JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Tomohiro YAMADA

On finiteness of odd superperfect numbers

Tome 32, nº 1 (2020), p. 259-274.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2020__32_1_259_0>

© Société Arithmétique de Bordeaux, 2020, tous droits réservés.

L'accès aux articles de la revue « Journal de Théorie des Nombres de Bordeaux » (http://jtnb.centre-mersenne.org/), implique l'accord avec les conditions générales d'utilisation (http://jtnb.centre-mersenne.org/legal/). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.centre-mersenne.org/

Journal de Théorie des Nombres de Bordeaux **32** (2020), 259–274

On finiteness of odd superperfect numbers

par Tomohiro YAMADA

Résumé. On montre de nouveaux résultats sur l'équation $\sigma(N) = aM$, $\sigma(M) = bN$. On en déduit, comme corollaire, qu'il n'existe qu'un nombre fini de nombres impairs superparfaits ayant un nombre fixé de facteurs premiers distincts.

Abstract. Some new results concerning the equation $\sigma(N) = aM$, $\sigma(M) = bN$ are proved. As a corollary, there are only finitely many odd superperfect numbers with a fixed number of distinct prime factors.

1. Introduction

As usual, $\sigma(N)$ shall denote the sum of positive divisors of a positive integer N and call a positive integer N to be perfect if $\sigma(N) = 2N$. Though it is not known whether or not an odd perfect number exists, many conditions which must be satisfied by such a number are known.

Analogous to this notion, Suryanarayana [17] called N to be superperfect if $\sigma(\sigma(N)) = 2N$. Suryanarayana showed that if N is even superperfect, then $N = 2^m$ with $2^{m+1} - 1$ prime, and if N is odd superperfect, then N must be square and have at least two distinct prime factors.

Dandapat, Hunsucker and Pomerance [3] showed that if $\sigma(\sigma(N)) = kN$ for some integer k and $\sigma(N)$ is a prime power, then N is even superperfect or $N = 21, k = 3$. Later Pomerance [12] called N to be super multiply perfect if $\sigma(\sigma(N)) = kN$ for some integer k and showed that if $p^m \mid \sigma(N)$ and $N \mid \sigma(p^m)$ for some prime power p^m, then $N = 2^{n-1}$ or $2^n - 1$ with $2^n - 1$ prime or $N = 15, 21, 1023$.

In the West Coast Number Theory Conference 2005, the author posed the question whether there exist only finitely many odd integers N such that $N \mid \sigma(\sigma(N))$ and $\omega(\sigma(N)) = s$ for each fixed s [19], where $\omega(n)$ denotes the number of distinct prime factors of n. The above-mentioned result of Dandapat, Hunsucker and Pomerance answers the special case $s = 1$ of this question affirmatively.

Manuscrit reçu le 28 juin 2019, révisé le 6 février 2020, accepté le 21 février 2020.

2020 *Mathematics Subject Classification.* 11A25, 11A05, 11D61, 11J86.

Mots-clefs. Odd perfect numbers, multiperfect numbers, superperfect number; the sum of divisors, arithmetic functions, exponential diophantine equations.

Concerning the unitary divisor sum $\sigma^*(N)$ (d is called a unitary divisor of N if $d \mid N$ and $d, N/d$ are coprime), the author already proved that $N = 9, 165$ are all the odd integers satisfying $\sigma^*(\sigma^*(N)) = 2N$ [20].

In this paper, although we cannot prove the above-mentioned conjectures, some results are proved. Before stating our results, we introduce the notation $C_i(\cdot)$ for $i = 0, 1, 2, \ldots$, each of which denotes some effectively computable positive constant depending only on its arguments.

Theorem 1.1. *If a quadruple of integers N, M, a, b satisfies $\sigma(N) = aM$, $\sigma(M) = bN$ and $\omega(\sigma(N)) \leq k$, then we have $a, b < C_0(k)$ for some effectively computable constant $C_0(k)$ depending only on k.*

Theorem 1.2. *Additionally to the condition described in Theorem 1.1, assume that both M and N are odd. Then each of M and N must have a prime factor smaller than some effectively computable constant $C_1(k)$ depending only on k.*

Remark 1.3. The additional condition would be necessary. Indeed, if we allow M or N to be even and take $N = 2^m, M = 2^{m+1}-1$, then Theorem 1.2 would imply that there exist only finitely many Mersenne primes, contary to the widely believed conjecture!

Theorem 1.4. *For any given integers a, b, k, k', there are only finitely many pairs of odd integers M, N satisfying $\sigma(N) = aM, \sigma(M) = bN$ with $\omega_1(N) \leq k'$ and $\omega(M) \leq k$, where $\omega_1(N)$ denotes the number of primes dividing N only once. Moreover, such integers M, N can be bounded by some effectively computable constant $C_2(a, b, k, k')$ depending only on a, b, k, k'.*

Using Suryanarayana's result that an odd superperfect number must be square, the latter result gives the following corollary, which implies our conjecture in the case $\sigma(\sigma(N)) = 2N$. Moreover, we observe that if N is an odd superperfect number, then $\sigma(N)$ must be a product of a square and a prime power p^e with $p \equiv e \equiv 1 \pmod 4$ and $\sigma(\sigma(\sigma(N))) = \sigma(2N) = 3\sigma(N)$. Hence, we have the following finiteness result.

Corollary 1.5. *For each fixed k, there exists some effectively computable constant B_k such that, if N is an odd superperfect number with either $\omega(N) \leq k$ or $\omega(\sigma(N)) \leq k$, then $N \leq B_k$.*

The corresponding result for odd perfect numbers has been known for years. Dickson [4] proved that there exist only finitely many odd perfect numbers N with $\omega(N) \leq k$ for each fixed k and an effective upper bound was given by Pomerance [13], improved by Nielsen [10, 11].

