\theta/n}} \right),$$

where the error term comes from the substitution term by trivial estimation. The first term on the right-hand side is, by using Theorem 6 and a simple change of variables, asymptotically

$$\sim \left(1 + \frac{1}{n}\right) \cdot r_f \cdot H_f(1) \int_1^{X^\theta} \frac{1}{x^{(n+1)/n}} \sin^2\left(\frac{\pi x^{1/n}}{L}\right) \, \mathrm{d}x$$

$$\sim \left(1 + \frac{1}{n}\right) \cdot r_f \cdot H_f(1) \int_{1/L}^{X^{\theta/n}/L} \frac{1}{(yL)^{n+1}} \sin^2(\pi y) \cdot L^n n y^{n-1} \, \mathrm{d}y$$

$$\sim \frac{r_f \cdot H_f(1) \cdot (n+1)}{L} \int_{1/L}^{X^{\theta/n}/L} \frac{\sin^2(\pi y)}{y^2} \, \mathrm{d}y$$

$$\sim \frac{r_f \cdot H_f(1) \cdot (n+1)}{L} \cdot \frac{\pi^2}{2},$$

where the last estimate follows from the identity

$$\int_0^\infty \frac{\sin^2(\pi y)}{y^2} \, \mathrm{d}y = \frac{\pi^2}{2}$$

together with the estimates

$$\int_0^{1/L} \frac{\sin^2(\pi y)}{y^2} \, \mathrm{d}y \ll \frac{1}{L},$$

$$\int_{X^{\theta/n}/L}^\infty \frac{\sin^2(\pi y)}{y^2} \, \mathrm{d}y \ll \frac{L}{X^{\theta/n}}$$

provided that $L = L(X) \longrightarrow \infty$ as $X \longrightarrow \infty$ and $L \ll X^{\theta/n - \varepsilon}$ for some fixed $\varepsilon > 0$.

An analogous computation shows that the second term on the right-hand side of (4.10) is

$$\sim -\frac{\pi^2}{2} \cdot \frac{r_f \cdot H_f(1)}{L}.$$

Together these imply the claimed asymptotics.   □

5. Proof of Theorem 1

Our main strategy in the proof is inspired by the method of Lester [19].

Let f be a non-trivial Hecke–Maass cusp form for the group $\mathrm{SL}(n, \mathbb{Z})$ with Hecke eigenvalues $A(m, 1, \ldots, 1)$. Let $0 < \theta \leq 1$ and define

$$P(x; \theta) := \frac{x^{(n-1)/2n}}{\pi \sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1, \ldots, 1, m)}{m^{(n+1)/2n}} \cos\left(2\pi n (mx)^{1/n} + \frac{(n-3)}{4}\pi\right)$$

for $X \leq x \leq 2X$.

Let us write

$$E(x;\theta) := \left(\sum_{m \leq x} A(m,1,\ldots,1) \right) - P(x;\theta).$$

We remark that arguments similar to those in Section 7 of [13] show that

(5.1) $$E(x;\theta) \ll_\varepsilon x^{1-(1+\theta)/n+\vartheta+\varepsilon},$$

where ϑ is the exponent towards the Ramanujan–Petersson conjecture for $\mathrm{GL}(n)$ Maass cusp forms. Notice that this is worse than the trivial bound unless ϑ is small. The pointwise bound (5.1) is too weak to establish Theorem 1, but it will be shown that on average $E(x + x^{(n-1)/n}/L;\theta) - E(x;\theta)$ is much smaller than what this bound implies, of course under certain assumptions.

The proof has three main steps. The first two are formulated in the following propositions. The first one evaluates the mean square of the expected main term for the sums of Hecke eigenvalues asymptotically over the short interval $[x, x + x^{(n-1)/n}/L]$ for a suitable $L = L(X)$.

Proposition 8. *Let f be a non-trivial Hecke–Maass cusp form for the group $\mathrm{SL}(n,\mathbb{Z})$ normalised so that $A(1,\ldots,1) = 1$. Let $0 \leq \theta \leq \frac{1}{n-1}$ and suppose that $2 \leq L \ll_\varepsilon X^{1/(n(n-1))-\varepsilon}$ for some small fixed $\varepsilon > 0$. Then we have*

$$\frac{1}{X} \int_X^{2X} \left| P\left(x + \frac{x^{(n-1)/n}}{L}; \theta \right) - P(x;\theta) \right|^2 \mathrm{d}x \sim \frac{X^{(n-1)/n}}{L} \cdot C_f,$$

where C_f is as in (2.1).

The other proposition shows that on average $P(x;\theta)$ is a sufficiently good approximation for the sum of Hecke eigenvalues $A(m,1,\ldots,1)$ over the interval $[x, x + x^{(n-1)/n}/L]$ under the assumption of the generalised Lindelöf hypothesis for the Godement–Jacquet L-function in the t-aspect and a weak version of the Ramanujan–Petersson conjecture. This is better than the pointwise upper bounds for the error term one gets from the $\mathrm{GL}(n)$ Voronoi summation formula.

Proposition 9. *Let f be a non-trivial Hecke–Maass cusp form for the group $\mathrm{SL}(n,\mathbb{Z})$ normalised so that $A(1,\ldots,1) = 1$. Suppose that $0 < \theta < 1/(n-1+2n\vartheta)$, where ϑ is the exponent towards the Ramanujan–Petersson conjecture, and assume also that $0 \leq \vartheta < 1/2 - 1/n$. Furthermore, suppose that the generalised Lindelöf hypothesis for the Godement–Jacquet L-function attached to the underlying Hecke–Maass cusp form holds in the*

t-aspect. Then we have

$$\frac{1}{X}\int_X^{2X}\left|E\left(x+\frac{x^{(n-1)/n}}{L};\theta\right)-E(x;\theta)\right|^2\mathrm{d}x\ll_\varepsilon X^{1-(1+\theta)/n+\varepsilon}$$

for every $\varepsilon>0$.

