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On eigenvalues of the kernel $\frac{1}{2} + \lfloor \frac{1}{xy} \rfloor - \frac{1}{xy}$

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Résumé. Nous montrons que le noyau $K(x,y) = \frac{1}{2} + \lfloor \frac{1}{xy} \rfloor - \frac{1}{xy}$ $(0 < x, y \leq 1)$ possède une infinité de valeurs propres positives et une infinité de valeurs propres négatives. Notre intérêt pour ce noyau est motivé par l'apparition de la forme quadratique $\sum_{m=1}^{N} \mu(m) \sum_{n=1}^{N} \mu(n) K(m/N, n/N)$ dans une identité pour la fonction de Mertens.

Abstract. We show that the kernel $K(x,y) = \frac{1}{2} + \lfloor \frac{1}{xy} \rfloor - \frac{1}{xy}$ $(0 < x, y \leq 1)$ has infinitely many positive eigenvalues and infinitely many negative eigenvalues. Our interest in this kernel is motivated by the appearance of the quadratic form $\sum_{m=1}^{N} \mu(m) \sum_{n=1}^{N} \mu(n) K(m/N, n/N)$ in an identity involving the Mertens function.

1. Introduction

For $0 < x, y \leq 1$, put

$$K(x,y) = \frac{1}{2} - \left\{ \frac{1}{xy} \right\},$$

where $\{t\} \in [0,1)$ denotes the fractional part of $t \in \mathbb{R}$ (i.e. $\{t\} = t - \lfloor t \rfloor$, where $\lfloor t \rfloor = \max\{m \in \mathbb{Z} : m \leq t\}$). When $0 \leq x, y \leq 1$ and $xy = 0$, put $K(x,y) = 0$. The function K thus defined on $[0,1] \times [0,1]$ is (in the terminology of [3]) a symmetric, non-null L_2-kernel. It is shown in [3, Section 3.8] that every such kernel has at least one eigenvalue λ. That is, there exists a number $\lambda \neq 0$, and an associated "eigenfunction" $\phi(x)$ (with $\infty > \int_0^1 |\phi(x)|^2 \mathrm{d}x > 0$), satisfying

$$\phi(x) = \lambda \int_0^1 K(x,y) \phi(y) \mathrm{d}y \tag{1.1}$$

almost everywhere, with respect to the Lebesgue measure, in $[0,1]$.

Since K is symmetric (i.e. satisfies $K(x,y) = K(y,x)$), all eigenvalues of K are real, and so there is no essential loss of generality in considering just those eigenfunctions of K that are real valued (i.e. at least one of the pair of real functions $\mathrm{Re}(\phi)$, $\mathrm{Im}(\phi)$ may be substituted for ϕ in (1.1)).

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In this paper we have the option of working only with eigenfunctions $\phi : [0,1] \to \mathbb{R}$ that satisfy (1.1) for all $x \in [0,1]$ (when $\lambda \in \mathbb{R}\backslash\{0\}$ is the appropriate eigenvalue). Choosing to do so would not be overly restrictive, for if $\phi(x)$ is any eigenfunction of K, with associated eigenvalue λ, then (1.1) holds almost everywhere in $[0,1]$, and the term $\lambda \int_0^1 K(x,y)\phi(y)\mathrm{d}y$ (occurring in (1.1)) is an eigenfunction of K that has the required property (the last part of this following, via the Cauchy–Schwarz inequality, from the observation that, for all $x \in [0,1]$, the integral $\int_0^1 (K(x,y))^2 \mathrm{d}y$ exists, and is finite). Although this is an option that is of no consequence with regard to the proof of our main result (Theorem 1.1, below), we shall find it helpful when discussing certain incidental matters.

By the general theory set out in [3, Section 3.8], the set

$$\mathcal{S}(K) = \{\lambda \,:\, \lambda \text{ is an eigenvalue of } K\}$$

is countable (in the sense that does not preclude its being finite). It is not hard to see that $\mathcal{S}(K)$ cannot be a finite set (for a sketch of a proof of this, see our Remark 2.3, following Lemma 2.1 below). This paper is devoted to proving the following stronger result.