It will be relatively easy to show that there exist only finitely many odd superperfect numbers N with *both* $\omega(N)$ and $\omega(\sigma(N))$ fixed, using Dickson's argument. However, if we only require that $\omega(N)$ *or* $\omega(\sigma(N))$ is fixed, we have only known that $\omega(N) \geq 2$ from [18] and $\omega(\sigma(N)) \geq 2$ from [3], which is implied by the result of [12] mentioned above. Our corollary states that, even under this restriction, we can obtain an effective upper bound for an odd superperfect number N.

Our argument in this paper is based upon the one in [20]. In [20], we used the fact that if $\sigma^*(\sigma^*(N)) = 2N$, then N must be factored into $N = \prod_i p_i^{e_i}$ with $p_i^{e_i} + 1 = 2^{a_i} q^{b_i}$ for some integers a_i, b_i, q. This means that $p_i^{e_i}$'s must be distributed very thin and therefore the product of $\sigma^*(p_i^{e_i})/p_i^{e_i}$'s must be small.

However, we deal with the σ function in this paper. For a small prime p, $\sigma(p^e)/p^e$ must be fairly large and therefore our argument from [20] does not work.

We introduce some preliminary notations. In order to prove Theorems 1.1, 1.2 and 1.4, we consider slightly more general situation. Assume that N is an integer such that $\omega(\sigma(N)) = k$ and we let $q_1 < q_2 < \cdots < q_k$ to be the prime divisors of $\sigma(N)$. For each $1 \leq r \leq k$ and a prime power l, let $S_{r,l} = S_{r,l}(q_1, q_2, \ldots, q_r)$ denote the set of prime divisors p of N such that $p^e \parallel N$ with $l \mid (e+1)$ and

$$\frac{p^l - 1}{p - 1} = \sigma(p^{l-1}) = \prod_{i=1}^{r} q_i^{a_i} \tag{1.1}$$

for some integers $a_i (1 \leq i \leq r)$ with $a_r \neq 0$ and let $S_r = \bigcup_l S_{r,l}$, where l runs over all prime powers. Clearly, each prime divisor of N must belong to a set $S_{r,l}$ for some $1 \leq r \leq k$ and a prime power l.

In Section 4, using a lower bound for linear forms of logarithms, we shall show that each S_r contains at most r small primes. Combined with Lemma 3.3, which states that the contribution of large prime factors to the size of $\sigma(M)\sigma(N)/MN$ must be very small, we shall prove the following fact.

Theorem 1.6. *Let N be an integer such that $\omega(\sigma(N)) = k$ and let $q_1 < q_2 < \cdots < q_k$ be the prime divisors of $\sigma(N) = aM$ as introduced above. Then, for every $r = 1, 2, \ldots, k$, N has at most r prime factors in S_r below $C_4(r, q_r) = \exp(C_3(r)(\log q_r / \log\log q_r)^{1/2(r+1)})$ and*

$$\sum_{\substack{p \in S_r, \\ p \geq C_4(r, q_r)}} \frac{1}{p} < \exp\left(-C_5(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right). \tag{1.2}$$

This theorem allows us to overcome the above-mentioned obstacle. Indeed, it is not difficult to derive Theorem 1.1 from Theorem 1.6, as shown in Section 5. With the aid of a diophantine inequality shown in Section 6, we shall prove Theorems 1.2 and 1.4.

2. Preliminary lemmas

In this section, we introduce some preliminary lemmas. The first lemma is a special case of Matveev [9, Theorem 2.2], which gives a lower bound for linear forms of logarithms. We use this lemma to prove our gap principle in Section 4. The second lemma describes an elementary property of values of cyclotomic polynomials.

Lemma 2.1. *Let $a_1, a_2, \ldots, a_n$ be positive integers with $a_1 > 1$. For each $j = 1, \ldots, n$, let $A_j \geq \max\{0.16, \log a_j\}$. Let $b_1, b_2, \ldots, b_n$ be arbitrary integers.*

Put

$$\begin{aligned} B &= \max\{1, |b_1|\, A_1/A_n, |b_2|\, A_2/A_n, \ldots, |b_n|\}, \\ \Omega &= A_1 A_2 \ldots A_n, \\ C'(n) &= 4.4n + 5.5 \log n + 7, \qquad (2.1) \\ C(n) &= \frac{16}{n!} e^n (2n+3)(n+2)(4(n+1))^{n+1} \left(\frac{1}{2} en\right) C'(n) \end{aligned}$$

and

$$\Lambda = b_1 \log a_1 + \cdots + b_n \log a_n. \tag{2.2}$$

Then we have $\Lambda = 0$ or

$$\log|\Lambda| > -C(n)(1 + \log 3 - \log 2 + \log B) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{2.3}$$

Remark 2.2. The assumption that $a_1 > 1$ is added in order to ensure that $\log a_1, \log a_2, \ldots, \log a_r$ are linearly independent over the integers for some $r (1 \leq r \leq n)$. We note that we do not need recent results for linear forms in logarithms. We can see that a lower bound of the form $\log|\Lambda| > -B^{1/g(n)} \log^{f(n)} A$, where $f(n), g(n)$ are effectively computable functions of n such that $g(n) > 1$, is strong enough for our purpose. Such an estimate would increase the right hand side of (4.2) but still give an estimate that $\log n_r < \log^{h(r)} m_{r+1}$ for some effectively computable function $h(r)$. Thus even an old estimate such as Fel'dman [7] suffices.

