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Stephan BAIER et Marc TECHNAU

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On the distribution of αp modulo one in imaginary quadratic number fields with class number one

par Stephan BAIER et Marc TECHNAU

Résumé. Nous étudions la répartition de αp modulo un dans les corps quadratiques imaginaires $\mathbb{K} \subset \mathbb{C}$ dont le nombre de classes est égal à un, où p parcourt l'ensemble des idéaux premiers de l'anneau des entiers $\mathcal{O} = \mathbb{Z}[\omega]$ de $\mathbb{K}$. Par analogie avec un résultat classique dû à R. C. Vaughan, nous obtenons que l'inégalité $\|\alpha p\|_\omega < \mathrm{N}(p)^{-1/8+\epsilon}$ est satisfaite pour une infinité de p, où $\|\varrho\|_\omega$ mesure la distance de $\varrho \in \mathbb{C}$ à $\mathcal{O}$ et $\mathrm{N}(p)$ est la norme de p.

La preuve est basée sur la méthode du crible de Harman et utilise des analogues pour les corps de nombres d'idées classiques dues à Vinogradov. De plus, nous introduisons un lissage qui nous permet d'utiliser la formule sommatoire de Poisson.

Abstract. We investigate the distribution of αp modulo one in imaginary quadratic number fields $\mathbb{K} \subset \mathbb{C}$ with class number one, where p is restricted to prime elements in the ring of integers $\mathcal{O} = \mathbb{Z}[\omega]$ of $\mathbb{K}$. In analogy to classical work due to R. C. Vaughan, we obtain that the inequality $\|\alpha p\|_\omega < \mathrm{N}(p)^{-1/8+\epsilon}$ is satisfied for infinitely many p, where $\|\varrho\|_\omega$ measures the distance of $\varrho \in \mathbb{C}$ to $\mathcal{O}$ and $\mathrm{N}(p)$ denotes the norm of p.

The proof is based on Harman's sieve method and employs number field analogues of classical ideas due to Vinogradov. Moreover, we introduce a smoothing which allows us to make conveniently use of the Poisson summation formula.

1. Introduction

Dirichlet's classical approximation theorem asserts that, given some real irrational α, there are infinitely many rational integers a, q $(q \neq 0)$ with

$$|\alpha - a/q| < q^{-2},$$

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or (equivalently) on writing $\|\rho\| = \min_{x\in\mathbb{Z}}|\rho - x|$ for the distance to a nearest integer,

$$\|q\alpha\| < q^{-1} \quad \text{for infinitely many } q. \tag{1.1}$$

Albeit individual values of α may allow for significantly sharper approximation by rational numbers, Hurwitz's approximation theorem implies that the exponent -1 in (1.1) is optimal in the sense that it cannot be decreased without the resulting new inequality failing to admit infinitely many solutions for some real irrational α (see, e.g., [5, Theorems 193 and 194]).

A natural variation on the question about the solubility of (1.1) is to impose the additional restriction that q be a rational prime and ask for which exponent θ one is able to establish that, for any real irrational α,

$$\|p\alpha\| < p^{-\theta} \quad \text{for infinitely many rational primes } p. \tag{1.2}$$

In this direction I. M. Vinogradov [21] obtained (1.2) with $\theta = \frac{1}{5} - \epsilon$, a result which has since then been improved by a number of researchers (see Table 1.1) culminating in the work of Matomäki [15] who obtained $\theta = 1/3 - \epsilon$. This exponent is considered to be the limit of the current technology (see the comments in [10]).

TABLE 1.1. Improvements on the admissible exponent θ in (1.2).

Date	Author(s)	θ	
1978	Technau [20]	$1/4 - \epsilon$	$= 0.25 - \epsilon$
1983	Harman [6]	$3/10$	$= 0.3$
1993	Jia [12]	$4/13$	$= 0.3076\ldots$
1996	Harman [7]	$7/22$	$= 0.31\overline{81}$
2000	Jia [13]	$9/28$	$= 0.3214\ldots$
2002	Heath-Brown–Jia [10]	$16/49$	$= 0.3265\ldots$
2009	Matomäki [15]	$1/3 - \epsilon$	$= 0.3\overline{3} - \epsilon$

In view of the above, the first named author [1] proposed to study the analogue of (1.2) for the Gaussian integers. The approach in [1] rests upon Harman's sieve method [6, 7, 8] and the required "arithmetical input" is obtained using novel Gaussian integer analogues of classical ideas due to Vinogradov [21, Lemma 8a].

In this paper, we consider the more general problem of proving analogues of (1.2) for imaginary quadratic number fields. It turns out that, in our opinion, this setting also has the pleasant side effect of painting a clearer picture of the Diophantine arguments that underpin the aforementioned arithmetical information. Preliminary results in this direction were obtained in the second author's doctoral dissertation [19]. A novel aspect of the present work is our additional use of smoothing directly incorporated into Harman's sieve method.

2. Main results

Before stating our results, we shall introduce some notation which is used throughout the rest of the article. We fix some imaginary quadratic number field $\mathbb{K}$ with distinguished embedding into the complex numbers $\mathbb{C}$ by means of which we shall regard $\mathbb{K}$ as a subfield of $\mathbb{C}$. By $\mathcal{O}$ we denote the *ring of integers* of $\mathbb{K}$, i.e., the integral closure of $\mathbb{Z}$ in $\mathbb{K}$. As $\mathbb{K}$ is a quadratic extension of $\mathbb{Q}$, it follows from well-known results from elementary algebraic number theory that $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank 2 and there is some $\omega \in \mathcal{O}$ such that $\{1, \omega\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}$. Since, by assumption, $\mathbb{K} \not\subseteq \mathbb{R}$, and $\mathbb{K}$ being the field of fractions of $\mathcal{O}$, it follows that $\Im\omega \neq 0$. In particular, $\{1, \omega\}$ turns out to be an $\mathbb{R}$-basis of $\mathbb{C}$ and, given some $\varrho \in \mathbb{C}$, we write $\Re_\omega \varrho$ and $\Im_\omega \varrho$ for the unique real numbers satisfying

$$\varrho = \Re_\omega \varrho + (\Im_\omega \varrho)\omega.$$

With this notation, we put

$$\|\varrho\|_\omega = \max\{\|\Re_\omega \varrho\|, \|\Im_\omega \varrho\|\}.$$

The natural notion of "size" of an element $m \in \mathcal{O}$ is furnished by its *norm* $\mathrm{N}(m)$, that is, the number of elements in the factor ring $\mathcal{O}/m\mathcal{O}$. It can be shown that $\mathrm{N}(m) = |m|^2$, where $|m|$ is the usual absolute value of m considered as a complex number.

The question we ask may now be enunciated as follows:

> *Given some imaginary quadratic number field $\mathbb{K} \subset \mathbb{C}$ with ring of integers $\mathcal{O}$, a choice $\{1, \omega\}$ of $\mathbb{Z}$-basis of $\mathcal{O}$, and given some $\alpha \in \mathbb{C} \setminus \mathbb{K}$, for which $\theta > 0$, does one have*
>
> $$\|p\alpha\|_\omega < \mathrm{N}(p)^{-\theta}$$
>
> *for infinitely many irreducible (or prime) elements $p \in \mathcal{O}$?*

As unique factorisation underpins the sieve method we employ to tackle the above question, we are forced to restrict our considerations to only those $\mathbb{K}$ with class number 1 (which, in this setting, is equivalent to $\mathcal{O}$ being a unique factorisation domain). The full determination of all such $\mathbb{K}$ is provided by the celebrated *Baker–Heegner–Stark theorem* [2, 11, 16, 17]:

Theorem (Baker–Heegner–Stark). *The imaginary quadratic number fields $\mathbb{K}$ with class number* 1 *are (up to isomorphism) precisely those $\mathbb{Q}(\sqrt{d})$ with $-d$ from the finite list* 1, 2, 3, 7, 11, 19, 43, 67, 163.

Our main result states that in the above question any $\theta < 1/8$ is admissible, provided that $\mathbb{K}$ has class number 1. This is the precise analogue of Vaughan's exponent $1/4 - \epsilon$ for the classical case, obtained in [20]. Note that in the class number 1 setting, the notions of *prime* and *irreducible* coincide.

Theorem 2.1. *Let $\mathbb{K} \subset \mathbb{C}$ be an imaginary quadratic number field with class number* 1 *and let $\mathcal{O}$ be its ring of integers with $\mathbb{Z}$-basis $\{1, \omega\}$. Suppose that α is a complex number such that $\alpha \notin \mathbb{K}$. Then, for any $\epsilon > 0$, there exists an infinite sequence of distinct prime elements $p \in \mathcal{O}$ such that*

$$\|p\alpha\|_\omega \leq \mathrm{N}(p)^{-1/8+\epsilon}. \tag{2.1}$$

Our approach to proving Theorem 2.1 involves counting prime elements with a certain smooth weight attached to them. On the other hand, one can also use sharp cut-offs and obtain a less fuzzy quantitative result at the cost of having to restrict to smaller values of θ in the above question. We prove the following:

Theorem 2.2. *Let $\mathbb{K} \subset \mathbb{C}$ be an imaginary quadratic number field with class number* 1 *and let $\mathcal{O}$ be its ring of integers with $\mathbb{Z}$-basis $\{1, \omega\}$. Suppose that $\epsilon > 0$ is sufficiently small. Let α be a complex number not contained in $\mathbb{K}$. Furthermore, suppose that one has coprime $a, q \in \mathcal{O}$ such that*

$$q \neq 0, \quad \frac{a}{q} \notin \mathcal{O}, \quad \text{and} \quad \gamma = \alpha - \frac{a}{q} \quad \text{satisfies} \quad |\gamma| \leq \frac{C}{\mathrm{N}(q)} \tag{2.2}$$

for some constant $C > 0$ and put $x = \mathrm{N}(q)^{28/5}$. Then, for any δ such that

$$x^{-1/28+\epsilon} \leq \delta < \tfrac{1}{2}, \tag{2.3}$$

we have

$$\left| \sum_{\substack{x/2 \leq \mathrm{N}(p) < x \\ \|p\alpha\|_\omega < \delta}} 1 - 4\delta^2 \sum_{x/2 \leq \mathrm{N}(p) < x} 1 \right| \ll_\epsilon C^2 \mathrm{N}(\omega)^7 \delta^2 x^{1-\epsilon},$$

where the summation variable p (as throughout) only assumes prime elements of $\mathcal{O}$ and the implied constant depends on ϵ alone.

Remark. (1) Instead of considering the homogeneous condition $\|p\alpha\|_\omega < \delta$ in the above theorem, one can also consider a shifted version, namely $\|p\alpha + \beta\|_\omega < \delta$, where β is an arbitrary complex number. In fact, the authors [6, 7, 12, 13, 20] listed in Table 1.1 also consider the shifted analogue of (1.2), but the innovation introduced by Heath-Brown [10] has, as they remark, the defect of entailing the restriction to $\beta = 0$. Regardless of this, the methods pursued by us in the present paper are perfectly capable of handling shifts β and we merely chose not to implement this for cosmetic reasons.

(2) In his recent preprint [9] on the case $\mathbb{K} = \mathbb{Q}(i)$, Harman achieved the result in Theorem 2.2 with the exponent 7/44, which corresponds to the exponent 7/22 in his classical result [7] mentioned above. To this end, he

didn't use a smoothing but introduced a number of novelties to overcome obstacles that are present in the non-smoothed approach. In particular, he was able to handle linear exponential sums over certain regions in $\mathbb{C}$ in an efficient way. It is likely that these novelties can be carried over to all imaginary-quadratic fields of class number 1, but we here confine ourselves to the simplest possible treatment, thus obtaining the exponent 1/28 for all fields of this kind. The way we overcome the said obstacles is to introduce a smoothing which allows us to use the Poisson summation formula conveniently. This leads us to Theorem 2.1, where we achieve the exponent 1/8 corresponding to Vaughan's result for the classical case [20]. To achieve the larger exponent 7/44 in full generality for the said fields, it would be required to carry over Harman's lower bound sieve, established in [7], to them. This is a task we aim to undertake in a separate paper since the proof of the aforementioned sieve result is technically complicated and therefore requires a large amount of extra work. In particular, it involves a number of numerical calculations which are not required for the proof of the basic version which we are using here.

Still assuming $\mathbb{K}$ to have class number 1, and appealing to Landau's prime ideal theorem one easily deduces that

$$\sum_{x/2\le \mathrm{N}(p)<x} 1 = \delta_{\mathcal{O}} \frac{x}{\log x}(1+o_{\mathcal{O}}(1)) \quad \text{as } x \longrightarrow \infty, \tag{2.4}$$

where $\delta_{\mathcal{O}} > 0$ is some constant only depending on $\mathcal{O}$.

Therefore, Theorem 2.2 implies Theorem 2.1 with the exponent $-1/8$ in (2.1) replaced with $-1/28$ provided one is able to verify the existence of infinitely many a and q as required by the theorem; however, the latter problem is already solved by Hilde Gintner [4]. In this regard, let Λ be the *fundamental parallelogram* spanned by 1 and ω,

$$\Lambda = \{\lambda_1 + \lambda_2\omega : \lambda_1, \lambda_2 \in [0,1)\}. \tag{2.5}$$

Lemma (Gitner). *Let α be a complex number not contained in $\mathbb{K}$. Then there are infinitely many $a, q \in \mathcal{O}$ satisfying* (2.2) *with $C = \pi^{-1}\sqrt{6}\,\mathrm{area}\,\Lambda$ and Λ given by* (2.5)*.*

Albeit the above lemma does not assert that a, q be coprime, if $\mathbb{K}$ has class number 1, then one can appeal to unique factorisation and cancel any potential non-trivial common factors from a and q. So, indeed, one has the aforementioned relation between Theorem 2.2 and Theorem 2.1.

3. Outline of the method

3.1. The sieve method. For the detection of the prime elements in Theorem 2.1 and Theorem 2.2 we use a sieve result due to Harman (with additional smoothing and adapted to our number field setting) which has the pleasant feature of keeping our exposition reasonably tidy. The underlying sieve method itself and its various refinements are also capable of yielding lower bounds instead of asymptotic formulae, in exchange for the prospect of increasing the admissible range for δ in (2.3) (or θ in the main question), but we do not implement this here. The interested reader is referred to Harman's exposition of his method [8]. The following special case suffices for our purposes; we write $d_k(\mathfrak{a})$ for the number of ways in which an ideal $\mathfrak{a} \subseteq \mathcal{O}$ can be written as a product of k ideals of $\mathcal{O}$, and we write $d(\mathfrak{a}) = d_2(\mathfrak{a})$.

Theorem 3.1 (Weighted version of Harman's sieve for $\mathcal{O}$)**.** *Suppose that $\mathcal{O}$ has class number 1 and let $x \geq 3$ be real. Let $w, \widetilde{w}\colon \mathcal{O} \to [0,1]$ be two functions such that, for both $\omega = w$ and $\omega = \widetilde{w}$,*

$$\lim_{R\to\infty} \sum_{\substack{r\in\mathcal{O}\\ \mathrm{N}(r)<R}} d_4(r\mathcal{O})\omega(r) \leq X \tag{3.1}$$

for some $X \geq 1$ and assume that $\omega(r) = \omega(\widetilde{r})$ for any pair of associate elements $r, \widetilde{r} \in \mathcal{O}$. Suppose further that one has numbers $Y > 1$, $0 < \mu < 1$, $0 < \kappa \leq \frac{1}{2}$, and $M \in (x^\mu, x)$ with the following property:

For any sequences $(a_m)_{m\in\mathcal{O}}$, $(b_n)_{n\in\mathcal{O}}$ of complex numbers with $|a_m| \leq 1$ and $|b_n| \leq d(n\mathcal{O})$, one has

$$\left| \sum\sum_{\substack{m,n\in\mathcal{O}\setminus\{0\}\\ 0<\mathrm{N}(m)<M}} a_m(w(mn) - \widetilde{w}(mn)) \right| \leq Y, \tag{3.2}$$

$$\left| \sum\sum_{\substack{m,n\in\mathcal{O}\setminus\{0\}\\ x^\mu<\mathrm{N}(m)<x^{\mu+\kappa}}} a_m b_n(w(mn) - \widetilde{w}(mn)) \right| \leq Y. \tag{3.3}$$

Then

$$|S(w, x^\kappa) - S(\widetilde{w}, x^\kappa)| \ll Y(\log(xX))^3,$$

where

$$S(\omega, x^\kappa) = \sum_{\substack{r\in\mathcal{O}\setminus\{0\}\\ \text{prime } p|r \Rightarrow \mathrm{N}(p)\geq x^\kappa}} \omega(r), \tag{3.4}$$

and the implied constant is absolute. (Here all infinite series appearing in Eqs. (3.2) *to* (3.4) *are guaranteed to be absolutely convergent by* (3.1)*.)*

Remark 3.2. We comment briefly on how we apply the above theorem: informally speaking, the goal is to choose the weight functions w and $\widetilde{w}$ in such a way that essentially only elements r with $\mathrm{N}(r) < x$ contribute in the definition of $S(w, x^\kappa)$ and $S(\widetilde{w}, x^\kappa)$. Then, choosing $\kappa = \frac{1}{2}$, these r are guaranteed to be prime, and, for $\omega \in \{w, \widetilde{w}\}$,

$$S(\omega, \sqrt{x}) \approx \sum_{\substack{\text{prime } p \in \mathcal{O} \\ \sqrt{x} \leq \mathrm{N}(p) < x}} \omega(p).$$

The choice of w is made as to guarantee that $S(w, \sqrt{x})$ is essentially a known quantity by an appeal to (2.4), and $\widetilde{w}$ is tailored to enforce a restriction such as $\|p\alpha\|_\omega \leq \delta$ as in Theorem 2.2. Finally, assuming that one proves suitably strong versions of (3.2) and (3.3), Theorem 3.1 asserts that $S(\widetilde{w}, \sqrt{x})$ must be of similar magnitude as $S(w, \sqrt{x})$, and, therefore, one ascertains information about the abundance of prime elements with the desired properties as encoded in the weight function $\widetilde{w}$.

