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Petersson norms of Eisenstein series and Kohnen–Zagier's formula

par Yoshinori MIZUNO

Dedicated to Professor Hisashi Kojima on the occasion of his 70th birthday

Résumé. Les normes de Petersson régularisées des séries d'Eisenstein de poids entier et demi-entier sont calculées. Nous utilisons ces résultats pour établir la formule de Kohnen–Zagier pour les séries d'Eisenstein.

Abstract. The regularized Petersson norms of Eisenstein series of integral and half-integral weight are computed. We use these results to establish Kohnen–Zagier's formula for Eisenstein series.

1. Introduction

In this paper, we formulate Kohnen–Zagier's formula including Eisenstein series by means of the regularized Petersson norms. As observed in [19, p. 218] by Koblitz, a Kohnen–Zagier type relation holds true for Eisenstein series, except for a certain proportional constant consisting of Petersson norms and elementary factors. It is mentioned there that one may think of this fact as a "prototype" of the Waldspurger–Kohnen–Zagier formula, but this Eisenstein series case is not included in that formula, because one cannot define their Petersson norms. The present paper is motivated by this comment due to Koblitz.

In [39], Zagier defined the regularized Petersson norm of Eisenstein series of integral weight, and he computed it exactly. In his computation via a generalization of Rankin's formula, an explicit form of the Rankin convolution of the Eisenstein series in terms of the Riemann zeta function is required in order to determine its residue. This is the same in the computations done by Pasol and Popa [30]. It seems that such a suitable explicit form of the Rankin convolution is not available in the case of half-integral weight Eisenstein series. However, adapting an idea from the theory of harmonic Maass forms, one can compute the regularized Petersson norms of Eisenstein series. Accordingly, this exact evaluation makes us possible to formulate the Kohnen–Zagier formula for Eisenstein series.

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The method used to compute the norms is well developed and now standard in the theory of harmonic Maass forms. We refer to the book [4] and the papers [1, 3, 5, 6, 7, 10, 11, 18], etc. But this is the first time that the method is applied to Kohnen–Zagier's formula for Eisenstein series. When we were finishing this computation, the paper [34] by Tsuyumine has became available, in which the regularized Petersson norms of Eisenstein series are computed by a quite different way. In our computation of the norms, analytic continuations of the relevant Eisenstein series are not required. In addition, our treatment is simpler than Tsuyumine's more general treatment, since our computations are worked out inside the plus space of half-integral weight modular forms. In a forthcoming paper [14], we generalize Tsuyumine's approach to Jacobi forms, and establish Gross–Kohnen–Zagier's formula (cf. [13]) for Eisenstein series by means of regularized Petersson norms.

Notation. Let $\mathbb{H} := \{\tau = u + iv \in \mathbb{C}; v > 0\}$ be the upper half-plane. The action of $\mathrm{GL}_2^+(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}); \det(g) > 0\}$ on $\mathbb{H}$ is denoted by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := \frac{a\tau+b}{c\tau+d}$. The branch of $z^\alpha := e^{\alpha \log z}$ for $z \in \mathbb{C} \setminus \{0\}$ is taken so that $\log z := \log|z| + i \arg z$ with $-\pi < \arg z \leq \pi$. Put $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}); c \equiv 0 \pmod{N}\}$ for $N \in \mathbb{N}$. Put $e(x) := e^{2\pi i x}$ for $x \in \mathbb{C}$, $q = e(\tau)$, $q^n = e(n\tau)$ for $\tau \in \mathbb{H}$, $n \in \mathbb{Z}$, and denote by $\lfloor \alpha \rfloor$ the integer such that $\lfloor \alpha \rfloor \leq \alpha < \lfloor \alpha \rfloor + 1$ for $\alpha \in \mathbb{R}$. For $s \in \mathbb{C}$ and $D \in \mathbb{Z}$, the Cohen function is defined in [8, 38] by

$$L_D(s) := \begin{cases} \zeta(2s-1), & \text{if } D = 0, \\ L(s, \chi_K) \sum_{a|f} \mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a), & \\ & \text{if } D \neq 0, D \equiv 0, 1 \pmod{4}, \\ 0, & \text{if } D \equiv 2, 3 \pmod{4}. \end{cases}$$

Here the natural number f is defined by $D = d_K f^2$ with the discriminant d_K of $K := \mathbb{Q}(\sqrt{D})$, $\chi_K := (\frac{d_K}{\cdot})$ is the Kronecker symbol,[1] μ is the Möbius function, $\zeta(s)$ is the Riemann zeta function, $L(s, \chi)$ is the Dirichlet L-function and $\sigma_s(n) := \sum_{d|n} d^s$. We denote by B_k the k-th Bernoulli number, and by $\Gamma(s)$ the gamma function.

2. Statement of the main result

2.1. Harmonic Maass forms and the extended Petersson paring.

Definition 2.1 (Modular forms)**.** Let $k \in \frac{1}{2}\mathbb{Z}$. Let $N = 1$ or 4 according as $k \in \mathbb{Z}$ or $k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$. A weight k modular form on $\Gamma = \Gamma_0(N)$ is any holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following properties:

[1] If D is a square of a natural number, we take $d_K = 1$, and χ_K is understood as the principal character, that is, the trivial character modulo 1.

(i) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$f(\gamma\tau) = \begin{cases} (c\tau + d)^k f(\tau), & \text{if } k \in \mathbb{Z}, \\ \epsilon_d^{-2k} \left(\frac{c}{d}\right) (c\tau + d)^k f(\tau), & \text{if } k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}. \end{cases}$$

Here $\epsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$ and $\left(\frac{c}{d}\right)$ is Shimura's quadratic residue symbol defined in [19, p. 147], [31].

(ii) f is holomorphic at every cusps of Γ.

We let M_k denote the $\mathbb{C}$-vector space of all weight k modular forms on Γ. We let $S_k \subset M_k$ denote the subspace of all cusp forms, namely, any $f \in S_k$ takes zeroes at every cusps of Γ. For $f, g \in M_k$, where at least one of f or g is a cusp form, the Petersson paring is defined by

$$\langle f, g \rangle_N := \int_{D(N)} f(\tau)\overline{g(\tau)} v^k \, \mathrm{d}\mu, \quad \mathrm{d}\mu := \frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

Here $D(N) := \Gamma \setminus \mathbb{H}$ is a fundamental domain with respect to $\Gamma = \Gamma_0(N)$. Usually, we assume that either $f \in S_k$ or $g \in S_k$ to guaranty the convergence of the integral.

Definition 2.2 (Harmonic Maass forms (cf. [4, Definition 4.2, p. 62]))**.** Let $k \in \frac{1}{2}\mathbb{Z}$. Let $N = 1$ or 4 according as $k \in \mathbb{Z}$ or $k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$. A weight k harmonic Maass form on $\Gamma = \Gamma_0(N)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying (i) in Definition 2.1 together with the following properties:

(ii) We have that $\Delta_k(f) = 0$, where

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) \quad (\tau = u + iv).$$

(iii) There exists $p_f(\tau) \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - p_f(\tau) = O(e^{-\epsilon v})$ as $v \to \infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps of Γ.[2]

We let $H_k(\Gamma)$ denote the $\mathbb{C}$-vector space of all weight k harmonic Maass forms on Γ. If we replace (iii) by $f(\tau) = O(e^{\epsilon v})$ as $v \to \infty$ for some $\epsilon > 0$ together with analogous conditions at all cusps of Γ, we get the space denoted by $H_k^!(\Gamma)$.

