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An introduction to oddly tame number fields.

par Guillermo MANTILLA-SOLER

Résumé. Il résulte des généralités sur les formes quadratiques que la classe spinorielle de la trace intégrale d'un corps de nombres détermine la signature et le discriminant du corps. Dans cet article, nous définissons une famille de corps de nombres, qui contient, entre autres, tous les corps galoisiens de degré impair modérément ramifiés, pour lesquels la réciproque est vraie. Autrement dit, pour un corps de nombres K de cette famille, on montre que la classe spinorielle de la trace intégrale ne contient pas d'informations sur K autres que celles qui sont fournies par le discriminant et la signature.

Abstract. It follows from generalities of quadratic forms that the spinor class of the integral trace of a number field determines the signature and the discriminant of the field. In this paper we define a family of number fields, that contains among others all odd degree Galois tame number fields, for which the converse is true. In other words, for a number field K in such family we prove that the spinor class of the integral trace carries no more information about K than the discriminant and the signature do.

1. Introduction

Let K be a number field, O_K its maximal order and let r_K and s_K be the number of real and complex places respectively. The *integral trace form* of K is the isometry class of the $\mathbb{Z}$-quadratic module

$$\langle O_K, \mathrm{t}_K \rangle$$

associated to the trace pairing

$$O_K \times O_K \to \mathbb{Z}; \quad (x, y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy).$$

Since $(r_K+s_K, s_K) = \mathrm{Sign}(\langle O_K, \mathrm{t}_K \rangle)$ and $\mathrm{disc}(K) = \det(\langle O_K, \mathrm{t}_K \rangle)$ (see [9]) it follows from generalities of quadratic forms (see [2, IX §4]) that the spinor genus of $\langle O_K, \mathrm{t}_K \rangle$ determines the signature and the discriminant of K. Besides the trivial case of fields of degree less than 3, there are some interesting instances where the converse of the above holds:

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Theorem ([4, Theorem 3.2])**.** Let K and L be cubic number fields. Then $\langle O_K, \mathrm{t}_K\rangle$ and $\langle O_L, \mathrm{t}_L\rangle$ belong to the same spinor genus if and only if $\mathrm{disc}(K) = \mathrm{disc}(L)$.

Recall that for cubic fields the sign of the discriminant determines the signature of the field, hence it is not necessary to mention signatures in the above theorem. There are examples that show that the above theorem cannot be extended to isometry classes; take for instance (see [7, Theorem 5.2] and [4, Theorem 3.2]) two non isomorphic cubic fields of positive fundamental discriminant, e.g., any two of the four cubic fields of discriminant 32009. There are some cases in which the isometry class of the trace is determined by the discriminant and the signature:

Theorem ([1, Theorems 4.2, 4.5])**.** Let n be a positive integer and let K, L be two tame $\mathbb{Z}/n\mathbb{Z}$-number fields. Then

$$\langle O_K, \mathrm{t}_K\rangle \cong \langle O_L, \mathrm{t}_L\rangle \text{ if and only if } \mathrm{disc}(K) = \mathrm{disc}(L).$$

For odd n there is no need to mention the signature since the fields are totally real. As it turns out, in the above situation, for n even the equality between discriminants imply the equality of the signatures. The following example shows that the theorem above cannot be extended to arbitrary Galois groups, not even abelian ones. All the examples in this paper have been obtained with the help of John Jones' tables of number fields [3].

Example 1.1. Let K and L be the number fields defined respectively by the polynomials $f_K = x^4 - 41x^2 + 144$ and $f_L = x^4 - x^3 - 46x^2 - 115x - 35$. Both fields are quartic V_4-Galois fields with discriminant equal to $5^2 \cdot 13^2 \cdot 17^2$. Since in the first field $p = 5$ has one prime factor, while in the second it has two, we see using [4, Proposition 2.9] that the integral traces of K and L are not in the same spinor genus, thus they are not isometric.

In this paper we define a class of number fields for which the discriminant and the signature are necessary and sufficient to determine the spinor class of the integral trace.

Definition 1.2. Let K be a number field, let p be a rational prime and let $e_1, \ldots, e_g$ be the ramification indices of p in K. The prime p is called *uniformly ramified in* K if $e_1 = \cdots = e_g$.

The definition of uniformly ramified prime is inspired by the behaviour of ramification in Galois number fields; however there are non Galois number fields in which every prime is uniformly ramified, e.g., a field in which every ramified prime is totally ramified. For instance, the number field defined by the polynomial $x^4 - x^3 - 7x^2 + 11x + 3$ is an S_4-quartic field in which $p = 59$ is the only ramified prime and such prime is totally ramified.

Let K be a number field. A prime p is called an *exceptional prime* for K if

- there is an even number of primes above p,
- or p is not uniformly ramified,
- or p is uniformly ramified with common ramification degree e but, $\frac{[K:\mathbb{Q}]}{e}$ is even.

Definition 1.3. A number field K is called oddly tame if the following conditions hold:

(1) The field K is tame, i.e., every ramified prime p is not wildly ramified.
(2) There is at most one odd ramified prime that is exceptional for K.