The proof of Theorem 3.1 is essentially identical to the usual proof of Harman's sieve in the setting of $\mathbb{Z}$ [8, §§3.2–3.3]; it uses the sieve of Eratosthenes–Legendre to relate $S(w, x^\kappa) - S(\widetilde{w}, x^\kappa)$ to certain sums with more variables and applies Buchstab's identity multiple times to produce variables in the correct ranges for Eqs. (3.2) and (3.3) to become applicable. In the process of doing so, one has to remove certain cross-conditions between summation variables (a feat which is accomplished using a variant of Perron's formula, see Lemma 7.1 below). In our number field setting, this results in a slight complication which is not present when working over the rational integers: at some point one is faced with having to remove a condition of the type $\mathrm{N}(p) \leq \mathrm{N}(\widetilde{p})$ from a double sum with summation variables p and $\widetilde{p}$ assuming only non-associate prime elements as values (see the arguments around (7.12) below). However, this problem can be overcome. For the convenience of the reader we provide a detailed proof of Theorem 3.1 in Section 7 below.

3.2. Outline of the rest of the paper. Apart from proving Theorem 3.1 in Section 7, we proceed as follows: by the outline given in Remark 3.2, the bulk of the remaining work lies in the verification of (3.2) and (3.3) for the two choices of w and $\widetilde{w}$ that we use below (non-smoothed and smoothed). Both arguments ultimately hinge on distribution results related to the sequence $\|n\alpha\|_\omega$ ($n \in \mathcal{O}$). We devote our attentions to establishing such results first. This is done in Section 4, with its principal results being stated in Theorem 4.3 and Theorem 4.4.

Section 5 is devoted to proving Theorem 2.2, with this goal being achieved in Section 5.7. The results concerning (3.2) and (3.3) are recorded in Proposition 5.4 and Proposition 5.2 respectively.

In Section 6 we undertake proving Theorem 2.1. The analogues of the aforementioned propositions are Proposition 6.6 and Proposition 6.7 and their proof is largely parallel to the proof of their non-smoothed counterparts. The main innovation here is contained in Lemma 6.4, which takes advantage of the smooth weights by means of Poisson's summation formula.

4. Exponential sum estimates

In Section 5 we need estimates for sums of the shape

$$\sum_{n} \sum_{m} \mathrm{e}(\Im_\omega(mn\alpha)),$$

where the summation over $m \in \mathcal{O}$ is restricted to some annulus $x(n) \le \mathrm{N}(m) \le y(n)$ with bounds $x(n)$ and $y(n)$ depending on $n \in \mathcal{O}$. In Section 4.2 we estimate the inner summation and in Section 4.3 we deal with the additional summation over n. The corresponding proof is then carried out in the subsequent sections. Moreover, using the tools developed in Section 4.4, we also establish a closely related result which is useful in Section 6 (see Theorem 4.4).

In what follows, we sometimes have expressions like $1/\|\Re_\omega \alpha\|$; if a division by zero occurs there, then the result is understood to mean $+\infty$.

4.1. Some facts about quadratic extensions. Before being able to tackle the problem outlined above, we take the opportunity to record here some basic facts about quadratic extensions which we use throughout.

Lemma 4.1. *Let $\mathbb{K} \subset \mathbb{C}$ be an imaginary quadratic number field with ring of integers $\mathcal{O}$ and generator ω of $\mathcal{O}$, that is, $\mathcal{O} = \mathbb{Z}[\omega]$. Then the following statements hold:*

1. *The number of units in $\mathcal{O}$ is bounded by six.*
2. $|\Im\omega| \ge \sqrt{3}/2$.
3. $2\Re\omega$ *is an integer.*
4. *For $y \ge 1$, $\#\{n \in \mathcal{O} : \mathrm{N}(n) \le y\} \ll y$, where the implied constant is absolute.*

Proof. The first assertion may be found in [5].

For the second assertion just observe that there is some negative squarefree rational integer d such that $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. Letting $D = 4d$ if $d \not\equiv 1 \bmod 4$, and $D = d$ otherwise, we have $\mathcal{O} = \mathbb{Z}[\widetilde{\omega}]$, where $\widetilde{\omega} = \frac{1}{2}(D + \sqrt{D})$. An elementary calculation shows that ω is of the form $k \pm \widetilde{\omega}$ for some rational integer k. This already proves (3). Moreover, from this and a short computation one immediately obtains (2) (with equality being attained for $d = -3$).

Concerning the last assertion, we note that the quantity bounded therein, $\#\{(n_1, n_2) \in \mathbb{Z}^2 : |n_1 + n_2\omega|^2 \le y\}$, counts points inside some ellipse. Elementary arguments suffice to show that this is asymptotically equal to $\pi y/|\Im\omega|$ and the uniform lower bound for $1/|\Im\omega|$ furnished by the second assertion finishes the proof. ∎

4.2. Basic estimates for linear exponential sums.

Lemma 4.2. *Let α be a complex number and suppose that one has numbers x, y such that $0 \le x \le y$. Then*

$$\left|\sum_{x\le \mathrm{N}(m)\le y} \mathrm{e}(\Im_\omega(m\alpha))\right| \ll \mathrm{N}(\omega)\sqrt{y}\min\left\{\sqrt{y}, \frac{1}{\|\Re_\omega\alpha\|}, \frac{1}{\|\Im_\omega\alpha\|}\right\}. \tag{4.1}$$

Proof. We may assume $y \ge 1$, for (4.1) is trivial otherwise. We denote the sum on the left hand side of (4.1) by $\mathrm{Lin}(x, y)$. On writing

$$\omega^2 = \xi_1 + \xi_2\omega \quad (\xi_1 = \Re_\omega\omega^2, \quad \xi_2 = \Im_\omega\omega^2), \tag{4.2}$$

$m = m_1+m_2\omega$ and $\ell_1 = m_1+m_2\xi_2$, we obtain the following two expressions for $\Im_\omega(m\alpha)$:

$$\Im_\omega(m\alpha) = m_2(\Re_\omega\alpha + \xi_2\Im_\omega\alpha) + m_1\Im_\omega\alpha = \ell_1\Im_\omega\alpha + m_2\Re_\omega\alpha.$$

Therefore,

$$\mathrm{Lin}(0, y) = \sum_{\substack{m_1 \quad m_2 \\ 0\le \mathrm{N}(m_1+m_2\omega)\le y}}\!\!\sum \mathrm{e}(m_2(\Re_\omega\alpha + \xi_2\Im_\omega\alpha))\,\mathrm{e}(m_1\Im_\omega\alpha) \tag{4.3}$$

and

$$\mathrm{Lin}(0, y) = \sum_{\substack{\ell_1 \quad m_2 \\ 0\le \mathrm{N}(\ell_1-m_2\xi_2+m_2\omega)\le y}}\!\!\sum \mathrm{e}(\ell_1\Im_\omega\alpha)\,\mathrm{e}(m_2\Re_\omega\alpha). \tag{4.4}$$

So, on recalling the well-known bound

$$\left|\sum_{a\le j\le b}\mathrm{e}(j\rho)\right| \le \frac{1}{2\|\rho\|} \quad (a, b, \rho \in \mathbb{R}),$$

and using the triangle inequality on the outer summations in (4.3) and (4.4), we obtain

$$|\mathrm{Lin}(0, y)| \le \frac{\#\{m_2 : \exists\, m_1 \text{ s.t. } 0 \le \mathrm{N}(m_1 + m_2\omega) \le y\}}{2\|\Im_\omega\alpha\|}, \tag{4.5}$$

$$|\mathrm{Lin}(0, y)| \le \frac{\#\{\ell_1 : \exists\, m_2 \text{ s.t. } 0 \le \mathrm{N}(\ell_1 - m_2\xi_2 + m_2\omega) \le y\}}{2\|\Re_\omega\alpha\|}. \tag{4.6}$$

The numerators here are bounded easily: indeed, since

$$\mathrm{N}(m_1 + m_2\omega) \ge m_2^2(\Im\omega)^2,$$

using $y \ge 1$ and Lemma 4.1(2), one easily bounds the numerator in (4.5) by $4\sqrt{y}$. On the other hand for $|\ell_1| > c\sqrt{y}$ with $c = (2 + \frac{2}{\sqrt{3}})\mathrm{N}(\omega)$ there is no m_2 such that

$$\begin{aligned} y &\ge \mathrm{N}(\ell_1 - m_2\xi_2 + m_2\omega) \\ &\ge \max\{|\ell_1 - m_2\xi_2 + m_2\Re\omega|^2,\, m_2^2(\Im\omega)^2\}, \end{aligned} \tag{4.7}$$

for otherwise it would follow that

$$\sqrt{y} > c\sqrt{y} - |m_2| \cdot |\xi_2 - \Re\omega|$$

so that $|m_2| \cdot |\xi_2 - \Re\omega| > (c-1)\sqrt{y}$, but then

$$|m_2 \Im\omega| > \frac{c-1}{|\xi_2| + |\Re\omega|}\sqrt{y} \geq \frac{(1+2/\sqrt{3})|\omega|^2}{|\omega|^2 + |\omega|}\sqrt{y} = \frac{1+2/\sqrt{3}}{1+|\omega|^{-1}}\sqrt{y} \geq \sqrt{y},$$

in contradiction to (4.7). Hence, the numerator in (4.6) is bounded by $2c\sqrt{y} + 1 \leq 8|\omega|^2\sqrt{y}$. Thus,

$$|\mathrm{Lin}(0,y)| \leq 8|\omega|^2\sqrt{y}\min\left\{\frac{1}{2\|\Im_\omega\alpha\|}, \frac{1}{2\|\Re_\omega\alpha\|}\right\}$$

and, together with the trivial bound $|\mathrm{Lin}(x,y)| \ll y$ from Lemma 4.1(4), this is clearly satisfactory to establish (4.1) for $x = 0$, and the case $x > 0$ follows from this bound and

$$\mathrm{Lin}(x,y) = \mathrm{Lin}(0,y) - \lim_{\epsilon \searrow 0} \mathrm{Lin}(0, x-\epsilon). \qquad \square$$

4.3. Distribution of fractional parts. In view of Lemma 4.2 and recalling the goal stated at the beginning of Section 4, we are faced with the problem of estimating sums of the shape

$$\sum_{n \in \mathscr{X}} E(n, M), \tag{4.8}$$

where $M \geq 2$, $\mathscr{X}$ is some subset of $\{n \in \mathcal{O} : 1 \leq \mathrm{N}(n) < x\}$ with $x \geq 1$, and

$$E(n,M) = \min\left\{M, \frac{1}{\|\Re_\omega(n\alpha)\|}, \frac{1}{\|\Im_\omega(n\alpha)\|}\right\}. \tag{4.9}$$

The usual attack against such a problem is to replace α by some Diophantine approximation a/q. Subsequently, after bounding the error introduced from the approximation, one is able to control averages of $E(n, M)$ with n constrained to boxes (say) not too large in terms of $|q|$. By splitting the full range of n in (4.8) into such boxes, one derives a bound for (4.8) of the shape seen in Theorem 4.3 below.

Theorem 4.3. *Let $\mathscr{X}$ be a subset of all $n \in \mathcal{O}$ with $1 \leq \mathrm{N}(n) < x$ and suppose that one has coprime $a, q \in \mathcal{O}$ satisfying* (2.2)*. Put*

$$S = \sum_{n \in \mathscr{X}} \min\left\{M, \frac{1}{\|\Re_\omega(n\alpha)\|}, \frac{1}{\|\Im_\omega(n\alpha)\|}\right\}.$$

Then, assuming $M \geq 2$,

$$S \ll (1 + C^2\mathrm{N}(\omega^2)x/\mathrm{N}(q))(M + \mathrm{N}(q\omega)\log M). \tag{4.10}$$

Furthermore, if $x \leq \mathrm{N}(q)/(12C\mathrm{N}(\omega))^2$, then

$$S \ll \mathrm{N}(q\omega)\log \mathrm{N}(q\omega). \tag{4.11}$$

The proof of this result follows the outline given above and is undertaken in the next two subsections. Similarly, we also obtain the following result:

Theorem 4.4. *Suppose that one has coprime $a, q \in \mathcal{O}$ satisfying* (2.2)*. For $0 < \Delta \leq \frac{1}{2}$ let*

$$H_\alpha(x, \Delta) = \#\{n \in \mathcal{O} : 0 < \mathrm{N}(n) \leq x,\ \|n\alpha\|_\omega \leq \Delta\}.$$

Then

$$H_\alpha(x, \Delta) \ll_{C,\omega} (1 + x|q|^{-2})(1 + \Delta^2|q|^2).$$

Moreover, $H_\alpha(x, \Delta)$ vanishes if $\Delta < 1/(4|q\omega|)$ and $x \leq |q|^2/(12C|\omega|^2)^2$.

4.4. Diophantine lemmas. As a first step, we show that

$$\|\alpha\|_\omega = \max\{\|\Re_\omega\alpha\|, \|\Im_\omega\alpha\|\}$$

cannot be too small if $\alpha \in \mathbb{K}$ is not an algebraic integer. The results in this section are probably already known in one form or another. However, we were unable to find a suitable reference and, therefore, provide full proofs for the reader's convenience.

Lemma 4.5. *For non-zero $a, q \in \mathcal{O}$ such that $\alpha = a/q \notin \mathcal{O}$, it holds that $\|\alpha\|_\omega \geq 1/(2|q\omega|)$.*

Proof. Pick $m = m_1 + m_2\omega \in \mathcal{O}$ such that

$$\|\Re_\omega\alpha\| = |\Re_\omega\alpha - m_1| \quad \text{and} \quad \|\Im_\omega\alpha\| = |\Im_\omega\alpha - m_2|.$$

Now certainly it holds that

$$\begin{aligned} |\alpha - m| &= |(\Re_\omega\alpha - m_1) + (\Im_\omega\alpha - m_2)\omega| \\ &\leq 2|\omega| \max\{|\Re_\omega\alpha - m_1|, |\Im_\omega\alpha - m_2|\} \\ &= 2|\omega| \max\{\|\Re_\omega\alpha\|, \|\Im_\omega\alpha\|\}. \end{aligned}$$

Therefore, to prove the lemma, it suffices to give a suitable lower bound for $|\alpha - m|$, which, upon noting that $\alpha \notin \mathcal{O}$, is quite easy:

$$|\alpha - m| = |q|^{-1}|a - qm| \geq |q|^{-1} \min_{0 \neq r \in \mathcal{O}} |r| = |q|^{-1}. \qquad \square$$

Next, we intend to derive a result similar to Lemma 4.5, when α is slightly perturbed:

Lemma 4.6. *Let α be a complex number and a, q be such that* (2.2) *holds. Furthermore, suppose that $n \in \mathcal{O}$ satisfies $|n| \leq |q|/(12C|\omega|^2)$ and na be indivisible by q. Then $\|n\alpha\|_\omega \geq 1/(4|q\omega|)$.*

Proof. First, we separate the perturbation γ from the rest: we have

$$\begin{aligned} \|\Im_\omega(n\alpha)\| &= \min_{k \in \mathbb{Z}} |\Im_\omega(na/q) - k + \Im_\omega(n\gamma)| \\ &\geq \min_{k \in \mathbb{Z}} |\Im_\omega(na/q) - k| - |\Im_\omega(n\gamma)| \\ &= \|\Im_\omega(na/q)\| - |\Im_\omega(n\gamma)|, \end{aligned}$$

and the same holds when one replaces $\Im_\omega$ by $\Re_\omega$. The last term therein is bounded easily: using $|\xi| \geq |\Im_\omega \xi| \cdot |\Im\omega|$, Lemma 4.1(2) and writing $N = C|n|/|q|^2$ for the moment, we have

$$|\Im_\omega(n\gamma)| \leq \max_{\substack{\xi\in\mathbb{C} \\ |\xi|\leq N}} |\Im_\omega \xi| \leq \frac{N}{|\Im\omega|} \leq \frac{2N}{\sqrt{3}}.$$