An important tool in the theory of harmonic Maass forms is the differential operator ξ_κ introduced by Bruinier and Funke ([4, (5.11), p. 74], [6]). For each $\kappa \in \mathbb{R}$, it is defined by

$$\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}, \quad \frac{\partial}{\partial \overline{\tau}} := \frac{1}{2} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right).$$

[2] In this paper, we are concerned with forms on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, or forms on $\Gamma = \Gamma_0(4)$ satisfying the "plus condition". Hence, we need to take care of the cusp $i\infty$ only.

We need the incomplete Gamma function $\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t$ ($a \in \mathbb{R}$, $x > 0$) to describe the Fourier expansions of elements of $H_k(\Gamma)$ and $H_k^!(\Gamma)$ ([4, Lemma 4.3, p. 64]). The function $\Gamma(a,x)$ is a special case of the W-Whittaker function in the sense that $W_{\mu-1/2,\mu}(y) = e^{y/2}y^{1/2-\mu}\Gamma(2\mu,y)$ ([29, 13.18, p. 338]).

Definition 2.3 (The regularized Petersson paring (cf. [5, 10, 39]))**.** Let $k \in \frac{1}{2}\mathbb{Z}$. Let $N = 1$ or 4 according as $k \in \mathbb{Z}$ or $k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$, and put $\Gamma = \Gamma_0(N)$. When $k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$, let $M_k^+ \subset M_k$ be the plus space consisting of modular forms whose d-th Fourier coefficients vanish unless $(-1)^{k-1/2}d \equiv 0, 1 \pmod 4$ (cf. [21]). For any f, $g \in M_k$ with $k \in \mathbb{Z}$ (resp. f, $g \in M_k^+$ with $k \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$) having the constant term $a_f(0)$, $a_g(0)$ in the Fourier expansion at the cusp $i\infty$, the regularized Petersson paring is defined by

$$\langle f, g\rangle_N^{\mathrm{reg}} := \lim_{Y\to\infty}\left(\int_{D(N)_Y} f(\tau)\overline{g(\tau)}v^{k-2}\,\mathrm{d}u\mathrm{d}v - c_k \cdot \frac{a_f(0)\overline{a_g(0)}}{k-1}Y^{k-1}\right).$$

Here $c_k = 1$ (resp. $c_k = 1 + 2^{-2k+1}$) and $D(N)_Y$ is the truncated fundamental domain with respect to Γ as described in the paper [39] (resp. [10]). We refer to Section 3 (resp. Section 4.4) for more detailed description of $D(N)_Y$. When at least one of f or g is a cusp form, it gives the usual Petersson paring on M_k (resp. M_k^+).

2.2. Kohnen–Zagier formula including Eisenstein series. Let $k \geq 2$ be an integer. Let D be a fundamental discriminant (including 1) such that $(-1)^k D > 0$. For any $g \in M_{k+\frac{1}{2}}^+$ with the Fourier expansion

$$g(\tau) = \sum_{\substack{d\geq 0 \\ (-1)^k d \equiv 0,1 \pmod 4}} c_g(d)e(d\tau),$$

the Shimura correspondence is defined by

$$\mathcal{S}_D^+(g)(\tau) := \frac{L_D(1-k)}{2}c_g(0) + \sum_{n\geq 1}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{k-1}c_g\left(\frac{n^2|D|}{d^2}\right)\right)e(n\tau).$$

Here $\left(\frac{D}{\cdot}\right)$ is the Kronecker symbol. In [21, 23], Kohnen and Zagier established the following.

(i) We have $\mathcal{S}_D^+(g) \in M_{2k}$.

(ii) For any prime p, let $T_{2k}(p)$ (resp. $T_{k+\frac{1}{2}}^+(p^2)$) be the Hecke operator acting on the space M_{2k} (resp. $M_{k+\frac{1}{2}}^+$). Then, for all primes p, it holds that

$$T_{2k}(p)(\mathcal{S}_D^+(g)) = \mathcal{S}_D^+(T_{k+\frac{1}{2}}^+(p^2)(g)).$$

(iii) For any normalized Hecke eigenform $f \in M_{2k}$, we can take $g \in M^+_{k+\frac{1}{2}}$ satisfying $\mathcal{S}^+_D(g) = c_g(|D|)f$ and $T^+_{k+\frac{1}{2}}(p^2)(g) = a_f(p)g$ for all primes p. In addition, if $f \in S_{2k}$ is a cusp form, then $g \in S^+_{k+\frac{1}{2}} := S_{k+\frac{1}{2}} \cap M^+_{k+\frac{1}{2}}$.

(iv) There is a linear combination $\alpha_1\mathcal{S}^+_{D_1} + \alpha_2\mathcal{S}^+_{D_2} + \cdots + \alpha_r\mathcal{S}^+_{D_r}$ ($r \in \mathbb{N}, \alpha_j \in \mathbb{C}, 1 \leq j \leq r$) giving

$$M^+_{k+\frac{1}{2}} \cong M_{2k}, \qquad S^+_{k+\frac{1}{2}} \cong S_{2k}.$$

We refer to [21, Theorem 1, p. 250], [21, Lemma, p. 256] and [23, p. 182] for these facts.

For any normalized Hecke eigenform $f \in S_{2k}$ and a cusp form $g \in S^+_{k+\frac{1}{2}}$ as stated in (iii) above, the Kohnen–Zagier formula established in [23, Theorem 1, p. 177] claims that

$$\frac{|c_g(|D|)|^2}{\frac{1}{6}\langle g, g\rangle_4} = \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(f, D, k)}{\langle f, f\rangle_1}.$$

Here $L(f, D, s) := \sum_{n=1}^{\infty}\left(\frac{D}{n}\right)a_f(n)n^{-s}$ ($\Re(s) \gg 0$) is the L-function of f twisted by the Kronecker symbol $\left(\frac{D}{\cdot}\right)$, and the value at $s = k$ is evaluated after analytic continuation.

Let $G_{k+\frac{1}{2}} \in M^+_{k+\frac{1}{2}}$ be the Eisenstein series discovered by Cohen [8] (cf. [19, Proposition 6, p. 193]). It has the Fourier expansion[3]

$$G_{k+\frac{1}{2}}(\tau) = \sum_{\substack{(-1)^k d \geq 0 \\ d \equiv 0,1 \pmod 4}} h_{k+\frac{1}{2}}(|d|)e(|d|\tau).$$

As explained in [21, p. 259, l. −5] and [23, p. 185, paragraph at l. 8], the function $G_{k+\frac{1}{2}}$ corresponds to the normalized Eisenstein series $G_{2k} \in M_{2k}$ defined by

$$G_{2k}(\tau) := -\frac{B_{2k}}{4k}\cdot\frac{1}{2}\sum_{c,d\in\mathbb{Z},\ (c,d)=1}\frac{1}{(c\tau+d)^{2k}}$$

through the map $\mathcal{S}^+_D$. Indeed,

$$\mathcal{S}^+_D(G_{k+\frac{1}{2}}) = h_{k+\frac{1}{2}}(|D|)G_{2k}, \quad T^+_{k+\frac{1}{2}}(p^2)(G_{k+\frac{1}{2}}) = \sigma_{2k-1}(p)G_{k+\frac{1}{2}}$$

for all primes p.

By means of the regularization of the Petersson norms, the Kohnen–Zagier formula for Eisenstein series can be formulated as follows.

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[3]Here $h_{k+\frac{1}{2}}(|D|) = L_D(1-k)$ with $L_D(s)$ defined in Section 1.