Lemma 2.3. *If a, l are positive integers with $a \geq 2, l \geq 3$ and $(a, l) \neq (2, 6)$, then $(a^l - 1)/(a - 1)$ must have at least $\tau(l) - 1$ distinct prime factors, where $\tau(l)$ denotes the number of divisors of l. Moreover, at least one of such prime factors is congruent to $1 \pmod{l}$.*

Proof. A well known result of Zsigmondy [21] states that if $a \geq 2, n \geq 3$ and $(a, n) \neq (2, 6)$, then $a^n - 1$ has a prime factor which does not divide $a^m - 1$ for any $m < n$. Applying this result to each divisor $d > 1$ of l, we see that $(a^l - 1)/(a - 1)$ must have at least $\tau(l) - 1$ prime factors. In particular, applying with $n = l$, we obtain a prime factor q such that $a \pmod{q}$ has order l and therefore $q \equiv 1 \pmod{l}$. □

3. The distribution of large primes in $S_{r,l}$

In this section, we shall give an upper bound for the sum $\sum_{p\in S_{r,l},\, p>X} 1/p$ for each fixed r, l.

Lemma 3.1. *Let p_0, p_1, p_2 be distinct primes with p_0 odd, l and f be positive integers and put $H_i = \lfloor f\log p_0/\log p_i\rfloor$ for $i = 1, 2$. If the congruence*

$$p_i^l \equiv 1 \pmod{p_0^f} \tag{3.1}$$

holds for $i = 1, 2$, then

$$\frac{1}{2}H_1H_2 \le \gcd(l, p_0^{f-1}(p_0-1)). \tag{3.2}$$

Proof. It is clear that $p_1^{a_1}p_2^{a_2}$ takes distinct values modulo p_0^f for all non-negative integers a_1 and a_2 with $0 \le a_1\log p_1 + a_2\log p_2 < f\log p_0$. So that $p_1^{a_1}p_2^{a_2}$ takes at least $H_1H_2/2$ distinct values modulo p_0^f. But these can take at most $\gcd(l, p_0^{f-1}(p_0-1))$ distinct values since both p_1 and p_2 have residual orders dividing $\gcd(l, p_0^{f-1}(p_0-1))$ modulo p_0^f by (3.1). Hence, we obtain $H_1H_2/2 \le \gcd(l, p_0^{f-1}(p_0-1))$. □

Lemma 3.2. *Let p_0, p_1, p_2 be distinct primes with $p_2 > p_1$ and $p_0 > 2$ and q, s be positive integers. If l is an integer greater than $5s^2$ and there are integers f_1, f_2 such that $p_0^{f_i} \mid \sigma(p_i^{l-1})$ and $p_0^{f_i} \ge \sigma(p_i^{l-1})^{1/s}$ for $i = 1, 2$, then*

$$\log p_2 > \frac{5}{4}\log p_1. \tag{3.3}$$

Proof. Assume that p_1, p_2 are two distinct primes satisfying the assumption in the lemma but $\log p_2 \le (5/4)\log p_1$, contrary to the statement of the lemma. Let $f = \min\{f_1, f_2\}$ and $H_i = \lfloor f\log p_0/\log p_i\rfloor$. We observe that $p_0^{f_i} \ge \sigma(p_i^{l-1})^{1/s} > p_i^{(l-1)/s}$ for $i = 1, 2$ and therefore $f\log p_0 > (l-1)\log p_1/s$. Since we have assumed that $l > 5s^2$, we have

$$H_1 \ge \left\lfloor \frac{l-1}{s}\right\rfloor \ge 5s \tag{3.4}$$

and

$$H_2 \ge \left\lfloor \frac{(l-1)\log p_1}{s\log p_2}\right\rfloor \ge \left\lfloor \frac{4(l-1)}{5s}\right\rfloor \ge 4s. \tag{3.5}$$

By the definition of H_i, we can easily see that

$$H_1 > \frac{5sf\log p_0}{(5s+1)\log p_1} \ge \frac{5f\log p_0}{6\log p_1} \tag{3.6}$$

and

$$H_2 > \frac{4sf\log p_0}{(4s+1)\log p_2} \ge \frac{4f\log p_0}{5\log p_2}. \tag{3.7}$$

Hence, Lemma 3.1 gives

$$l\log p_1\log p_2>\frac{1}{3}f^2\log^2 p_0>\frac{(l-1)^2}{3s^2}\log^2 p_1. \tag{3.8}$$

Recalling the assumption that $l>5s^2$, we have

$$\frac{\log p_2}{\log p_1}>\frac{(l-1)^2}{3ls^2}\geq\frac{25s^2}{3(5s^2+1)}\geq\frac{25}{18}>\frac{5}{4}, \tag{3.9}$$

which contradicts to the assumption. Hence, we have $p_2>p_1^{5/4}$. □

Using this result, we obtain the following inequality.

Lemma 3.3. *For any set $S_{r,l}$ defined in the introduction and $X>2$, we have*

$$\sum_{p>X,p\in S_{r,l}}\frac{1}{p}<\frac{C_6(r)\log^r X}{X}. \tag{3.10}$$

Remark 3.4. It is well known that, for fixed integers $l>2,r$ and fixed primes $q_1,q_2,\ldots,q_r$, there exist only finitely many primes p and integers $a_1,a_2,\ldots,a_r$ satisfying (1.1). Combining Coates' theorem [2] and Schinzel's theorem [14], it follows that such integers and, consequently, the elements of $S_{r,l}$ are bounded by an effectively computable constant depending on l and the q_i's. For details of the history of the largest prime factor of polynomial values, see Shorey and Tijdeman's book [16, Chapter 7]. Furthermore, two theorems of Evertse [5, 6] imply that $|S_{r,l}|$ is bounded by an effectively computable constant depending on r,l. However, in this paper, we need a result depending only on r.

Proof. First we note that $S_{r,l}$ can be divided into r sets $S_{r,l,j}(1\leq j\leq r)$ so that if $p\in S_{r,l,j}$, then $q_j^{f_j}\mid\sigma(p^{l-1})$ for an integer f_j such that $q_j^{f_j}\geq\sigma(p^{l-1})^{1/r}$.

