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A logarithmic improvement in the Bombieri–Vinogradov theorem

par Alisa SEDUNOVA

Résumé. Le théorème de Bombieri–Vinogradov est l'un des outils fondamentaux de la théorie analytique de nombres; ses applications sont nombreuses et ne se limitent pas à ce seul domaine. Dans cet article, nous améliorons la meilleure version actuellement connue du théorème, établie par Dress–Iwaniec–Tenenbaum [4], en remplaçant $(\log x)^2$ par $(\log x)^{5/2}$. Nous utilisons une version pondérée de l'identité de Vaughan, ce qui nous permet de faire une troncature lisse, et une estimation de Barban–Vehov [2], généralisée par Graham [6], qui est liée au crible de Selberg. Nous donnons des versions effective et non effective du résultat. En excluant les petits modules, cela nous permet de déduire un théorème de Bombieri–Vinogradov complètement effective pour $q \le x^{1/2-\varepsilon}$.

Abstract. The Bombieri–Vinogradov theorem is one of the standard, basic tools of an analytic number theorist; its applications are many, and not limited to the field. In this paper, we improve on the strongest version to date by Dress–Iwaniec–Tenenbaum [4], getting $(\log x)^2$ instead of $(\log x)^{5/2}$. We use a weighted form of Vaughan's identity, allowing a smooth truncation inside the procedure, and an estimate of Barban–Vehov [2] (later generalized by Graham [6]), which is related to Selberg's sieve. We give effective and non-effective versions of the result. Using that and excluding the small moduli one can derive the fully effective Bombieri–Vinogradov theorem for $q \le x^{1/2-\varepsilon}$.

1. Introduction

For integer number a and $q \ge 1$, let

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a (\operatorname{mod} q)}} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function. The Bombieri–Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all q up to $x^{1/2}$, or, rather almost up to $x^{1/2}$.

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Theorem 1.1 (Bombieri–Vinogradov)**.** *Let $B > 0$ be given and $A = A(B)$. Then for $Q \leq x^{1/2}/(\log x)^A$ we have*

$$\sum_{q\leq Q} \max_{2\leq y\leq x} \max_{\substack{a \\ (a,q)=1}} \left|\psi(y;q,a) - \frac{y}{\varphi(q)}\right| \ll_B \frac{x}{(\log x)^B}.$$

The implied constant in this theorem is not effective, since we have to take care of characters associated with those q that have small prime factors. At the same time, effective versions — in which the effect of an exceptional character is avoided in one way or another — have been known since Lenstra–Pomerance [9], Timofeev [14], and, very recently, Liu [10].

In the rest of the introduction we state the results of the present paper.

Theorem 1.2 (Bombieri–Vinogradov, ineffective)**.** *Let $A > 2$ and $Q \leq x^{1/2}/(\log x)^A$. Then we have the following bound*

$$\sum_{q\leq Q} \max_{2\leq y\leq x} \max_{\substack{a \\ (a,q)=1}} \left|\psi(y;q,a) - \frac{y}{\varphi(q)}\right| \ll_A \frac{x}{(\log x)^{A-2}}.$$

The implied constant in Theorem 1.2 is ineffective.

Previously, the best result of the type of Theorem 1.2 in the literature followed from [4]; it had $A - 5/2$ instead of $A - 2$. While [4] does not state the result in full — focusing on estimating th crucial sum — a complete form can be found in [13] (together with a fully explicit version). It is

$$\sum_{\substack{q\leq Q \\ l(q)>Q_1}} \max_{2\leq y\leq x} \max_{\substack{a \\ (a,q)=1}} \left|\psi(y;q,a) - \frac{\psi(y)}{\varphi(q)}\right| \leq C\left(\frac{x}{Q_1} + x^{\frac{1}{2}}Q + x^{\frac{2}{3}}Q^{\frac{1}{2}} + x^{\frac{5}{6}}\log\frac{Q}{Q_1}\right)(\log x)^{\frac{7}{2}},$$

where C is an explicit absolute constant (a similar fully explicit result was proven in Akbary–Hambrook [1] with $(\log x)^{9/2}$ instead of $(\log x)^{7/2}$) and $l(q)$ denotes the biggest prime factor of q. Another effective variant without explicit constants is given by Lenstra–Pomerance [9, Lemma 11.2] (with a larger power of log) in their work on Gaussian periods.

The key tool for the proof of Theorem 1.4 is Vaughan's identity, which we have to get in an explicit version for our goal. Let

$$\psi(y,\chi) = \sum_{n\leq y} \Lambda(n)\chi(n),$$

be the twisted summatory function of the von Mangoldt function Λ and a Dirichlet character χ modulo q. The key tool in getting Theorem 1.2 is the following estimate.