Theorem 2.4. *Let $k \geq 2$ be an integer and D a fundamental discriminant such that $(-1)^k D > 0$. Then*

$$\frac{h_{k+\frac{1}{2}}(|D|)^2}{\frac{1}{6}\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}} = \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(G_{2k}, D, k)}{\langle G_{2k}, G_{2k}\rangle_1^{\mathrm{reg}}}.$$

Both regularizations of the norms for Eisenstein series are indeed extensions of the usual Petersson norms for cusp forms. Recalling $M_{2k} = \mathbb{C}G_{2k} \oplus S_{2k}$ and $M^+_{k+\frac{1}{2}} = \mathbb{C}G_{k+\frac{1}{2}} \oplus S^+_{k+\frac{1}{2}}$ (cf. [21, p. 259, l. -7]), we have a suitable formulation of the Kohnen–Zagier formula including the Eisenstein series. Notice that $h_{k+\frac{1}{2}}(|D|) \in \mathbb{Q}$ and thus $h_{k+\frac{1}{2}}(|D|)^2 = |h_{k+\frac{1}{2}}(|D|)|^2$ holds.[4]

2.3. Some remarks.

(1). Let $k \geq 2$ be an integer. One has $\omega^-_{G_{2k}}\omega^+_{G_{2k}}/(i\langle G_{2k}, G_{2k}\rangle_1^{\mathrm{reg}}) \in \mathbb{Q}$ as shown in [40, p. 455, l. 3], where the periods $\omega^-_{G_{2k}}$, $\omega^+_{G_{2k}}$ attached to G_{2k} are given as follows. For any $f \in M_{2k}$ and $s \in \mathbb{C}$ with $\Re(s) \gg 0$, we put

$$\begin{aligned} L^*(f,s) &:= \int_0^\infty (f(iv) - a_f(0))v^{s-1}\mathrm{d}v = (2\pi)^{-s}\Gamma(s)L(f,s), \\ L(f,s) &:= \sum_{n=1}^\infty a_f(n)n^{-s}, \end{aligned}$$

which can be continued meromorphically to the whole complex s-plane. According to [40, p. 452], we define

$$r_f(X) := \frac{a_f(0)}{2k-1}(X^{2k-1}+X^{-1}) + \sum_{n=0}^{2k-2} i^{1-n}\binom{2k-2}{n}L^*(f,n+1)X^{2k-2-n}.$$

For $f = G_{2k}$, [40, Proposition, p. 453] tells us that $r_{G_{2k}}(X) = \omega^-_{G_{2k}}p^-_{2k}(X) + \omega^+_{G_{2k}}p^+_{2k}(X)$. Here $p^+_{2k}(X) := X^{2k-2}-1$, and $p^-_{2k}(X)$ is given explicitly there ($p^-_{2k}(X)X \in \mathbb{Q}[X]$, $\deg(p^-_{2k}(X)X) = 2k$), the periods attached to G_{2k} are given by

$$\omega^-_{G_{2k}} = -\frac{(2k-2)!}{2}, \quad \omega^+_{G_{2k}} = \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}\omega^-_{G_{2k}}.$$

In the proof of Theorem 2.4, we see that $\langle G_{2k}, G_{2k}\rangle_1^{\mathrm{reg}}$, $\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}$ are non-zero real numbers and that $\langle G_{2k}, G_{2k}\rangle_1^{\mathrm{reg}} = \alpha\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}$,

[4] Instead the absolute value symbol in the left-hand side of the Kohnen–Zagier formula can be removed, if we normalize $g \in S^+_{k+\frac{1}{2}}$ suitably in the sense that βg ($\beta \in \mathbb{C}^\times$) has real algebraic Fourier coefficients as noted before [23, Theorem 1, p. 177].

where $\alpha := 2^{k-1}(-1)^{k+\lfloor\frac{k+1}{2}\rfloor}/6 \in \mathbb{Q}$. Hence, we obtain an Eisenstein analogue of [23, Corollary 3, p. 180], that is,

$$\frac{\omega^{+}_{G_{2k}}}{i\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}} \in \mathbb{Q}.$$

(2). Let f be any natural number, $k \geq 2$ an integer and D a fundamental discriminant such that $(-1)^k D > 0$. For $s \in \mathbb{C}$ with $\Re(s) \gg 0$, let us define

$$L(G_{2k}, Df^2, s) := \sum_{m=1}^{\infty} \epsilon_{Df^2}(m)\sigma_{2k-1}(m)m^{-s}$$

according to [23, pp. 180-181]. Here $\epsilon_{Df^2}(m)$ are the Dirichlet coefficients of $L_{Df^2}(s) = \sum_{m=1}^{\infty} \epsilon_{Df^2}(m)m^{-s}$. Analogous to [23, Corollary 4, p. 181], Theorem 2.4 holds true for general discriminants Df^2 with $(-1)^k Df^2 > 0$. In fact, the proof in [23, p. 188-189] works for the Hecke eigenform G_{2k}.

(3). We can discuss an analogy of [23, Corollary 5, p. 181] by the method in [27] (cf. [28]), although this weight identical case is excluded from [27].

(4). Let $k \geq 2$ be an integer. For any $g \in S_{2k}$, there exists a harmonic Maass form $P_g \in H_{2-2k}(\mathrm{SL}_2(\mathbb{Z}))$ such that $\xi_{2-2k}(P_g) = g$ by [4, Theorem 5.10, p. 74]. According to [4, Lemma 4.3, p. 64], it has the Fourier expansion of the form

$$P_g(\tau) = \sum_{n \gg -\infty} c^{+}_{P_g}(n)e(n\tau) + \sum_{n<0} c^{-}_{P_g}(n)\Gamma(2k-1, -4\pi n v)e(n\tau).$$

By the same computation used in Section 3, we have the formula (cf. [5, Section 4.1])

$$\langle f, g\rangle_1 = \langle f, g\rangle_1^{\mathrm{reg}} = \sum_{1 \leq n \ll +\infty} a_f(n)c^{+}_{P_g}(-n) \quad (f, g \in S_{2k}).$$

(5). Let $k \geq 2$ be an integer. We denote by $H^{+}_{\frac{3}{2}-k} \subset H_{\frac{3}{2}-k}(\Gamma_0(4))$ the subspace consisting of harmonic Maass forms whose d-th Fourier coefficients (with respect to $e(du)$) vanish unless $(-1)^{1-k}d \equiv 0, 1 \pmod 4$ (cf. [10]). By the same computation given in [10], a formula similar to the above remark (4) holds true for $\langle f, g\rangle_4$ on $S^{+}_{k+\frac{1}{2}}$. The required surjectivity of $\xi_{\frac{3}{2}-k}$ from $H^{+}_{\frac{3}{2}-k}$ onto $S^{+}_{k+\frac{1}{2}}$ can be seen as follows. In terms of the notation in [17, Theorem 4.4, p. 2580], for $2 < k + \frac{1}{2} \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$ and any integer $m < 0$ with $(-1)^{1-k}m \equiv 0, 1 \pmod 4$, it is not difficult to see

$$\begin{aligned}&\xi_{\frac{3}{2}-k}(F^{+}_{m,\frac{3}{2}-k,4}(\tau, (k+1/2)/2)) \\ &\qquad = -(4\pi|m|)^{k-\frac{1}{2}}(k-1/2)F^{+}_{-m,k+\frac{1}{2},4}(\tau, (k+1/2)/2) \in S^{+}_{k+\frac{1}{2}}.\end{aligned}$$

Note that $\xi_{\frac{3}{2}-k}$ is antilinear and that any $g \in S^+_{k+\frac{1}{2}}$ is a finite linear combination of holomorphic Poincaré series $F^+_{-m,k+\frac{1}{2},4}(\tau,(k+1/2)/2)$ $(m<0$, $(-1)^{k-1}m \equiv 0,1 \pmod 4)$ (cf. [22] and the proof of [24, Proposition 7, p. 172]).