Remark 10. *Notice that this bound is superior compared to the bound $\ll_\varepsilon X^{2-2(1+\theta)/n+2\vartheta+\varepsilon}$, which follows from (5.1).*

Once these have been established, the proof can be completed as follows. For now, let $\varepsilon>0$ be small but fixed. For notational simplicity, we set

$$S(x,L):=\sum_{x\le m\le x+x^{(n-1)/n}/L}A(m,1,\ldots,1)$$

and

$$Q(x,L,\theta):=P\left(x+\frac{x^{(n-1)/n}}{L};\theta\right)-P(x;\theta).$$

Then by making use of the elementary identity

$$|S|^2=|Q|^2+|S-Q|^2+2\Re\left(Q(\overline{S}-\overline{Q})\right)$$

we obtain

$$\frac{1}{X}\int_X^{2X}\left|\sum_{x\le m\le x+x^{(n-1)/n}/L}A(m,1,\ldots,1)\right|^2\mathrm{d}x$$

$$=\frac{1}{X}\int_X^{2X}|Q(x,L;\theta)|^2\,\mathrm{d}x+O\left(\frac{1}{X}\int_X^{2X}|S(x,L)-Q(x,L;\theta)|^2\,\mathrm{d}x\right)$$

$$+O\left(\frac{1}{X}\int_X^{2X}|S(x,L)-Q(x,L;\theta)|\cdot|Q(x,L;\theta)|\,\mathrm{d}x\right).$$

By Proposition 8, the first term on the right-hand side is

$$\sim C_f\cdot\frac{X^{(n-1)/n}}{L}$$

assuming $L\ll_\varepsilon X^{1/(n(n-1))-\varepsilon}$ and the second term is, say, $\ll_\varepsilon X^{1-(1+\theta)/n+\varepsilon/2}$ by Proposition 9 provided that $0<\theta<1/(n-1+2n\vartheta)$ and $\vartheta<1/2-1/n$. For the last term, an application of the Cauchy–Schwarz inequality yields

$$\ll_\varepsilon\frac{1}{X}\cdot\left(\frac{X^{(2n-1)/n}}{L}\right)^{1/2}\cdot(X^{2-(1+\theta)/n+\varepsilon/2})^{1/2}$$

$$\ll_\varepsilon\frac{X^{1-(2+\theta)/2n+\varepsilon/4}}{L^{1/2}}.$$

Notice that this is smaller than the main term due to the assumption $L \ll_\varepsilon X^{\theta/n-\varepsilon}$. This completes the proof of Theorem 1. The next two subsections are devoted to the proofs of Propositions 8 and 9.

5.1. Proof of Proposition 8. We start by writing

$$P\left(x + \frac{x^{(n-1)/n}}{L}; \theta\right) - P(x; \theta)$$

$$= \underbrace{P\left(x + \frac{x^{(n-1)/n}}{L}; \theta\right) - P\left(\left(x^{1/n} + \frac{1}{nL}\right)^n; \theta\right)}_{=:I_1(x,L;\theta)}$$

$$+ \underbrace{P\left(\left(x^{1/n} + \frac{1}{nL}\right)^n; \theta\right) - P(x; \theta)}_{=:I_2(x,L;\theta)}.$$

The idea here is that $I_2(x, L; \theta)$ is easier to handle than the original difference and intuitively $I_1(x, L; \theta)$ should be small on average, which turns out to be the case. Indeed, for $I_2(x, L; \theta)$ the resulting trigonometric polynomial simplifies so that its mean-square can be evaluated by using Lemma 7.

We have

$$(5.2) \quad \frac{1}{X} \int_X^{2X} \left| P\left(x + \frac{x^{(n-1)/n}}{L}; \theta\right) - P(x; \theta) \right|^2 \, \mathrm{d}x$$

$$= \frac{1}{X} \int_X^{2X} |I_1(x, L; \theta)|^2 \, \mathrm{d}x + \frac{1}{X} \int_X^{2X} |I_2(x, L; \theta)|^2 \, \mathrm{d}x$$

$$+ O\left(\frac{1}{X} \int_X^{2X} |I_1(x, L; \theta) I_2(x, L; \theta)| \, \mathrm{d}x \right).$$

The proof of the proposition now proceeds by estimating the first two terms on the right-hand side separately. The second term is treated in Lemma 11 and the first term in Lemma 12. The error term is then handled by an application of the Cauchy–Schwarz inequality. Once we have shown that the contribution of the first term on the right-hand side is $\ll_\varepsilon L^{-4} X^{(n-1)/n+(3-n)/(n(n-1))-\varepsilon}$ and the contribution of the second term is $\ll X^{(n-1)/n}/L$, it follows that the error term contributes

$$\ll_\varepsilon \frac{1}{X} \left(\frac{X^{(2n-1)/n+(3-n)/(n(n-1))-\varepsilon}}{L^4} \right)^{1/2} \left(\frac{X^{2-1/n}}{L} \right)^{1/2}$$

$$\ll_\varepsilon \frac{X^{(n-1)/n+(3-n)/(2n(n-1))-\varepsilon/2}}{L^{5/2}},$$

which is small enough as $n \geq 3$.

Lemma 11. *Suppose that $0 < \theta \leq 1/(n-1) - \varepsilon$ for some small fixed $\varepsilon > 0$. Then we have*

$$\frac{1}{X} \int_X^{2X} |I_2(x, L; \theta)|^2 \, dx \sim C_f \cdot \frac{X^{(n-1)/n}}{L}.$$

Proof. To estimate the difference $I_2(x, L; \theta)$ we are reduced to understanding terms of the form

$$(x + \Xi)^{(n-1)/2n} \cos\left(2\pi n(mx)^{1/n} + \frac{2\pi m^{1/n}}{L} + \frac{(n-3)}{4}\pi\right)$$

$$- x^{(n-1)/2n} \cos\left(2\pi n(mx)^{1/n} + \frac{(n-3)}{4}\pi\right),$$

where Ξ is given by the equation $x + \Xi = (x^{1/n} + 1/nL)^n$. Now, the relevant observation is that

$$\left|(x + \Xi)^{(n-1)/2n} - x^{(n-1)/2n}\right| \asymp \left|\int_x^{x+\Xi} y^{-(n+1)/2n} \, dy\right| \ll x^{-(n+1)/2n} \cdot \Xi.$$

But by the binomial theorem we have

$$\Xi \ll \frac{x^{(n-1)/n}}{L},$$

and so

$$\left|(x + \Xi)^{(n-1)/2n} - x^{(n-1)/2n}\right| \ll \frac{x^{(n-3)/2n}}{L}.$$

This shows that

$$(x + \Xi)^{(n-1)/2n} \cos\left(2\pi n(mx)^{1/n} + \frac{2\pi m^{1/n}}{L} + \frac{(n-3)}{4}\pi\right)$$

$$- x^{(n-1)/2n} \cos\left(2\pi n(mx)^{1/n} + \frac{(n-3)}{4}\pi\right)$$

$$= x^{(n-1)/2n} \left(\cos\left(2\pi n(mx)^{1/n} + \frac{2\pi m^{1/n}}{L} + \frac{(n-3)}{4}\pi\right)\right.$$

$$\left. - \cos\left(2\pi n(mx)^{1/n} + \frac{(n-3)}{4}\pi\right)\right)$$

$$+ O\left(\frac{x^{(n-3)/2n}}{L} \cos\left(2\pi n(mx)^{1/n} + \frac{2\pi m^{1/n}}{L} + \frac{(n-3)}{4}\pi\right)\right).$$