Proposition 1.3 (Vaughan's inequality, improved)**.** *For $x \geq 4$ and any $\varepsilon > 0$ we have*

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi(q)}{}^{*}\max_{y\leq x}|\psi(y,\chi)| \ll_{\varepsilon} \left(x+Q^2x^{\frac{1}{2}}+Qx^{\frac{13}{14}+\varepsilon}\right)(\log x)^2,$$

where Q is any positive real number and $\sum^{*}{}_{\chi(q)}$ stands for a sum over all primitive characters $\chi(\operatorname{mod} q)$.

The improvement here consists in having a factor of $(\log x)^2$, rather than $(\log x)^{5/2}$. In order to prove Proposition 1.3 we use the weighted version of Vaughan's identity (see Lemma 1.5) and a corollary of Graham [6], which was also given in a less general form in an earlier work of Barban–Vehov [2]. While Graham uses the Siegel–Walfisz theorem, there is an effective (and explicit) version of it in Helfgott [7, Chapter 5]. We follow methods developed in [7, Chapter 5] in the proof.

Analogously to [13] we formulate an effective version of Proposition 1.3 in the following corollary.

Corollary 1.4 (Bombieri–Vinogradov, with exceptional character taken out)**.** *Let $x \geq 4$, $1 \leq Q_1 \leq Q \leq x^{1/2}$. Denote by $l(q)$ the smallest prime divisor of q. Then for any positive $\varepsilon > 0$ we have*

$$\sum_{\substack{q\leq Q\\ l(q)>Q_1}}\max_{2\leq y\leq x}\max_{\substack{a\\(a,q)=1}}\left|\psi(y;q,a)-\frac{\psi(y)}{\varphi(q)}\right| \ll_{\varepsilon} x^{\frac{1}{2}}Q(\log x)^2+\frac{x}{Q_1}(\log x)^3+x^{\frac{13}{14}+\varepsilon}(\log x)^4. \tag{1.1}$$

The implied constant is effective and can be made explicit using [7, Chapter 5] together with the best available constant in Pólya–Vinogradov inequality given by Frolenkov–Soundararajan [5, Theorem 2]. The effectivity is attained by getting rid of those moduli that have small prime divisors, thus of a possible exceptional character. Analogously to [1] one can state the corollary above for $\pi(y;q,a)$ instead of $\psi(y;q,a)$, where $\pi(y;q,a)$ counts the number of primes up to x congruent to $a \bmod q$.

Proposition 1.3 allows us to prove the Bombieri–Vinogradov theorem in the form of Theorem 1.2 and, hence, Corollary 1.4. In addition to Theorem 1.4, the proof uses the Siegel–Walfisz theorem, which states that

$$\psi(x,\chi)-\delta(\chi)x \ll_A xe^{-c\sqrt{\log x}}$$

uniformly for $q \leq (\log x)^A$. Here $A > 0$ is a fixed real number, c is an absolute positive constant, and $\delta(\chi) = 1$ if χ is principal and is zero otherwise. The implied constant in the Bombieri–Vinogradov theorem is ineffective since the implied constant in the Siegel–Walfisz theorem is ineffective. To prove Corollary 1.2 we use the Siegel–Walfisz theorem to deal with moduli

$q \le Q$ having small prime divisors and Theorem 1.4 to deal with the sum over the remaining moduli.

The recent work of Liu [10] gives us a genuinely effective Bombieri–Vinogradov theorem. This is ultimately due to the fact that we can use an effective Landau–Page result (see Page [12], Landau [8] and also Vinogradov [15, Chapter 10]), which is non-trivial up to $(\log x)^2$ instead of making a standard ineffective step on applying Siegel–Walfisz theorem.

In [10] various applications of the statement are considered, such as an asymptotic formula for the representation of a large integer as the sum of two squares and a prime and Titchmarsh divisor problem (both results obviously become effective).

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Auxiliary lemmas

We start with a so-called weighted Vaughan identity. It allows us to get cancellation in type II sums.

Lemma 1.5 (Weighted Vaughan identity). *Let $U, V \ge 1$. Let $\eta(t) : \mathbb{Z}^+ \to \mathbb{R}$ be any function such that $\eta(t) = 1$ for $t \le V$. We have*

$$\Lambda(n) = \lambda_0(n) + \lambda_1(n) + \lambda_2(n) + \lambda_3(n),$$

where

$$\lambda_0(n) = \begin{cases} \Lambda(n), & n \le U, \\ 0, & n > U, \end{cases} \qquad \lambda_1(n) = \sum_{d|n} \mu(d)\eta(d) \log \frac{n}{d},$$
$$\lambda_2(n) = -\sum_{c \le U} \sum_{dc|n} \mu(d)\Lambda(c)\eta(d), \quad \lambda_3(n) = \sum_{c > U} \sum_{dc|n} \mu(d)\Lambda(c)(1 - \eta(d)).$$