A similar calculation also bounds the corresponding $\Re_\omega$-term:

$$\begin{aligned} |\Re_\omega(n\gamma)| &\leq \max\{|\rho_1| : \rho_1, \rho_2 \in \mathbb{R},\, (\rho_1 + \rho_2 \Re\omega)^2 + (\rho_2 \Im\omega)^2 \leq N^2\} \\ &\leq \max\{|\theta| + |\rho_2 \Re\omega| : \theta, \rho_2 \in \mathbb{R},\, \theta^2 + (\rho_2\Im\omega)^2 \leq N^2\} \\ &\leq (1 + |\Re\omega/\Im\omega|)N \\ &\leq (1 + 2/\sqrt{3})|\omega|N. \end{aligned}$$

Thus, using Lemma 4.5,

$$\begin{aligned} \|n\alpha\|_\omega &= \max\{\|\Re_\omega(n\alpha)\|, \|\Im_\omega(n\alpha)\|\} \\ &\geq \max\{\|\Re_\omega(na/q)\|, \|\Im_\omega(na/q)\|\} - 3|\omega|N \\ &\geq \frac{1}{2|q\omega|} - 3|\omega|N = \frac{1}{|q|}\left(\frac{1}{2|\omega|} - 3C|\omega|\frac{|n|}{|q|}\right). \end{aligned}$$

Now, by assumption, the term in the parentheses is $\geq (4|\omega|)^{-1}$, and the assertion of the lemma follows. □

4.5. Proof of Theorem 4.3. Assume the hypotheses of Theorem 4.3 and let n and $\widetilde{n}$ be two distinct algebraic integers in $\mathcal{O}$ which coincide modulo q. Then there is some non-zero $m \in \mathcal{O}$ such that $n - \widetilde{n} = mq$ and, hence, $|n - \widetilde{n}| = |m||q| \geq |q|$. Assuming a and q to be coprime and $|n - \widetilde{n}| < |q|$, we conclude that $(n - \widetilde{n})a$ is divisible by q if and only if $n = \widetilde{n}$. Consequently, if $\mathscr{R} \subseteq \mathbb{C}$ is some set with

$$\operatorname{diam} \mathscr{R} \leq \frac{|q|}{12C|\omega|^2}, \tag{4.12}$$

then, according to Lemma 4.6, any two distinct points $n\alpha$, $\widetilde{n}\alpha$ ($n, \widetilde{n} \in \mathscr{X} \cap \mathscr{R}$) satisfy the spacing condition

$$\max\{\|\Re_\omega((n-\widetilde{n})\alpha)\|, \|\Im_\omega((n-\widetilde{n})\alpha)\|\} \geq \frac{1}{4|q\omega|}.$$

Therefore, for $0 < \Delta_1, \Delta_2 \leq \frac{1}{2}$, the sum

$$\sum_{\substack{n\in\mathscr{X}\cap\mathscr{R} \\ \|\Re_\omega(n\alpha)\|\leq\Delta_1 \\ \|\Im_\omega(n\alpha)\|\leq\Delta_2}} 1$$

is bounded by

$$\sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \{\Re_\omega(n\alpha)\}\le\Delta_1\\ \{\Im_\omega(n\alpha)\}\le\Delta_2}} 1 + \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \{\Re_\omega(n\alpha)\}\ge 1-\Delta_1\\ \{\Im_\omega(n\alpha)\}\le\Delta_2}} 1 + \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \{\Re_\omega(n\alpha)\}\le\Delta_1\\ \{\Im_\omega(n\alpha)\}\ge 1-\Delta_2}} 1 + \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \{\Re_\omega(n\alpha)\}\ge 1-\Delta_1\\ \{\Im_\omega(n\alpha)\}\ge 1-\Delta_2}} 1$$

which in turn is bounded by four times the maximum number of points of pairwise maximum norm distance $\ge (4|\omega||q|)^{-1}$ that can be put in a rectangle with side lengths Δ_1 and Δ_2, i.e.,

$$\sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \|\Re_\omega(n\alpha)\|\le\Delta_1\\ \|\Im_\omega(n\alpha)\|\le\Delta_2}} 1 \ll (1+|q\omega|\Delta_1)(1+|q\omega|\Delta_2). \tag{4.13}$$

Moving on, let $L\in\mathbb{N}$ be a parameter at our disposal. Then the sum

$$S(\mathscr{R}) = \sum_{n\in\mathscr{X}\cap\mathscr{R}} E(n,M)$$

with $E(n,M)$ as defined in (4.9) admits a decomposition

$$\begin{aligned} S(\mathscr{R}) \le & \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \|\Re_\omega(n\alpha)\|\le 2^{-L}\\ \|\Im_\omega(n\alpha)\|\le 2^{-L}}} M + \sum\sum\sum_{\substack{2\le k_1,k_2\le L,\, n\in\mathscr{X}\cap\mathscr{R}\\ 2^{-k_1}<\|\Re_\omega(n\alpha)\|\le 2^{1-k_1}\\ 2^{-k_2}<\|\Im_\omega(n\alpha)\|\le 2^{1-k_2}}} \min\{2^{k_1},2^{k_2}\} \\ & + \sum_{2\le k\le L}\left\{ \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ 2^{-k}<\|\Re_\omega(n\alpha)\|\le 2^{1-k}\\ \|\Im_\omega(n\alpha)\|\le 2^{-L}}} + \sum_{\substack{n\in\mathscr{X}\cap\mathscr{R}\\ \|\Re_\omega(n\alpha)\|\le 2^{-L}\\ 2^{-k}<\|\Im_\omega(n\alpha)\|\le 2^{1-k}}} \right\} 2^k \\ = & S_1(\mathscr{R}) + S_2(\mathscr{R}) + S_3(\mathscr{R}), \quad \text{say.} \end{aligned}$$

By (4.13) and using $(a+b)^2 \le 2a^2+2b^2$ $(a,b\ge 1)$,

$$S_1(\mathscr{R}) \ll (1+|q\omega|2^{-L})^2 M \ll M + |q\omega|^2 2^{-2L} M.$$

Moreover, using $\min\{2^{k_1},2^{k_2}\} \le \sqrt{2^{k_1+k_2}}$ and (4.13),

$$S_2(\mathscr{R}) \ll \left(\sum_{2\le k\le L} 2^{k/2}(1+|q\omega|2^{-k})\right)^2 \ll 2^L + |q\omega|^2.$$

Similarly,

$$S_3(\mathscr{R}) \ll \sum_{2\le k\le L} 2^k(1+|q\omega|2^{-k})(1+|q\omega|2^{-L}) \ll 2^L + |q\omega|^2 L.$$

Assuming $M\ge 2$, we take $L = \lceil \frac{1}{2\log 2}\log(2M)\rceil$ to obtain

$$S(\mathscr{R}) \ll M + |q\omega|^2\log(2M) \ll M + \mathrm{N}(q\omega)\log M. \tag{4.14}$$

Additionally, if $x \leq |q|^2/(12C|\omega|^2)^2$, then we take $L = \lceil \frac{1}{\log 2} \log(4|q\omega|) \rceil$. In this case Lemma 4.6 shows that $S_1(\mathscr{R})$ vanishes and, consequently, we have

$$S(\mathscr{R}) \ll |q\omega|^2 \log(2|q\omega|) \ll \mathrm{N}(q\omega) \log \mathrm{N}(q\omega). \tag{4.15}$$

Finally, we note that the set $\mathscr{X}$ can be covered by fewer than

$$\left(1 + \frac{2\sqrt{x}}{\{\text{diameter bound}\}/\sqrt{2}}\right)^2 \ll 1 + C^2|\omega|^4 x|q|^{-2} \tag{4.16}$$

squares $\mathscr{R}$ with diameter (4.12). Together with (4.14) this proves (4.10), and together with (4.15) we obtain (4.11). This proves Theorem 4.3.

Proof of Theorem 4.4. The assertion concerning the vanishing of $H_\alpha(x, \Delta)$ is contained in Lemma 4.6. As for the bound for $H_\alpha(x, \Delta)$, cover the set $\mathscr{X} = \{n \in \mathcal{O} : 0 < \mathrm{N}(n) \leq x\}$ with rectangles as above and employ (4.13). □

5. The non-smoothed version

Here we tackle the problem of verifying the assumptions of Theorem 3.1 in a setting suitable for proving Theorem 2.2. Throughout, we assume the hypotheses of Theorem 2.2, although $x \geq 3$ may be considered arbitrary until Section 5.7, where we take $x = \mathrm{N}(q)^{28/5}$.

5.1. Setting up linear and bilinear forms. Let $\mathscr{B} = \{n \in \mathcal{O} : x/2 \leq \mathrm{N}(n) < x\}$ and $\mathscr{A} = \{n \in \mathscr{B} : \|n\alpha\|_\omega < \delta\}$. Concerning Theorem 3.1, we choose $\widetilde{w}$ to be $\mathbb{1}_{\mathscr{A}}$, the characteristic function of the set $\mathscr{A}$, and $w = 4\delta^2 \mathbb{1}_{\mathscr{B}}$. Given these definitions, the limit in (3.1) is actually attained for every $R \geq x$ and trivial estimates suffice to show that X therein may be taken $\ll x^5$.[1] Moving on, we shall want to compare sums of the type

$$\sum\sum_{mn \in \mathscr{A}} a_m b_n \quad \text{and} \quad 4\delta^2 \sum\sum_{mn \in \mathscr{B}} a_m b_n, \tag{5.1}$$

where the summation indices m, n vary through $\mathcal{O}$ and the coefficient sequences $(a_m)_m$ and $(b_n)_n$ consist of complex numbers and satisfy $|a_n| \leq 1$ and $|b_n| \leq d(n\mathcal{O})$.

To be more specific, for parameters $\mu > 0$ and $0 < \kappa \leq \frac{1}{2}$, there are two types of sums we would like to estimate

- *Type I*: $b_n = 1$ in the above and $(a_m)_m$ is supported only on m with $0 < \mathrm{N}(m) < M$ for some M with $x^\mu < M \leq x$ (see (3.2)).
- *Type II*: $(a_m)_m$ is supported only on m with $x^\mu \leq \mathrm{N}(m) < x^{\mu+\kappa}$ (see (3.3)).

[1] Of course, for such divisor sums much better estimates are available (see, e.g., Lemma 5.3 below for d_4 replaced with d_2). However, since in Theorem 3.1, only the logarithm of X enters in the final error term, we can be very sloppy here.

Each type requires a different treatment, but for now it is convenient to start by transforming (5.1) without restricting to either of the above types. We start with the following result which furnishes a finite Fourier approximation to the *saw-tooth function* ψ given by

$$\psi(t) = t - \lfloor t \rfloor - \tfrac{1}{2} \quad (t \in \mathbb{R}).$$

Lemma 5.1. *For all real x and $J \geq 1$, we have*

$$\psi(x) = \sum_{1 \leq |j| < J} (2\pi i j)^{-1} \, \mathrm{e}(-jx) + O\biggl(\min\biggl\{\log 2J, \frac{1}{J\|x\|}\biggr\}\biggr).$$

Proof. This is Lemma 4.1.2 in [3]. □

We now derive a useful expansion of the characteristic function $\mathbb{1}_{\mathscr{A}}$ of $\mathscr{A}$ evaluated at algebraic integers. For an element y of $\mathcal{O}$ we put

$$x_{1,y} = \Re_\omega(y\alpha) \quad \text{and} \quad x_{2,y} = \Im_\omega(y\alpha).$$

Furthermore, let

$$\mathcal{O}_{\delta,\alpha} = \{y \in \mathcal{O} : x_{1,y} \in \pm\delta + \mathbb{Z} \text{ or } x_{2,y} \in \pm\delta + \mathbb{Z}\}.$$

We now consider the characteristic function $\mathbb{1}_{\mathscr{A}} \colon \mathcal{O} \to \{0,1\}$ of the set $\mathscr{A}$. For any $y \in \mathscr{B} \setminus \mathcal{O}_{\delta,\alpha}$ we have the expansion

$$\begin{aligned}
\mathbb{1}_{\mathscr{A}}(y) &= \prod_{k=1,2} (2\delta + (\psi(-x_{k,y} - \delta) - \psi(-x_{k,y} + \delta))) \\
&= 4\delta^2 + 2\delta \sum_{k=1,2} (\psi(-x_{k,y} - \delta) - \psi(-x_{k,y} + \delta)) \\
&\qquad + \prod_{k=1,2} (\psi(-x_{k,y} - \delta) - \psi(-x_{k,y} + \delta)) \\
&= 4\delta^2 + 2\delta \sum_{k=1,2} \Xi_k(y) + \Xi_3(y), \quad \text{say.}
\end{aligned} \tag{5.2}$$

Note here that the first equality in (5.2) may not hold for $y \in \mathcal{O}_{\delta,\alpha}$. Nevertheless, the last line in (5.2) remains bounded even in that case. Therefore one can, as we do below, also use the last line of (5.2) as a substitute for $\mathbb{1}_{\mathscr{A}}(y)$ even when $y \in \mathcal{O}_{\delta,\alpha}$. This only introduces an error bounded by a constant times for how many $y \in \mathcal{O}_{\delta,\alpha}$ is this applied.

For $k = 1, 2, 3$ we consider the sums

$$\mathit{\Sigma}_k = \sum_{m,\,n}\sum a_m b_n \Xi_k(mn).$$

For $k=1,2$, on applying Lemma 5.1 with some $J\geq 1$ to be specified later (see (5.26) below), for *any* choice of summation ranges for m,n, we have

$$\begin{aligned}\Sigma_k &= \sum\sum_{m,n} a_m b_n \sum_{1\leq |j|<J} (2\pi i j)^{-1}(\mathrm{e}(-j\delta)-\mathrm{e}(j\delta))\,\mathrm{e}(j(-x_{k,mn})) + O(G)\\ &\ll \sum_{1\leq |j|<J} \Pi(j)\cdot \Big|\sum\sum_{m,n} a_m b_n\,\mathrm{e}(jx_{k,mn})\Big| + G,\end{aligned}$$

where

$$\Pi(j)=\min\{|j|^{-1},\delta\} \tag{5.3}$$

and

$$G := \sum_{k=1}^{2}\sum_{l=0}^{1}\sum\sum_{m,n}|a_m b_n| \min\Big\{\log 2J, \frac{1}{J\|(-1)^l\delta - x_{k,mn}\|}\Big\}. \tag{5.4}$$

Similarly, we obtain

$$\Sigma_3 \ll \sum_{\substack{1\leq |j_1|<J\\ 1\leq |j_2|<J}} \Pi(j_1)\Pi(j_2)\cdot\Big|\sum\sum_{m,n} a_m b_n\,\mathrm{e}(-j_1x_{1,mn}-j_2x_{2,mn})\Big| + (\log 2J)G,$$

where we have used the trivial estimates

$$\sum_{1\leq |j|<J}(2\pi i j)^{-1}(\mathrm{e}(-j\delta)-\mathrm{e}(j\delta))\,\mathrm{e}(j(-x_{k,mn})) \ll \log 2J$$

and

$$\min\Big\{\log 2J, \frac{1}{J\|(-1)^l\delta - x_{k,mn}\|}\Big\} \ll \log 2J.$$

Now consider

$$E = \sum_{mn\in\mathscr{A}}{}^{*}\sum a_m b_n - 4\delta^2 \sum_{mn\in\mathscr{B}}{}^{*}\sum a_m b_n, \tag{5.5}$$

where the star in the summation indicates that the range of m is to be restricted to a Type I or Type II range. The first sum may be written as

$$\sum_{mn\in\mathscr{A}}{}^{*}\sum a_m b_n = \sum_{mn\in\mathscr{A}}{}^{*}\sum a_m b_n \mathbb{1}_{\mathscr{A}}(mn)$$

and we can apply (5.2) to all those terms where $mn\notin\mathcal{O}_{\delta,\alpha}$. On the other hand, we may use the last line of (5.2) as a substitute for $\mathbb{1}_{\mathscr{A}}(mn)$ for *all* terms mn in the above at the cost of an error $O(G)$ (see the comment just below (5.2)). Then, combining this with our analysis of the sums Σ_k from above, we find that

$$E \ll (\log 2J)G + \delta \max_{k=1,2} \sum_{1\le|j|<J} \Pi(j)\left|\sideset{}{^*}\sum_{mn\in\mathscr{B}}\sum a_m b_n\, \mathrm{e}(-jx_{k,mn})\right| \tag{5.6}$$
$$+ \sum_{\substack{1\le|j_1|<J\\1\le|j_2|<J}}\sum \Pi(j_1)\Pi(j_2)\left|\sideset{}{^*}\sum_{mn\in\mathscr{B}}\sum a_m b_n\, \mathrm{e}(-j_1x_{1,mn} - j_2x_{2,mn})\right|.$$

5.2. Removing the weights: dyadic intervals. Here we shall remove the weights (5.3) attached to the sums in (5.6). This may be achieved by splitting the summation over j (or j_1, j_2) into dyadic intervals: indeed, for any non-negative $f : \mathbb{Z}^2 \to \mathbb{R}$, letting

$$F(J_1, J_2) = \sum_{\substack{0\le|j_1|<J_1\\0\le|j_2|<J_2\\(j_1,j_2)\ne(0,0)}}\sum f(j_1, j_2), \tag{5.7}$$

we find that

$$\sum_{\substack{1\le|j_1|<J\\1\le|j_2|<J}}\sum \Pi(j_1)\Pi(j_2)f(j_1, j_2) \ll (\log J)^2 \max_{\substack{1\le J_1\le J\\1\le J_2\le J}} \Pi(j_1)\Pi(j_2)F(J_1, J_2),$$
$$\sum_{1\le|j|<J} \Pi(j)f(j, 0) \ll (\log J) \max_{1\le J_1\le J} \Pi(J_1)F(J_1, 1),$$
$$\sum_{1\le|j|<J} \Pi(j)f(0, j) \ll (\log J) \max_{1\le J_2\le J} \Pi(J_2)F(1, J_2).$$

Of course, we shall apply this with

$$f(j_1, j_2) = \left|\sideset{}{^*}\sum_{mn\in\mathscr{B}}\sum a_m b_n\, \mathrm{e}(-j_1x_{1,mn} - j_2x_{2,mn})\right|. \tag{5.8}$$

Now assume for the moment that we have bounds

$$F(J_1, J_2) \ll \mathcal{F}(J_1, J_2), \tag{5.9}$$

where the right-hand side is symmetric in both arguments and does not depend on the particular choice of the coefficients in (5.8) (but, of course, still subject to the Type I/II conditions presented in Section 5.1); the reader may wish to glance at Proposition 5.2 and Proposition 5.4 below, where we furnish such bounds for the Type II and Type I sums respectively.