Theorem 1.1. *The sets $\mathcal{S}(K) \cap (-\infty, 0)$ and $\mathcal{S}(K) \cap (0, \infty)$ are infinite.*

We prove this theorem in Section 3, after some necessary preliminaries.

Our particular interest in the kernel $K(x,y)$ is motivated by a connection with the Möbius function $\mu(n)$ and its associated summatory function $M(x) = \sum_{m \le x} \mu(n)$ (known as the Mertens function). This connection is apparent in our recent joint work [1] with Huxley, where it is (in effect) noted that for each positive integer N one has

$$\frac{M\left(N^2\right)}{N^2} + \frac{1}{N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} K\left(\frac{m}{N}, \frac{n}{N}\right) \mu(m)\mu(n) = \frac{M(N)\left(M(N)+4\right)}{2N^2} - \left(\sum_{m=1}^{N} \frac{\mu(m)}{m}\right)^2 \tag{1.2}$$

(this following directly from [1, Equations (3)–(5) and (37)]). In the preprint [4] it is shown that the sum over m and n on the left-hand side of (1.2) may be approximated, reasonably well, by sums involving the numbers $\mu(1), \ldots, \mu(N)$, certain of the (smaller) eigenvalues of K and the values that the corresponding eigenfunctions have at the points $x = \frac{m}{N}$ $(m = 1, \ldots, N)$.

The author is indebted to the anonymous referee for pointing out that F. Mertens himself showed (in 1897) that, for all positive integers n, one has

$$M(n) = 2M\left(\sqrt{n}\right) - \sum_{r,s\leq\sqrt{n}}\sum \mu(r)\mu(s)\left\lfloor \frac{n}{rs} \right\rfloor .$$

The proof of this appeared in [2, Section 3]. This result of Mertens contains the "principal case" of [1, Equations (3)–(5)], from which we have derived the equation (1.2), and is equivalent to that subcase of the "principal case" of [1, Theorem 1] in which one has $d = 2$ and $N_1 = N_2 = \lfloor\sqrt{K}\rfloor$ (with $K = n$).

Acknowledgement. The author wishes to thank the anonymous referee for pointing out the relevance of the work [2] of Mertens, and for several other comments that have helped to improve this paper.

2. Notation and some Hilbert–Schmidt Theory

We denote by $L^2([0,1])$ the semimetric space of functions $f : [0,1] \to \mathbb{R}$ that are measurable (in the sense of Lebesgue) and satisfy the condition $\int_0^1 (f(x))^2 \mathrm{d}x < \infty$.

For each eigenvalue $\lambda \in \mathcal{S}(K)$, the corresponding set of eigenfunctions satisfying (1.1) for all $x \in [0,1]$ spans a finite dimensional subspace of $L^2([0,1])$: we follow [3] in referring to the dimension of this subspace, r_λ (say), as the "index" of λ. We put $\omega = \sum_{\lambda\in\mathcal{S}(K)} r_\lambda$, so that $\omega \in \mathbb{N}$ if $\mathcal{S}(K)$ is finite, while $\omega = \infty$ otherwise. Since K is symmetric, any two eigenfunctions of K corresponding to eigenvalues λ, μ (say) with $\lambda \neq \mu$ are an orthogonal pair with respect to the (semi-definite) inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x \qquad (f, g \in L^2([0,1])). \tag{2.1}$$

See [3, Sections 2.3 and 3.1] regarding this matter. In [3, Section 3.8] it is shown that there exists a system ϕ_j ($j \in \mathbb{N}$ and $j \leq \omega$) of eigenfunctions of K that is orthonormal, so that one has

$$\langle \phi_j, \phi_k\rangle := \int_0^1 \phi_j(x)\phi_k(x)\mathrm{d}x = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

whenever $j, k \in \mathbb{N}$ satisfy $j, k \leq \omega$, and that is (at the same time) maximal, so that the corresponding sequence λ_j ($j \in \mathbb{N}$ and $j \leq \omega$) of eigenvalues of K is such that one has $|\{j \in \mathbb{N} : j \leq \omega \text{ and } \lambda_j = \lambda\}| = r_\lambda$ for all $\lambda \in \mathcal{S}(K)$.