Proof. Let $n > U$, since otherwise the statement is trivial. Define the following quantities

$$\begin{aligned} \Lambda_1(n) &= \sum_{\substack{d|n \\ d \le V}} \mu(d) \log \frac{n}{d} \\ &= \lambda_1(n) - \sum_{\substack{d|n \\ d > V}} \mu(d)\eta(d) \log \frac{n}{d} = \lambda_1(n) + \lambda_1'(n), \end{aligned}$$

$$\begin{aligned}
\Lambda_2(n) &= -\sum_{c\le U}\sum_{\substack{dc|n\\ d\le V}}\mu(d)\Lambda(c)\\
&= \lambda_2(n) + \sum_{c\le U}\sum_{\substack{dc|n\\ d>V}}\mu(d)\Lambda(c)\eta(d) = \lambda_2(n)+\lambda_2'(n),\\
\Lambda_3(n) &= \sum_{c>U}\sum_{\substack{dc|n\\ d>V}}\mu(d)\Lambda(c)\\
&= \lambda_3(n) + \sum_{c>U}\sum_{\substack{dc|n\\ d>V}}\mu(d)\Lambda(c)\eta(d) = \lambda_3(n)+\lambda_3'(n).
\end{aligned}$$

Vaughan's identity in its classical form is

$$\Lambda(n) = \Lambda_1(n)+\Lambda_2(n)+\Lambda_3(n),$$

so it remains to show that $\lambda_1'(n)+\lambda_2'(n)+\lambda_3'(n)=0$ for every n. Let us rewrite this sum

$$\begin{aligned}
\sum_{i=1}^{3}\lambda_i'(n) &= \sum_{\substack{d|n\\ d>V}}\Bigg(-\mu(d)\eta(d)\log\frac{n}{d} + \sum_{\substack{c|n/d\\ c\le U}}\mu(d)\Lambda(c)\eta(d) + \sum_{\substack{c|n/d\\ c>U}}\mu(d)\Lambda(c)\eta(d)\Bigg)\\
&= \sum_{\substack{d|n\\ d>V}}\Bigg(-\mu(d)\eta(d)\log\frac{n}{d} + \mu(d)\eta(d)\sum_{c|n/d}\Lambda(c)\Bigg) = 0,
\end{aligned}$$

where in the last equality we used the fact that $\sum_{a|b}\Lambda(a)=\log b$. □

Lemma 1.6 (Graham [6]). *Let $1\le N_1\le N_2\le N$ and define*

$$f_i(d)=\begin{cases}\mu(d)\log(N_i/d), & d\le N_i,\\ 0, & d>N_i.\end{cases}$$

We have

$$\sum_{n=1}^{N}\Bigg(\sum_{d_1|n}f_1(d_1)\Bigg)\Bigg(\sum_{d_2|n}f_2(d_2)\Bigg) = N\log N_1 + O(N).$$

From the lemma above one can deduce the following.

Corollary 1.7. *Define a function $\eta(t)$, that is equal to 1 for $t\le V$, to 0 for $t>V_0$ and*

$$\eta(t)=\frac{\log(V_0/t)}{\log(V_0/V)},\quad V<t\le V_0.$$

Then

$$\sum_{k\le Y}\Bigg|\sum_{d|k}\mu(d)\eta(d)\Bigg|^2 \ll \frac{Y}{\log(V_0/V)}.$$

The constant here can be made explicit using [7, Chapter 5].

We also need the large sieve inequality as stated in a classical form in (see for example, Montgomery [11, p. 561]),

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\left|\sum_{m=m_0+1}^{m_0+M}a_m\chi(m)\right|^2\le (M+Q^2)\sum_{m=m_0+1}^{m_0+M}|a_m|^2,\qquad m_0,\ M\in\mathbb{N}, \tag{1.2}$$

from which we deduce the following.

Lemma 1.8 (Large sieve inequality)**.** *Let a_m, b_n be arbitrary complex numbers with $m_0\le m\le M, n_0\le n\le N$ and $m_0,n_0,M,N\in\mathbb{N}$. Then*

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\max_y\left|\sum_{m=m_0}^{M}\sum_{\substack{n=n_0\\ mn\le y}}^{N}a_mb_n\chi(mn)\right|$$
$$\le c_3(M'+Q^2)^{\frac12}(N'+Q^2)^{\frac12}\left(\sum_{m=m_0}^{M}|a_m|^2\right)^{\frac12}\left(\sum_{n=n_0}^{N}|b_n|^2\right)^{\frac12}\log(2MN),$$

where $c_3=2.64\ldots$ and $M'=M-m_0+1$, $N'=N-n_0+1$ are the number of terms in the sums over m and n respectively.