Then, using (5.6) we have

$$\frac{|E|}{J^\epsilon} \ll_\epsilon \max_{1\le J_1\le J} \delta\Pi(J_1)\mathcal{F}(J_1, 1) + \max_{\substack{1\le J_1\le J\\1\le J_2\le J}} \Pi(J_1)\Pi(J_2)\mathcal{F}(J_1, J_2) + G. \tag{5.10}$$

We shall return to this in Section 5.7 and now focus on establishing the aforementioned bounds of the shape (5.9).

5.3. Transforming the argument in the exponential term. In the proof of the bounds for the Type I and Type II sums we need to combine variables in $\mathcal{O}$ (see Sections 5.4 and 5.5 below). Having this goal in mind, the shape of the argument of the exponential in (5.8) appears to be, at a superficial glance, a technical obstruction.

However, this putative problem vanishes after a simple variable transformation that we shall now describe: by definition of $x_{k,mn}$,

$$-j_1x_{1,mn} - j_2x_{2,mn} = -j_1\Re_\omega(mn\alpha) - j_2\Im_\omega(mn\alpha). \tag{5.11}$$

Letting ξ_2 be given as in (4.2) and writing $\ell = \ell_1+\ell_2\omega$, a short computation yields

$$\Im_\omega(\ell\rho) = \ell_2\Re_\omega\rho + (\ell_1 + \ell_2\xi_2)\Im_\omega\rho \quad (\rho \in \mathbb{C}). \tag{5.12}$$

Then, via the equivalence

$$\begin{pmatrix} 0 & -1 \\ -1 & -\xi_2 \end{pmatrix}\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \iff \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} \xi_2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} j_1 \\ j_2 \end{pmatrix},$$

and assuming $(j_1, j_2) \neq (0, 0)$, we observe that (5.11) equals $\Im_\omega(\ell mn\alpha)$ when (ℓ_1, ℓ_2) is calculated via the above formula. We let $\mathscr{L}(J_1, J_2)$ be the set of algebraic integers $\ell \in \mathcal{O}$ arising from (j_1, j_2) via the above formula, that is, $\mathscr{L}(J_1, J_2)$ is the set

$$\{(\xi_2 j_1 - j_2) - j_1\omega : |j_1| < J_1,\ |j_2| < J_2,\ (j_1, j_2) \neq (0, 0)\}.$$

Consequently, if $F(J_1, J_2)$ is given by (5.7) with f given by (5.8), then

$$F(J_1, J_2) = \sum_{\ell\in\mathscr{L}(J_1,J_2)} \Big|\sum\nolimits^{*}\sum_{mn\in\mathscr{B}} a_m b_n \,\mathrm{e}(\Im_\omega(\ell mn\alpha))\Big|. \tag{5.13}$$

For a later extension of the summation over ℓ, we note that, using Lemma 4.1(2), $\mathscr{L}(J_1, J_2)$ can be seen to be contained in the set of all ℓ satisfying

$$1 \leq \mathrm{N}(\ell) < 5\mathrm{N}(\omega^2)(J_1 + J_2)^2. \tag{5.14}$$

The reader will note that this set potentially contains many more elements than $\mathscr{L}(J_1, J_2)$, for we obviously have

$$\#\mathscr{L}(J_1, J_2) \leq (2J_1 + 1)(2J_2 + 1) \leq 9\, J_1J_2; \tag{5.15}$$

In any case, we require both (5.14) and (5.15).

5.4. The Type II sums. In this section, we establish the following.

Proposition 5.2 (Type II bound)**.** *Consider F from* (5.7) *with f given by* (5.8) *subject to*

$$\sum_{mn\in\mathscr{B}}{}^{*}\sum = \sum_{\substack{x/2\le \mathrm{N}(mn)<x\\ x^{\mu}<\mathrm{N}(m)<x^{\mu+\kappa}}}\sum,$$

where $\mu\in(0,1]$, $\kappa\in(0,\frac{1}{2}]$ and $x\ge 3$. For the coefficients in (5.8) *assume that $|a_m|\le 1$ and $|b_n|\le d(n\mathcal{O})$. Moreover, suppose that a, q, γ and C are as in* (2.2)*. Then, for any $\epsilon\in(0,\frac{1}{2}]$,*

$$F(J_1,J_2)\ll_\epsilon C\mathrm{N}(\omega)^{7/2}x^{\epsilon}\Big(J_1J_2x^{(1+\mu+\kappa)/2}+(J_1J_2)^{1/2+\epsilon}\times\big((J_1+J_2)x\mathrm{N}(q)^{-1/2}+(J_1+J_2)x^{1-\mu/4}+\mathrm{N}(q)^{1/2}x^{(2+\mu+\kappa)/4}\big)\Big).\tag{5.16}$$

In the course of the proof of Proposition 5.2 and at other places we need the following lemma to control trivial sums over m and n.

Lemma 5.3. *Let $\mathbb{K}$ be a fixed quadratic number field and $\mathcal{O}$ its ring of integers. For an ideal $\mathfrak{a}\subseteq\mathcal{O}$ let $d(\mathfrak{a})$ denote the number of ideals $\mathfrak{b}\supseteq\mathfrak{a}$, and fix $\epsilon>0$ and some integer $\ell\ge 2$. Then, for $x\ge 2$,*

(1) $\displaystyle\sum_{N\mathfrak{a}\le x} d(\mathfrak{a})^{\ell}\ll_{\mathcal{O},\ell} x(\log x)^{2^{2\ell}-1}$,

(2) $d(\mathfrak{a})\ll_{\mathcal{O},\epsilon}(N\mathfrak{a})^{\epsilon}$,

where the implied constants depend at most on $\mathcal{O}$, ℓ and ϵ.

Proof. The first assertion is a direct consequence of [14]. On the other hand, the second assertion is immediate from the first. □

Using Lemma 4.1(1) and Lemma 5.3(1), we have

$$\sum_{mn\in\mathscr{B}}\sum 1\le \#\{\text{units in }\mathcal{O}\}\cdot\sum_{N\mathfrak{a}<x} d(\mathfrak{a})\ll x(\log x)^3.$$

(Note that here the dependence on $\mathcal{O}$ in Lemma 5.3 can be neglected, as we are dealing only with the finitely many imaginary quadratic number fields $\mathbb{K}$ with class number 1.)

Proof of Proposition 5.2. Looking at (5.13), we may split the summation over m into "dyadic annuli," getting

$$F(J_1,J_2)\ll(\log x)\max_{\substack{x^{\mu}<K,K'\le x^{\mu+\kappa}\\ K\le K'<2K}} F(J_1,J_2,K,K'),\tag{5.17}$$

where, upon employing the transformation described in Section 5.3 along the way, $\widehat{F} = F(J_1, J_2, K, K')$ may be taken to be

$$\sum_{\substack{\ell \in \mathscr{L}(J_1,J_2) \\ K \leq \mathrm{N}(m) < K'}}\!\!\sum \left| \sum_{x/2 \leq \mathrm{N}(nm) < x} b_n \, \mathrm{e}(\Im_\omega(\ell m n \alpha)) \right|.$$

(Here and in the following we are always assuming J_1, J_2, K and K' to be positive integers such that $K \leq K' < 2K$.) By (5.15) and Lemma 4.1 (4),

$$\sum_{\substack{\ell \in \mathscr{L}(J_1,J_2) \\ K \leq \mathrm{N}(m) < K'}}\!\!\sum 1 \ll J_1 J_2 K.$$

Hence, letting

$$Q = (\widehat{F})^2 / (J_1 J_2 K), \tag{5.18}$$

Cauchy's inequality gives

$$Q \leq \sum_{\substack{\ell \in \mathscr{L}(J_1,J_2) \\ K \leq \mathrm{N}(m) < K'}}\!\!\sum \left| \sum_{x/2 \leq \mathrm{N}(nm) < x} b_n \, \mathrm{e}(\Im_\omega(\ell m n \alpha)) \right|^2,$$

which, upon expanding the square and rearranging, yields

$$Q \leq \sum_{\substack{\ell \in \mathscr{L}(J_1,J_2) \\ x/(2K') \leq \mathrm{N}(n) < x/K \\ x/(2K') \leq \mathrm{N}(\tilde{n}) < x/K}}\!\!\sum\!\!\sum |b_n \overline{b_{\tilde{n}}}| \left| \sum_m{}^{*} \mathrm{e}(\Im_\omega(\ell m (n - \tilde{n}) \alpha)) \right|,$$

where $\sum_m^*$ restricts the summation to those m with

$$\max\{K, x/2\mathrm{N}(n), x/2\mathrm{N}(\tilde{n})\} \leq \mathrm{N}(m) < \min\{K', x/\mathrm{N}(n), x/\mathrm{N}(\tilde{n})\}.$$

Next, we isolate the "diagonal contribution" Δ, that is, those terms where $n = \tilde{n}$, for in this case the sum over m can only be bounded trivially. Using Lemma 5.3 (1), (5.15) and Lemma 4.1 (4), this is found to be

$$\begin{aligned} \Delta &= \sum_{x/(2K') \leq \mathrm{N}(n) < x/K} |b_n|^2 \sum_{\ell \in \mathscr{L}(J_1,J_2)} \sum_m{}^{*} 1 \\ &\ll x K^{-1} (\log x)^{15} J_1 J_2 K \\ &\ll J_1 J_2 x (\log x)^{15}. \end{aligned} \tag{5.19}$$

Moreover, using (5.14), Lemma 5.3 (2), Lemma 4.2 and (5.14), we have

$$Q \ll_\epsilon \Delta + \mathrm{N}(\omega)(x/K)^{2\epsilon} \sqrt{K'} \sum_{1 \leq \mathrm{N}(j) < U} c_j E(j, \sqrt{K'}),$$

where

$$U = 20\mathrm{N}(\omega^2)(J_1+J_2)^2 x K^{-1},$$

$$c_j = \sum_{\substack{\ell\in\mathscr{L}(J_1,J_2)\\ \ell|j}} \sum\sum_{\substack{x/(2K')\le \mathrm{N}(n)<x/K\\ x/(2K')\le \mathrm{N}(\tilde n)<x/K\\ j/\ell=(n-\tilde n)}} 1 \ll_\epsilon U^\epsilon \cdot x/K \tag{5.20}$$

and E is given by (4.9). Thus, using (5.19) and Theorem 4.3, and recalling (5.18),

$$(\widehat{F})^2 \ll_\epsilon (J_1J_2)^2 xK(\log x)^{15} + \mathrm{N}(\omega)(x/K)^{2\epsilon}U^\epsilon J_1J_2x\sqrt{K} \\ \times (1+C^2\mathrm{N}(\omega^2)U/\mathrm{N}(q))(\sqrt{K}+\mathrm{N}(q\omega)\log(2\sqrt{K})).$$

Upon taking the square root, and simplifying the resulting expressions,

$$\widehat{F} \ll_\epsilon C\mathrm{N}(\omega)^{7/2}x^{2\epsilon}\Big(J_1J_2\sqrt{xK} + (J_1J_2)^{1/2+\epsilon} \\ \times \big((J_1+J_2)x\mathrm{N}(q)^{-1/2} + (J_1+J_2)xK^{-1/4} + \mathrm{N}(q)^{1/2}x^{1/2}K^{1/4}\big)\Big).$$

Recalling (5.17), we infer (5.16) after adjusting ϵ. □

5.5. The Type I sums. The next step is to estimate the Type I sums. We establish the following.

Proposition 5.4 (Type I bound)**.** *Consider F from* (5.7) *with f given by* (5.8) *subject to*

$$\sideset{}{^*}\sum_{mn\in\mathscr{B}}\sum = \sum\sum_{\substack{x/2\le \mathrm{N}(mn)<x\\ \mathrm{N}(m)<M}},$$

where $M \le x$ and $x \ge 3$. For the coefficients in (5.8) *assume that $|a_m| \le 1$ and $b_n = \mathbb{1}_{\{1\le \mathrm{N}(n)<x\}}$. Moreover, suppose that a, q, γ and C are as in* (2.2)*. Then, for any $\epsilon \in (0, \frac{1}{2}]$,*

$$F(J_1,J_2) \ll_\epsilon C^2\mathrm{N}(\omega)^7(J_1+J_2)^\epsilon(x\mathrm{N}(q))^\epsilon \\ \times\big((J_1+J_2)^2x/\mathrm{N}(q) + (J_1+J_2)^2x^{1/2}M^{1/2} + x^{1/2}\mathrm{N}(q)\big). \tag{5.21}$$

Proof. As we did with the Type II sums in the proof of Proposition 5.2, we may split the summation over m into dyadic annuli, getting

$$F(J_1,J_2) \ll (\log x) \max_{\substack{1\le K,K'\le M\\ K\le K'<2K}} \widetilde{F}(J_1,J_2,K,K'), \tag{5.22}$$

where $\widetilde{F} = \widetilde{F}(J_1,J_2,K,K')$ is given by

$$\sum\sum_{\substack{\ell\in\mathscr{L}(J_1,J_2)\\ K\le \mathrm{N}(m)<K'}} \Bigg|\sum_{x/2\le \mathrm{N}(nm)<x} \mathrm{e}(\Im_\omega(\ell mn\alpha))\Bigg|.$$

Letting $R = \sqrt{x/K}$ and employing Lemma 4.2 as well as Lemma 5.3 (2), we infer

$$\begin{aligned}\widetilde{F} &\ll \mathrm{N}(\omega) R \sum\sum_{\substack{\ell \in \mathscr{L}(J_1,J_2)\\ K \le \mathrm{N}(m) < K'}} E(\ell m, R)\\ &\ll_\epsilon \mathrm{N}(\omega) U^\epsilon R \sum_{K \le \mathrm{N}(j) < U} E(j, R),\end{aligned}$$

with E given by (4.9) and U by (5.20) with K' in place of x/K. Theorem 4.3 now shows that

$$\begin{aligned}\widetilde{F} &\ll_\epsilon U^\epsilon \mathrm{N}(\omega) R (1 + C^2 \mathrm{N}(\omega^2) U / \mathrm{N}(q))(R + \mathrm{N}(q\omega)\log(2R))\\ &\ll_\epsilon \mathrm{N}(\omega^2)(J_1+J_2)^{2\epsilon} x^\epsilon (xK^{-1}) + C^2 \mathrm{N}(\omega^7)(J_1+J_2)^{2\epsilon} x^\epsilon\\ &\qquad \times ((J_1+J_2)^2 x/\mathrm{N}(q) + (J_1+J_2)^2 x^{1/2} K^{1/2} + x^{1/2} \mathrm{N}(q) K^{-1/2}).\end{aligned}$$

Herein, for very small K, the term xK^{-1} becomes problematic. To circumvent this, we note that Theorem 4.3 also furnishes the bound

$$\widetilde{F} \ll_\epsilon \mathrm{N}(\omega)^{7/2} (J_1 + J_2)^{2\epsilon} x^{1/2+\epsilon} \mathrm{N}(q)^{1+\epsilon} K^{-1/2},$$

provided that

$$U \le \mathrm{N}(q)/(12C\mathrm{N}(\omega))^2. \tag{5.23}$$

On the other hand, if (5.23) fails to hold, then, recalling (5.20), we have

$$xK^{-1} = 10\mathrm{N}(\omega^2)(J_1 + J_2)^2 x U^{-1} \ll \mathrm{N}(\omega^4) C^2 (J_1 + J_2)^2 x/\mathrm{N}(q).$$