By [3, Section 3.10, (8)], we have

$$\sum_{\substack{j\in\mathbb{N} \\ j\leq\omega}} \frac{1}{\lambda_j^2} = \int_0^1 \int_0^1 (K(x,y))^2 \,\mathrm{d}x\mathrm{d}y < \frac{1}{4}. \tag{2.2}$$

It follows that either $\mathcal{S}(K)$ is finite, or else one has $|\lambda_j| \to \infty$ as $j \to \infty$. Therefore, as in [3, Section 3.8], we may assume that the ϕ_j's (and associated λ_j's) are numbered in such a way that the absolute values of the associated eigenvalues form a sequence, $|\lambda_j|$ ($j \in \mathbb{N}$ and $j \leq \omega$), that is monotonic increasing.

We now develop some notation in which there is a clear distinction between positive and negative eigenvalues (and between the corresponding eigenfunctions). Let ω^+ (resp. ω^-) be the number of positive (resp. negative) terms in the sequence λ_j ($j \in \mathbb{N}$ and $j \leq \omega$), so that $\omega^+, \omega^- \in \mathbb{N} \cup \{0, \infty\}$ and $\omega^+ + \omega^- = \omega$. If all of the negative eigenvalues are removed from the sequence λ_j ($j \in \mathbb{N}$ and $j \leq \omega$) then what remains is some monotonic increasing subsequence λ_{m_k} ($k \in \mathbb{N}$ and $k \leq \omega^+$) in which each positive eigenvalue of K appears. If one instead removes the positive eigenvalues then what remains is some monotonic decreasing subsequence λ_{n_k} ($k \in \mathbb{N}$ and $k \leq \omega^-$) in which each negative eigenvalue of K appears. For $k \in \mathbb{N}$ satisfying $k \leq \omega^+$ (resp. $k \leq \omega^-$) we put $\lambda_k^+ = \lambda_{m_k}$ and $\phi_k^+ = \phi_{m_k}$ (resp. $\lambda_k^- = \lambda_{n_k}$ and $\phi_k^- = \phi_{n_k}$): note this has the consequence that (1.1) holds when λ and ϕ are λ_k^+ and ϕ_k^+ (resp. λ_k^- and ϕ_k^-), respectively. As every eigenvalue of K is real and non-zero (and so either positive or negative), it is clear that the sets $\{\phi_j : j \in \mathbb{N}\}$ and $\{\phi_k^+ : k \in \mathbb{N} \text{ and } k \leq \omega^+\} \cup \{\phi_k^- : k \in \mathbb{N} \text{ and } k \leq \omega^-\}$ are equal, and so we know (in particular) that the elements of the latter set form an orthonormal system.

Lemma 2.1. *Let $\phi, \psi \in L^2([0,1])$ and put*

$$J = J(\phi, \psi) = \int_0^1 \int_0^1 K(x,y)\phi(x)\psi(y)\mathrm{d}x\mathrm{d}y.$$

Then

$$J = \sum_{\substack{j \in \mathbb{N} \\ j \leq \omega}} \frac{\langle \phi, \phi_j \rangle \langle \phi_j, \psi \rangle}{\lambda_j} = \sum_{\substack{k \in \mathbb{N} \\ k \leq \omega^+}} \frac{\langle \phi, \phi_k^+ \rangle \langle \phi_k^+, \psi \rangle}{\lambda_k^+} + \sum_{\substack{k \in \mathbb{N} \\ k \leq \omega^-}} \frac{\langle \phi, \phi_k^- \rangle \langle \phi_k^-, \psi \rangle}{\lambda_k^-}.$$

Proof. See [3, Section 3.11], where this result is proved in greater generality (i.e. for an arbitrary symmetric L_2-kernel) by applying a theorem of Hilbert and Schmidt (for which see [3, Section 3.10]). □