Assume that $l>5r^2$. If $p_1<p_2$ are two primes belonging to $S_{r,l,j}$, then $\log p_2>(5/4)\log p_1$ by Lemma 3.2. Hence, we obtain

$$\sum_{\substack{p>X,\\ p\in S_{r,l,j}}}\frac{1}{p}<\sum_{i=0}^{\infty}\frac{1}{X^{(5/4)^i}}<\frac{4}{X} \tag{3.11}$$

and therefore

$$\sum_{\substack{p>X,\\ p\in S_{r,l}}}\frac{1}{p}<\sum_{j=1}^{r}\sum_{\substack{p>X,\\ p\in S_{r,l,j}}}\frac{1}{p}<\frac{2r}{X}<\frac{C_6(r)\log^r X}{X}. \tag{3.12}$$

Next assume that $l\leq 5r^2$. It is clear that the number of primes $p<x$ belonging to $S_{r,l}$ is at most $(l\log x)^r/\prod_{i=1}^r\log q_i$ and partial summation

gives

$$(3.13) \qquad \sum_{p>X, p\in S_{r,l}} \frac{1}{p} < \frac{1}{X} + \int_X^\infty \frac{(l\log t)^r \mathrm{d}t}{t^2 \prod_{i=1}^r \log q_i} < C_7(r)\frac{(l\log X)^r}{X}.$$

Since $l \le 5r^2$, we have $(l\log X)^r \le (5r^2\log X)^r$ and therefore

$$(3.14) \qquad \sum_{p>X, p\in S_{r,l}} \frac{1}{p} < C_6(r)\frac{\log^r X}{X}. \qquad \square$$

4. Main theory – proof of Theorem 1.6

In this section, we shall prove Theorem 1.6, which plays the most essential role in this paper. We begin by proving the following lemma.

Lemma 4.1. *Let $r, l_1, \ldots, l_{r+1}$ and $n_1 < n_2 < \cdots < n_r$ be positive integers. Let $m_1 < m_2 < \cdots < m_{r+1}$ be distinct primes. Assume that there exist integers $a_{ij} (1 \le i \le s+1, 1 \le j \le s)$ such that*

$$(4.1) \qquad \frac{m_j^{l_j}-1}{m_j-1} = \prod_{i=1}^r n_i^{a_{ij}}$$

for $j = 1, 2, \ldots, r+1$ and $a_{ij} > 0$ for the index i for which n_i assumes the maximum. Then we have

$$(4.2) \qquad \log n_r < C_8(r)(\log^{2(r+1)} m_{r+1})(\log\log m_{r+1}).$$

Proof. We put

$$(4.3) \quad \Lambda_j = -l_j \log m_j + \log(m_j - 1) + \sum_i a_{ij}\log n_i = \log\left(\frac{m_j^{l_j}-1}{m_j^{l_j}}\right) \ne 0$$

for each $j = 1, 2, \ldots, r+1$. Since $\Lambda_j \ne 0$, using Matveev's lower bound given in Lemma 2.1 we obtain

$$(4.4) \qquad \log|\Lambda_j| > -C(r+2)\log\left(\frac{3el_j\log m_j}{2\log n_r}\right)\log^2 m_j \prod_{i=1}^r \log n_i.$$

Observing that $|\Lambda_j| < 1/(m_j^{l_j}-1)$, we have

$$(4.5) \qquad l_j < C_9(r)\log m_j \log^r n_r \log(\log^2 m_j \log^{r-1} n_r)$$

and therefore

$$(4.6) \qquad a_{ij} < l_j\frac{\log m_j}{\log n_i} < C_{10}(r)\log^2 m_j \log^r n_r \log(\log m_j \log n_r).$$

Putting $A = C_{10}(r)\log^2 m_{r+1}\log^r n_r \log(\log m_{r+1}\log n_r)$, we see that (4.1) ensures the existence of integers $g_1, \dots, g_{r+1}$ not all zero with absolute values at most $((r+1)^{1/2}A)^r$ such that

$$\prod_{j=1}^{r+1}\left(\frac{m_j^{l_j}-1}{m_j-1}\right)^{g_j}=1 \tag{4.7}$$

by an improved form of Siegel's lemma (the original form of Siegel's lemma gives the upper bound $1+((r+1)A)^r$. For detail, see [15, Chapter I]).

We put

$$\Lambda=\sum_{j=1}^{r+1} g_j l_j \log m_j - g_j \log(m_j-1). \tag{4.8}$$

Since

$$\Lambda=\sum_{j=1}^{r+1} g_j \log\left(\frac{m_j^{l_j}}{m_j-1}\right)=\sum_{j=1}^{r+1} g_j \log\left(\frac{m_j^{l_j}}{m_j^{l_j}-1}\right) \tag{4.9}$$

and $m_j^{l_j}-1$ must be divisible by n_r by assumption, we have

$$\begin{aligned}|\Lambda| &< \sum_{j=1}^{r+1}\frac{g_j}{m_j^{l_j}-1}\le\frac{2(r+1)(rA)^r}{n_r}\\ &< 2(r+1)^2(rC_{10}(r))^r\times\frac{\log^{2r} m_{r+1}\log^{r^2} n_r \log^r(\log m_{r+1}\log n_r)}{n_r}.\end{aligned} \tag{4.10}$$

We observe that Λ does not vanish since $e^\Lambda=\prod_{j=1}^{r+1} m_j^{l_j g_j}/(m_j-1)^{g_j}$ must be divisible by the largest prime m_t among m_j's for which $l_t g_t \neq 0$. Hence, taking $G=\max\{|g_j l_j \log m_j/\log m_{r+1}| \mid 1\le j\le r+1\}$, for which we have

$$G < C_{11}(r)(r+1)^{r/2}\log^{2r+3} m_{r+1}\log^{r^2+1} n_r \tag{4.11}$$

from (4.6), we can apply Matveev's theorem to Λ and obtain

$$\begin{aligned}\log|\Lambda| &\ge -C(2(r+1))\left(\log\left(\frac{3}{2}eG\right)\right)\prod_{j=1}^{r+1}(\log m_j)^2\\ &\ge -C_{12}(r)\log(\log m_{r+1}\log n_r)\log^{2(r+1)} m_{r+1}.\end{aligned} \tag{4.12}$$