By using the formula for the difference of two cosines, $\cos(\alpha) - \cos(\beta) = -2\sin((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$, it follows that

$$I_2(x, L; \theta)$$

$$= \frac{-2x^{(n-1)/2n}}{\pi\sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} \sin\left(\frac{\pi m^{1/n}}{L}\right)$$

$$\cdot \sin\left(2\pi n m^{1/n}\left(x^{1/n} + \frac{1}{2nL}\right) + \frac{(n-3)}{4}\pi\right)$$

$$+ O\left(\frac{x^{(n-3)/2n}}{L}\left|\sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} \cos\left(2\pi n m^{1/n}\left(x^{1/n} + \frac{1}{nL}\right) + \frac{(n-3)}{4}\pi\right)\right|\right)$$

$$=: M(x, L; \theta) + R(x, L; \theta).$$

Hence,

$$(5.3) \quad \frac{1}{X}\int_X^{2X} |I_2(x, L; \theta)|^2 \, \mathrm{d}x$$

$$= \frac{1}{X}\int_X^{2X} |M(x, L; \theta)|^2 \, \mathrm{d}x + \frac{1}{X}\int_X^{2X} |R(x, L; \theta)|^2 \, \mathrm{d}x$$

$$+ O\left(\frac{1}{X}\int_X^{2X} |M(x, L; \theta)R(x, L; \theta)| \, \mathrm{d}x\right).$$

The main term can be written as

$$M(x, L; \theta) = -\frac{x^{(n-1)/2n}}{\pi i \sqrt{n}}\left(\sum_{m \leq X^\theta} a_m^+ e(n(mx)^{1/n}) - \sum_{m \leq X^\theta} a_m^- e(-n(mx)^{1/n})\right),$$

where

$$a_m^\pm := \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} e\left(\pm\frac{m^{1/n}}{2L} \pm \frac{(n-3)}{8}\right) \sin\left(\frac{\pi m^{1/n}}{L}\right).$$

Let us first evaluate the mean square of $M(x, L; \theta)$. Notice that

$$(5.4) \quad |M(x, L; \theta)|^2$$

$$= \frac{x^{(n-1)/n}}{n\pi^2}\left(\left|\sum_{m \leq X^\theta} a_m^+ e(n(mx)^{1/n})\right|^2 + \left|\sum_{m \leq X^\theta} a_m^- e(-n(mx)^{1/n})\right|^2\right)$$

$$- \frac{2x^{(n-1)/n}}{n\pi^2}\Re\left(\left(\sum_{m \leq X^\theta} \overline{a_m^+} e(-n(mx)^{1/n})\right)\left(\sum_{m \leq X^\theta} a_m^- e(-n(mx)^{1/n})\right)\right).$$

We consider the first two terms on the right-hand side simultaneously as their treatment is identical due to the fact that $|a_m^+| = |a_m^-|$. By opening the absolute squares these split into diagonal and off-diagonal terms.

By the first derivative test [10, Section 5.1] the non-diagonal terms give a contribution

$$\ll X^{(n-2)/n} \sum_{\substack{1 \le m, \ell \le X^\theta \\ m > \ell}} \frac{|a_m^+ a_\ell^+|}{m^{1/n} - \ell^{1/n}}$$

$$\ll X^{(n-2)/n} \sum_{\substack{1 \le m, \ell \le X^\theta \\ m > \ell}} \frac{|a_m^+ a_\ell^+| m^{(n-1)/n}}{|m - \ell|}$$

$$\ll X^{(n-2)/n} X^{\theta(n-1)/n} \log X \sum_{1 \le m \le X^\theta} |a_m^+|^2,$$

where the last estimate follows from the elementary estimate $ab \ll a^2 + b^2$.

The total contribution coming from the diagonal terms is

$$\frac{(2^{(2n-1)/n} - 1)}{(2 - 1/n)n\pi^2} \left(\sum_{m \le X^\theta} |a_m^+|^2 + \sum_{m \le X^\theta} |a_m^-|^2 \right) X^{(n-1)/n}$$

$$= \frac{2(2^{(2n-1)/n} - 1)}{(2 - 1/n)n\pi^2} \cdot X^{(n-1)/n} \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \sin^2 \left(\frac{\pi m^{1/n}}{L} \right).$$

For the third term in (5.4) we observe that it can be estimated similarly by using the first derivative test as the off-diagonal terms above. Therefore it follows that

$$\frac{1}{X} \int_X^{2X} |M(x, L; \theta)|^2 \, \mathrm{d}x$$

$$= \frac{2}{n\pi^2} \cdot \frac{2^{(2n-1)/n} - 1}{2 - 1/n} X^{(n-1)/n} \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \sin^2 \left(\frac{\pi m^{1/n}}{L} \right)$$

$$+ O \left(X^{(n-2)/n + \theta(n-1)/n} \log X \sum_{m \le X^\theta} |a_m^+|^2 \right).$$

By using Lemma 7 we infer that

$$\sum_{m \le X^\theta} |a_m^+|^2 = \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots 1)|^2}{m^{(n+1)/n}} \sin^2 \left(\frac{\pi m^{1/n}}{L} \right) \sim \frac{r_f \cdot H_f(1) \cdot n}{L} \cdot \frac{\pi^2}{2}.$$

The assumption $\theta < 1/(n-1) - \varepsilon$ guarantees that the error term is smaller than the main term. The mean square of the remainder term $R(x, L, \theta)$ is treated similarly: it is

$$\ll \frac{1}{X} \cdot \frac{1}{L^2} \cdot X^{(2n-3)/n} \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \ll \frac{X^{(n-3)/n}}{L^2}$$

by using (4.7).