Remark 2.2. The proof supplied in [3] shows that each sum over j, or k, in Lemma 2.1 is absolutely convergent (when not finite or empty). This may also be deduced directly from Bessel's inequality, since, for $j \in \mathbb{N}$ with $j \leq \omega$, one has $|\lambda_j| \geq |\lambda_1| > 0$ and so, by the inequality of arithmetic and geometric means, $|\langle \phi, \phi_j \rangle \langle \phi_j, \psi \rangle / \lambda_j| \leq \left(|\langle \phi, \phi_j \rangle|^2 + |\langle \phi_j, \psi \rangle|^2 \right) / (2|\lambda_1|)$.

Remark 2.3. Given what was noted in the third paragraph of Section 1, the eigenfunctions ϕ_j ($j \in \mathbb{N}$ and $j \leq \omega$) can be chosen in such a way that each has $\phi_j(x) = \lambda_j \int_0^1 K(x,y)\phi_j(y)\mathrm{d}y$ for all $x \in [0,1]$. Assume that such a choice has been made. One can show that it follows that each ϕ_j is continuous on the interval $(0,1]$: we leave the proof of this as an exercise for the reader, and remark that the eigenfunctions in question can also be shown to be continuous at the point $x = 0$ (this last fact, however, is not relevant to our main concern here). Therefore if $\omega \neq \infty$ (so that $\omega \in \mathbb{N}$) then, by applying Lemma 2.1 for suitably chosen functions $\phi(x)$ and $\psi(y)$ that are supported in intervals $[a-\varepsilon, a]$ and $[b-\varepsilon, b]$ (respectively), we find (letting $\varepsilon \to 0+$, and using the right-continuity of the real function $t \mapsto \{t\}$) that $K(a,b) = \sum_{j=1}^{\omega} \phi_j(a)\phi_j(b)/\lambda_j$ when one has $0 < a, b \leq 1$. This last identity would imply that $K(x,y)$ is continuous on $(0,1] \times (0,1]$, whereas one has (for example) $\lim_{x \to (\frac{1}{2})+} K(x,x) = -\frac{1}{2}$, but $K(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. By this *reductio ad absurdum* we may conclude that $\omega = \infty$. It follows that $\mathcal{S}(K)$ is not a finite set.

3. Negative (resp. positive) eigenvalues of K

In this section we prove Theorem 1.1 by showing that $\omega^+ = \omega^- = \infty$. In doing so we shall make use of both Lemma 2.1 and the following purely number-theoretic result.

Lemma 3.1. *Let $u \in \{1, -1\}$, let $Q \in [5, \infty)$ and let N be a non-negative integer. Then there exist $N+1$ distinct primes $p_1, \ldots, p_{N+1}$, all greater than Q, and an integer n satisfying*

$$3n^2 \equiv m_j \pmod{p_j^2} \qquad (j = 1, \ldots, N+1), \tag{3.1}$$

where m_j denotes the least positive integer satisfying both

$$2m_j \equiv 3 \pmod{p_j} \qquad \text{and} \qquad m_j \equiv u \pmod 3. \tag{3.2}$$

One has here

$$0 < m_j < 3p_j \qquad (j = 1, \ldots, N+1), \tag{3.3}$$

and the integer n may be chosen so as to satisfy

$$P^2 < n < 2P^2, \tag{3.4}$$

where P is the product of the primes $p_1, \ldots, p_{N+1}$.

Proof. It is a corollary of Dirichlet's theorem on primes in arithmetic progressions that the set $\{p : p \text{ is prime}, \ p \equiv \pm 1 \pmod 8 \text{ and } p > Q\}$ is infinite: we take $p_1, \ldots, p_{N+1}$ to be any $N+1$ distinct elements of this set. As $Q > 3$, and as distinct positive primes are coprime to one another, we have $(p_j, 2) = (p_j, 3) = 1$ $(j = 1, \ldots, N+1)$, and $(p_j, p_k) = 1$ $(1 \leq j < k \leq N+1)$. It therefore follows by the Chinese Remainder Theorem that, for $j = 1, \ldots, N+1$, the simultaneous congruences in (3.2) are soluble (for the integer m_j): since $(u, 3) = (\pm 1, 3) = 1 = (3, p_j)$, the set of all integer solutions of these congruences is one of the residue classes mod $3p_j$ that are prime to $3p_j$, and so there is a least positive integer solution m_j, and this solution must satisfy both $m_j \le 3p_j$ (by its minimality) and $m_j \neq 3p_j$ (as $(m_j, 3p_j) = 1$), so that the inequalities in (3.3) will be satisfied.