(6). It seems possible to pursue a similar computation for general levels. Indeed, certain basis of the space of holomorphic Eisenstein series, or their preimages under ξ-operator are partly available in [15] for integral weight case, and in [36, Chapter 7] for half-integral weight case.

2.4. Outline of this paper. In Section 3, we recall a computation of $\langle G_{2k}, G_{2k}\rangle_1^{\mathrm{reg}}$, which has been done in the theory of harmonic Maass forms. In Section 4, we compute $\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}$ along the lines of the paper [10]. The non-holomorphic Cohen–Eisenstein series $F(k,\sigma,\tau)$ plays an important role to work out an analogy of Section 3. Theorem 2.4 is proved in Section 5. In Section 6, we confirm that our result and Tsuyumine's result are consistent.

3. Integral weight case

Let $k \geq 4$ be even. For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and any function f on $\mathbb{H}$, the slash operator is defined by

$$(f|_k g)(\tau) := f(g\tau)(c\tau+d)^{-k}.$$

The Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ is

$$E_k(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k\gamma = \frac{1}{2}\sum_{c,d\in\mathbb{Z},\ (c,d)=1} \frac{1}{(c\tau+d)^k} \in M_k.$$

Here $\Gamma_\infty := \{\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right); n\in\mathbb{Z}\}$. Following [4, p. 104] and [18, (1.3.2), p. 18], put

$$\begin{aligned} P_{E_k}(\tau) &:= \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} v^{k-1}|_{2-k}\gamma \\ &= \frac{1}{2}\sum_{c,d\in\mathbb{Z},\ (c,d)=1} \frac{v^{k-1}}{(c\tau+d)(c\overline{\tau}+d)^{k-1}} \in H^!_{2-k}(\mathrm{SL}_2(\mathbb{Z})). \end{aligned}$$

We refer the reader to [4, Theorem 6.15, p. 104] for basic properties of P_{E_k}. By a computation using the intertwining relation between ξ_{2-k} and the action of $\mathrm{SL}_2(\mathbb{R})$ (cf. [4, Lemma 5.2, p. 68]),[5] or directly from the Fourier expansions (Remark 3.2 below), we know as in [18, Theorem 1.3.6, p. 18] and [4, Corollary 6.16, p. 106] that

$$\xi_{2-k}(P_{E_k}) = (k-1)E_k,$$

[5] [4, Lemma 5.2, p. 68] treats the case $\mathrm{SL}_2(\mathbb{Z})$, but this holds for $\mathrm{SL}_2(\mathbb{R})$.

or in other words,

$$\overline{E_k(\tau)}v^{k-2} = -\frac{2i}{k-1} \cdot \frac{\partial}{\partial\overline{\tau}} P_{E_k}(\tau).$$

Here, it is easy to confirm that one can differentiate the definition (and the Fourier series) of P_{E_k} term by term.

The truncated fundamental domain $D(1)_Y$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ introduced in Definition 2.3 is given for $Y > 1$ by $D(1)_Y = \{\tau = u + iv \in \mathbb{H}; -\frac{1}{2} \leq u \leq \frac{1}{2}, |\tau| \geq 1, v \leq Y\}$. Following the papers [5, 7], in particular [5, Section 4.1] and [7, Theorem 4.1, p. 682], we compute

$$\begin{aligned}\int_{D(1)_Y} E_k(\tau)\overline{E_k(\tau)}v^{k-2}\,\mathrm{d}u\mathrm{d}v &= -\frac{2i}{k-1}\int_{D(1)_Y} E_k(\tau)\frac{\partial}{\partial\overline{\tau}}P_{E_k}(\tau)\,\mathrm{d}u\mathrm{d}v \\ &= -\frac{1}{k-1}\int_{\partial(D(1)_Y)} E_k(\tau)P_{E_k}(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{k-1}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} E_k(\tau)P_{E_k}(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where $\partial(D(1)_Y)$ denotes a positively oriented boundary of $D(1)_Y$, the path of the last integral is the horizontal line from $-\frac{1}{2} + iY$ to $\frac{1}{2} + iY$. At the second equal sign, we used the Stokes formula and the holomorphy of E_k on $\mathbb{H}$. Then, at the third equal sign, we used the fact that $E_kP_{E_k}$ is of weight 2, therefore the remaining part of the integral cancels out.

In order to compute the last integral, we use the Fourier expansions (cf. [18, Theorem 1.3.6, p. 18], [4, Theorem 6.15, p. 104], or Remark 3.2 given below)

$$\begin{aligned}E_k(\tau) &= \sum_{n\geq 0} c(n)e(n\tau), \\ P_{E_k}(\tau) &= a^+(0) + a^-(0)v^{k-1} \\ &\qquad + \sum_{n>0} a^+(n)e(n\tau) + \sum_{n>0} a^-(n)\Gamma(k-1, 4\pi nv)e(-n\tau).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} E_k(\tau)P_{E_k}(\tau)\,\mathrm{d}\tau \\ &\qquad = c(0)\{a^+(0) + a^-(0)Y^{k-1}\} + \sum_{n>0} c(n)a^-(n)\Gamma(k-1, 4\pi nY).\end{aligned}$$

In view of $\Gamma(a, x) \sim e^{-x}x^{a-1}$ as $0 < x \to \infty$ (cf. [4, (4.6), p. 64], [20, Lemma 6, p. 196]) together with $c(0) = a^-(0) = 1$ and the polynomial growth estimate of the Fourier coefficients $c(n)$, $a^\pm(n)$ with respect to $|n|$

(cf. Remark 3.2), we see that the term $\frac{1}{k-1}Y^{k-1}$ gives the reason that the integral $\int_{D(1)_Y} E_k(\tau)\overline{E_k(\tau)}v^{k-2}\,\mathrm{d}u\mathrm{d}v$ diverges as $Y \to \infty$.

Hence the value $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$ (cf. Definition 2.3) is finite and we get

$$\langle E_k, E_k\rangle_1^{\mathrm{reg}} = \frac{1}{k-1}c(0)a^+(0).$$

The explicit Fourier expansions of E_k and P_{E_k} tell us that

$$c(0) = 1, \quad a^+(0) = \frac{2\cdot k!}{B_k}\cdot\frac{\zeta(k-1)}{(4\pi)^{k-1}},$$

and finally we obtain

$$\langle E_k, E_k\rangle_1^{\mathrm{reg}} = \frac{1}{k-1}\cdot\frac{2\cdot k!}{B_k}\cdot\frac{\zeta(k-1)}{(4\pi)^{k-1}}.$$

Remark 3.1. As in Section 2.2, put

$$G_k := -\frac{B_k}{2k}E_k$$

for any even integer $k \geq 4$. By $\zeta(k) = -(2\pi i)^k B_k/(2\cdot k!)$, we get

$$\langle G_k, G_k\rangle_1^{\mathrm{reg}} = (-1)^{\frac{k}{2}-1}\frac{(k-1)!(k-2)!\zeta(k)\zeta(k-1)}{2^{3k-2}\pi^{2k-1}}.$$

This coincides with the formula in [39, p. 435] (cf. [40, (13), p. 454]), which has been obtained from the explicit formula of the Rankin convolution of G_k.[6]