If there is an odd ramified exceptional prime for K, then we call it *the* exceptional prime of K.

Remark 1.4. Notice that the collection of oddly tame number fields contains all tame Galois number fields of odd degree and all tame number fields in which all ramified primes are totally ramified. However, there can be oddly tame number fields that are of even degree, that are not Galois or in which no ramified prime is totally ramified. One such example is the sextic field K defined by the polynomial $x^6-2x^5+3x^4-9x^3+8x^2-7x-5$. This is a D_{12}-sextic number field with discriminant $3^3\cdot 23^3$ such that $3O_K=\mathcal{P}_0^2$ and $23O_K=(\mathcal{P}_1\mathcal{P}_2)^2$ for some prime ideals $\mathcal{P}_i$. This is the unique, up to conjugacy, oddly tame number field of signature $(2,2)$ and discriminant $3^3\cdot 23^3$. However, as verified by Example 1.5, there are pairs of non-isomorphic oddly tame number fields of the same signature and discriminant.

Our main result states that for oddly tame number fields the signature and the discriminant are necessary and sufficient to determine the spinor genus of the integral trace. Moreover, for non totally real fields this can be improved to isometry class.

Theorem (cf. Theorem 2.4)**.** Let K, L be two oddly tame number fields. Suppose that the set formed by exceptional primes of K and L contains at most one element. Then $\langle O_K,\mathrm{t}_K\rangle$ and $\langle O_L,\mathrm{t}_L\rangle$ belong to the same spinor genus if and only if $\mathrm{disc}(K)=\mathrm{disc}(L)$ and $\mathrm{Sign}(\langle O_K,\mathrm{t}_K\rangle)=\mathrm{Sign}(\langle O_L,\mathrm{t}_L\rangle)$. Furthermore, If K is not totally real

$$\begin{aligned}\langle O_K,\mathrm{t}_K\rangle\cong\langle O_L,\mathrm{t}_L\rangle \quad &\text{if and only if}\\ &\mathrm{disc}(K)=\mathrm{disc}(L) \text{ and } \mathrm{Sign}(\langle O_K,\mathrm{t}_K\rangle)=\mathrm{Sign}(\langle O_L,\mathrm{t}_L\rangle)\end{aligned}$$

Example 1.5. Consider the sextic fields K and L defined by the polynomials $f_K:=x^6-x^5-2x^4+x^3+7x^2-6x+4$ and $f_L:=x^6-3x^5+10x^4-15x^3+19x^2-12x+3$. Both fields have signature $(r,s)=(0,3)$, discriminant $-3^3\cdot 107^2$ and Galois closure with Galois group D_{12}. The fields are not isomorphic; in K the prime $p = 3$ has only one prime lying over it, while in L it has three. In both fields the prime $p = 3$ is uniformly ramified with ramification index $e = 2$. In particular, K and L are oddly tame number fields that are neither Galois or of odd degree or with totally ramified primes. Moreover, the prime $q = 107$ is exceptional in both fields. Since K is not totally real we know by Theorem 2.4 that $\langle O_K, \mathrm{t}_K\rangle \cong \langle O_L, \mathrm{t}_L\rangle$.

2. Proofs of our results

One of the main ingredients we use is the set of α-invariants of a number field; for definitions and properties see [6].

Definition 2.1. Let K be a number and let p be an odd prime. Let g be the number of prime factors of p in K and let $(e_1, \dots, e_g)$ and $(f_1, \dots, f_g)$ be the ramification and residue degrees of p over K. The first ramification invariant of p in K is the integer

$$\alpha_p^K := \left(\prod_{i=1}^{g} e_i^{f_i}\right) u_p^{(F-g)},$$

where $F = \sum f_i$ and $u_p \in \{1, \dots, p-1\}$ is the first non quadratic residue modulo p.

Among the useful properties of such invariants we have that they determine the genus of the integral trace which in degree at least 3, thanks to [5], is the same as the spinor genus. The main relation about α-invariants and the integral trace is the following:

Proposition 2.2 ([4, Proposition 2.9])**.** *Let K, L be tame number fields of degree $n \geq 3$. The forms $\langle O_K, \mathrm{t}_K\rangle$ and $\langle O_L, \mathrm{t}_L\rangle$ belong to the same spinor genus if and only if the following conditions hold:*

(1) $\mathrm{disc}(K) = \mathrm{disc}(L)$,
(2) $s_K = s_L$,
(3) *For every odd prime p that divides the common discriminant of K and L we have that*

$$\left(\frac{\alpha_p^K}{p}\right) = \left(\frac{\alpha_p^L}{p}\right).$$

In oddly tame number fields the α_p invariant of a non-exceptional ramified prime can be expressed, up to squares, in terms of the degree and the discriminant. More explicitly:

Lemma 2.3. *Let K be a degree n oddly tame number field of discriminant d. Let p be an odd non-exceptional ramified prime in K. Then,*

$$\alpha_p^K = \frac{n}{n - v_p(d)} \mod (\mathbb{Z}_p^*)^2,$$

where v_p denotes the standard p-adic valuation in $\mathbb{Q}$.