Since the numbers $p_1, \dots, p_{N+1}$ are pairwise coprime, the Chinese Remainder Theorem shows also that a solution $n \in \mathbb{Z}$ for the simultaneous congruences in (3.1) may be found, provided only that each one of those congruences is soluble (for n). Given any $j \in \{1, \dots, N+1\}$, we note that, as $(p_j, 3) = (p_j, m_j) = 1$, the congruence $3n^2 \equiv m_j \pmod{p_j^2}$ is soluble (for n) if and only if $3m_j$ is a quadratic residue mod p_j^2. Since p_j is an odd prime, this last condition on $3m_j$ will be satisfied if and only if $3m_j$ is a quadratic residue mod p_j. By the first congruence in (3.2), we do have $6m_j \equiv 3^2 \pmod{p_j}$, so that $6m_j = (2)(3m_j)$ is a quadratic residue mod p_j. We deduce that the congruence $3n^2 \equiv m_j \pmod{p_j^2}$ is soluble if and only if 2 is a quadratic residue mod p_j. Given that $p_j \equiv \pm 1 \pmod 8$, the solubility of the congruence $3n^2 \equiv m_j \pmod{p_j^2}$ therefore follows as a consequence of the well-known fact that, for all odd primes p, the Legendre symbol $(\frac{2}{p})$ equals $(-1)^{(p^2-1)/8}$ (and so equals 1 when $p \equiv \pm 1 \pmod 8$). This completes the proof of the solubility of the simultaneous congruences in (3.1): as $(m_j, p_j) = 1$ $(j = 1, \dots, N+1)$, it follows that the set of all integers n satisfying these simultaneous congruences contains a residue class mod P^2 that is prime to P^2, and so must contain at least one element n that lies strictly between P^2 and $P^2 + P^2$, as in (3.4). $\square$

Proof of Theorem 1.1. As was mentioned earlier, each eigenvalue of K has a finite index (this is, for example, a corollary of the relations in (2.2)). It follows directly from this fact that ω^+ will be some non-negative integer if the set $\mathcal{S}(K) \cap (0, \infty)$ is not infinite. Similarly, if the set $\mathcal{S}(K) \cap (-\infty, 0)$ is not infinite, then ω^- is a non-negative integer (and so not equal to ∞). Therefore Theorem 1.1 will follow if we can show that $\omega^+ = \omega^- = \infty$. We shall achieve this through "proof by contradiction".

Suppose it is not the case that $\omega^+ = \omega^- = \infty$. Then either $\omega^+ \in \mathbb{N} \cup \{0\}$, or else $\omega^+ = \infty$ and $\omega^- \in \mathbb{N} \cup \{0\}$. In the former case we put $N = \omega^+$ and $u = -1$; in the latter case we put $N = \omega^-$ and $u = 1$.

Appealing to Lemma 3.1, we put $Q = 5(N+1)^{1/2}$, and choose $N+1$ distinct primes $p_1, \dots, p_{N+1} > Q \ge 5$, with associated integers $m_1, \dots, m_{N+1}$ and n, in such a way that the conditions (3.1)–(3.4) are all satisfied. We then put

$$x_j = \frac{p_j}{n} \qquad (j = 1, \dots, N+1).$$

By (3.4), we have $\{x_1, \dots, x_{N+1}\} \subset (0, P/P^2) \subseteq (0, 1/p_1) \subseteq (0, 1/7)$.