Now, combining inequalities (4.10) and (4.12), we have

$$\log n_r < C_8(r)(\log^{2(r+1)} m_{r+1})(\log\log m_{r+1}), \tag{4.13}$$

which proves the lemma. □

We see that the former part of Theorem 1.6 is an immediate consequence of this lemma. Indeed, taking $p_1 < p_2 < \cdots < p_{r+1}$ to be any $r+1$ prime factors of N and applying Lemma 4.1 with $m_i = p_i$ for $i = 1, 2, \dots, r+1$ and $n_j = q_j$ for $j = 1, 2, \dots, r$, we must have $\log p_{r+1} > C_3(r)(\log q_r / \log\log q_r)^{1/2(r+1)}$. Thus it remains to prove (1.2). By Lemma 3.3, we have, for each r, l,

$$\sum_{\substack{p \in S_{r,l},\\ p \geq C_4(r,q_r)}} \frac{1}{p} < \exp\left(-C_{13}(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right). \tag{4.14}$$

Since l must be a prime power dividing one of $(q_i - 1)$'s $(1 \leq i \leq r)$ by Lemma 2.3, there exist at most $\log^r q_r$ choices for l. Hence, we obtain

$$\begin{aligned}\sum_{\substack{p \in S_r,\\ p \geq C_4(r,q_r)}} \frac{1}{p} &< (\log q_r)^r \exp\left(-C_{13}(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right)\\ &< \exp\left(-C_5(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right).\end{aligned} \tag{4.15}$$

This completes the proof of Theorem 1.6.

5. Proof of Theorem 1.1

We may assume that $\sigma(N)$ has exactly k distinct prime factors. By Theorem 1.6, there exist at most $k(k+1)/2$ prime factors p of N for which $p \in S_r$ and $p < C_4(r, q_r)$ for some r. Let T be the set of such primes. Then, summing (1.2) over $r = 1, 2, \dots, k$, we obtain

$$\sum_{p \mid N, p \notin T} \frac{1}{p} = \sum_{r=1}^{k} \sum_{\substack{p \in S_r,\\ p \geq C_4(r,q_r)}} \frac{1}{p} < \exp\left(-C_{14}(k)\left(\frac{\log q_1}{\log\log q_1}\right)^{\frac{1}{2(k+1)}}\right). \tag{5.1}$$

Since the sum of reciprocals of the first $k(k+1)/2$ primes is $< C_{15} \log\log k$, we have

$$\begin{aligned}\sum_{p \mid N} \frac{1}{p} &= \sum_{p \in T} \frac{1}{p} + \sum_{p \mid N, p \notin T} \frac{1}{p}\\ &< C_{15} \log\log k + \exp\left(-C_{14}(k)\left(\frac{\log q_1}{\log\log q_1}\right)^{\frac{1}{2(k+1)}}\right).\end{aligned} \tag{5.2}$$

Hence, $\sum_{p \mid N}(1/p) < C_{16}(k)$. Clearly we have $\sum_{p \mid M}(1/p) < C_{17}(k)$ since M has at most k distinct prime factors. Now Theorem 1.1 immediately follows from the observation that

$$\sigma(N)/N < \prod_{p|N} p/(p-1) < \exp\left(\sum_{p|N} 1/(p-1)\right) < \exp\left(\sum_{p|N} (2/p)\right).$$

6. Approximation of rational numbers

In this section, we shall prove a lemma concerning diophantine approximation which is used to prove Theorem 1.2 and 1.4. We shall begin with introducing some notations. For each prime p, we let $h(p^g) = \sigma(p^g)/p^g$ for $g = 1, 2, \ldots$ and $h(p^\infty) = p/(p-1)$. Moreover, for not necessarily distinct primes $p_1, p_2, \ldots, p_k$ and $e_1, e_2, \ldots, e_k \in \{0, 1, 2, \ldots, \infty\}$, we let $h(p_1^{e_1}, p_2^{e_2}, \ldots, p_k^{e_k}) = \prod_{i=1}^{k} h(p_i^{e_i})$.

We observe that if $p_1, p_2, \ldots, p_k$ are distinct primes and $e_1, e_2, \ldots, e_k$ are nonnegative integers, then

$$h(p_1^{e_1}, p_2^{e_2}, \ldots, p_k^{e_k}) = \frac{\sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})}{p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}}. \tag{6.1}$$

For brevity, we write $h(p_1^{x_1}, p_2^{x_2}, \ldots, p_k^{x_k}) = h_k(p^x)$ and $h(p_1^\infty, p_2^\infty, \ldots, p_k^\infty) = h_k(p^\infty)$.

For a rational number α and (not necessarily distinct) primes $p_1, \ldots, p_k$, let $s_k(\alpha, p) = s(\alpha, p_1, \ldots, p_k)$ be the infimum of numbers of the form $\alpha - h_k(p^e)$ with $e_1, \ldots, e_k$ such that $h_l(p^e) = \alpha$ for some (not necessarily distinct) primes $p_{k+1}, p_{k+2}, \ldots, p_l$ and exponents $e_{k+1}, e_{k+2}, \ldots, e_l$. Moreover, let $s(\alpha; k)$ be the infimum of $s_k(\alpha, p)$ with $p_1, \ldots, p_k$ running over all primes.

We shall prove that $s(\alpha; k)$ can be bounded from below by an effectively computable positive constant depending only on α and k. This result is essentially included in [13, Theorem 4.2]. But we reproduce the proof of this lemma since our lemma allows duplication of primes and, as Pomerance notes in [13, p. 204], the proof can be much shortened when $p_1, p_2, \ldots, p_k$ are all odd.