Therefore, after joining both bounds,

$$\begin{aligned}\widetilde{F} &\ll_\epsilon C^2 \mathrm{N}(\omega)^7 (J_1 + J_2)^{2\epsilon} (x\mathrm{N}(q))^\epsilon\\ &\qquad \times ((J_1 + J_2)^2 x \mathrm{N}(q)^{-1} + (J_1 + J_2)^2 x^{1/2} K^{1/2} + x^{1/2} \mathrm{N}(q) K^{-1/2}).\end{aligned}$$

Upon plugging this into (5.22), we obtain (5.21) after adjusting ϵ. □

5.6. Estimation of *G*. The final task is to bound the error term G, defined in (5.4). We shall establish the following.

Proposition 5.5. *Consider G from* (5.4)*, and suppose that a, q, γ and C are as in* (2.2)*. Then we have*

$$G \ll_\epsilon C^2 \mathrm{N}(\omega)^3 (xJ)^\epsilon \Big(\mathrm{N}(q)^{1/2} + \frac{\mathrm{N}(q)}{J} + \frac{x}{\mathrm{N}(q)^{1/2}} + \frac{x}{J} \Big). \tag{5.24}$$

Proof. Using the definition of $x_{k,mn}$, writing $r = mn$ and using Lemma 5.3, we obtain

$$G \ll_\epsilon (xJ)^\epsilon \sum_{l=0}^{1} \sum_{1\le \mathrm{N}(r)\le x} \left(\min\left\{1, \frac{1}{J\|(-1)^l\delta - \Im_\omega(r\theta)\|}\right\} + \min\left\{1, \frac{1}{J\|(-1)^l\delta - \Re_\omega(r\theta)\|}\right\}\right).$$

We shall bound

$$\sum_{1\le \mathrm{N}(r)\le x} m_r, \quad \text{where} \quad m_r = \min\left\{1, \frac{1}{J\|\delta - \Im_\omega(r\theta)\|}\right\}$$

and treat the remaining three sums of this type similarly. To this end, similarly as in Section 4.5, we cover the set of r's in question,

$$\mathscr{X} = \{r \in \mathcal{O} \,:\, 1 \le \mathrm{N}(r) \le x\},$$

by $O(1 + C^2N(\omega)^2x/N(q))$ many rectangles $\mathscr{R}$ with diameter satisfying (4.12) (see also (4.16)), so that

$$\mathscr{X} \subset \mathcal{O} \cap \bigcup_{\mathscr{R}} \mathscr{R}.$$

Furthermore, we write

$$\sum_{r\in\mathscr{X}\cap\mathscr{R}} m_r \ll \sum_{j=0}^{J} \min\left\{1, \frac{1}{J\|j/J\|}\right\} \sum_{\substack{r\in\mathscr{X}\cap\mathscr{R} \\ j/J\le\{\delta-\Im_\omega(r\theta)\}\le(j+1)/J}} 1.$$

Similarly as in Section 4.5 (see (4.13)), we establish that

$$\sum_{\substack{r\in\mathscr{X}\cap\mathscr{R} \\ j/J\le\{\delta-\Im_\omega(r\theta)\}\le(j+1)/J}} 1 \ll \mathrm{N}(q\omega)^{1/2}\left(1 + \frac{\mathrm{N}(q\omega)^{1/2}}{J}\right).$$

It follows that

$$\sum_{1\le \mathrm{N}(r)\le x} m_r \ll_\epsilon \left(1 + \frac{C^2N(\omega)^2x}{N(q)}\right) J^\epsilon \mathrm{N}(q\omega)^{1/2}\left(1 + \frac{\mathrm{N}(q\omega)^{1/2}}{J}\right)$$
$$= J^\epsilon\left(\mathrm{N}(q\omega)^{1/2} + \frac{\mathrm{N}(q\omega)}{J} + \frac{C^2\mathrm{N}(\omega)^{5/2}x}{\mathrm{N}(q)^{1/2}} + \frac{C^2\mathrm{N}(\omega)^3x}{J}\right).$$

Treating the remaining three sums of this type similarly, we obtain (5.24) after adjusting ϵ. □

5.7. Assembling the parts. Finally, we are in a position to use (5.10). Assume the hypotheses of Proposition 5.2. Recall (5.3). Set

$$\widetilde{G} := \mathrm{N}(q)^{1/2} + \mathrm{N}(q)J^{-1} + x\mathrm{N}(q)^{-1/2} + xJ^{-1},$$

the term in the brackets on the right-hand side of (5.24). Then, looking at (5.16), we use (5.10) and (5.24) together with the inequalities

$$\Pi(H)\cdot H \ll 1, \quad \Pi(H)\cdot H^{3/2} \le J^{1/2}, \quad \Pi(H)\cdot H^{1/2} \le \delta^{1/2},$$

$$\Pi(H_1)\Pi(H_2)\cdot H_1H_2 \ll 1, \quad \Pi(H_1)\Pi(H_2)\cdot (H_1H_2)^{1/2}(H_1+H_2) \ll (\delta J)^{1/2}$$

and

$$\Pi(H_1)\Pi(H_2)\cdot (H_1H_2)^{1/2} \ll \delta$$

if $H, H_1, H_2 \le J$, to bound the error in the Type II sums (see (5.5)) as

$$\frac{|E|}{C^2\mathrm{N}(\omega)^{7/2}(xJ)^{\epsilon}} \ll_{\epsilon} x^{(1+\mu+\kappa)/2} + (\delta J)^{1/2}x\mathrm{N}(q)^{-1/2} + (\delta J)^{1/2}x^{1-\mu/4} + \delta\mathrm{N}(q)^{1/2}x^{(2+\mu+\kappa)/4} + \widetilde{G},$$

ϵ being sufficiently small.

Moving on to the Type I sums, accordingly assuming the hypotheses of Proposition 5.4 and looking at (5.21), we use (5.10) and (5.24) together with the inequalities

$$\Pi(H)\cdot H^2 \ll J, \quad \Pi(H) \ll \delta,$$

$$\Pi(H_1)\Pi(H_2)\cdot (H_1^2+H_2^2) \ll \delta J \quad \text{and} \quad \Pi(H_1)\Pi(H_2) \ll \delta^2$$

to infer the estimate

$$\frac{|E|}{C^2\mathrm{N}(\omega)^{7}(xJ)^{\epsilon}} \ll_{\epsilon} \mathrm{N}(q)^{\epsilon}(\delta Jx\mathrm{N}(q)^{-1} + \delta Jx^{1/2}M^{1/2} + \delta^2x^{1/2}\mathrm{N}(q)) + \widetilde{G}$$

for the error in the Type I sums.

On recalling (5.5) and plugging the above bounds into Theorem 3.1, we find that the error

$$\widetilde{E} = \frac{|S(w,x^{\kappa}) - S(\widetilde{w},x^{\kappa})|}{C^2\mathrm{N}(\omega)^7}$$

satisfies the bound

$$\begin{aligned}\frac{\widetilde{E}}{(xJ)^{\epsilon}} \ll_{\epsilon} &\ (\mathrm{N}(q)^{1/2} + \mathrm{N}(q)J^{-1} + x\mathrm{N}(q)^{-1/2} + xJ^{-1})\\ &+ \mathrm{N}(q)^{\epsilon}(\delta Jx\mathrm{N}(q)^{-1} + \delta Jx^{1/2}M^{1/2} + \delta^2x^{3/4}\mathrm{N}(q)^{1/2} + \delta^2x^{1/2}\mathrm{N}(q))\\ &+ x^{(1+\mu+\kappa)/2} + (\delta J)^{1/2}x\mathrm{N}(q)^{-1/2}\\ &+ (\delta J)^{1/2}x^{1-\mu/4} + \delta\mathrm{N}(q)^{1/2}x^{1/2+(\mu+\kappa)/4}.\end{aligned}$$

Evidently, this bound is increasing with κ and to detect primes, we must take $\kappa = \frac{1}{2}$. In view of (2.4), we shall aim for a bound of the type

$$|\widetilde{E}| \ll_\epsilon (\delta^2 x) x^{-\epsilon} \tag{5.25}$$

with δ in some range (w.r.t. x) as large as possible. With this constraint in mind, and given q, we take $x = \mathrm{N}(q)^{28/5}$ (as was stated in Theorem 2.2) so that $\mathrm{N}(q) = x^{5/28}$ and, moreover,

$$J = \lceil \delta^{-2} x^{2\epsilon} \rceil, \quad M = x^{1/2}, \quad \mu = \tfrac{5}{14}. \tag{5.26}$$

Then, under the additional assumption that $J \le x$, we obtain

$$\widetilde{E} \ll_\epsilon \delta^2 x^{1-\epsilon} + x^{5\epsilon}\Big(\delta x^{45/56} + x^{13/14} + \delta^{-1/2} x^{51/56} + \delta^{-1} x^{23/28}\Big),$$

provided ϵ is sufficiently small. This implies that (5.25) holds for sufficiently small ϵ and

$$\delta \ge x^{-1/28+3\epsilon},$$

which concludes the proof of Theorem 2.2 after adjusting ϵ.

6. The smoothed version

Here, in a similar vein to Section 5, we work on providing the details for what was outlined in Remark 3.2. However, this time the aim is to prove Theorem 2.1.

6.1. The modified setup. Throughout the rest of Section 6 we make the following assumptions: $\epsilon > 0$ is supposed to be sufficiently small and fixed. $\mathbb{K}$ is an imaginary quadratic number field with class number 1. The number $x \ge 3$ is assumed to be sufficiently large and α, a, q, C are as in Theorem 2.2. Moreover, we suppose that

$$\tfrac{1}{2} \ge \delta \ge x^{-1000}. \tag{6.1}$$

The exact lower bound here is of no particular consequence, as our final results even fall short of being non-trivial for $\delta \le x^{-1/16}$. However, in the course of getting there, we need to have bounds of the shape $\delta^{-1} e^{-x^\epsilon} \ll_A x^{-A}$ for any $A > 0$ as $x \to \infty$. Such bounds are used (often tacitly) throughout.

Furthermore, we write

$$N = x^{1-\epsilon} \quad \text{and} \quad f_N(z) = e^{-\pi |z|^2/N} \tag{6.2}$$

and define the weight function w to be used in conjunction with Theorem 3.1 by

$$w(z) = \delta^2 f_N(z).$$

To define $\widetilde{w}$, we let

$$W_\delta(\vartheta) = \sum_{n \in \mathbb{Z}} e^{-\pi(\vartheta - n)^2/\delta^2}$$

which by Poisson summation formula implies

$$W_\delta(\vartheta) = \delta \sum_{j\in\mathbb{Z}} e^{-\pi\delta^2 j^2}\,\mathrm{e}(j\vartheta).$$

Then let

$$\widetilde{w}(z) = f_N(z) W_\delta(\Im_\omega(z\alpha)) W_\delta(\Re_\omega(z\alpha) + \xi_2 \Im_\omega(z\alpha))$$

with ξ_2 from (4.2).

6.2. Removing the weights. Our next immediate goal is to see that w and $\widetilde{w}$ are actually suitable weights for the type of argument outlined in Remark 3.2. This is contained in Corollary 6.3 below, but first we need two lemmas. We use the notation $S(\,\cdot\,,\sqrt{x})$ from (3.4).

Lemma 6.1. $S(w,\sqrt{x}) \gg_\epsilon \delta^2 \dfrac{N}{\log N}$.

Proof. The claim follows at once from

$$S(w,\sqrt{x}) = \sum_{\substack{r\in\mathcal{O}\setminus\{0\}\\ \text{prime } p\mid r \Rightarrow \mathrm{N}(p)\geq\sqrt{x}}} w(r) \geq \sum_{\substack{\text{prime } p\in\mathcal{O}\\ \sqrt{x}\leq \mathrm{N}(p)\leq N}} w(p) \geq \delta^2 e^{-\pi} \sum_{\substack{\text{prime } p\in\mathcal{O}\\ \sqrt{x}\leq \mathrm{N}(p)\leq N}} 1$$

after applying the prime number theorem for $\mathcal{O}$ (see (2.4)). □

Lemma 6.2. $S(\widetilde{w},\sqrt{x}) \ll_{\epsilon,\omega} \displaystyle\sum_{\substack{\text{prime } p\in\mathcal{O}\\ \mathrm{N}(p)<x\\ \|p\alpha\|_\omega<\delta x^\epsilon}} e^{-\pi \mathrm{N}(p)/N} + 1.$

Proof. We may split up $S(\widetilde{w},\sqrt{x})$ as follows:

$$\Bigg\{ \sum_{\substack{\text{prime } p\in\mathcal{O}\\ \sqrt{x}\leq\mathrm{N}(p)<x\\ \|p\alpha\|_\omega<\delta x^\epsilon}} + \sum_{\substack{\text{prime } p\in\mathcal{O}\\ \sqrt{x}\leq\mathrm{N}(p)<x\\ \|p\alpha\|_\omega\geq\delta x^\epsilon}} \Bigg\}\{\text{terms}\} + \{\text{error term}\},$$

where the terms being summed are

$$W_\delta(\Im_\omega(p\alpha)) W_\delta(\Re_\omega(p\alpha) + \xi_2\Im_\omega(p\alpha)) e^{-\pi\mathrm{N}(p)/N}$$

and

$$\{\text{error term}\} \ll_\epsilon 1 + \sum_{\substack{r\in\mathcal{O}\\ \mathrm{N}(r)\geq x}} e^{-\pi\mathrm{N}(r)/N} \ll_\epsilon 1.$$

The first sum is bounded using the trivial estimate $W_\delta(\,\cdot\,) \ll 1$.

Now if $\|p\alpha\|_\omega \geq \delta x^\epsilon$, then $\|\Im_\omega(p\alpha)\| \geq \delta x^\epsilon/2\xi_2$, or $\|\Im_\omega(p\alpha)\| \leq \delta x^\epsilon/2\xi_2$ *and* $\|\Re_\omega(p\alpha)\| \geq \delta x^\epsilon$. In the first case, using the inequality

$$(\|\vartheta\| - m)^2 \geq m^2/4 \quad (\vartheta\in\mathbb{R},\, m\in\mathbb{Z}\setminus\{0\}), \tag{6.3}$$

we have

$$W_\delta(\Im_\omega(p\alpha)) = \sum_{m\in\mathbb{Z}} e^{-\pi(\|\Im_\omega(p\alpha)\|-m)^2/\delta^2} \ll e^{-\pi x^{2\epsilon}/(2\xi_2)^2} + \sum_{m=1}^{\infty} e^{-\frac{\pi}{4}m^2/\delta^2}$$
$$\ll_{A,\epsilon,\omega} \delta^2 x^{-A-1},$$

where for the last estimate we employ (6.1). In the second case, the assumptions ensure that $\|\Re_\omega(p\alpha) + \xi_2\Im_\omega(p\alpha)\| \geq \delta x^\epsilon/2$. Therefore, by arguing as before, we have

$$W_\delta(\Re_\omega(p\alpha) + \xi_2\Im_\omega(p\alpha)) \ll e^{-\frac{\pi}{4}x^{2\epsilon}} + \sum_{m=1}^{\infty} e^{-\frac{\pi}{4}m^2/\delta^2} \ll_{A,\epsilon} \delta^2 x^{-A-1}.$$

Thus, altogether we have

$$\sum_{\substack{\text{prime } p\in\mathcal{O}\\ \mathrm{N}(p)<x\\ \|p\alpha\|_\omega\geq\delta x^\epsilon}} W_\delta(\Im_\omega(p\alpha))W_\delta(\Re_\omega(p\alpha) + \xi_2\Im_\omega(p\alpha))e^{-\pi\mathrm{N}(p)/N}$$
$$\ll_{A,\epsilon,\omega} \delta^2 x^{-A-1} \sum_{\substack{r\in\mathcal{O}\\ \mathrm{N}(r)<x}} 1 \ll_{A,\epsilon,\omega} \delta^2 x^{-A}.$$

This proves the lemma. □

Corollary 6.3. *Still assuming the hypotheses from Section 6.1, suppose that one knows that*

$$|S(w,\sqrt{x}) - S(\widetilde{w},\sqrt{x})| = o_{\epsilon,\omega}\Big(\delta^2 \frac{N}{\log N}\Big),$$

where $S(\,\cdot\,,\sqrt{x})$ is defined as in (3.4)*. Then*

$$\sum_{\substack{\text{prime } p\in\mathcal{O}\\ \mathrm{N}(p)<x\\ \|p\alpha\|_\omega<\delta x^\epsilon}} e^{-\pi\mathrm{N}(p)/N} \gg_{\epsilon,\omega} \delta^2 \frac{N}{\log N}.$$

In particular, for any sufficiently large x, there is a prime element $p \in \mathcal{O}$ such that $\mathrm{N}(p) < x$ and $\|p\alpha\|_\omega < \delta x^\epsilon$.

6.3. Estimation of smoothed sums. The next result is a smoothed analogue of Lemma 4.2. Later, this is used in combination with Theorem 4.4. (The reader may contrast this with our use of Theorem 4.3 as the underlying tool for proving Proposition 5.4 and Proposition 5.2.)