Finally, the cross-terms in (5.3) are handled by a single application of the Cauchy–Schwarz inequality; they contribute

$$\ll_\varepsilon \frac{1}{X} \left(\frac{X^{(2n-3)/n}}{L^2} \right)^{1/2} \left(\frac{X^{(2n-1)/n+\varepsilon}}{L} \right)^{1/2}$$

$$\ll_\varepsilon \frac{X^{(n-2)/n+\varepsilon/2}}{L^{3/2}},$$

which is smaller than the main term if ε is small enough in terms of n. This completes the proof of the lemma. $\qquad\square$

Lemma 12. *Assume that $0 < \theta < 1/(n-1) - \varepsilon$ for some fixed $\varepsilon > 0$. Then we have*

$$\frac{1}{X} \int_X^{2X} |I_1(x, L; \theta)|^2 \, dx \ll_\varepsilon \frac{X^{(n-1)/n+(3-n)/(n(n-1))-\varepsilon}}{L^4} + \frac{X^{(n-5)/n}}{L^4}.$$

Proof. For simplicity, we set

$$x_1 := x^{1/n} + \frac{1}{nL} \qquad \text{and} \qquad x_2 := \left(x + \frac{x^{(n-1)/n}}{L} \right)^{1/n}.$$

Then

$$I_1(x, L; \theta) = x_2^{(n-1)/2} \Sigma(x_2) - x_1^{(n-1)/2} \Sigma(x_1),$$

where we have set

$$\Sigma(x) := \frac{1}{\pi\sqrt{n}} \sum_{m \le X^\theta} \frac{A(1, \ldots, 1, m)}{m^{(n+1)/2n}} \cos\left(2\pi n x m^{1/n} + \frac{(n-3)}{4}\pi \right).$$

By the triangle inequality we get

$$\left| x_2^{(n-1)/2} \Sigma(x_2) - x_1^{(n-1)/2} \Sigma(x_1) \right|$$

$$= \left| \left(x_1^{(n-1)/2} - x_2^{(n-1)/2} \right) \Sigma(x_1) + x_2^{(n-1)/2} \left(\Sigma(x_1) - \Sigma(x_2) \right) \right|$$

$$\ll \left| x_1^{(n-1)/2} - x_2^{(n-1)/2} \right| \cdot |\Sigma(x_1)| + x_2^{(n-1)/2} \left| \Sigma(x_1) - \Sigma(x_2) \right|.$$

By the mean value theorem we have

$$\left| x_1^{(n-1)/2} - x_2^{(n-1)/2} \right| \asymp \int_{x_1}^{x_2} t^{(n-3)/2} \, dt \asymp |x_1 - x_2| X^{(n-3)/2n}.$$

For the second term we observe that

$$\Sigma(x_1) - \Sigma(x_2)$$
$$= \frac{1}{\pi\sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} \left(\cos\left(2\pi n m^{1/n} x_1 + \frac{(n-3)}{4}\pi \right) \right.$$
$$\left. - \cos\left(2\pi n m^{1/n} x_2 + \frac{(n-3)}{4}\pi \right) \right)$$
$$\ll \sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} \cdot m^{1/n} |x_1 - x_2|$$
$$= \sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n-1)/2n}} |x_1 - x_2|,$$

as

$$\left| \cos\left(2\pi n m^{1/n} x_1 + \frac{(n-3)}{4}\pi \right) - \cos\left(2\pi n m^{1/n} x_2 + \frac{(n-3)}{4}\pi \right) \right|$$
$$\asymp \left| m^{1/n} \int_{x_1}^{x_2} \sin\left(2\pi n m^{1/n} t + \frac{(n-3)}{4}\pi \right) \mathrm{d}t \right|$$
$$\ll m^{1/n} |x_1 - x_2|.$$

Thus we have

$$\left| x_1^{(n-1)/2} \Sigma(x_1) - x_2^{(n-1)/2} \Sigma(x_2) \right|$$
$$\ll |x_1 - x_2| \left(X^{(n-1)/2n} \sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n-1)/2n}} + X^{(n-3)/2n} |\Sigma(x_1)| \right).$$

But as, say,

$$\sum_{m \leq X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n-1)/2n}} \ll_\varepsilon X^{\theta(n+1)/2n + \varepsilon/2n}$$

by partial summation, and

$$|x_1 - x_2| \asymp \left| \int_{x+\Xi}^{x + x^{(n-1)/n}/L} t^{1/n - 1} \mathrm{d}t \right|$$
$$\ll \left| \frac{x^{(n-1)/n}}{L} - \Xi \right| (x + \Xi)^{(1-n)/n}$$
$$\ll \frac{x^{(n-2)/n}}{L^2} \cdot X^{(1-n)/n}$$
$$\ll \frac{1}{L^2 X^{1/n}},$$

it follows that this can be further estimated to be

$$\ll_\varepsilon \frac{1}{L^2 X^{1/n}} \left(X^{(n-1)/2n+\theta(n+1)/2n+\varepsilon/2n} + X^{(n-3)/2n} |\Sigma(x_1)| \right).$$

By using the inequality $ab \ll a^2 + b^2$ we infer

$$\frac{1}{X} \int_X^{2X} |I_1(x, L; \theta)|^2 \, \mathrm{d}x$$

$$\ll_\varepsilon \frac{X^{(n-1)/n+\theta(1+1/n)+\varepsilon/n}}{L^4 X^{2/n}} + \frac{X^{(n-3)/n}}{L^4 X^{2/n}} \cdot \frac{1}{X} \int_X^{2X} |\Sigma(x_1)|^2 \, \mathrm{d}x$$

$$\ll_\varepsilon \frac{X^{(n-3)/n+\theta(n+1)/n+\varepsilon/n}}{L^4} + \frac{X^{(n-5)/n}}{L^4}.$$

The claim follows from this by recalling that $\theta < 1/(n-1) - \varepsilon$. In the last step we have used the fact that

$$\frac{1}{X} \int_X^{2X} |\Sigma(x_1)|^2 \, \mathrm{d}x \ll 1.$$

This follows by opening the absolute square and integrating termwise. The off-diagonal contributes

$$\ll_\varepsilon X^{-1/n+\theta(n-1)/n+\varepsilon(n-1)/n} \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}}$$

$$\ll_\varepsilon X^{-1/n+\theta(n-1)/n+\varepsilon(n-1)/n}$$

$$\ll 1$$

by using the first derivative test and the assumption $\theta < 1/(n-1) - \varepsilon$. The diagonal term is obviously

$$\ll \sum_{m \le X^\theta} \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}} \ll 1.$$

This completes the proof. □

Now, as Lemmas 11 and 12 are proved, the proof of Proposition 8 is completed by the discussion above. □

5.2. Proof of Proposition 9. Recall that

$$E(x; \theta) = \left(\sum_{m \le x} A(m, 1, \ldots, 1) \right) - P(x; \theta).$$

Throughout the proof, let $\varepsilon > 0$ be small enough in terms of n but fixed. We start by estimating

$$\frac{1}{X} \int_X^{2X} \left| E\left(x + \frac{x^{(n-1)/n}}{L}; \theta\right) - E(x; \theta) \right|^2 \, \mathrm{d}x$$

$$\ll \frac{1}{X} \int_X^{2X} \left| E\left(x + \frac{x^{(n-1)/n}}{L}; \theta\right) \right|^2 \, \mathrm{d}x + \frac{1}{X} \int_X^{2X} |E(x; \theta)|^2 \, \mathrm{d}x.$$

The analysis of both terms on the right-hand side is similar and hence we concentrate on the latter term

$$\frac{1}{X} \int_X^{2X} |E(x; \theta)|^2 \, \mathrm{d}x.$$

As usual, the starting point is the truncated Perron's formula which gives, for a small enough fixed $\delta > 0$,

$$\sum_{m \le x} A(m, 1, \ldots, 1) = \frac{1}{2\pi i} \int_{1+\delta-iX}^{1+\delta+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} + O(X^{\vartheta + \varepsilon/2})$$

uniformly for $X \le x \le 2X$.