Lemma 6.1. *For any rational number $\alpha = n/d > 1$ and primes $p_1, p_2, \ldots, p_k$, we have $s_k(n/d, p) > \delta_k(n, p)$, where $\delta_k(n, p) = \delta(n, p_1, p_2, \ldots, p_k)$ is an effectively computable positive constant depending only on $p_1, p_2, \ldots, p_k$ and n.*

Proof. For $k = 0$, we clearly have $s(\alpha) = \alpha - 1 \geq 1/d > 1/n$. Now we shall give a lower bound for $s_k(\alpha, p)$ in terms of $\alpha, p_1, p_2, \ldots, p_k$ and $s_{k-1}(\alpha, p)$. This inductively prove the lemma.

We first see that $h_k(p^\infty) \neq \alpha$. Indeed, the denominator of $h_k(p^\infty)$ is even while the denominator of $\alpha = h_k(p^e)$ must be odd. So that it suffices to deal two cases $h_k(p^\infty) < \alpha$ and $h_k(p^\infty) > \alpha$.

In the former case, we see that

$$\alpha - h_k(p^e) > \alpha - h_k(p^\infty) \geq 1/\left(d\prod_i (p_i - 1)\right) > 1/\left(n\prod_i (p_i - 1)\right).$$

Thus we have

$$s_k(n/d, p) \geq 1/\left(n\prod_i (p_i - 1)\right)$$

in this case.

In the latter case, letting $x_j = \lfloor \log(2kn\prod_i (p_i - 1))/\log p_j \rfloor$, we see that $e_j < x_j$ for some j since

$$\begin{aligned}
h_k(p^x) &= \prod_i h_k(p^\infty) \prod_i \left(1 - \frac{1}{p_i^{x_i+1}}\right) \\
&\geq \prod_i h_k(p^\infty) \left(1 - \sum_i \frac{1}{p_i^{x_i+1}}\right) \\
&> \left(\alpha + \frac{1}{d\prod_i (p_i - 1)}\right)\left(1 - \frac{1}{2n\prod_i (p_i - 1)}\right) \\
&= \alpha\left(1 + \frac{1}{n\prod_i (p_i - 1)}\right)\left(1 - \frac{1}{2n\prod_i (p_i - 1)}\right) \\
&> \alpha.
\end{aligned} \tag{6.2}$$

In this case, we have

$$s_k(\alpha, p) \geq \inf\left\{ s(\alpha/h(p_i^{e_i}), p_1, p_2, \ldots, \widehat{p_i}, p_k) \,\middle|\, \begin{array}{l} 1 \leq i \leq k, \\ 1 \leq e_j < x_j \ (1 \leq j \leq k, j \neq i) \end{array} \right\}.$$

Observing that the reduced numerator of $\alpha/h(p_i^{e_i})$ divides $np_i^{e_i}$, we can take

$$\begin{aligned}
\delta_k(n, p) = \min\Bigg\{ & 1/\left(n\prod_i (p_i - 1)\right), \\
& \inf\left\{ \delta(np_i^{e_i}, p_1, p_2, \ldots, \widehat{p_i}, p_k) \,\middle|\, \begin{array}{l} 1 \leq i \leq k, \\ 1 \leq e_j < x_j \ (1 \leq j \leq k) \end{array} \right\}\Bigg\}.
\end{aligned}$$

By induction, this completes the proof. $\square$

Lemma 6.2. *Let n, d be integers with d odd, $p_1, \ldots, p_s$ any odd (not necessarily distinct) primes and $e_1, \ldots, e_s$ non-negative integers. Assume that $h(p_1^{e_1}, p_2^{e_2}, \ldots, p_s^{e_s}) < n/d$ but $h(p_1^{e_1}, p_2^{e_2}, \ldots, p_l^{e_l}) = n/d$ for some (not necessarily distinct) primes $p_{s+1}, p_{s+2}, \ldots, p_l$ and positive integral exponents $e_{s+1}, e_{s+2}, \ldots, e_l$. Then the inequality*

$$\frac{n}{d} - \prod_{i=1}^{s} h(p_i^{e_i}) > C_{18}(s, n) \tag{6.3}$$

holds for effectively computable constants $C_{18}(s, n)$ depending only on s, n.

Proof. For $s=0$, we have a trivial estimate $C_{18}(0,n) \geq n/(n-1)-1 > 1/n$.

Next we shall show that we can compute $C_{18}(s+1,n)$ in term of $C_{18}(s,n)$. This gives the lemma by induction. If

$$p_i > \frac{2n}{dC_{18}(s,n)} - 1 \tag{6.4}$$

for some i, then we have

$$h(p_i^{e_i}) < \left(1-\frac{1}{p_i}\right)^{-1} < \left(1-\frac{\frac{C_{18}(s,n)}{2}}{\frac{n}{d}-\frac{C_{18}(s,n)}{2}}\right)^{-1} = \frac{\frac{n}{d}-\frac{C_{18}(s,n)}{2}}{\frac{n}{d}-C_{18}(s,n)} \tag{6.5}$$

and therefore the inductive hypothesis yields that

$$\prod_{i=1}^{s+1} h(p_i^{e_i}) \leq h(p_i^{e_i})\left(\frac{n}{d}-C_{18}(s,n)\right) < \frac{n}{d}-\frac{C_{18}(s,n)}{2}. \tag{6.6}$$

If (6.4) does not hold for any i, then we have

$$\prod_{i=1}^{s+1} h(p_i^{e_i}) < \frac{n}{d} - \min \delta(n,p_1,\dots,p_{s+1}), \tag{6.7}$$

where $p_1,\dots,p_{s+1}$ run all primes below $2n/(dC_{18}(s,n))$. Hence, we have

$$\frac{n}{d} - \prod_{i=1}^{s+1} h(p_i^{e_i}) > \min\left\{\frac{C_{18}(s,n)}{2}, \bar{\delta}(s,n)\right\}, \tag{6.8}$$

where $\bar{\delta}(s,n)$ denotes the minimum value of $\delta(n,p_1,\dots,p_{s+1})$ with $p_i \leq 2n/C_{18}(s,n)$. Now, Lemma 6.1 ensures that $C_{18}(s+1,n)$ is positive and effectively computable. □