We observe now that, as $p_j^2 \equiv 1 \pmod 3$, for $j = 1, \dots, N+1$, the congruences (3.1) and (3.2) imply that we have $3n^2 = m_j + (3v_j - u)p_j^2$, for

$j = 1, \ldots, N+1$, where each v_j is integer valued. By this, we obtain:

$$\begin{aligned} K(x_j, x_j) &= \frac{1}{2} - \left\{ \frac{n^2}{p_j^2} \right\} \\ &= \frac{1}{2} - \left\{ \frac{m_j}{3p_j^2} - \frac{u}{3} \right\} = -\left(\frac{u}{6} + \frac{m_j}{3p_j^2} \right) \qquad (j = 1, \ldots, N+1), \end{aligned}$$

with the final equality following due to our having $0 < m_j/(3p_j^2) < 1/p_j \leq 1/7 < 1/3$ (as a consequence of (3.3)) and $u \in \{-1, 1\}$.

We may note also that (3.1) and (3.2) imply that one has $6n^2 \equiv 2m_j \equiv 3 \pmod{p_j}$, and so $2n^2 \equiv 1 \pmod{p_j}$, for $j = 1, \ldots, N+1$. It follows that, when $j, k \in \{1, \ldots, N+1\}$ and $j \neq k$ (so that $(p_j, p_k) = 1$), one will have $2n^2 \equiv 1 \pmod{p_j p_k}$, and so $2n^2 = 1 + (1 + 2w_{jk})p_j p_k$, where w_{jk} is some integer. We may therefore deduce that

$$\begin{aligned} K(x_j, x_k) &= \frac{1}{2} - \left\{ \frac{n^2}{p_j p_k} \right\} \\ &= \frac{1}{2} - \left\{ \frac{1}{2p_j p_k} + \frac{1}{2} \right\} = -\frac{1}{2p_j p_k} \qquad (1 \leq j, k \leq N+1,\ j \neq k), \end{aligned}$$

since we have $0 < 1/(2p_j p_k) \leq 1/154 < 1/2$ here.

Let now

$$\Delta = \log\left(\frac{t}{1 - e^{-t}} \right) \qquad \text{and} \qquad \delta = t - \Delta,$$

where $t > 0$ is to be specified at a later point in this proof. By this we have

$$e^{\Delta} - e^{-\delta} = t = \Delta + \delta. \tag{3.5}$$

Note that

$$e^t > \frac{te^t}{e^t - 1} = \frac{t}{1 - e^{-t}} = \frac{(t/2)e^{t/2}}{\sinh(t/2)} > \frac{(t/2)\cosh(t/2)}{\sinh(t/2)} > 1,$$

so that

$$0 < \Delta, \delta < t.$$

For $j = 1, \ldots, N+1$ and $0 \leq x \leq 1$, we define $\psi_j(x)$ by:

$$\psi_j(x) = \begin{cases} 1/\sqrt{tx_j} & \text{if } e^{-\delta} < x/x_j < e^{\Delta}, \\ 0 & \text{otherwise.} \end{cases}$$

The functions $\psi_1(x), \ldots, \psi_{N+1}(x)$ so defined are elements of the space $L^2([0,1])$. Assuming that

$$t \leq \frac{2}{\max\{p_1, \ldots, p_{N+1}\}}, \tag{3.6}$$

we have

$$\left|\log\left(\frac{x_k}{x_j}\right)\right| = \left|\log\left(\frac{p_k}{p_j}\right)\right|$$
$$> \frac{|p_k - p_j|}{\max\{p_k, p_j\}} \geq t = \Delta + \delta \qquad (1 \leq j, k \leq N+1,\ j \neq k),$$

so that, by virtue of the pairwise disjointness of the sets that are their supports, the functions $\psi_1, \ldots, \psi_{N+1}$ are pairwise orthogonal with respect to the inner product (2.1). By (3.4) and (3.6), we have also $e^{\Delta} x_j < e^t p_j / n < p_j^{-1} \exp(2/p_j) \leq (1/7)\exp(2/7) < 1$, for $j = 1, \ldots, N+1$, and so it follows (using (3.5)) that $\{\psi_1, \ldots, \psi_{N+1}\}$ is an orthonormal subset of $L^2([0,1])$.