Lemma 6.4. *Let $R \geq 1$ and $\gamma \in \mathbb{R}$. Then, for every $\epsilon > 0$ and f_R defined as in* (6.2),

$$\Big|\sum_{m\in\mathcal{O}} f_R(m)\,\mathrm{e}(\Im_\omega(m\vartheta))\Big| \ll_{\epsilon,\omega} \begin{cases} R & \text{if } \|\vartheta\|_\omega < x^\epsilon/\sqrt{R} \\ Re^{-x^\epsilon} & \text{otherwise.}\end{cases}$$

Proof. Let A_ω be the invertible (2×2)-matrix $\left(\begin{smallmatrix}1 & \Re\omega\\ 0 & \Im\omega\end{smallmatrix}\right)$ and write $A_\omega^{-\top} = (A_\omega^{-1})^\top$. Moreover, recall that $\omega^2 = \xi_1 + \xi_2\omega$ and let $v_\vartheta = (\Im_\omega\vartheta, \xi_2\Im_\omega\vartheta + \Re_\omega\vartheta)$. Then, writing $g(u,v) = e^{-\pi(u^2+v^2)}$, we have

$$\Sigma \coloneqq \sum_{m\in\mathcal{O}} f_R(m)\,\mathrm{e}(\Im_\omega(m\vartheta)) = \sum_{m\in\mathbb{Z}^2} g(R^{-1/2}A_\omega m)\,\mathrm{e}(\langle m, v_\vartheta\rangle).$$

By the Poisson summation formula (see, e.g., [18]) and a change of variables,

$$\begin{aligned}\Sigma &= \sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}^2} g(R^{-1/2}A_\omega m)\,\mathrm{e}(\langle m, v_\vartheta\rangle)\,\mathrm{e}(-\langle m, n\rangle)\,\mathrm{d}m\\ &= \frac{R}{|\det A_\omega|}\sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}^2} g(y)\,\mathrm{e}(-\langle R^{1/2}A_\omega^{-1}y, n - v_\vartheta\rangle)\,\mathrm{d}y\\ &= \frac{R}{|\det A_\omega|}\sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}^2} g(y)\,\mathrm{e}(-\langle y, R^{1/2}A_\omega^{-\top}(n - v_\vartheta)\rangle)\,\mathrm{d}y.\end{aligned}$$

Using the fact that g is its own Fourier transform, we have

$$\Sigma = \frac{R}{|\Im\omega|}\sum_{m\in A^{-\top}(\mathbb{Z}^2 - v_\vartheta)} g(R^{1/2}m).$$

A quick computation shows that

$$A_\omega^{-\top}\left(\begin{pmatrix}n_1\\ n_2\end{pmatrix} - v_\vartheta\right) = \begin{pmatrix} n_1 - \Im_\omega\vartheta\\ \frac{n_2 - \Re_\omega\vartheta - (2\Re\omega)\Im_\omega\vartheta - (n_1 - \Im_\omega\vartheta)\Re\omega}{\Im\omega}\end{pmatrix}.$$

Consequently, after a linear change of summation variables,

$$\Sigma = \frac{R}{|\Im\omega|}\sum\sum_{k_1,k_2\in\mathbb{Z}} e^{-\pi R((k_1-\kappa_1)^2 + (k_2 - \kappa_2(k_1))^2/(\Im\omega)^2)},$$

where

$$\kappa_1 = \|\Im_\omega\vartheta\|,\quad \kappa_2(k_1) = \|\Re_\omega\vartheta - (2\Re\omega)(\Im_\omega\vartheta) - (k_1 - \|\Im_\omega\vartheta\|)\Re\omega\|.$$

As the bound $\Sigma \ll R$ is trivial, we now assume that

$$\tfrac{1}{2} \ge \max\{\|\Re_\omega\vartheta\|, \|\Im_\omega\vartheta\|\} = \|\vartheta\|_\omega \ge x^\epsilon/\sqrt{R}.$$

In particular, we have $R \ge 4x^{2\epsilon}$. To bound Σ we split off the term for $k_1 = k_2 = 0$, namely

$$\Sigma_0 = \frac{R}{|\Im\omega|} e^{-\pi R(\kappa_1^2 + \kappa_2(0)^2/(\Im\omega)^2)},$$

from the rest, that is,

$$\Sigma = \Sigma_0 + \Sigma_*,\quad \Sigma_* = \frac{R}{|\Im\omega|}\sum\sum_{\substack{k_1,k_2\in\mathbb{Z}\\ (k_1,k_2)\ne(0,0)}} e^{-\pi R((k_1-\kappa_1)^2 + (k_2-\kappa_2(k_1))^2/(\Im\omega)^2)}.$$

Using (6.3), we have

$$\Sigma_* \ll R \sum_{\substack{k_1,k_2\in\mathbb{Z}\\(k_1,k_2)\neq(0,0)}}\sum e^{-\pi R(k_1^2+k_2^2/(\Im\omega)^2)/4} \ll R\sum_{k=1}^{\infty}\sum_{\substack{k_1,k_2\in\mathbb{Z}\\k_1^2+k_2^2=k}}\sum e^{-\pi Rk/\max\{4,4(\Im\omega)^2\}}$$
$$\ll e^{-\pi R/\max\{8,8(\Im\omega)^2\}} \ll_{\epsilon,\omega} e^{-x^\epsilon}.$$

To bound Σ_0, we put $r=\max\{1,2|\Re\omega|\}$. By assumption, $\|\vartheta\|_\omega \geq x^\epsilon/\sqrt{R}$, so that we either have $\|\Im_\omega\vartheta\| \geq x^\epsilon/(r\sqrt{R})$, or $\|\Im_\omega\vartheta\| < x^\epsilon/(r\sqrt{R})$ *and* $\|\Re_\omega\vartheta\| \geq x^\epsilon/\sqrt{R}$. In the former case,

$$\Sigma_0 \ll Re^{-\pi R\kappa_1^2} \ll Re^{-\pi x^{2\epsilon}/r^2} \ll_{\epsilon,\omega} Re^{-x^\epsilon}.$$

In the latter case, we first note that $2\Re\omega$ is an integer (see Lemma 4.1(3)), so that

$$\kappa_2(0) = \|\Re_\omega\vartheta - ((2\Re\omega)(\Im_\omega\vartheta) - \|\Im_\omega\vartheta\|\Re\omega)\| = \|\Re_\omega\vartheta - (\|\Im_\omega\vartheta\|\Re\omega)\|$$

and, hence,

$$\Sigma_0 \ll Re^{-\pi R\kappa_2(0)^2/(\Im\omega)^2} \ll Re^{-\pi x^{2\epsilon}/(2\Im\omega)^2} \ll_\epsilon Re^{-x^\epsilon}.$$

Recalling that the original sum under consideration, Σ, is $\Sigma_0+\Sigma_*$, the assertion of the lemma follows. □

6.4. Cutting off.

Lemma 6.5. *Consider the sum*

$$\Sigma = \sum_{m\in\mathcal{O}}^{*}\sum_{n\in\mathcal{O}} a_m b_n (w(mn)-\widetilde{w}(mn)),$$

where a_m and b_n are arbitrary complex coefficients satisfying $|a_m|\leq 1$, $a_0=b_0=0$, $|b_n|\leq \#\{$divisors of $n\}$, and $\sum_m^*$ means some arbitrary restriction on the summation over $m\in\mathcal{O}$. Then, for every $A,\epsilon>0$,

$$|\Sigma| \ll_{A,\epsilon,\omega} \delta^2 \sum_{\substack{j\in\mathcal{O}\setminus\{0\}\\ \mathrm{N}(j)\leq\delta^{-2}x^\epsilon}}^{*} \left|\sum_{m\in\mathcal{O}}\sum_{n\in\mathcal{O}} a_m b_n f_N(mn)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right| + \delta^2 x^{-A}.$$

Proof. Recalling (5.12), we have

$$\widetilde{w}(z)-w(z)$$
$$= f_N(z)\delta^2 \sum_{\substack{j_1,j_2\in\mathbb{Z}\\(j_1,j_2)\neq(0,0)}}\sum e^{-\pi\delta^2(j_1^2+j_2^2)}\,\mathrm{e}(j_1\Im_\omega(z\alpha)+j_2(\Re_\omega(z\alpha)+\xi_2\Im_\omega(z\alpha)))$$
$$= f_N(z)\delta^2 \sum_{\substack{j_1,j_2\in\mathbb{Z}\\(j_1,j_2)\neq(0,0)}}\sum e^{-\pi\delta^2(j_1^2+j_2^2)}\,\mathrm{e}(\Im_\omega((j_1+j_2\omega)z\alpha)).$$

Consequently,

$$(6.4)\quad |\Sigma| \leq \delta^2 \sum_{\substack{j_1,j_2\in\mathbb{Z}\\(j_1,j_2)\neq(0,0)\\j:=j_1+j_2\omega}} e^{-\pi\delta^2(j_1^2+j_2^2)} \left| \sideset{}{^*}\sum_{m\in\mathcal{O}} a_m \sum_{n\in\mathcal{O}} b_n f_N(mn)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right|.$$

The inner-most sum over n is bounded by $e^{-\pi\mathrm{N}(m)/N}$ multiplied by

$$\sum_{n\in\mathcal{O}} |b_n| e^{-\pi\mathrm{N}(n)/N} \ll \sum_{r=1}^{\infty} e^{-\pi r/N} \sum_{\substack{n\in\mathcal{O}\\ \mathrm{N}(n)=r}} |b_n| \ll \sum_{r=1}^{\infty} e^{-\pi r/N} r^2 \ll N^3.$$

Thus,

$$\left| \sideset{}{^*}\sum_{m\in\mathcal{O}} \sum_{n\in\mathcal{O}} a_m b_n f_N(mn)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right| \ll N^5 \ll x^5.$$

A short computation shows that

$$\mathrm{N}(j_1+j_2\omega) \leq \delta^{-2}x^\epsilon \implies \max\{|j_1|,|j_2|\} \leq c\delta^{-1}x^{\epsilon/2},$$

where $c = (1+|\Re\omega|)/|\Im\omega|$. Consequently,

$$\sum_{\substack{j_1,j_2\in\mathbb{Z}\\ \max\{|j_1|,|j_2|\}>c\delta^{-1}x^{\epsilon/2}}} e^{-\pi\delta^2(j_1^2+j_2^2)} \ll \sum_{\substack{r=1\\ r>c^2\delta^{-2}x^\epsilon}}^{\infty} re^{-\pi\delta^2 r} \ll \delta^{-4}x(1+c^2)e^{-\pi c^2 x^{2\epsilon}} \ll_{A,\epsilon,\omega} \delta^2 x^{-A-5}.$$

Hence, we obtain the claimed result from (6.4) after estimating the contribution from all terms with $\mathrm{N}(j) > \delta^{-2}x^\epsilon$ via the above and using the trivial inequality $e^{-\pi\delta^2(j_1^2+j_2^2)} \leq 1$ on the remaining terms. □

6.5. Type I estimates.

Proposition 6.6 (Type I estimate)**.** *Consider the sum Σ from Lemma 6.5 with $b_n = 1$ for all $n \in \mathcal{O}\setminus\{0\}$ and*

$$\sideset{}{^*}\sum_{m} = \sum_{0<\mathrm{N}(m)\leq M}$$

for some positive M. Then

$$|\Sigma| \ll_{C,\epsilon,\omega} \delta^2 N \cdot x^{5\epsilon}(|q|^2 x^{-1} + \delta^{-2}|q|^{-2} + \delta^{-2}Mx^{-1}).$$

Proof. By Lemma 6.5 we have

$$|\Sigma| \ll_{\epsilon,\omega} \delta^2 \sum_{\substack{j\in\mathcal{O}\setminus\{0\}\\ \mathrm{N}(j)\leq\delta^{-2}x^\epsilon}} \sum_{0<\mathrm{N}(m)\leq M} \left| \sum_{n\in\mathcal{O}} f_{N/\mathrm{N}(m)}(n)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right| + Mx^\epsilon.$$

Therefore, by Lemma 6.4,

$$|\Sigma| \ll_{\epsilon,\omega} \delta^2 N \sum_{\substack{0<\mathrm{N}(j)\le\delta^{-2}x^{\epsilon} \\ 0<\mathrm{N}(m)\le M \\ \|jm\alpha\|_\omega<\sqrt{\mathrm{N}(m)/x^{1-3\epsilon}}}}\sum (\mathrm{N}(m))^{-1} + Mx^{\epsilon}.$$

Moreover, by the second part of Theorem 4.4, we find that no terms with $\mathrm{N}(m) < L_0$ contribute to the above sum if we set

$$L_0 := \min\left\{\frac{x^{1-3\varepsilon}}{(4|q\omega|)^2}, \frac{\delta^2|q|^2}{(12C|\omega|^2)^2}\right\}.$$

Then, upon splitting the summation over m into dyadic annuli, we obtain

$$|\Sigma| \ll_{\epsilon,\omega} \delta^2 N(\log x) \sup_{L_0\le L\le M} L^{-1} \sum_{\substack{0<\mathrm{N}(j)\le\delta^{-2}x^{\epsilon} \\ L/2<\mathrm{N}(m)\le L \\ \|jm\alpha\|_\omega<\sqrt{L/x^{1-3\epsilon}}}}\sum 1 + Mx^{\epsilon}.$$

Finally, using Theorem 4.4 and $N = x^{1-\varepsilon}$,

$$\begin{aligned}|\Sigma| &\ll_{\epsilon,\omega} \delta^2 N x^{\epsilon} \sup_{L_0\le L\le M} (L^{-1} + \delta^{-2}x^{\epsilon}|q|^{-2})(1 + L|q|^2x^{-1+3\epsilon}) + Mx^{\epsilon} \\ &\ll_{C,\epsilon,\omega} \delta^2 N x^{5\epsilon}(|q|^2x^{-1} + \delta^{-2}|q|^{-2} + \delta^{-2}Mx^{-1}).\end{aligned}$$

□

6.6. Type II estimates.

Proposition 6.7 (Type II estimate)**.** *Consider the sum* Σ *from Lemma 6.5 with*

$$\sum_m{}^{*} = \sum_{x^{\mu}<\mathrm{N}(m)\le x^{\mu+\kappa}},$$

for some $\mu, \kappa \in (0,1)$*. Then, for any* $\epsilon \in (0,\mu)$*,*

$$\begin{aligned}|\Sigma| \ll_{C,\epsilon,\omega} \delta^2 N \cdot x^{7\epsilon}(&\delta^{-2}\min\{x^{-1/4}, x^{(\mu+\kappa-1)/2}\} \\ &+ \delta^{-2}|q|^{-1} + \delta^{-1}|q|x^{-1/2} + \delta^{-1}x^{-\mu/2} + \delta^{-1}x^{(\mu+\kappa-1)/2}).\end{aligned}$$

Proof. By Lemma 6.5 we have

$$|\Sigma| \ll_{A,\epsilon,\omega} \delta^2 \sum_{\substack{j\in\mathcal{O}\setminus\{0\} \\ \mathrm{N}(j)\le\delta^{-2}x^{\epsilon}}} \left|\sum_{\substack{x^{\mu}<\mathrm{N}(m)\le x^{\mu+\kappa} \\ n\in\mathcal{O}}}\sum a_m b_n f_N(mn)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right| + \delta^2 x^{-A}.$$

Upon splitting the summation over m into dyadic annuli, we obtain

$$\begin{aligned}|\Sigma| &\ll_{A,\epsilon,\omega} \delta^2 x^{-A} + \delta^2(\log x) \\ &\quad\times \sup_{x^{\mu}\le L\le x^{\mu+\kappa}} \sum_{\substack{j\in\mathcal{O}\setminus\{0\} \\ \mathrm{N}(j)\le\delta^{-2}x^{\epsilon}}} \left|\sum_{\substack{L<\mathrm{N}(m)\le 2L \\ n\in\mathcal{O}}}\sum a_m b_n f_N(mn)\,\mathrm{e}(\Im_\omega(jmn\alpha))\right|.\end{aligned}$$

By similar arguments as in Section 6.2, we see that one can restrict the summation over n to $\mathrm{N}(n) \le x/L$ at the cost of an error $\ll_{A,\epsilon,\omega} \delta^2 x^{-A}$. Thus,

$$|\Sigma| \ll_{A,\epsilon,\omega} \delta^2 (\log x) \sup_{x^\mu \le L \le x^{\mu+\kappa}} \Sigma_L + \delta^2 x^{-A}, \tag{6.5}$$

where

$$\Sigma_L = \sum_{\substack{j \in \mathcal{O}\setminus\{0\} \\ \mathrm{N}(j) \le \delta^{-2} x^\epsilon}} \left| \sum_{\substack{L < \mathrm{N}(m) \le 2L \\ 0 < \mathrm{N}(n) \le x/L}} \sum a_m b_n f_N(mn) \, \mathrm{e}(\Im_\omega(jmn\alpha)) \right|.$$

We write

$$\Sigma_L = \sum_{\substack{j \in \mathcal{O}\setminus\{0\} \\ \mathrm{N}(j) \le \delta^{-2} x^\epsilon}} c_j \sum_{\substack{L < \mathrm{N}(m) \le 2L \\ 0 < \mathrm{N}(n) \le x/L}} \sum a_m b_n f_N(mn) \, \mathrm{e}(\Im_\omega(jmn\alpha))$$

with c_j being suitable complex coefficients satisfying $|c_j| = 1$.