Proof. Let $p \neq 2$ be a ramified prime in K, not exceptional. By definition of α invariants, and since p is uniformly ramified, $\alpha_p^K = e^F u_p^{F-g}$ where e is the ramification invariant of p over K. By hypothesis g is odd, and since $n = eF$ we have, by hypothesis as well, that F is odd. Thus, $\alpha_p^K = e$ mod $(\mathbb{Z}_p^*)^2$. On the other hand, since p is tame, thanks to [8, Chapter III, Proposition 13], we have that

$$v_p(d) = (e-1)F = n - F = n - \frac{n}{e}$$

from where the result follows. $\square$

Theorem 2.4. *Let $K$and L be two oddly tame number fields. Suppose that the set formed by exceptional primes of K and L contains at most one element. Then $\langle O_K, \mathrm{t}_K\rangle$ and $\langle O_L, \mathrm{t}_L\rangle$ belong to the same spinor genus if and only if* $\mathrm{disc}(K) = \mathrm{disc}(L)$ *and* $\mathrm{Sign}(\langle O_K, \mathrm{t}_K\rangle) = \mathrm{Sign}(\langle O_L, \mathrm{t}_L\rangle)$. *Furthermore, If K is not totally real*

$$\begin{aligned} \langle O_K, \mathrm{t}_K\rangle \cong \langle O_L, \mathrm{t}_L\rangle \quad & \textit{if and only if} \\ & \mathrm{disc}(K) = \mathrm{disc}(L) \ \textit{and} \ \mathrm{Sign}(\langle O_K, \mathrm{t}_K\rangle) = \mathrm{Sign}(\langle O_L, \mathrm{t}_L\rangle) \end{aligned}$$

Proof. We may assume that K and L have degree at least 3. We show the non trivial implication. Since for number fields of degree at least 3 the genus and the spinor genus of the integral trace form coincide, see [5, Theorem 2.12], it is enough to show that $\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{R} \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{R}$ and that $\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Z}_p \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Z}_p$ for all prime p. Since K and L have the same signature it follows from a result of Taussky [9] that

$$\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{R} \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{R}.$$

Since for odd primes p the isometry class of a quadratic forms over $\mathbb{Z}_p$, with invertible discriminant, is determined by the discriminant and dimension of the form (see [2, Chapter 8, Lemma 3.4]) we have that

$$\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Z}_p \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Z}_p$$

for every odd unramified prime p. Given that both fields are tame it follows from [4, Proposition 2.7] that

$$\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Z}_2 \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Z}_2.$$

Thanks to Lemma 2.3

$$\left(\frac{\alpha_p^K}{p}\right) = \left(\frac{\alpha_p^L}{p}\right)$$

for every odd prime p that divides the common discriminant and that is not exceptional.

Hence, by [6, Theorem 3.14]

$$\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Z}_p \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Z}_p$$

for every prime p with one possible exception; an exceptional prime if it exists. Since $\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{R} \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{R}$ it follows from the product formula for Hilbert symbols that

$$\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Q}_p \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Q}_p$$

for every prime p. Let p be the possible exceptional prime. For $\alpha = \alpha_p^K$ or α_p^L, since $p \nmid \alpha$ one can check that

$$\left(\frac{\alpha}{p}\right) = (\alpha, p)_p$$

where $(\cdot,\cdot)_p$ is the p-adic Hilbert symbol. Combining [6, Theorem 2.13] with [6, Lemma 2.9] and [4, Lemma 2.1] then shows that $\langle O_K, \mathrm{t}_K\rangle \otimes \mathbb{Q}_p \cong \langle O_L, \mathrm{t}_L\rangle \otimes \mathbb{Q}_p$ implies that

$$\left(\frac{\alpha_p^K}{p}\right) = \left(\frac{\alpha_p^L}{p}\right)$$

for every prime p. The result then follows from Proposition 2.2 and for the isometry class in the non totally real case from [4, Theorem 2.13]. $\square$

The following example shows, thanks to Theorem 2.4, that the Galois structure of number fields with the same integral trace can be different.

Example 2.5. Consider the quartic fields K and L defined by the polynomials $f_K := x^4 - x^3 + 36x^2 - 36x + 281$ and $f_L := x^4 - 2x^3 + 14x^2 - 13x + 6$. Both fields have signature $(r,s) = (0,2)$, discriminant $5^3 \cdot 29^2$. The field K is a $\mathbb{Z}/4\mathbb{Z}$-number field. In contrast L is not Galois, and its Galois closure has Galois group D_8. In particular, the fields are not isomorphic. In both fields the prime $p = 5$ is totally ramified, hence K and L are oddly tame number fields. The prime $q = 29$ is exceptional in both fields. Since K is not totally real $\langle O_K, \mathrm{t}_K\rangle \cong \langle O_L, \mathrm{t}_L\rangle$.

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