The error term is admissible as we assume that $\vartheta < 1/2 - 1/n$. We shift the line segment of integration first to the line $\sigma = 1/2$. The Phragmén-Lindelöf principle tells that in the strip $1/2 \le \sigma \le 1 + \delta$, the estimate of the form $L(s, f) \ll_\varepsilon (1 + |t|)^{\varepsilon/4}$ holds under the assumption of the generalised Lindelöf hypothesis. By using this, the horizontal line segments from the shift contribute

$$\ll \int_{1/2}^{1+\delta} |L(\sigma \pm iX, f)| \, x^\sigma \frac{\mathrm{d}\sigma}{\sigma \pm iX}$$

$$\ll_\varepsilon X^{\varepsilon/4 - 1/2} + X^{\delta + \varepsilon/4}$$

$$\ll_\varepsilon X^{\delta + \varepsilon/4}.$$

It follows that

$$\sum_{m \le x} A(m, 1, \ldots, 1)$$

$$= \frac{1}{2\pi i} \int_{1/2-iY}^{1/2+iY} L(s, f) x^s \frac{\mathrm{d}s}{s} + \frac{1}{2\pi i} \int_{1/2-iX}^{1/2-iY} L(s, f) x^s \frac{\mathrm{d}s}{s}$$

$$+ \frac{1}{2\pi i} \int_{1/2+iY}^{1/2+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} + O\left(X^{\vartheta + \varepsilon/2}\right),$$

uniformly for $X \le x \le 2X$, where $0 < Y < X$ is a parameter chosen later.

Next we move the line segment of integration to the line $\sigma = -\delta$ in the first integral on the right-hand side. By using the convexity bound

$L(s, f) \ll (1 + |t|)^{(1+\delta-\sigma)n/2}$ in the vertical strip $-\delta \leq \sigma \leq 1 + \delta$ it follows that the horizontal line segments contribute

$$\ll \int_{-\delta}^{1/2} |L(\sigma \pm iY, f)| x^\sigma \frac{d\sigma}{\sigma \pm iY}$$

$$\ll_\varepsilon Y^{n/4+n\delta/2-1} X^{1/2} + Y^{n/2+n\delta-1} X^{-\delta}$$

$$\ll_\varepsilon X^{1/2-(1+\theta)/2n+\varepsilon/2}$$

as we will choose δ so that $2\delta\theta \leq \varepsilon$, where the last estimate follows from the assumption on θ together with the fact that we are going to choose Y such that it satisfies $Y \asymp X^{(1+\theta)/n}$.

Next, we treat the term

$$\frac{1}{2\pi i} \int_{-\delta-iY}^{-\delta+iY} L(s, f) x^s \frac{ds}{s}.$$

We are now in the position to apply the method used in [13]. Since we intend to apply Stirling's formula, we write

$$\frac{1}{2\pi i} \int_{-\delta-iY}^{-\delta+iY} L(s, f) \, x^s \, \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \left(\int_{-\delta-iY}^{-\delta-i\Lambda} + \int_{-\delta+i\Lambda}^{-\delta+iY} \right) L(s, f) \, x^s \, \frac{ds}{s} + O_{\delta,\Lambda}(1),$$

where $\Lambda := 1 + 2 \max_{1 \leq j \leq n} \{|\lambda_j(\nu)|, |\widetilde{\lambda}_j(\nu)|\}$. Now we may apply the functional equation of the Godement–Jacquet L-function (4.4), interchange the order of integration, and summation and apply Stirling's formula to get

$$(5.5) \quad \frac{1}{2\pi i} \left(\int_{-\delta-iY}^{-\delta-i\Lambda} + \int_{-\delta+i\Lambda}^{-\delta+iY} \right) L(s, f) \, x^s \, \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \left(\int_{-\delta-iY}^{-\delta-i\Lambda} + \int_{-\delta+i\Lambda}^{-\delta+iY} \right) \pi^{ns-n/2} \frac{G(1-s, \widetilde{f})}{G(s, f)} L(1-s, \widetilde{f}) \, x^s \, \frac{ds}{s}$$

$$= \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{A(1, \ldots, 1, m)}{m}$$

$$\cdot \left(\int_{-\delta-iY}^{-\delta-i\Lambda} + \int_{-\delta+i\Lambda}^{-\delta+iY} \right) \frac{1}{i} \pi^{ns-n/2} \, n^{ns-n/2}$$

$$\cdot \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} \left(1 + O(|s|^{-1})\right) m^s \, x^s \, \frac{ds}{s}.$$

In the region of integration, the quotient of the Γ-factors is $\ll t^{n/2+n\delta}$ by Stirling's formula, and so the series corresponding to the O-term can be

estimated to be

$$\ll \sum_{m=1}^{\infty} \frac{|A(1,\ldots,1,m)|}{m^{1+\delta}} \int_{\Lambda}^{Y} t^{n/2+n\delta}\, t^{-1}\, x^{-\delta}\, \frac{\mathrm{d}t}{t}$$

$$\ll x^{-\delta}\, Y^{n/2+n\delta-1}$$

$$\ll_{\varepsilon} X^{\varepsilon/2}\, Y^{n/2-1}$$

$$\ll_{\varepsilon} X^{1/2-(1+\theta)/n+\theta/2+\varepsilon/2}$$

$$\ll_{\varepsilon} X^{1/2-(1+\theta)/2n+\varepsilon/2},$$

provided that $2\theta\delta \leqslant \varepsilon$, by using (4.7), the assumption on θ, and the fact that $Y \asymp X^{(1+\theta)/n}$.