7. Proof of Theorem 1.2

First we shall show that M must have a prime factor smaller than $C_1(k)$. Let T be the same set as defined in Section 5. By Theorem 1.6, T contains at most $k(k+1)/2$ primes. Since N is odd, we can apply Lemma 6.2 taking $p_1,p_2,\dots,p_s$ to be the primes in T and $p_{s+1},\dots,p_l$ to be the prime factors of MN not in T, with primes dividing both M and N counted doubly, to obtain $\prod_{p\in T} h(p) < ab - C_{19}(k,ab)$. Hence, $\prod_{p\mid MN, p\notin T} h(p) > 1 + C_{20}(k,ab)$, implying

$$\sum_{p\mid MN, p\notin T} \frac{1}{p} > C_{21}(k,ab). \tag{7.1}$$

But, as in the proof of Theorem 1.1, we have

$$\sum_{p\mid N, p\notin T} \frac{1}{p} < \exp\left(-C_{14}(k)\left(\frac{\log q_1}{\log\log q_1}\right)^{\frac{1}{2(k+1)}}\right). \tag{7.2}$$

Hence, observing that $\omega(M) \leq \omega(\sigma(N)) \leq k$, we have $q_1 < C_{22}(k, ab)$. Since $a, b < C_0(k)$, we have $q_1 < C_1(k)$.

Next we shall show that N must have a prime factor smaller than $C_1(k)$. X shall denote the smallest prime factor of N. Let Q be an arbitrary real number which shall be chosen later and s be the index satisfying $q_s \leq Q < q_{s+1}$. Similarly to (4.15), there exist at most r primes below $C_4(r, q_r)$ in S_r and we have

$$\begin{aligned}\sum_{r=s+1}^{k} \sum_{\substack{p\in S_r,\\ p\geq C_4(r,q_r)}} \frac{1}{p} &< \exp\left(-C_5(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right)\\ &< \exp\left(-C_5(r)\left(\frac{\log Q}{\log\log Q}\right)^{\frac{1}{2(r+1)}}\right).\end{aligned} \tag{7.3}$$

Hence, for any real X, we obtain

$$\sum_{r=s+1}^{k} \sum_{p\in S_r, p\geq X} \frac{1}{p} < \frac{k(k+1)}{2X} + \exp\left(-C_5(k)\left(\frac{\log Q}{\log\log Q}\right)^{\frac{1}{2(k+1)}}\right). \tag{7.4}$$

Since $q_s \leq Q$, Lemma 3.3 gives that

$$\sum_{p\in S_r, p\geq X} \frac{1}{p} < \frac{(\log Q \log X)^r}{X} \tag{7.5}$$

for each $r \leq s$.

Hence, we have

$$\sum_{p\mid N} \frac{1}{p} < \frac{s(\log Q \log X)^s}{X} + \frac{k(k+1)}{2X} + \exp\left(-C_5(k)\left(\frac{\log Q}{\log\log Q}\right)^{\frac{1}{2(k+1)}}\right). \tag{7.6}$$

Taking Q so that $C_5(k)(\log Q/\log\log Q)^{1/2(k+1)} = \log X$, we have

$$\sum_{p\mid N} \frac{1}{p} < \frac{C_{23}(k) \log^{2k^2+k} X}{X}. \tag{7.7}$$

However, since $\sigma(M)\sigma(N)/MN = ab$ and M is odd with $\omega(M) \leq k$, Lemma 6.2 gives that $\sigma(M)/M < ab - C_{18}(k, ab)$ and therefore $\sigma(N)/N > 1 + C(k, ab)$, implying that

$$\sum_{p\mid N} \frac{1}{p} > C_{24}(k, ab). \tag{7.8}$$

Hence, we must have $X < C_{25}(k, ab)$. Since $a, b < C_0(k)$ by Theorem 1.1, we have $X < C_1(k)$, which proves Theorem 1.2.

8. Proof of Theorem 1.4

First we shall show that $q_s < C_{26}(a, b, k, k', s)$ by induction. The inductive base is that $q_1 < C_1(k)$, which is the former part of Theorem 1.2. Now, it suffices to prove that for any positive integer $s \le k-1$ we have $q_{s+1} < C_{26}(a, b, k, k', s+1)$ under the assumption that $q_1, \ldots, q_s < C_{26}(a, b, k, k', s)$.

We see that for each $r \le s$, S_r contains at most a bounded number of primes. Each $S_{r,l}$ with $l \ge 3$ contains at most $C_{27}(r, l)$ primes by two theorems of Evertse [5, 6]. Since $q_1, q_2, \ldots, q_s \le C_{26}(a, b, k, k', s)$ by the inductive hypothesis, we see that $l \le q_s$ is also bounded by $C_{28}(a, b, k, k', s)$. By assumption, for each $r \le s$, except at most k' primes, any prime in S_r belongs to some $S_{r,l}$ for some prime power $l \ge 3$. Hence, each S_r with $r \le s$ contains at most $C_{29}(a, b, k, k', r)$ primes. We note that, by virtue of the inductive assumption that $q_1, q_2, \ldots, q_s \le C_{26}(a, b, k, k', s)$, we can also use classical finiteness results such as Bugeaud and Győry [1], Coates [2] and Kotov [8].

Moreover, by Theorem 1.6, for $r > s$, S_r contains at most r prime factors below $C_4(r, q_r)$.