Remark 3.2. Suppose that $k \in \mathbb{Z}$ is even, $\tau \in \mathbb{H}$ and $\Re(s) + k/2 > 1$. In general,

$$E_k(\tau, s) := \frac{1}{2}\sum_{c,d\in\mathbb{Z},\ (c,d)=1}\frac{v^s}{(c\tau+d)^k|c\tau+d|^{2s}}$$

has the Fourier expansion $E_k(\tau, s) = \sum_{n\in\mathbb{Z}} c_n(v, s)e(nu)$ with

$$c_0(v, s) = v^s + i^{-k}2^{2-2s-k}\pi\frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)}\frac{\zeta(2s+k-1)}{\zeta(2s+k)}v^{1-s-k},$$

and

$$c_n(v, s)$$
$$= \frac{i^{-k}\pi^{s+k/2}|n|^{-s-k/2}\sigma_{2s+k-1}(n)}{\Gamma(s+(\mathrm{sgn}(n)+1)k/2)\zeta(2s+k)}v^{-k/2}W_{\mathrm{sgn}(n)k/2,s+k/2-1/2}(4\pi|n|v)$$

for $n \in \mathbb{Z}\setminus\{0\}$. Here the function $W_{\alpha,\beta}(v)$ is the W-Whittaker function. This Fourier expansion can be drawn from [25, Theorem 7.2.9, p. 284] or [2, Proposition 2.5, §2.3]. One has $E_k(\tau) = E_k(\tau, 0)$, $P_{E_k}(\tau) = E_{2-k}(\tau, k-1)$

[6] There is a misprint in [39]. The number 2^{3k-3} in the denominator should be 2^{3k-2} as mentioned in [40, p. 455], [26, p. 111].

for any even $k \geq 4$, and we can adopt $W_{\mu+1/2,\mu}(y) = e^{-y/2}y^{\mu+1/2}$, $W_{\mu-1/2,\mu}(y) = e^{y/2}y^{1/2-\mu}\Gamma(2\mu, y)$ ([29, 13.18, p. 338]) to determine $c(n)$, $a^{\pm}(n)$. Notice that $\sigma_s(n) = |n|^s\sigma_{-s}(n)$, and that if $\Re(s) > 1$, then $|\sigma_{-s}(n)| \leq \zeta(\Re(s))$ ([41, p. 260]).

4. Half-integral weight case

4.1. Setting. The following setting owes to [19, Chapter IV, §1].

Let G be the set of all pairs $(\alpha, \phi(\tau))$, where $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\phi(\tau)$ is a holomorphic function on $\mathbb{H}$ such that $\phi(\tau)^2 = t(\det\alpha)^{-\frac{1}{2}}(c\tau + d)$ for some $t \in \{\pm 1\}$. The set G is a group with the identity $(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1)$ by defining the product of elements $(\alpha, \phi(\tau))$ and $(\beta, \psi(\tau))$ of G as

$$(\alpha, \phi(\tau))(\beta, \psi(\tau)) = (\alpha\beta, \phi(\beta\tau)\psi(\tau)) \in G.$$

The inverse of an element of G is given by

$$(\alpha, \phi(\tau))^{-1} = (\alpha^{-1}, \phi(\alpha^{-1}\tau)^{-1}) \in G.$$

Let k be an odd integer. For any $g = (\alpha, \phi(\tau)) \in G$ and any function f on $\mathbb{H}$, the slash operator is defined by

$$f(\tau)|[g]_{\frac{k}{2}} := f(\alpha\tau)\phi(\tau)^{-k}.$$

We have the relation

$$(f|[g_1]_{\frac{k}{2}})|[g_2]_{\frac{k}{2}} = f|[g_1g_2]_{\frac{k}{2}}, \quad g_1, g_2 \in G.$$

Consider the subgroup defined by

$$\widetilde{\Gamma_0(4)} := \{\widetilde{\gamma} := (\gamma, j(\gamma, \tau)); \gamma \in \Gamma_0(4)\} \subset G,$$
$$j(\gamma, \tau) := \frac{\theta(\gamma\tau)}{\theta(\tau)} = \epsilon_d^{-1}\left(\frac{c}{d}\right)(c\tau + d)^{\frac{1}{2}}, \quad \theta(\tau) := \sum_{n\in\mathbb{Z}} e(n^2\tau),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and we refer to Definition 2.1 for ϵ_d and $\left(\frac{c}{d}\right)$. For $\gamma \in \Gamma_0(4)$, putting

$$\rho := \left(\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right), \sqrt{2\tau}\right) \in G, \quad \gamma_1 := \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)^{-1} \in \Gamma_0(4),$$

we have $\widetilde{\gamma_1} = \rho\widetilde{\gamma}\rho^{-1}$ (cf. [19, Problem 2, p. 184]). Hence, if f satisfies $f|[\widetilde{\gamma}]_{\frac{k}{2}} = f$ for any $\gamma \in \Gamma_0(4)$, it holds that $(f|[\rho]_{\frac{k}{2}})|[\widetilde{\gamma}]_{\frac{k}{2}} = f|[\rho]_{\frac{k}{2}}$ for any $\gamma \in \Gamma_0(4)$ (cf. [19, Problem 6, p. 184]).

Put

$$L := -2iv^2\frac{\partial}{\partial\overline{\tau}}, \quad \tau = u + iv \in \mathbb{H}.$$

Then the differential operator introduced in Section 2.1 is given by

$$\xi_{2-\frac{k}{2}}(f)(\tau) = 2iv^{2-\frac{k}{2}}\overline{\frac{\partial}{\partial\overline{\tau}}f(\tau)} = v^{-\frac{k}{2}}\overline{L(f)(\tau)}.$$

For any C^{∞}-function f on $\mathbb{H}$, and any $g \in G$, it is not difficult to see that

$$L(f|[g]_{\frac{k}{2}}) = L(f)|[g]_{\frac{k}{2}-2}.$$

See [4, Lemma 5.2, p. 68].[7]

4.2. Non-holomorphic Cohen–Eisenstein series. In this section, we summarize facts about the non-holomorphic Cohen–Eisenstein series, for which we use [16, §2] as a main reference. Also, we refer to [8, 9, 12, 19, 37] for related works.

Let k be an odd integer. For $\tau = u + iv \in \mathbb{H}$ and $\sigma \in \mathbb{C}$ such that $-k + 2\Re(\sigma) - 4 > 0$, the Eisenstein series is defined by

$$\begin{aligned} F(k,\sigma,\tau) &:= E(k,\sigma,\tau) \\ &\qquad + 2^{k/2-\sigma}(e(k/8) + e(-k/8))E(k,\sigma,-1/(4\tau))(-2i\tau)^{k/2}, \\ E(k,\sigma,\tau) &:= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} v^{\sigma/2}|[\widetilde{\gamma}]_{-\frac{k}{2}} \\ &= v^{\sigma/2} \sum_{d=1,\mathrm{odd}}^{\infty} \sum_{c=-\infty}^{\infty} \left(\frac{4c}{d}\right) \epsilon_d^{-k}(4c\tau + d)^{k/2}|4c\tau + d|^{-\sigma}. \end{aligned}$$

We note that $E(k,\sigma,\tau)|[\rho]_{-\frac{k}{2}}(-i)^{k/2} = E(k,\sigma,-1/(4\tau))(-2i\tau)^{k/2}$ and that $f \in \{E, E|[\rho]_{-\frac{k}{2}}, F\}$ satisfies $f|[\widetilde{\gamma}]_{-\frac{k}{2}} = f$ for any $\gamma \in \Gamma_0(4)$, where $E = E(k,\sigma,\tau)$, $F = F(k,\sigma,\tau)$. The function $F(k,\sigma,\tau)$ has the Fourier expansion

$$F(k,\sigma,\tau) = v^{\sigma/2} + v^{\sigma/2} \sum_{d=-\infty}^{\infty} c(d,\sigma,k)e(du)\tau_d\left(v, \frac{\sigma-k}{2}, \frac{\sigma}{2}\right),$$

where $\tau_d(v,\alpha,\beta)$ is defined by $\tau_d(v,\alpha,\beta) := \int_{-\infty}^{\infty} e(-du)\tau^{-\alpha}\overline{\tau}^{-\beta}\mathrm{d}u$ (with $d \in \mathbb{R}$, $\tau = u + iv \in \mathbb{H}$, α, $\beta \in \mathbb{C}$, $\Re(\alpha+\beta) > 1$) and $c(d,\sigma,k)$ is given by

$$c(d,\sigma,k) := 2^{k+3/2-2\sigma}e^{(-1)^{(k+1)/2}(\pi i/4)}\frac{L_{(-1)^{(k+1)/2}d}(\sigma - \frac{k+1}{2})}{\zeta(2\sigma-k-1)}$$

with $L_D(s)$ as in Section 1. Accordingly, $c(d,\sigma,k)$ can be continued to a meromorphic function on the whole complex σ-plane, and $F(k,\sigma,\tau)$ satisfies the "plus condition" saying that

$$c(d,\sigma,k) = 0 \text{ for } d \in \mathbb{Z} \text{ with } (-1)^{(k+1)/2}d \equiv 2, 3 \pmod 4.$$