We are going to transform the rest of the integral in (5.5) further by making a simple change of variables to rewrite it as

$$2\Re\left(\int_{\Lambda}^{Y} (\pi n)^{n(-\delta+it)-n/2}\, \frac{\Gamma\left(\frac{1-n(-\delta+it)}{2}\right)}{\Gamma\left(\frac{n(-\delta+it)-(n-1)}{2}\right)}\, (mx)^{-\delta+it}\, \frac{\mathrm{d}t}{-\delta+it} \right).$$

Using the elementary fact that

$$\frac{1}{-\delta+it} = \frac{1}{it} + O(t^{-2}),$$

this equals

$$2\Re\left(\int_{\Lambda}^{Y} (\pi n)^{n(-\delta+it)-n/2}\, \frac{\Gamma\left(\frac{1-n(-\delta+it)}{2}\right)}{\Gamma\left(\frac{n(-\delta+it)-(n-1)}{2}\right)}\, (mx)^{-\delta+it}\, \frac{\mathrm{d}t}{it} \right)$$

$$+ O(X^{\varepsilon/2}\, Y^{n/2-1}).$$

By Stirling's formula,

$$\frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} = \left(\frac{nt}{2}\right)^{n/2-n\sigma} \exp\left(-int\log\frac{nt}{2} + int + \frac{\pi n i}{4}\right)\left(1+O(t^{-1})\right).$$

Substituting this back to the last integral, and observing that the terms coming from the $O(t^{-1})$-term contribute $\ll_{\varepsilon} X^{\varepsilon/2}\, Y^{n/2-1}$, it takes the form

$$2\,(2\pi)^{-n/2}\,(2^n\pi^n mx)^{-\delta}$$

$$\cdot \Re\left(\int_{\Lambda}^{Y} t^{n/2+n\delta-1} \exp\left(-int\log\frac{t}{2\pi} + it\log(mx) + int + \frac{\pi(n-2)i}{4}\right)\mathrm{d}t \right).$$

The derivative of the phase is, up to a constant, given by

$$-n\log t + \log(2^n\,\pi^n\,m\,x),$$

and so the integrand has a unique saddle point at

$$t = 2\pi \, (mx)^{1/n}.$$

We will choose Y to be

$$Y := 2\pi \left(\left(X^\theta + \frac{1}{2} \right) x \right)^{1/n},$$

so that for the terms $m > X^\theta$, the integrands have no saddle points and are therefore oscillating.

First, we treat these high-frequency terms with $m > X^\theta$. By using the fact that $t \leq Y$, the derivative of the phase in the corresponding integrals is

$$\log \frac{2^n \pi^n mx}{t^n} \gg \log \frac{2^n \pi^n mx}{Y^n} = \log \frac{m}{X^\theta + \frac{1}{2}},$$

and so, by the first derivative test and (4.7), they contribute

$$\ll X^{-\delta} Y^{n/2+n\delta-1} \sum_{m > X^\theta} \frac{|A(1, \dots, 1, m)|}{m^{1+\delta}} \cdot \frac{1}{\log \frac{m}{X^\theta + \frac{1}{2}}}$$

$$\ll X^{-\delta} Y^{n/2+n\delta-1} \sum_{X^\theta < m \leqslant 2X^\theta} \frac{|A(1, \dots, 1, m)|}{m^{1+\delta} \left(\frac{m}{X^\theta + \frac{1}{2}} - 1 \right)} + X^{-\delta} Y^{n/2+n\delta-1}$$

$$\ll_\varepsilon X^{(1/2+\delta-1/n)(1+\theta)-\delta+\theta\vartheta+\varepsilon/2} + X^{(1/2+\delta-1/n)(1+\theta)-\delta}$$

$$\ll_\varepsilon X^{1/2-(1+\theta)/2n+\varepsilon/2},$$

where the elementary fact that $\log x \gg x - 1$ for $x \in \,]1, 2[$ is used in the second estimate, in the penultimate step we have used the fact that $Y \asymp X^{(1+\theta)/n}$, in the last estimate we have used that $\theta < 1/(n-1+2n\vartheta)$, and finally we have bounded the sum trivially by using the absolute values:

$$\sum_{X^\theta < m \leqslant 2X^\theta} \frac{|A(1, \dots, 1, m)|}{m^{1+\delta} \left(\frac{m}{X^\theta + \frac{1}{2}} - 1 \right)}$$

$$\ll X^{\theta\vartheta-\theta\delta} \sum_{X^\theta < m \leqslant 2X^\theta} \frac{1}{m - X^\theta - \frac{1}{2}} \ll_\varepsilon X^{\theta\vartheta+\varepsilon/2}.$$

Next, we will deal with the low-frequency terms, that is, terms with $m \leqslant X^\theta$. First, we extend the integrals over the line segments $[-\delta - iY, -\delta - i\Lambda]$ and $[-\delta + i\Lambda, -\delta + iY]$ to be over the whole line segment connecting $-\delta - iY$ to $-\delta + iY$ with an error $O_{\delta,\Lambda}(1)$. Similarly, we may replace the factor s^{-1} by $(s + \Lambda)^{-1}$ with the error $O(X^{\varepsilon/2} Y^{n/2-1})$. Thus, the terms that we are left to deal with are

$$(n\pi)^{-n/2} \sum_{m \leqslant X^\theta} \frac{A(1, \dots, 1, m)}{m} \cdot \frac{1}{2\pi i} \int_{-\delta-iY}^{-\delta+iY} \frac{\Gamma\left(\frac{1-ns}{2} \right)}{\Gamma\left(\frac{ns-(n-1)}{2} \right)} \left(\pi^n n^n mx \right)^s \frac{\mathrm{d}s}{s+\Lambda}.$$

The main terms come from using Corollary 5 on these integrals with the choice $y = \pi^n n^n m x$. These main terms are given by

$$\frac{x^{(n-1)/2n}}{\pi\sqrt{n}} \sum_{m \leqslant X^\theta} \frac{A(1,\ldots,1,m)}{m^{(n+1)/2n}} \cos\left(2n\pi(mx)^{1/n} + \frac{(n-3)}{4}\pi\right) = P(x;\theta).$$

Note that the condition $y < (nY/2)^n$ is satisfied by the choice of Y. The contribution coming from the error terms of Corollary 5 can be estimated as follows by using partial summation together with (4.6) and recalling the fact that $Y \asymp X^{(1+\theta)/n}$:

$$\sum_{m \leqslant X^\theta} \frac{|A(1,\ldots,1,m)|}{m}\left((mx)^{(n-3)/2n} + Y^{n/2-1+n\delta} + Y^{(n-2)/2}\frac{1}{\log\frac{Y^n}{2^n\pi^n mx}}\right)$$

$$\ll_\varepsilon X^{(n-3)(1+\theta)/2n+\varepsilon/2}$$

$$+ X^{((n-2)/2n+\delta)(1+\theta)} + X^{(n-2)(1+\theta)/2n} \sum_{m \leqslant X^\theta} \frac{|A(1,\ldots,1,m)|}{m}\frac{1}{\log\frac{X^\theta+\frac{1}{2}}{m}}$$