Let $\psi \in L^2([0,1])$ be defined by

$$\psi(x) = \sum_{j=1}^{N+1} \alpha_j \psi_j(x) \qquad (0 \leq x \leq 1),$$

where $\alpha_1, \ldots, \alpha_{N+1}$ denote certain real constants that we shall choose later. Then, as a consequence of Lemma 2.1 (combined with the fact that the square of any real number is real and non-negative), we find that

$$uJ(\psi, \psi) = u \int_0^1 \int_0^1 K(x,y)\psi(x)\psi(y)\mathrm{d}x\mathrm{d}y \geq -\sum_{1 \leq j \leq N} \frac{\langle \psi, \phi_j^{\pm}\rangle^2}{|\lambda_j^{\pm}|},$$

where each ambiguous sign "$\pm$" is such that one has $\pm u = -1$. We therefore will have

$$uJ(\psi, \psi) \geq 0 \tag{3.7}$$

if (when the sign "$\pm$" is as above) one has $\langle \psi, \phi_j^{\pm}\rangle = 0$ for each positive integer $j \leq N$. This last condition on ψ holds subject to a certain set of N homogeneous linear equations in variables $z_1, \ldots, z_{N+1}$ (say) being satisfied when, for $j = 1, \ldots, N+1$, one has $z_j = \alpha_j$. If $N \neq 0$ then the coefficients of this set of equations form an $N \times (N+1)$ real matrix, the columns of which are (necessarily) linearly dependent. We therefore have (3.7) for some choice of $\alpha_1, \ldots, \alpha_{N+1} \in \mathbb{R}$ that is distinct from the "trivial solution" $(\alpha_1, \ldots, \alpha_{N+1}) = (0, \ldots, 0)$. We assume such a choice of $\alpha_1, \ldots, \alpha_{N+1}$ in what follows. Thus (3.7) holds.

By our definitions of $\psi_1, \ldots, \psi_{N+1}$ and ψ, we find that

$$J(\psi, \psi) = \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} \Psi_{j,k} \alpha_j \alpha_k,$$

where

$$\begin{aligned}\Psi_{j,k} &= \int_0^1\int_0^1 K(x,y)\psi_j(x)\psi_k(y)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{t\sqrt{x_jx_k}}\int_{x_j/e^{\delta}}^{x_je^{\Delta}}\left(\int_{x_k/e^{\delta}}^{x_ke^{\Delta}} K(x,y)\mathrm{d}y\right)\mathrm{d}x.\end{aligned}$$

Within the last double integral we have always

$$\left|\frac{1}{xy}-\frac{1}{x_jx_k}\right| < \frac{e^{2t}-1}{x_jx_k} = \frac{\left(e^{2t}-1\right)n^2}{p_jp_k} \le \frac{\left(e^{2t}-1\right)n^2}{49}. \tag{3.8}$$

On the other hand, our earlier calculations of $K(x_j,x_k)$ (including that in the case $k=j$) make it plain that we have

$$\frac{1}{3} < \left\{\frac{1}{x_jx_k}\right\} = \frac{1}{2} - K\left(x_j,x_k\right) < \frac{2}{3}+\frac{1}{7} = \frac{17}{21} \qquad (1\le j,k\le N+1).$$

Therefore, provided that we choose $t>0$ so small as to satisfy

$$e^{2t}-1 \le \frac{28}{3n^2}, \tag{3.9}$$

it will then be the case that (3.8) implies the continuity of the kernel K at the point (x,y). Therefore, given the particulars of the definition of $K(x,y)$, we may deduce (subject to (3.9)) that

$$\Psi_{j,k} = \frac{1}{t\sqrt{x_jx_k}}\int_{x_j/e^{\delta}}^{x_je^{\Delta}}\left(\int_{x_k/e^{\delta}}^{x_ke^{\Delta}}\left(K\left(x_j,x_k\right)+\frac{1}{x_jx_k}-\frac{1}{xy}\right)\mathrm{d}y\right)\mathrm{d}x, \tag{3.10}$$

for $1\le j,k\le N+1$. Note that (3.9) implies $t\le 14/(3n^2)$, and so (by (3.4)), it is certainly a stronger condition on t than that in (3.6).