Let k be an odd integer. Let $K \subset \mathbb{C} \setminus \{(k+3)/2\}$ be any compact subset of the complex σ-plane. Then there exist two positive constants $M > 0$, $\delta > 0$ depending only on K such that $|\zeta(2\sigma-k-1)c(d,\sigma,k)| \le M|d|^{\delta}$ if $d \in \mathbb{Z} \setminus \{0\}$, $\sigma \in K$ (cf. [28, 32]).

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[7] [4, Lemma 5.2, p. 68] treats the case $\widetilde{\Gamma_0(4)}$. A direct computation shows that this relation holds for G.

The function $\tau_d(v,\alpha,\beta)$ has been well studied in [25, 32, 33]. It can be written using the Whittaker function $W_{\alpha,\mu}(v)$ (cf. [20, Lemma 5, p. 196]). The function $W_{\alpha,\mu}(v)$ $(v>0)$ satisfies a differential equation with respect to v, is holomorphic for all $(\alpha,\mu)\in\mathbb{C}^2$ and is of rapid decay as $v\to\infty$, uniformly in $(\alpha,\mu)\in K'$, where $K'\subset\mathbb{C}^2$ is any compact subset. For convenience, we record here the relation between $W_{\alpha,\mu}(v)$, the ω-function in [25, §7.2] and the V-function in [33, Appendix A3] in the form

$$\begin{aligned} W_{\alpha,\mu}(v) &= v^{\alpha}e^{-v/2}\omega(v;1/2+\alpha+\mu;1/2-\alpha+\mu) \\ &= v^{\alpha}V(v;1/2+\alpha+\mu;1/2-\alpha+\mu). \end{aligned}$$

A summary about these facts on $W_{\alpha,\mu}(v)$ are written, for example, in [20, §3], [28, §3.1] based on the works [25, 32, 33]. See [38, §4] and [28, §2.5] for the facts on $L_D(s)$. In this paper, it is sufficient to consider the case $-k+2\Re(\sigma)-4>0$, that is, the relevant series $F(k,\sigma,\tau)$ are absolutely convergent.

4.3. Specializations of $\boldsymbol{F(k,\sigma,\tau)}$. Let k be an odd integer such that $k\geq 5$. Let us define[8]

$$H_{\frac{k}{2}}(\tau) := F(-k,0,\tau), \quad P_{H_{\frac{k}{2}}}(\tau) := F(k-4,k-2,\tau).$$

The series defining $H_{\frac{k}{2}}$, $P_{H_{\frac{k}{2}}}$ are absolutely convergent, and we have that $H_{\frac{k}{2}}|[\widetilde{\gamma}]_{\frac{k}{2}} = H_{\frac{k}{2}}$ and $P_{H_{\frac{k}{2}}}|[\widetilde{\gamma}]_{2-\frac{k}{2}} = P_{H_{\frac{k}{2}}}$ for any $\gamma\in\Gamma_0(4)$. It is not difficult to see that $H_{\frac{k}{2}}\in M^+_{\frac{k}{2}}$ ([8], [19, §4.2]), and that $P_{H_{\frac{k}{2}}}\in H^!_{2-\frac{k}{2}}(\Gamma_0(4))$ satisfies the "plus condition" (cf. [4, Theorem 6.15, p. 104]).

Using the intertwining relation between the differential operator L and the action of G mentioned at the end of Section 4.1 combined with the fact that $\xi_{2-\frac{k}{2}}$ is antilinear, we see that (cf. [4, Theorem 6.15, p. 104])

$$\xi_{2-\frac{k}{2}}(P_{H_{\frac{k}{2}}}) = (k/2-1)\,H_{\frac{k}{2}}.$$

Indeed, it is easy to confirm that one can differentiate the definition of $E(k-4,k-2,\tau)$ term by term, and consequently we have $\xi_{2-\frac{k}{2}}(E(k-4,k-2,\tau)) = \left(\frac{k}{2}-1\right)E(-k,0,\tau)$, $\xi_{2-\frac{k}{2}}(G(k-4,k-2,\tau)) = -\left(\frac{k}{2}-1\right)G(-k,0,\tau)$, where $G(k,\sigma,\tau) := E(k,\sigma,\tau)|[\rho]_{-\frac{k}{2}}(-i)^{k/2}$.

[8] Note that the Eisenstein series $H_{\frac{k}{2}}$ defined in [19, §4.2] is $\zeta(2-k)F(-k,0,\tau)$, which coincides with $G_{\frac{k}{2}}$ in Section 2.2 of this paper.

Since $\tau_d\left(v, \frac{k}{2}, 0\right) = 0$ for any integer $d \leq 0$ (cf. [20, Lemma 5, p. 196]), the function $H_{\frac{k}{2}}$ has the Fourier expansion

$$H_{\frac{k}{2}}(\tau) = 1 + \sum_{d=1}^{\infty} c(d, 0, -k)e(du)\tau_d\left(v, \frac{k}{2}, 0\right),$$

and the function $P_{H_{\frac{k}{2}}}$ has the Fourier expansion

$$P_{H_{\frac{k}{2}}}(\tau) = v^{\frac{k}{2}-1} + v^{\frac{k}{2}-1} \sum_{d=-\infty}^{\infty} c(d, k-2, k-4)e(du)\tau_d\left(v, 1, \frac{k}{2} - 1\right).$$

Suppose that $D \in \mathbb{Z} \setminus \{0\}$ and $D \equiv 0, 1 \pmod 4$. By [28, Proposition 1, p. 144], we have $|L_D(s)| \leq \zeta(1+\eta)^2\zeta(1+2\eta)$ for $s = 1 + \eta + it$ ($\eta > 0$, $t \in \mathbb{R}$). Thus, for a given odd integer $k \geq 5$, there exists a constant $M > 0$ such that $|c(d, 0, -k)| \leq M$, $|c(d, k-2, k-4)| \leq M$ for all $d \in \mathbb{Z} \setminus \{0\}$.

Remark 4.1. The preimage of $H_{\frac{k}{2}}$ under the operator $\xi_{2-\frac{k}{2}}$ is also constructed in Wagner [35] by a different way for another purpose (cf. [4, the first Remark, p. 110]). Our construction should be consistent with Wagner's by means of the functional equation of $F(k, \sigma, \tau)$.