$$\ll_\varepsilon X^{(n-3)(1+\theta)/2n+\varepsilon/2} + X^{(n-2)(1+\theta)/2n+\varepsilon/4}\left(1 + \sum_{m \leqslant X^\theta} \frac{|A(1,\ldots,1,m)|}{X^\theta + \frac{1}{2} - m}\right)$$

$$\ll_\varepsilon X^{1/2-(1+\theta)/2n+\varepsilon/2}$$

if $2\delta(1+\theta) \leq \varepsilon$, where the last estimate follows simply by using the absolute values as before. Therefore we have shown that

$$\frac{1}{2\pi i}\int_{-\delta-iY}^{-\delta+iY} L(s,f)x^s\frac{ds}{s} = P(x;\theta) + O\left(X^{1/2-(1+\theta)/2n+\varepsilon/2}\right)$$

in the range $0 < \theta < 1/(n-1+2n\vartheta)$ assuming $\vartheta < 1/2 - 1/n$ and $2\delta(1+\theta) \leq \varepsilon$. Furthermore, it follows that

$$E(x;\theta) = \int_{1/2-iX}^{1/2-iY} L(s,f)x^s\frac{ds}{s} + \int_{1/2+iY}^{1/2+iX} L(s,f)x^s\frac{ds}{s}$$

$$+ O\left(X^{1/2-(1+\theta)/2n+\varepsilon/2}\right)$$

for $0 < \theta < 1/(n-1+2n\vartheta)$ and $\vartheta < 1/2 - 1/n$.

From this it follows that

$$\frac{1}{X}\int_X^{2X} |E(x;\theta)|^2 \, dx$$

$$\ll_\varepsilon X^{1-(1+\theta)/n+\varepsilon}$$

$$+ \frac{1}{X}\int_X^{2X}\left|\int_{1/2-iX}^{1/2-iY} L(s,f)x^s\frac{ds}{s} + \int_{1/2+iY}^{1/2+iX} L(s,f)x^s\frac{ds}{s}\right|^2 dx$$

for θ and ϑ in the same ranges as before.

Hence, we are now reduced to study the integral

$$\frac{1}{X}\int_X^{2X}\left|\int_{1/2-iX}^{1/2-iY}L(s,f)x^s\frac{\mathrm{d}s}{s}+\int_{1/2+iY}^{1/2+iX}L(s,f)x^s\frac{\mathrm{d}s}{s}\right|^2\mathrm{d}x.$$

Let us fix a smooth compactly supported non-negative weight function w majorising the characteristic function of the interval $[1,2]$.

Now we simply compute

$$\frac{1}{X}\int_X^{2X}\left|\int_{1/2-iX}^{1/2-iY}L(s,f)x^s\frac{\mathrm{d}s}{s}+\int_{1/2+iY}^{1/2+iX}L(s,f)x^s\frac{\mathrm{d}s}{s}\right|^2\mathrm{d}x$$

$$\leq\frac{1}{X}\int_{\mathbb{R}}\left|\int_{1/2-iX}^{1/2-iY}L(s,f)x^s\frac{\mathrm{d}s}{s}+\int_{1/2+iY}^{1/2+iX}L(s,f)x^s\frac{\mathrm{d}s}{s}\right|^2 w\left(\frac{x}{X}\right)\mathrm{d}x$$

$$\ll\frac{1}{X}\int_{\mathbb{R}}\left|\int_{1/2+iY}^{1/2+iX}L(s,f)x^s\frac{\mathrm{d}s}{s}\right|^2 w\left(\frac{x}{X}\right)\mathrm{d}x$$

$$=\frac{1}{X}\int_{\mathbb{R}}\int_{1/2+iY}^{1/2+iX}\int_{1/2+iY}^{1/2+iX}L(s_1,f)x^{s_1}\overline{L(s_2,f)}x^{\overline{s}_2}\frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}\overline{s}_2}{\overline{s}_2}w\left(\frac{x}{X}\right)\mathrm{d}x$$

$$=\frac{1}{X}\int_{\mathbb{R}}\int_Y^X\int_Y^X L\left(\frac{1}{2}+it,f\right)x^{1/2+it}\overline{L\left(\frac{1}{2}+iv,f\right)}x^{1/2-iv}$$

$$\cdot\frac{\mathrm{d}t\mathrm{d}v}{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}-iv\right)}w\left(\frac{x}{X}\right)\mathrm{d}x$$

$$=\int_Y^X\int_Y^X\frac{L\left(\frac{1}{2}+it,f\right)L\left(\frac{1}{2}-iv,\widehat{f}\right)}{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}-iv\right)}X^{1+i(t-v)}\left(\int_{\mathbb{R}}x^{1+i(t-v)}w(x)\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}v.$$

By repeated integration by parts, we see that the inner integral is negligible (i.e. $\ll_A X^{-A}$ for any $A\geq 1$) when $|t-v|\geq X^\eta$ for some fixed $0<\eta<(1+\theta)/n$. In the complementary range the inner integral is bounded. Using this we simply estimate the remaining part of the integral as

$$\ll X\int\int_{\substack{Y\leq t,v\leq X\\|t-v|\leq X^\eta}}\frac{\left|L\left(\frac{1}{2}+it,f\right)L\left(\frac{1}{2}-iv,\widehat{f}\right)\right|}{tv}\,\mathrm{d}t\,\mathrm{d}v$$

$$\ll_\varepsilon X\int\int_{\substack{Y\leq t,v\leq X\\|t-v|\leq X^\eta}}(tv)^{-1+\varepsilon n/4(1+\theta)}\,\mathrm{d}t\,\mathrm{d}v$$

$$\ll_\varepsilon X^{1+\eta}Y^{-1+\varepsilon n/4(1+\theta)}\int_{Y-X^\eta}^{X+X^\eta}t^{-1+\varepsilon/4(1+\theta)}\,\mathrm{d}t$$

$$\ll_\varepsilon X^{1+\eta+\varepsilon/4(1+\theta)}Y^{-1+\varepsilon/4(1+\theta)}$$

$$\ll_\varepsilon X^{1-(1+\theta)/n+\eta+\varepsilon/4n+\varepsilon/4(1+\theta)},$$

where, in the third step, we have used the fact that, for fixed t, the parameter v ranges over a set of measure $\asymp X^\eta$. The resulting upper bound is small enough if we choose $\eta = \varepsilon/6 > 0$. Here we have used the generalised Lindelöf hypothesis in the second step and the fact that $Y \asymp X^{(1+\theta)/n}$ in the last step. This finishes the proof. □