Next, we remove the factor $f_N(mn)$ by writing the Gaussian as an inverse Mellin transform in the form

$$e^{-x^2} = \frac{1}{2\pi i} \cdot \int_{c-i\infty}^{c+i\infty} x^{-s} \cdot \frac{\Gamma(s/2)}{2} \, \mathrm{d}s,$$

where $c > 0$. This implies

$$f_N(mn) = \frac{1}{4\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \left(\frac{\sqrt{\pi}|mn|}{\sqrt{N}} \right)^{-s} \Gamma\left(\frac{s}{2}\right) \mathrm{d}s$$

and hence

$$\Sigma_L = \frac{1}{4\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) N^{s/2} \Sigma_L(s) \, \mathrm{d}s, \tag{6.6}$$

where

$$\Sigma_L(s) \coloneqq \sum_{\substack{j \in \mathcal{O}\setminus\{0\} \\ \mathrm{N}(j) \le \delta^{-2} x^\epsilon}} c_j \sum_{\substack{L < \mathrm{N}(m) \le 2L \\ 0 < \mathrm{N}(n) \le x/L}} \sum a_m(s) b_n(s) \, \mathrm{e}(\Im_\omega(jmn\alpha))$$

with $a_m(s) \coloneqq a_m |m|^{-s}$ and $b_n(s) \coloneqq b_n |n|^{-s}$. We set $c \coloneqq 1/\log x$. Then $N^{s/2} = O(1)$ and

$$a_m(s) \ll |a_m| \quad \text{and} \quad b_n(s) \ll |b_n|$$

for all s with $\Re s = c$ and m and n in the relevant summation ranges.

Now we estimate $\Sigma_L(s)$. In the following we tacitly assume that $x^\mu \le L \le x^{\mu+\kappa}$. By Cauchy's inequality,

$$|\Sigma_L(s)|^2 \ll \delta^{-2} x^\epsilon L \sum_{\substack{j \in \mathcal{O}\setminus\{0\} \\ \mathrm{N}(j) \le \delta^{-2} x^\epsilon}} \sum_{L < \mathrm{N}(m) \le 2L} \left| \sum_{0 < \mathrm{N}(n) \le x/L} b_n(s) \, \mathrm{e}(\Im_\omega(jmn\alpha)) \right|^2,$$

which implies

$$|\Sigma_L(s)|^2 \ll \delta^{-2} x^{2\epsilon} L \sum_{\substack{k \in \mathcal{O}\setminus\{0\} \\ \mathrm{N}(k) \le 2x^\epsilon L \delta^{-2}}} \left| \sum_{0<\mathrm{N}(n)\le x/L} b_n(s)\, \mathrm{e}(\Im_\omega(kn\alpha)) \right|^2. \tag{6.7}$$

Next, we use the uniform bound $\mathbb{1}_{\{\mathrm{N}(k)\le 2L'\}}$ $(k \in \mathcal{O})$ for $L' := x^\epsilon L \delta^{-2}$ to extend the summation over k, getting

$$|\Sigma_L(s)|^2 \ll \delta^{-2} x^{2\epsilon} L \sum_{k \in \mathcal{O}} f_{L'}(k) \left| \sum_{0<\mathrm{N}(n)\le x/L} b_n(s)\, \mathrm{e}(\Im_\omega(kn\alpha)) \right|^2.$$

Upon expanding the square in $|\sum_n \ldots|^2$,

$$|\Sigma_L(s)|^2 \ll \delta^{-2} x^{2\epsilon} L \sum\sum_{\substack{0<\mathrm{N}(n_1)\le x/L \\ 0<\mathrm{N}(n_2)\le x/L}} b_{n_1}(s)\overline{b_{n_2}(s)} \sum_{k\in\mathcal{O}} f_{L'}(k)\, \mathrm{e}(\Im_\omega(k(n_1-n_2)\alpha)).$$

The subsum with $n_1 = n_2$ is $\ll \delta^{-4} x^{1+4\epsilon} L$. Thus, on writing $\ell = j(n_1 - n_2)$,

$$|\Sigma_L(s)|^2 \ll \delta^{-4} x^{1+4\epsilon} L + \delta^{-2} x^{3\epsilon} L \Sigma'_L, \tag{6.8}$$

where

$$\Sigma'_L = \sum_{\substack{\ell\in\mathcal{O}\setminus\{0\} \\ \mathrm{N}(\ell)\le 4x/L}} \sum\sum_{\substack{0<\mathrm{N}(n_1)\le x/L \\ 0<\mathrm{N}(n_2)\le x/L \\ \ell = n_1 - n_2}} \left| \sum_{k\in\mathcal{O}} f_{L'}(k)\, \mathrm{e}(\Im_\omega(m\ell\alpha)) \right|.$$

By Lemma 6.4, it follows that

$$\begin{aligned} \Sigma'_L &\ll_{A,\epsilon,\omega} L' \sum_{\substack{\ell\in\mathcal{O}\setminus\{0\} \\ \mathrm{N}(\ell)\le 4x/L \\ \|\ell\alpha\|_\omega < x^\epsilon/\sqrt{L'}}} \sum\sum_{\substack{0<\mathrm{N}(n_1)\le x/L \\ 0<\mathrm{N}(n_2)\le x/L \\ \ell = n_1 - n_2}} 1 + x^{-A} \\ &\ll_{A,\epsilon,\omega} x^{1+\epsilon}\delta^{-2} \sum_{\substack{\mathrm{N}(\ell)\le 4x/L \\ \|\ell\alpha\|_\omega < x^\epsilon/\sqrt{L'}}} 1 + x^{-A}. \end{aligned}$$

By Theorem 4.4,

$$\begin{aligned} \Sigma'_L &\ll_{C,A,\epsilon,\omega} x^{1+\epsilon}\delta^{-2}(1 + xL^{-1}|q|^{-2})(1 + x^{2\epsilon}L'^{-1}|q|^2) + x^{-A} \\ &\ll x^{1+2\epsilon}\delta^{-2}(1 + xL^{-1}|q|^{-2} + \delta^2 L^{-1}|q|^2 + \delta^2 x L^{-2}). \end{aligned}$$

Combining this with (6.8) and taking square root, we obtain the estimate

$$|\Sigma_L(s)| \ll x^{3\varepsilon}(\delta^{-2}x^{1/2}L^{1/2} + \delta^{-2}x|q|^{-1} + \delta^{-1}x^{1/2}|q| + \delta^{-1}xL^{-1/2}). \tag{6.9}$$

We may reverse the roles of m and n and estimate $|\Sigma_L(s)|^2$ by

$$|\Sigma_L(s)|^2 \ll \delta^{-2} x^{1+2\epsilon} L^{-1} \sum_{\substack{k\in\mathcal{O}\setminus\{0\} \\ \mathrm{N}(k)\le 2x^{1+\epsilon}L^{-1}\delta^{-2}}} \left| \sum_{0<\mathrm{N}(m)\le L} a_m(s)\, \mathrm{e}(\Im_\omega(km\alpha)) \right|^2$$

in place of (6.7). Then we can continue in a similar way as above. We arrive at the same estimate as in (6.9) but with L replaced by x/L, i.e.

$$|\Sigma_L(s)| \ll x^{3\varepsilon}\big(\delta^{-2}xL^{-1/2} + \delta^{-2}x|q|^{-1} + \delta^{-1}x^{1/2}|q| + \delta^{-1}x^{1/2}L^{1/2}\big). \tag{6.10}$$

Using (6.9) if $L \le x^{1/2}$ and (6.10) if $L > x^{1/2}$, and recalling that $x^\mu \le L \le x^{\mu+\kappa}$, we deduce that

$$\begin{aligned} |\Sigma_L(s)| \ll x^{3\varepsilon}\big(&\delta^{-2}\min\{x^{3/4}, x^{(\mu+\kappa+1)/2}\} + \delta^{-2}x|q|^{-1} \\ &+ \delta^{-1}x^{1/2}|q| + \delta^{-1}x^{1-\mu/2} + \delta^{-1}x^{(\mu+\kappa+1)/2}\big). \end{aligned}$$

Using (6.6) together with Stirling's approximation for the Gamma function, the same bound, up to a factor of $\log x$, holds for Σ_L. Plugging this into (6.5), we obtain the assertion of the proposition. □

6.7. Conclusion.

Proof of Theorem 2.1. We note that

$$\lim_{R\to\infty} \sum_{\substack{r\in\mathcal{O} \\ \mathrm{N}(r)<R}} d_4(r\mathcal{O})\widetilde{w}(r) \le \lim_{R\to\infty} \sum_{\substack{r\in\mathcal{O} \\ \mathrm{N}(r)<R}} d_4(r\mathcal{O})w(r) \le x^3$$

for all sufficiently large x depending on ϵ. (Mind though that the first two quantities depend implicitly on x by means of the definitions of $\widetilde{w}$, w, and N.) Then, for $\mu \in (0, \frac{1}{2})$, choosing $\kappa = \frac{1}{2}$ and $M = 2x^\mu$ in accordance with Theorem 3.1, Propositions 6.6 and 6.7 give

$$\begin{aligned} &|S(w, \sqrt{x}) - S(\widetilde{w}, \sqrt{x})| \\ &\quad \ll_{C,\epsilon,\omega} \delta^2 N \cdot x^{7\epsilon}\Big(|q|^2 x^{-1} + \delta^{-2}|q|^{-1} + \delta^{-1}|q|x^{-1/2} \\ &\qquad\qquad + \delta^{-2}x^{-1/4} + \delta^{-1}x^{-\mu/2} + \delta^{-2}x^{-1+\mu}\Big). \end{aligned}$$

Upon taking $x = |q|^3$ and $\mu = \frac{1}{4}$, we find

$$|S(w, \sqrt{x}) - S(\widetilde{w}, \sqrt{x})| \ll_{C,\epsilon,\omega} \delta^2 N \cdot x^{-\epsilon}$$

provided that $\frac{1}{2} \ge \delta \ge x^{-1/8+10\epsilon}$. By using Corollary 6.3, the theorem follows. □

7. Proof of the weighted version of Harman's sieve for $\mathcal{O}$

Before embarking on the proof of Theorem 3.1, we record the following useful lemma:

Lemma 7.1. *For any two distinct real numbers $\rho, \gamma > 0$ and $T \geq 1$ one has*

$$\left| \mathbb{1}_{\{\gamma<\rho\}} - \frac{1}{\pi} \int_{-T}^{T} e^{i\gamma t} \frac{\sin(\rho t)}{t} \,\mathrm{d}t \right| \ll \frac{1}{T|\gamma-\rho|},$$

where the implied constant is absolute.

Proof. See, for instance, [8, Lemma 2.2]. □

Proof of Theorem 3.1. We follow [8] quite closely. We assume that ω takes the values w and $\widetilde{w}$ and put $z = x^{\kappa}$.

We start by introducing some notation. For each prime ideal of $\mathcal{O}$ choose a generator $p \in \mathcal{O}$ and let $\mathbb{P}_{\mathcal{O}}$ be the set of all those p. For $Z \geq 0$ write

$$\mathbb{P}_{\mathcal{O}}(Z) = \{p \in \mathbb{P}_{\mathcal{O}} : \mathrm{N}(p) < Z\}$$

and

$$P_{\mathcal{O}}(Z) = \prod_{p \in \mathbb{P}_{\mathcal{O}}(Z)} p.$$

We also need to introduce some $\mathcal{O}$-version of the Möbius μ function: for a non-unit d, let $\mu(d)$ be defined as $(-1)^r$ if d is the product of precisely r non-associate prime elements and $\mu(d) = 0$ otherwise. If d is a unit, then put $\mu(d) = 1$. Then, by inclusion–exclusion, we have

$$S(\omega, z) = \sum_{r \in \mathcal{O}\setminus\{0\}} \omega(r) \sum_{\substack{m | P_{\mathcal{O}}(z) \\ m | r}} \mu(m) = \sum_{m | P_{\mathcal{O}}(z)} \mu(m) \sum_{n \in \mathcal{O}\setminus\{0\}} \omega(mn). \tag{7.1}$$

On writing

$$\Delta(m) = \sum_{n \in \mathcal{O}\setminus\{0\}} (w(mn) - \widetilde{w}(mn)), \tag{7.2}$$

applying (7.1) for $\omega = w$ and $\omega = \widetilde{w}$ yields

$$\begin{aligned} S(w,z) - S(\widetilde{w},z) &= \Bigg\{ \sum_{\substack{m | P_{\mathcal{O}}(z) \\ \mathrm{N}(m) < M}} + \sum_{\substack{m | P_{\mathcal{O}}(z) \\ \mathrm{N}(m) \geq M}} \Bigg\} \mu(m) \Delta(m) \\ &= S_{\mathrm{I}} + S_{\mathrm{II}}, \quad \text{say}. \end{aligned} \tag{7.3}$$

By (3.2) with $a_m = \mu(m) \mathbb{1}_{\{m | P_{\mathcal{O}}(z)\}}$ we infer $|S_{\mathrm{I}}| \leq Y$. Therefore, to prove the theorem, it remains to establish that

$$|S_{\mathrm{II}}| \ll Y (\log(xX))^3. \tag{7.4}$$

The next step is to arrange S_{II} into subsums according to the "size" of the prime factors in m (where m is the summation variable from (7.3)). To have some such notion of size, fix some total order $\prec$ on $\mathbb{P}_{\mathcal{O}}(z)$ such that $\mathrm{N}(p_2) \leq \mathrm{N}(p_1)$ whenever $p_2 \prec p_1$. (Clearly many such orders exist, but the precise choice must not concern us.) Moreover, for $p \in \mathbb{P}_{\mathcal{O}}(z)$, let

$$\Pi(p) = \prod_{q \prec p} q.$$

Now take $g\colon \mathcal{O} \to \mathbb{C}$ to be any function with $g(m) = g(\widetilde{m})$ whenever m and $\widetilde{m}$ are associates. Then, we may group the terms of the sum

$$S = \sum_{m | P_{\mathcal{O}}(z)} \mu(m) g(m)$$

according to the largest prime factor p_1 of m (w.r.t. $\prec$):

$$S = g(1) - \sum_{p_1 \in \mathbb{P}_{\mathcal{O}}(z)} \sum_{d | \Pi(p_1)} \mu(d) g(p_1 d). \tag{7.5}$$

Evidently, the process giving (7.5) also works if $P_{\mathcal{O}}(z)$ is replaced by $\Pi(p)$; for any $r \in \mathcal{O}$ one has

$$\sum_{d | \Pi(p_1)} \mu(d) g(rd) = g(r) - \sum_{p_2 \prec p_1} \sum_{d | \Pi(p_2)} \mu(d) g(r p_2 d). \tag{7.6}$$

Minding the inner most sum on the right hand side above, it is obvious that the above identity can be iterated if so desired. To describe for which sub-sums iteration is beneficial, we let

$$\begin{aligned} \mathbb{P}_{\mathcal{O}}(z) &= \{p_1 \in \mathbb{P}_{\mathcal{O}}(z) : \mathrm{N}(p_1) > x^\mu\} \,\dot\cup\, \{p_1 \in \mathbb{P}_{\mathcal{O}}(z) : \mathrm{N}(p_1) \leq x^\mu\} \\ &= \mathscr{P}_1 \,\dot\cup\, \mathscr{Q}_1, \quad \text{say}, \end{aligned}$$

and, inductively for $s = 2, 3, \ldots$,

$$\begin{aligned} \mathscr{Q}'_s &= \{(p_1, \ldots, p_{s-1}, p_s) \in (\mathbb{P}_{\mathcal{O}}(z))^s : p_s \prec p_{s-1},\ (p_1, \ldots, p_{s-1}) \in \mathscr{Q}_{s-1}\} \\ &= \mathscr{P}_s \,\dot\cup\, \mathscr{Q}_s, \end{aligned}$$

where

$$\begin{aligned} \mathscr{P}_s &= \{(p_1, \ldots, p_{s-1}, p_s) \in \mathscr{Q}'_s : N(p_1 \cdots p_{s-1} p_s) > x^\mu\}, \\ \mathscr{Q}_s &= \{(p_1, \ldots, p_{s-1}, p_s) \in \mathscr{Q}'_s : N(p_1 \cdots p_{s-1} p_s) \leq x^\mu\}. \end{aligned}$$