4.4. Computation of the inner product. Put $g_0 := \left(\begin{smallmatrix} 0 & -1/2 \\ 2 & 0 \end{smallmatrix}\right)$, $g_{1/2} := \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, and define three lines in $\mathbb{H}$ by $C_\infty : \Im(\tau) = Y$, $C_0 : \left|\tau - \frac{i}{8Y}\right| = \frac{1}{8Y}$, $C_{1/2} : \left|\tau - (\frac{1}{2} + \frac{i}{8Y})\right| = \frac{1}{8Y}$, where $Y > 2$. Then $g_0 \cdot i\infty = 0$, $g_{1/2} \cdot i\infty = 1/2$, and g_0 maps C_∞ to C_0, while $g_{1/2}$ maps C_∞ to $C_{1/2}$. Let $D(4)_Y$ be the truncated fundamental domain with respect to $\Gamma_0(4)$ as described in the paper [10] by Duke–Imamoḡlu–Tóth. It is obtained by cutting off neighborhoods of each cusps $i\infty$, 0, $1/2$ by the lines C_∞, C_0, $C_{1/2}$ from the specific fundamental domain for $\Gamma_0(4)$ used in [10]. By [10, Lemma 3.2] and the facts summarized in Sections 4.2 and 4.3 of the present paper, for $Y > 2$, one deduces

$$\begin{aligned}
&\int_{D(4)_Y} H_{\frac{k}{2}}(\tau)\overline{H_{\frac{k}{2}}(\tau)}v^{\frac{k}{2}-2}\,\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{\frac{k}{2}-1}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY}\left[H_{\frac{k}{2}}(\tau)P_{H_{\frac{k}{2}}}(\tau) + \frac{1}{2}H^{\mathrm{e}}_{\frac{k}{2}}(\tau)P^{\mathrm{e}}_{H_{\frac{k}{2}}}(\tau) + \frac{1}{2}H^{\mathrm{o}}_{\frac{k}{2}}(\tau)P^{\mathrm{o}}_{H_{\frac{k}{2}}}(\tau)\right]\mathrm{d}\tau,
\end{aligned}$$

where the path of the last integral is the horizontal line from $-\frac{1}{2} + iY$ to $\frac{1}{2} + iY$, and for

$$f(\tau) = \sum_{d \in \mathbb{Z}} b(d, v)e(du)$$

satisfying the "plus condition", we put

$$f^{\mathrm{e}}(\tau) = \sum_{d\equiv 0 \pmod 2} b\left(d, \frac{v}{4}\right) e\left(d\frac{u}{4}\right),$$

$$f^{\mathrm{o}}(\tau) = \sum_{d\equiv 1 \pmod 2} b\left(d, \frac{v}{4}\right) e\left(\frac{d}{8}\right) e\left(d\frac{u}{4}\right),$$

according to [10].

Inserting the Fourier expansions of $H_{\frac{k}{2}}$, $P_{H_{\frac{k}{2}}}$ given in Section 4.3, the integral of $H_{\frac{k}{2}}P_{H_{\frac{k}{2}}}$ is expressed as

$$\begin{aligned}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} & H_{\frac{k}{2}}(\tau)P_{H_{\frac{k}{2}}}(\tau)\,\mathrm{d}\tau \\ &= Y^{\frac{k}{2}-1} + Y^{\frac{k}{2}-1}c(0,k-2,k-4)\tau_0\left(Y,1,\frac{k}{2}-1\right) \\ &\quad + \sum_{\substack{d\geq 1 \\ (-1)^{(k-1)/2}d\equiv 0,1 \pmod 4}} c(d,0,-k)c(-d,k-2,k-4)T_d(Y,k),\end{aligned}$$

where $T_d(Y,k) := Y^{\frac{k}{2}-1}\tau_d\left(Y,\frac{k}{2},0\right)\tau_{-d}\left(Y,1,\frac{k}{2}-1\right)$. Analogous formulas hold for the integrals of $H^{\mathrm{e}}_{\frac{k}{2}}P^{\mathrm{e}}_{H_{\frac{k}{2}}}$ and $H^{\mathrm{o}}_{\frac{k}{2}}P^{\mathrm{o}}_{H_{\frac{k}{2}}}$ as

$$\begin{aligned}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} & H^{\mathrm{e}}_{\frac{k}{2}}(\tau)P^{\mathrm{e}}_{H_{\frac{k}{2}}}(\tau)\,\mathrm{d}\tau \\ &= \left(\frac{Y}{4}\right)^{\frac{k}{2}-1} + \left(\frac{Y}{4}\right)^{\frac{k}{2}-1} c(0,k-2,k-4)\tau_0\left(\frac{Y}{4},1,\frac{k}{2}-1\right) \\ &\quad + \sum_{\substack{d\geq 1 \\ (-1)^{(k-1)/2}d\equiv 0 \pmod 4}} c(d,0,-k)c(-d,k-2,k-4)T_d\left(\frac{Y}{4},k\right),\end{aligned}$$

$$\begin{aligned}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY} & H^{\mathrm{o}}_{\frac{k}{2}}(\tau)P^{\mathrm{o}}_{H_{\frac{k}{2}}}(\tau)\,\mathrm{d}\tau \\ &= \sum_{\substack{d\geq 1 \\ (-1)^{(k-1)/2}d\equiv 1 \pmod 4}} c(d,0,-k)c(-d,k-2,k-4)T_d\left(\frac{Y}{4},k\right).\end{aligned}$$

Applying the estimation of $c(d,0,-k)$, $c(d,k-2,k-4)$ indicated at the end of Section 4.3 and the formula $\tau_0\left(Y,1,\frac{k}{2}-1\right) = i^{\frac{k}{2}-2}2^{2-\frac{k}{2}}\pi\cdot Y^{1-\frac{k}{2}}$ together with an asymptotic nature of $\tau_d(v,\alpha,\beta)$ as $v\to\infty$ (cf. [20, Lemmas 5 and 6, p. 196]), we find that the term $\frac{1+2^{1-k}}{\frac{k}{2}-1}Y^{\frac{k}{2}-1}$ gives the reason that the integral $\int_{D(4)_Y} H_{\frac{k}{2}}(\tau)\overline{H_{\frac{k}{2}}(\tau)}v^{\frac{k}{2}-2}\,\mathrm{d}u\mathrm{d}v$ diverges as $Y \to \infty$. Note here that the quantity $\frac{1+2^{1-k}}{\frac{k}{2}-1}Y^{\frac{k}{2}-1}$ arises as a part of the contributions other than the infinite sums over $d \geq 1$ in the above integral formulas.

Hence, we see that the value $\langle H_{\frac{k}{2}}, H_{\frac{k}{2}}\rangle_4^{\mathrm{reg}}$ (cf. Definition 2.3) is finite and is given by

$$\langle H_{\frac{k}{2}}, H_{\frac{k}{2}}\rangle_4^{\mathrm{reg}} = \frac{3}{2\left(\frac{k}{2}-1\right)}c(0,k-2,k-4)\cdot i^{\frac{k}{2}-2}2^{2-\frac{k}{2}}\pi.$$

The explicit Fourier expansion of $F(k,\sigma,\tau)$ in Section 4.2 tells us that[9]

$$c(0,k-2,k-4) = 2^{\frac{3}{2}-k}e^{(-1)^{(k-3)/2}(\pi i/4)}\frac{\zeta(k-2)}{\zeta(k-1)},$$

and we obtain

$$\frac{1}{6}\langle H_{\frac{k}{2}}, H_{\frac{k}{2}}\rangle_4^{\mathrm{reg}} = \frac{2^{\frac{3}{2}(1-k)}\pi}{\frac{k}{2}-1}i^{\frac{k}{2}-2}e^{(-1)^{(k-3)/2}(\pi i/4)}\frac{\zeta(k-2)}{\zeta(k-1)}.$$

Notice that $G_{k+\frac{1}{2}}$ introduced in Section 2.2 can be written as $G_{k+\frac{1}{2}} = \zeta(1-2k)H_{k+\frac{1}{2}}$ for any integer $k \geq 2$. So, we have computed $\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}$.