6. Proof of Theorem 2

We start by observing that

$$(6.1) \quad \frac{1}{X} \int_X^{2X} \left| \sum_{x \leq m \leq x + \Delta} A(m, 1, \ldots, 1) \right|^2 \, \mathrm{d}x$$

$$= \frac{1}{X} \int_X^{2X} |P(x + \Delta; \theta) - P(x; \theta)|^2 \, \mathrm{d}x$$

$$+ \frac{1}{X} \int_X^{2X} |E(x + \Delta; \theta) - E(x; \theta)|^2 \, \mathrm{d}x$$

$$+ O\left(\frac{1}{X} \int_X^{2X} |P(x+\Delta; \theta) - P(x; \theta)| \cdot |E(x+\Delta; \theta) - E(x; \theta)| \, \mathrm{d}x \right)$$

for any $0 < \theta \leq 1$. In fact, we will suppose that $0 < \theta < 1/(n - 1 + 2n\vartheta)$. For the first term on the right-hand side we see that

$$(6.2) \quad \frac{1}{X} \int_X^{2X} |P(x + \Delta; \theta) - P(x; \theta)|^2 \, \mathrm{d}x$$

$$= \frac{1}{X} \int_X^{2X} |P(x + \Delta; \theta)|^2 \, \mathrm{d}x + \frac{1}{X} \int_X^{2X} |P(x; \theta)|^2 \, \mathrm{d}x$$

$$- \frac{1}{X} \int_X^{2X} \left[P(x + \Delta; \theta)\overline{P(x; \theta)} + P(x; \theta)\overline{P(x + \Delta; \theta)} \right] \, \mathrm{d}x.$$

By writing cosines in terms of exponentials we have

$$P(x; \theta) = \frac{x^{(n-1)/2n}}{2\pi\sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1, \ldots, 1, m)}{m^{(n+1)/2n}} e\left(n(mx)^{1/n} + \frac{(n-3)}{8} \right)$$

$$+ \frac{x^{(n-1)/2n}}{2\pi\sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1, \ldots, 1, m)}{m^{(n+1)/2n}} e\left(-n(mx)^{1/n} - \frac{(n-3)}{8} \right).$$

Arguing just as in the proof of Lemma 11 we see that

$$\frac{1}{X} \int_X^{2X} |P(x; \theta)|^2 \, \mathrm{d}x$$

$$\sim \frac{1}{2} \cdot \frac{1}{n\pi^2} \cdot \frac{2^{(2n-1)/n} - 1}{2 - 1/n} \cdot X^{(n-1)/n} \sum_{m=1}^\infty \frac{|A(m, 1, \ldots, 1)|^2}{m^{(n+1)/n}}$$

assuming $\theta < 1/(n-1) - \varepsilon$ for some small fixed $\varepsilon > 0$. The identical argument shows that

$$\frac{1}{X} \int_X^{2X} |P(x + \Delta; \theta)|^2 \, \mathrm{d}x$$

satisfies the same asymptotics under the additional assumption $\Delta = o(X)$.

On the other hand, in order to estimate the last remaining term in (6.2), a short calculation by writing cosines in terms of exponential functions shows that we need to estimate integrals of the form

$$\frac{1}{X} \int_X^{2X} (x(x+\Delta))^{(n-1)/2n}$$

$$e\left(\pm n \left((m(x+\Delta))^{1/n} \pm (\ell x)^{1/n}\right) \pm \mu \cdot \frac{(n-3)}{4}\right) \mathrm{d}x$$

with $\mu \in \{0, 1\}$.

Set $F(x) := (m(x+\Delta))^{1/n} - (\ell x)^{1/n}$. Using the easy observation that for $x \neq y$ we have $|x^{1/n} - y^{1/n}| \gg |x - y|(\max(x, y))^{(1-n)/n}$, it follows that

$$|F'(x)| \gg X^{(1-n)/n}|m - \ell|(\max(m, \ell))^{(1-n)/n}$$

for $m \neq \ell$. Also

$$(m(x+\Delta))^{1/n} - (mx)^{1/n} = m^{1/n} \int_x^{x+\Delta} t^{(1-n)/n} \, \mathrm{d}t \asymp m^{1/n} \Delta X^{(1-n)/n}.$$

Therefore, by applying the first derivative test, we have

$$\frac{1}{X} \int_X^{2X} \frac{(x(x+\Delta))^{(n-1)/2n}}{4\pi^2 n}$$

$$e\left(\pm n \left((m(x+\Delta))^{1/n} - (\ell x)^{1/n}\right) \pm \mu \cdot \frac{(n-3)}{4}\right) \mathrm{d}x$$

$$\ll \begin{cases} \frac{X^{(n-2)/n}(\max(m,\ell))^{(n-1)/n}}{|m-\ell|}, & \text{if } m \neq \ell \\ \frac{X^{(2n-2)/n}}{\Delta m^{1/n}}, & \text{if } m = \ell \end{cases}$$

Similarly,

$$\frac{1}{X} \int_X^{2X} \frac{(x(x+\Delta))^{(n-1)/2n}}{4\pi^2 n}$$

$$e\left(\pm n \left((m(x+\Delta))^{1/n} + (\ell x)^{1/n}\right) \pm \mu \cdot \frac{(n-3)}{4}\right) \mathrm{d}x$$

$$\ll \frac{X^{(n-2)/n}(\max(m, \ell))^{(n-1)/n}}{|m + \ell|}.$$

Hence, the non-diagonal terms in

$$\frac{1}{X}\int_X^{2X}|P(x+\Delta;\theta)P(x;\theta)|\,\mathrm{d}x$$

contribute

$$\ll_\varepsilon X^{(n-2)/n+\theta(n-1)/n+(n-2)\varepsilon/n}$$
$$\ll_\varepsilon X^{(n-1)/n-\varepsilon/n}$$

by using (4.7) and the assumption $\theta < 1/(n-1) - \varepsilon$. The diagonal contribution is estimated as

$$\ll \frac{X^{(2n-2)/n}}{\Delta}\sum_{m\leq X^\theta}\frac{|A(1,\ldots,1,m)|^2}{m^{(n+1)/n}}$$
$$\ll \frac{X^{(2n-2)/n}}{\Delta}$$
$$\ll_\varepsilon X^{(n-1)/n-\varepsilon},$$

provided that $\Delta \gg_\varepsilon X^{(n-1)/n+\varepsilon}$, again by using (4.7).

The term involving $E(x;\theta)$ is $\ll_\varepsilon X^{(n-1)/n-\varepsilon}$, which follows from the proof of Proposition 9 (here the generalised Lindelöf hypothesis is needed) for $0 < \theta < 1/(n-1+2n\vartheta)$ assuming $\vartheta < 1/2 - 1/n$. Finally, the error term in (6.1) is $\ll_\varepsilon X^{(n-1)/n-\varepsilon/2n}$ by the Cauchy–Schwarz inequality. This concludes the proof. □

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