Assuming that g vanishes on arguments r with $\mathrm{N}(r) \leq x^\mu$, and on applying (7.5) and (7.6),

$$\begin{aligned} S &= -\Bigg\{ \sum_{p_1 \in \mathscr{P}_1} + \sum_{p_1 \in \mathscr{Q}_1} \Bigg\} \sum_{d | \Pi(p_1)} \mu(d) g(p_1 d) \\ &= - \sum_{p_1 \in \mathscr{P}_1} \sum_{d | \Pi(p_1)} \mu(d) g(p_1 d) + \sum_{(p_1, p_2) \in \mathscr{P}_2} \sum_{d | \Pi(p_2)} \mu(d) g(p_1 p_2 d) \\ &\qquad + \sum_{(p_1, p_2) \in \mathscr{Q}_2} \sum_{d | \Pi(p_2)} \mu(d) g(p_1 p_2 d). \end{aligned}$$

On iterating this process (always applying (7.6) to the $\mathscr{Q}$-part) it transpires that

$$S = \sum_{s \le t} (-1)^s \sum_{(p_1, p_2, \ldots, p_s) \in \mathscr{P}_s} \sum_{d \mid \Pi(p_s)} \mu(d) g(p_1 p_2 \cdots p_s d) + (-1)^t \sum_{(p_1, p_2, \ldots, p_t) \in \mathscr{Q}_t} \sum_{d \mid \Pi(p_t)} \mu(d) g(p_1 p_2 \cdots p_t d)$$

for any $t \in \mathbb{N}$. Since the product of t prime elements has norm $\ge 2^t$, we have

$$\mathscr{Q}_t = \emptyset \quad \text{for} \quad t > \frac{\mu}{\log 2} \log x.$$

Hence,

$$S = \sum_{s \le t} (-1)^s \sum_{(p_1, p_2, \ldots, p_s) \in \mathscr{P}_s} \sum_{d \mid \Pi(p_s)} \mu(d) g(p_1 p_2 \cdots p_s d)$$

for (say)

$$t = \lfloor (\log x)/\log 2 \rfloor + 1 \ll \log x. \tag{7.7}$$

We apply this to S_{II} with $g(m) = \Delta(m) \mathbb{1}_{\{\mathrm{N}(m) \ge M\}}$. Note that, since $M > x^\mu$, we have $g(r) = 0$ for all r with $\mathrm{N}(r) \le x^\mu$, as was assumed in the above arguments. Thus,

$$S_{\mathrm{II}} = \sum_{s \le t} (-1)^s S_{\mathrm{II}}(s), \tag{7.8}$$

where

$$S_{\mathrm{II}}(s) = \sum_{\substack{(p_1, \ldots, p_s) \in \mathscr{P}_s \\ m := p_1 \cdots p_s}} \sum_{\substack{d \mid \Pi(p_s) \\ \mathrm{N}(md) \ge M}} \mu(d) \Delta(md).$$

Another application of (7.6) gives

$$\begin{aligned} S_{\mathrm{II}}(s) &= \sum_{\substack{(p_1, \ldots, p_s) \in \mathscr{P}_s \\ m := p_1 \cdots p_s \\ \mathrm{N}(m) \ge M}} \Delta(m) - \sum_{\substack{(p_1, \ldots, p_s) \in \mathscr{P}_s \\ m := p_1 \cdots p_s}} \sum_{p \prec p_s} \sum_{\substack{d \mid \Pi(p) \\ \mathrm{N}(mpd) \ge M}} \mu(d) \Delta(mpd) \\ &= S_{\mathrm{II},1}(s) - S_{\mathrm{II},2}(s), \quad \text{say.} \end{aligned} \tag{7.9}$$

Given $m = p_1 \cdots p_{s-1} p_s$ with

$$(p_1, \ldots, p_{s-1}, p_s) \in \mathscr{P}_s \quad \text{and} \quad (p_1, \ldots, p_{s-1}) \in \mathscr{Q}_{s-1},$$

and noting that $\mathrm{N}(p_s) \le \mathrm{N}(p_1) < z = x^\kappa$, we have

$$x^\mu < \mathrm{N}(m) = \mathrm{N}(p_1 \cdots p_{s-1}) \mathrm{N}(p_s) < x^\mu x^\kappa.$$

Using this, we find that $S_{\mathrm{II},1}(s)$ can be expressed as

$$\sum_{m, n \in \mathcal{O} \setminus \{0\}} \sum a_m (w(mn) - \widetilde{w}(mn)),$$

where the coefficients

$$a_m = \mathbb{1}_{\{\mathrm{N}(m)\geq M\}}\,\mathbb{1}_{\{p_1\cdots p_s:(p_1,\dots,p_s)\in\mathscr{P}_s\}}(m)$$

are only supported on m with $x^\mu < \mathrm{N}(m) < x^{\mu+\kappa}$. Hence, by (3.3),

$$|S_{\mathrm{II},1}(s)| \leq Y. \tag{7.10}$$

Moving on to $S_{\mathrm{II},2}(s)$, we expand the definition (7.2) of Δ, getting

$$S_{\mathrm{II},2}(s) = S_{\mathrm{II},2}(s,w) - S_{\mathrm{II},2}(s,\widetilde{w}),$$

where

$$\begin{aligned} S_{\mathrm{II},2}(s,\omega) &= \sum_{\substack{(p_1,\dots,p_s)\in\mathscr{P}_s\\ m:=p_1\cdots p_s}} \sum_{p\prec p_s} \sum_{\substack{d\mid\Pi(p)\\ \mathrm{N}(mpd)\geq M}} \mu(d) \sum_{\ell\in\mathcal{O}\setminus\{0\}} \omega(m\ell pd) \\ &= \sum_{\substack{(p_1,\dots,p_s)\in\mathscr{P}_s\\ m:=p_1\cdots p_s}} \sum_{n\in\mathcal{O}\setminus\{0\}} \sum_{p\prec p_s} \sum_{\substack{d\mid\Pi(p)\\ \ell pd=n\\ \mathrm{N}(mpd)\geq M}}\sum \mu(d)\omega(mn). \end{aligned}$$

In order to apply (3.3), we must disentangle the variables m and n in the above summation. To this end, split

$$\sum_{p\prec p_s} = \sum_{\substack{p\prec p_s\\ \mathrm{N}(p)=\mathrm{N}(p_s)}} + \sum_{\substack{p\prec p_s\\ \mathrm{N}(p)<\mathrm{N}(p_s)}} \tag{7.11}$$

to obtain a decomposition

$$S_{\mathrm{II},2}(s,\omega) = S^{=}_{\mathrm{II},2}(s,\omega) + S^{<}_{\mathrm{II},2}(s,\omega), \quad \text{say.} \tag{7.12}$$

For $S^{<}_{\mathrm{II},2}(s,\omega)$ we have

$$S^{<}_{\mathrm{II},2}(s,\omega) = \sum_{\substack{(p_1,\dots,p_s)\in\mathscr{P}_s\\ m:=p_1\cdots p_s}} \sum_{n\in\mathcal{O}\setminus\{0\}} \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)} \sum_{\substack{d\mid\Pi(p)\\ \ell pd=n}}\sum \mu(d)\chi(m,d,p,p_s)\omega(mn),$$

where

$$\chi(m,d,p,p_s) = \mathbb{1}_{\{\mathrm{N}(mpd)\geq M\}}\mathbb{1}_{\{\mathrm{N}(p)<\mathrm{N}(p_s)\}},$$

and the sum $S^{=}_{\mathrm{II},2}(s,\omega)$ can be expressed similarly, but needs a little more care: by basic ramification theory, the first summation on the right hand side of (7.11) contains at most one term and we shall write $\mathscr{P}'_s$ for the set of $(p_1,\dots,p_s)\in\mathscr{P}_s$ for which there is such a term, that is, some $p\prec p_s$ with $\mathrm{N}(p)=\mathrm{N}(p_s)$. Furthermore, let $\mathbb{P}_{\mathcal{O}}(z)'$ denote the set of the p's just mentioned, i.e.,

$$\mathbb{P}_{\mathcal{O}}(z)' = \{p\in\mathbb{P}_{\mathcal{O}}(z) : \exists\, p_s \text{ s.t. } p\prec p_s,\ \mathrm{N}(p)=\mathrm{N}(p_s)\}.$$

Thus,

$$S^{=}_{\mathrm{II},2}(s,\omega) = \sum_{\substack{(p_1,\dots,p_s)\in\mathscr{P}'_s\\ m:=p_1\cdots p_s}} \sum_{n\in\mathcal{O}\setminus\{0\}} \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)'} \sum_{\substack{d\mid\Pi(p)\\ \ell pd=n}}\sum \mu(d)\widetilde{\chi}(m,d,p,p_s)\omega(mn),$$

where

$$\begin{aligned}\widetilde{\chi}(m,d,p,p_s) &= \mathbb{1}_{\{\mathrm{N}(mpd)\geq M\}}\mathbb{1}_{\{\mathrm{N}(p)=\mathrm{N}(p_s)\}} \\ &= \mathbb{1}_{\{\mathrm{N}(mpd)\geq M\}}\mathbb{1}_{\{\mathrm{N}(p)\leq\mathrm{N}(p_s)\}} - \chi(m,d,p,p_s).\end{aligned}\tag{7.13}$$

To disentangle m-dependent quantities ($\mathrm{N}(m)$ and $\mathrm{N}(p_s)$) from n-dependent quantities ($\mathrm{N}(pd)$ and $\mathrm{N}(p)$) in the above, we employ Lemma 7.1. We pick some real number ϱ (depending only on M) with $|\varrho| \leq \frac{1}{2}$ such that $\{M+\varrho\} = \frac{1}{2}$ and for $m,p,d \in \mathcal{O}$ the condition $\mathrm{N}(mpd) \geq M$ is equivalent to $\log N(mpd) \geq \log(M+\varrho)$. Then

$$|\log N(mpd) - \log(M+\varrho)| \geq \log\frac{x+1}{x+\frac{1}{2}} \geq \frac{1}{3x}.$$

Therefore, Lemma 7.1 shows that

$$\mathbb{1}_{\{\mathrm{N}(mpd)\geq M\}} = 1 - \frac{1}{\pi}\int_{-T}^{T}(N(mpd))^{it}\sin(t\log(M+\varrho))\frac{\mathrm{d}t}{t} + O(x/T)$$

for every $T \geq 1$. Similarly,

$$\begin{aligned}\mathbb{1}_{\{\mathrm{N}(p)<\mathrm{N}(p_s)\}} &= \frac{1}{\pi}\int_{-T}^{T} e^{\frac{it}{2}}e^{it\mathrm{N}(p)}\sin(t\mathrm{N}(p_s))\frac{\mathrm{d}t}{t} + O(1/T), \\ \mathbb{1}_{\{\mathrm{N}(p)\leq\mathrm{N}(p_s)\}} &= \frac{1}{\pi}\int_{-T}^{T} e^{-\frac{it}{2}}e^{it\mathrm{N}(p)}\sin(t\mathrm{N}(p_s))\frac{\mathrm{d}t}{t} + O(1/T).\end{aligned}$$

Thus,

$$\begin{aligned}S_{\mathrm{II},2}^{<}(s,\omega) &= \frac{1}{\pi}\int_{-T}^{T}\sum\sum_{m,n\in\mathcal{O}\setminus\{0\}} a_m(t)b_n(t)\omega(mn)\frac{\mathrm{d}t}{t} \\ &\quad - \frac{1}{\pi^2}\int_{-T}^{T}\int_{-T}^{T}\sum\sum_{m,n\in\mathcal{O}\setminus\{0\}} a_m(t,\tau)b_n(t,\tau)\omega(mn)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}t}{t} \\ &\quad + O\left(\frac{x}{T} + \frac{1}{T}\int_{-T}^{T}|\sin(\tau\log(M+\varrho))|\frac{\mathrm{d}\tau}{|\tau|}\right) \\ &\qquad \times O\left(\sum_{\substack{(p_1,\ldots,p_s)\in\mathscr{P}_s \\ m:=p_1\cdots p_s}}\ \sum_{n\in\mathcal{O}\setminus\{0\}}\ \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)}\ \sum\sum_{\substack{d|\Pi(p) \\ \ell pd=n}}\omega(mn)\right),\end{aligned}\tag{7.14}$$

with coefficients

$$\begin{aligned}a_m(t) &= \begin{cases}\sin(t\mathrm{N}(p_s)) & \text{if } \exists\ (p_1,\ldots,p_s)\in\mathscr{P}_s \text{ s.t. } m = p_1\cdots p_s, \\ 0 & \text{otherwise,}\end{cases} \\ b_n(t) &= \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)}\ \sum\sum_{\substack{d|\Pi(p) \\ \ell pd=n\neq 0}} e^{\frac{it}{2}}e^{it\mathrm{N}(p)}\mu(d),\end{aligned}\tag{7.15}$$

as well as

$$\begin{aligned} a_m(t,\tau) &= a_m(t)(N(m))^{i\tau}\sin(\tau\log(M+\varrho)), \\ b_n(t,\tau) &= \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)} \sum_{\substack{d\mid\Pi(p) \\ \ell pd=n\neq 0}}\sum e^{\frac{it}{2}} e^{it\mathrm{N}(p)}\mu(d)(N(pd))^{i\tau}. \end{aligned} \tag{7.16}$$

We proceed by gathering some intermediate information before applying (3.3): in the definition of the coefficients b_n, neither of the summations over p and d includes associates. Thus,

$$|b_n(t)|, |b_n(t,\tau)| \le d(n\mathcal{O}).$$

For the other coefficients we always have

$$|a_m(t)|, |a_m(t,\tau)| \le 1,$$

yet if t and τ are small, one can (and must) do better: indeed,

$$|a_m(t)| \le \min\{1,|t|\delta_1\}, \quad |a_m(t,\tau)| \le \min\{1,|t|\delta_1,|\tau|\delta_2,|t\tau|\delta_1\delta_2\}, \tag{7.17}$$

where

$$\delta_1 := x^{1/2} \quad\text{and}\quad \delta_2 := \log\Big(x+\frac{1}{2}\Big).$$

In view of this, we must deal with functions $f\colon \mathbb{R}\times(0,1)\to\mathbb{R}$ of the shape

$$f(t,\delta) = \begin{cases} \delta t & \text{if } |t|\le\delta^{-1}, \\ 1 & \text{otherwise} \end{cases}$$

and their integrals

$$\int_{-T}^{T} f(t,\delta)\frac{\mathrm{d}t}{|t|} \ll \delta\int_0^{\delta^{-1}}\mathrm{d}t + \left|\int_{\delta^{-1}}^{T}\frac{\mathrm{d}t}{t}\right| \ll 1+|\log(T\delta)|. \tag{7.18}$$

Lastly, we note that, by Lemma 4.1 and (3.1),

$$\sum_{\substack{(p_1,\ldots,p_s)\in\mathscr{P}_s \\ m:=p_1\cdots p_s}} \sum_{n\in\mathcal{O}\setminus\{0\}} \sum_{p\in\mathbb{P}_{\mathcal{O}}(z)} \sum_{\substack{d\mid\Pi(p) \\ \ell pd=n}}\sum \omega(mn) \ll \sum_{r\in\mathcal{O}\setminus\{0\}} d_4(r\mathcal{O})\omega(r) \ll X. \tag{7.19}$$

Collecting what we have gathered so far, we may derive a bound for

$$\mathcal{E} = |S_{\mathrm{II},2}^{<}(s,w) - S_{\mathrm{II},2}^{<}(s,\widetilde{w})|$$

as follows: after applying (7.14) with $\omega = w$ and $\omega = \widetilde{w}$, the $O(\ldots)$-terms are treated directly with (7.19) and (7.18), whereas for the rest one may apply (3.3). Here it is important to use (7.17) for small $|t|$ respectively $|\tau|$ first (prior to applying (3.3)) and (7.18) then bounds the integrals. Therefore, after some computations, we infer

$$\mathcal{E} \ll Y\log(Tx)(1+\log(T\log(x+\tfrac{1}{2}))) + XT^{-1}(x+\log(T\log(x+\tfrac{1}{2}))). \tag{7.20}$$

Of course, the same arguments also apply to $S^{=}_{\mathrm{II},2}(s,\omega)$; in view of (7.13) we have to apply them twice, but in both cases the coefficients corresponding to (7.15) and (7.16) obey the same bounds we used to derive (7.20). Consequently, (7.20) also holds with $S^{=}_{\mathrm{II},2}$ in place of $S^{<}_{\mathrm{II},2}$. In total, recalling (7.9), (7.10) and (7.12), we have

$$|S_{\mathrm{II}}(s)| \ll Y + \{\text{the bound from (7.20)}\}$$

and it transpires that choosing $T = xX$ suffices to yield a bound $\ll Y(\log(xX))^2$. On plugging this into (7.8) and recalling (7.7), we infer (7.4). Hence, the theorem is proved. □

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Stephan BAIER
Ramakrishna Mission Vivekananda Educational Research Institute
Department of Mathematics
G. T. Road, PO Belur Math, Howrah
West Bengal 711202, India
E-mail: email_baier@yahoo.de
URL: https://www.researchgate.net/profile/Stephan_Baier2

Marc TECHNAU
Graz University of Technology
Institute of Analysis and Number Theory
Kopernikusgasse 24/II
8010 Graz, Austria
E-mail: mtechnau@math.tugraz.at
URL: https://www.math.tugraz.at/~mtechnau/