Now, let U_s be the set of prime factors p dividing N at least twice for which $p \ge C_4(r, q_r)$ and $p \in S_r$ for some $r > s$. It follows from the above observations that there exist at most $C_{30}(a, b, k, k', s) + s(s+1)/2 + k + k'$ primes outside U_s dividing MN and therefore Lemma 6.2 yields that

$$\sum_{p \in U_s} \frac{1}{p} = \sum_{r=s+1}^{k} \sum_{\substack{p \in S_r,\\ p \ge C_4(r, q_r)}} \frac{1}{p} > C_{31}(a, b, k, k', s). \tag{8.1}$$

However, (1.2) gives that

$$\sum_{\substack{p \in S_r,\\ p \ge C_4(r, q_r)}} \frac{1}{p} < \exp\left(-C_5(r)\left(\frac{\log q_r}{\log\log q_r}\right)^{\frac{1}{2(r+1)}}\right) \tag{8.2}$$

and therefore

$$\sum_{p \in U_s} \frac{1}{p} = \sum_{r=s+1}^{k} \sum_{\substack{p \in S_r,\\ p \ge C_4(r, q_r)}} \frac{1}{p} < \exp\left(-C_{32}(k)\left(\frac{\log q_{s+1}}{\log\log q_{s+1}}\right)^{\frac{1}{2(k+1)}}\right). \tag{8.3}$$

In order that both (8.1) and (8.3) simultaneously hold, we must have $q_{s+1} < C_{26}(a, b, k, k', s+1)$, which completes our inductive argument to prove that $q_j < C_{26}(a, b, k, k', j)$ for every $j = 1, 2, \ldots, k$.

Now, by virtue of Lemma 2.3,

$$\frac{p^l - 1}{p - 1} = q_1^{e_1} q_2^{e_2} \ldots q_k^{e_k} \tag{8.4}$$

implies that $l < q_k < C_{26}(a, b, k, k', j)$ and, using classical finiteness results such as Bugeaud and Győry [1], Coates [2] and Kotov [8], we finally obtain $p < C_{33}(l, q_1, q_2, \ldots, q_k) < C_{34}(a, b, k, k')$. This proves the theorem.

9. Concluding remarks

Our proof of Theorem 1.4 exhibited in the last section indicates that we can explicitly give the upper bound for N in terms of a, b, k, k'; although Evertse's results [5, 6] are not effective for the *size* of solutions, these results give an effective upper bounds for the *number* of solutions. However, the upper bound which our proof yields would become considerably large due to its inductive nature exhibited in the last section. For sufficiently large k, our proof yields that

$$N < \exp\exp\cdots\exp(a + b + k + k'), \tag{9.1}$$

where the number of iterations of the exponential function is $\ll k$ and $\gg k$.

References

[1] Y. Bugeaud & K. Győry, "Bounds for the solutions of Thue-Mahler equations and norm form equations", *Acta Arith.* **74** (1996), no. 3, p. 273-292.

[2] J. Coates, "An effective p-adic analogue of a theorem of Thue. II: The greatest prime factor of a binary form", *Acta Arith.* **16** (1969), p. 399-412.

[3] G. G. Dandapat, J. L. Hunsucker & C. Pomerance, "Some new results on odd perfect numbers", *Pac. J. Math.* **57** (1975), p. 359-364.

[4] L. E. Dickson, "Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors", *American J.* **35** (1913), p. 413-422.

[5] J.-H. Evertse, "On equations in S-units and the Thue-Mahler equation", *Invent. Math.* **75** (1984), p. 561-584.

[6] ———, "The number of solutions of the Thue–Mahler equation", *J. Reine Angew. Math.* **482** (1997), p. 121-149.

[7] N. I. Fel'dman, "Improved estimate for a linear form of the logarithms of algebraic numbers", *Mat. Sb.* **77** (1968), p. 423-436, translation in *Math. USSR, Sb.* **6** (1968), p. 393-406.

[8] S. V. Kotov, "Greatest prime factor of a polynomial", *Mat. Zametki* **13** (1973), p. 515-522, translation in *Math. Notes* **13** (1973), p. 313-317.

[9] E. M. Matveev, "An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II", *Izv. Ross. Akad. Nauk, Ser. Mat.* **64** (2000), no. 6, p. 125-180, translation in *Izv. Math.* **64** (2000), no. 6, p. 127–169.

[10] P. P. Nielsen, "An upper bound for odd perfect numbers", *Integers* **3** (2003), article ID A14 (9 pages).

[11] ———, "Odd perfect numbers, diophantine equation, and upper bounds", *Math. Comput.* **84** (2015), p. 2549-2567.

[12] C. Pomerance, "On multiply perfect numbers with a special property", *Pac. J. Math.* **57** (1975), p. 511-517.

[13] ———, "Multiply perfect numbers, Mersenne primes and effective computability", *Math. Ann.* **226** (1977), p. 195-206.

[14] A. Schinzel, "On two theorems of Gelfond and some of their applications", *Acta Arith.* **13** (1967), p. 177-236.

[15] W. A. Schmidt, *Diophantine approximations and diophantine equations*, Lecture Notes in Mathematics, vol. 1467, Springer, 1991.

[16] T. N. Shorey & R. Tijdeman, *Exponential diophantine equations*, Cambridge Tracts in Mathematics, vol. 87, Cambridge University Press, 1986.

[17] D. SURYANARAYANA, "Super perfect numbers", *Elem. Math.* **24** (1969), p. 16-17.
[18] ———, "There is no odd super perfect number of the form $p^{2\alpha}$", *Elem. Math.* **28** (1973), p. 148-150.
[19] T. YAMADA, "Problem 005:10", 2005, Western Number Theory Problems, `http://www.ma.utexas.edu/users/goddardb/WCNT11/problems2005.pdf`.
[20] ———, "Unitary super perfect numbers", *Math. Pannonica* **19** (2008), p. 37-47.
[21] K. ZSIGMONDY, "Zur Theorie der Potenzreste", *Monatsh. Math.* **3** (1892), p. 265-284.

Tomohiro YAMADA
Center for Japanese language and culture, Osaka University,
562-8558, 8-1-1, Aomatanihigashi, Minoo, Osaka, Japan
E-mail: `tyamada1093@gmail.com`
URL: `http://tyamada1093.web.fc2.com/math/index.html`