5. Proof of Theorem 2.4

Let $k \geq 2$ be an integer. Let D be a fundamental discriminant such that $(-1)^k D > 0$. The Eisenstein series $G_{k+\frac{1}{2}} \in M^+_{k+\frac{1}{2}}$ in Section 2.2 has the Fourier expansion

$$\begin{aligned} G_{k+\frac{1}{2}}(\tau) &= \zeta(1-2k)H_{k+\frac{1}{2}}(\tau) \\ &= \sum_{\substack{d \geq 0 \\ (-1)^k d \equiv 0,1 \pmod 4}} L_{(-1)^k d}(1-k)e(d\tau). \end{aligned}$$

The well-known fact is that the L-function of $G_{2k} \in M_{2k}$ is given by

$$L(G_{2k}, D, s) = \sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\sigma_{2k-1}(n)n^{-s} = L_D(s)L_D(s+1-2k).$$

By the functional equation of $L_D(s)$, we have

$$L(G_{2k}, D, k) = L_D(k)L_D(1-k) = h_{k+\frac{1}{2}}(|D|)^2\cdot \pi^{k-\frac{1}{2}}|D|^{\frac{1}{2}-k}\frac{\Gamma\left(\frac{1-k+\delta_D}{2}\right)}{\Gamma\left(\frac{k+\delta_D}{2}\right)},$$

[9]Also see [16, Proposition 2.3, p. 279], and use $\sqrt{2}e^{(-1)^{(k-3)/2}(\pi i/4)} = 1+i^{-k}$ for odd k.

where $\delta_D = 0$ if $D > 0$, and $\delta_D = 1$ if $D < 0$. Thus $\delta_D = 0$ if k is even, and $\delta_D = 1$ if k is odd. Recall the values

$$\langle G_{2k}, G_{2k} \rangle_1^{\mathrm{reg}} = (-1)^{k-1} \frac{\Gamma(2k)\Gamma(2k-1)\zeta(2k)\zeta(2k-1)}{2^{6k-2}\pi^{4k-1}},$$

$$\frac{1}{6}\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}} = \zeta(1-2k)^2 \cdot \frac{2^{-3k+1}\pi}{2k-1} i^{k-\frac{3}{2}} e^{(-1)^{k-1}(\pi i/4)} \frac{\zeta(2k-1)}{\zeta(2k)}$$

given in Sections 3 and 4.4. It follows from $\zeta(2k)\Gamma(2k)/\zeta(1-2k) = (-1)^k \pi^{2k} 2^{2k-1}$ and $i^{k-\frac{3}{2}} e^{(-1)^{k-1}(\pi i/4)} = -(-1)^{\lfloor \frac{k+1}{2} \rfloor}$ that

$$\frac{\langle G_{2k}, G_{2k} \rangle_1^{\mathrm{reg}}}{\frac{1}{6}\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}}} = 2^{k-1}(-1)^{k+\lfloor \frac{k+1}{2} \rfloor}.$$

We complete the proof of Theorem 2.4 by using

$$\frac{\Gamma\left(\frac{1-k+\delta_D}{2}\right)}{\Gamma\left(\frac{k+\delta_D}{2}\right)} = \pi^{\frac{1}{2}} \frac{2^{k-1}}{\Gamma(k)} (-1)^{k+\lfloor \frac{k+1}{2} \rfloor}.$$

6. Comparison with Tsuyumine's regularized pairing

Let $k \geq 2$ be an integer. In [34, p. 5], Tsuyumine defined the regularized Petersson pairing $\langle\langle f, g \rangle\rangle_{\Gamma_0(4)}$ on $M_{k+\frac{1}{2}}$.[10] It agrees with our Definition 2.3; $\langle\langle f, g \rangle\rangle_{\Gamma_0(4)} = \langle f, g \rangle_4^{\mathrm{reg}}$ for $f, g \in M_{k+\frac{1}{2}}^+$. Here we chose $T^{(i\infty)} = T^{(1/2)} = Y$, $T^{(0)} = 4Y$ in the first displayed formula in the proof of [34, Lemma 2.2, p. 14]. Notice that the constant terms of $H_{k+\frac{1}{2}}$ in the Fourier expansions at each cusps, which are required to describe $\langle\langle f, g \rangle\rangle_{\Gamma_0(4)}$, can be determined by Kohnen's method [21] (cf. [10, §3], [27, Proposition 5, p. 10]).

By [34, (25), p. 16] (resp. the first sentence of [34, p. 19]), Tsuyumine's $E_{k+1/2, \chi_{-4}^k}(\tau, 0)$ (resp. $E_{k+1/2}^{\chi_{-4}^k}(\tau, 0)$) coincides with our $E(-2k-1, 0, \tau)$ (resp. $\tau^{-k-1/2} E(-2k-1, 0, -1/(4\tau))$) defined in Section 4.2 of this paper. Thus, by definition, we have

$$H_{k+\frac{1}{2}}(\tau) = E_{k+1/2, \chi_{-4}^k}(\tau, 0) + 2^{-2k-1/2} E_{k+1/2}^{\chi_{-4}^k}(\tau, 0) i^{(-1)^k/2}.$$

In [34, p. 35-36], one finds the values

$$\langle\langle E_{k+1/2, \chi_{-4}^k}(\tau, 0), E_{k+1/2, \chi_{-4}^k}(\tau, 0) \rangle\rangle_{\Gamma_0(4)} = -(-1)^{\frac{k(k+1)}{2}} \beta_k,$$

$$\langle\langle E_{k+1/2}^{\chi_{-4}^k}(\tau, 0), E_{k+1/2}^{\chi_{-4}^k}(\tau, 0) \rangle\rangle_{\Gamma_0(4)} = -(-1)^{\frac{k(k+1)}{2}} 2^{2k+1} \beta_k,$$

$$i^{(-1)^k/2} \langle\langle E_{k+1/2}^{\chi_{-4}^k}(\tau, 0), E_{k+1/2, \chi_{-4}^k}(\tau, 0) \rangle\rangle_{\Gamma_0(4)} = -(-1)^{\lfloor \frac{k+1}{2} \rfloor} 2^{3/2} (2^{2k-1} - 1) \beta_k,$$

[10] We denote Tsuyumine's paring by $\langle\langle f, g \rangle\rangle_{\Gamma_0(4)}$ instead of his notation $\langle f, g \rangle_{\Gamma_0(4)}$ in [34].

where we put

$$\beta_k := \frac{2^{-k+2}\pi\zeta(2k-1)}{(2^{2k}-1)(2k-1)\zeta(2k)}.$$

Using the obvious relations

$$\langle\langle f, g\rangle\rangle_{\Gamma_0(4)} = \overline{\langle\langle g, f\rangle\rangle_{\Gamma_0(4)}},$$

$$\langle\langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle\rangle_{\Gamma_0(4)} = \alpha_1\langle\langle f_1, g\rangle\rangle_{\Gamma_0(4)} + \alpha_2\langle\langle f_2, g\rangle\rangle_{\Gamma_0(4)} \quad (\alpha_1, \alpha_2 \in \mathbb{C}),$$

together with $(-1)^{\frac{k(k+1)}{2}} = (-1)^{\lfloor\frac{k+1}{2}\rfloor}$, we obtain

$$\langle\langle H_{k+\frac{1}{2}}, H_{k+\frac{1}{2}}\rangle\rangle_{\Gamma_0(4)} = -(-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{3\cdot 2^{-3k+2}\pi\zeta(2k-1)}{(2k-1)\zeta(2k)}.$$

This is consistent with our result for $\langle H_{k+\frac{1}{2}}, H_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}$.

Remark 6.1. It is common in both methods that the values arise from the constant terms of non-holomorphic Eisenstein series. Because of the vanishing of the relevant part, these values are invisible when we look at only holomorphic Eisenstein series.

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