By (3.5), our result in (3.10) simplifies to:

$$\Psi_{j,k} = t\sqrt{x_jx_k}K\left(x_j,x_k\right) \qquad (1\le j,k\le N+1).$$

By this, together with our earlier calculations of $K(x_j,x_k)$ (including that in the case $j=k$), we find that

$$\begin{aligned}J(\psi,\psi) &= t\sum_{j=1}^{N+1}\sum_{k=1}^{N+1} K\left(x_j,x_k\right)\left(\alpha_j\sqrt{x_j}\right)\left(\alpha_k\sqrt{x_k}\right) \\ &= (-t)\sum_{j=1}^{N+1}\left(\frac{u}{6}+\frac{m_j}{3p_j^2}\right)\alpha_j^2x_j + (-t)\sum\sum_{1\le j<k\le N+1}\frac{\alpha_j\alpha_k\sqrt{x_jx_k}}{p_jp_k},\end{aligned}$$

and so

$$uJ(\psi,\psi) = -\frac{t}{n}\left(\sum_{j=1}^{N+1}\left(\frac{p_j}{6}+\frac{um_j}{3p_j}\right)\alpha_j^2 + u\sum\sum_{1\le j<k\le N+1}\frac{\alpha_j\alpha_k}{\sqrt{p_jp_k}}\right).$$

Using the inequality $|\alpha_j \alpha_k / \sqrt{p_j p_k}| \leq (\alpha_j^2 p_j^{-1} + \alpha_k^2 p_k^{-1})/2$, together with the triangle inequality, one can show that the sum over j and k occurring in the last equation has absolute value less than or equal to $\frac{1}{2} N \sum_{j=1}^{N+1} \alpha_j^2 p_j^{-1}$. By this and (3.3), one obtains the upper bound

$$uJ(\psi, \psi) \leq -\frac{t}{n} \sum_{j=1}^{N+1} \left(\frac{p_j}{6} - \left(1 + \frac{N}{2p_j} \right) \right) \alpha_j^2. \tag{3.11}$$

We have $\min\{p_1, \ldots, p_{N+1}\} > Q \geq 5 \max\{1, \sqrt{N}\}$. Therefore

$$1/\left(\frac{p_j}{6}\right) = \frac{6}{p_j} \leq \frac{6}{7} \quad \text{and} \quad \left(\frac{N}{2p_j}\right) / \left(\frac{p_j}{6}\right) = \frac{3N}{p_j^2} \leq \frac{3N}{Q^2} \leq \frac{3}{25} < \frac{1}{8},$$

for $j = 1, \ldots, N+1$, and so it follows from (3.11) and the "non-triviality" of $(\alpha_1, \ldots, \alpha_{N+1}) \in \mathbb{R}^{N+1}$ that we have

$$uJ(\psi, \psi) \leq -\frac{t}{n} \sum_{j=1}^{N+1} \frac{p_j \alpha_j^2}{336} \leq -\frac{t}{48n} \sum_{j=1}^{N+1} \alpha_j^2 < 0.$$

Since the last of these inequalities is strict, we find that (3.7) is contradicted, and so complete our "proof by contradiction" that $\omega^+ = \omega^- = \infty$. ☐

Remark 3.2. The idea for the above proof came after reading some of H. Weyl's paper [5]: in particular, his proof of "Satz 1" there.

Remark 3.3. By elaborating upon the above proof one can obtain lower bounds for the terms in the sequence (λ_k^-), and upper bounds for the terms in the sequence (λ_k^+). These bounds are, however, extremely weak: the most that I have been able to show, in respect of positive eigenvalues of K, is that one has

$$\lambda_k^+ \leq 2772\big(918(k+1)\log(k+1)\big)^{6(k+1)} \qquad (k \in \mathbb{N}).$$

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