For the proof see [1, Lemma 6.1] or [3, Theorem 8.3.3].

2. Proof of Proposition 1.3

We proceed now with the proof of Proposition 1.3. Fix arbitrary real numbers $Q>0$ and $x\ge 4$. Without loss of generality we can assume that $2\le Q\le x^{1/2}$ and decompose the von Mangoldt function using a weighted form of Vaughan's identity, namely Lemma 1.5.

$$\Lambda(n)=\lambda_0(n)+\lambda_1(n)+\lambda_2(n)+\lambda_3(n),$$

where $\lambda_i(n)$, $i=0,1,2,3$ are as in the statement of the lemma and U,V,V_0 are parameters. Notice also that we are free to choose $\eta(t)$ as we wish, we only need to fulfill the conditions stated in Lemma 1.5.

Assume $y\le x$, $q\le Q$, and χ is a primitive character mod q. We use the above decomposition to write

$$\psi(y,\chi)=s_0+s_1+s_2+s_3,$$

where

$$s_i=\sum_{n\le y}\lambda_i(n)\chi(n).$$

Denote the contributions to our main sum by

$$S_i = \sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \max_{y\le x} |s_i|. \tag{2.1}$$

Easily we obtain

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \max_{y\le x} |\psi(y,\chi)| \le S_0 + S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
s_1 &= \sum_{d\le y} \mu(d)\chi(d)\eta(d) \sum_{h\le \frac{y}{d}} \chi(h)\log h,\\
s_2 &= \sum_{\substack{dcr\le y\\ c\le U}} \chi(dcr)\mu(d)\Lambda(c)\eta(d),\\
s_3 &= \sum_{n\le y} \chi(n) \sum_{c>U} \sum_{dc|n} \mu(d)\Lambda(c)(1-\eta(d)).
\end{aligned}$$

and $S_0 \ll UQ^2$ by the Chebyshev's estimate

$$|s_0| \le \sum_{n\le U} \Lambda(n) \ll U.$$

In what follows we choose $\eta(\,\cdot\,)$ from the paper by Graham [6], i. e.

$$\eta(d) = \frac{\log(V_0/d)}{\log(V_0/V)}, \quad V\le d\le V_0. \tag{2.2}$$

We remind that $\eta(d) = 1$ for $d \le V$ and $\eta(d) = 0$ for $d > V_0$. This choice immediately allows us to win $(\log x)^{\frac{1}{2}}$ in the last sum, that is of type II.

Type I sums. We start with linear sums among s_i and work with s_1 first. Write

$$\sum_{h\le y/d} \chi(h)\log h = \sum_{h\le y/d} \chi(h) \int_1^h \frac{\mathrm{d}u}{u}$$

and exchange the sum and the integral

$$\begin{aligned}
s_1 &= \sum_{d\le V_0} \mu(d)\chi(d)\eta(d) \int_1^{y/d} \sum_{u\le h\le y/d} \chi(h)\frac{\mathrm{d}u}{u}\\
&= \int_1^y \sum_{d\le \min(V_0, y/u)} \mu(d)\chi(d)\eta(d) \sum_{u\le h\le y/d} \chi(h)\frac{\mathrm{d}u}{u}.
\end{aligned}$$

Now since for $d > y/u$ the internal sum over h is empty we drop the condition $d \leq y/u$ and get

$$\begin{aligned} s_1 &= \int_1^y \Bigg(\sum_{d \leq V_0} \mu(d)\chi(d)\eta(d) \sum_{u \leq h \leq y/d} \chi(h) \Bigg) \frac{\mathrm{d}u}{u} \\ &= \int_1^y \Bigg(\sum_{d \leq V} \mu(d)\chi(d) \sum_{u \leq h \leq y/d} \chi(h) \Bigg) \frac{\mathrm{d}u}{u} \\ &\qquad + \frac{1}{\log(V_0/V)} \int_1^y \Bigg(\sum_{V \leq d \leq V_0} \mu(d)\chi(d) \log \frac{V_0}{d} \sum_{u \leq h \leq y/d} \chi(h) \Bigg) \frac{\mathrm{d}u}{u}, \end{aligned}$$

where in the last equality we used the definition of η given in (2.2). Denote the summands σ_1 and σ_2. Then for σ_1 we have

$$|\sigma_1| \leq \sum_{d \leq V} \max_{1 \leq u \leq y} \Bigg| \sum_{u \leq h \leq y/d} \chi(h) \Bigg| \int_1^y \frac{\mathrm{d}u}{u} \leq (\log y) \sum_{d \leq V} \max_{1 \leq u \leq y} \Bigg| \sum_{u \leq h \leq y/d} \chi(h) \Bigg|.$$

If $q = 1$, then we only have the trivial character modulo q and

$$|\sigma_1| \leq y(\log y) \sum_{d \leq V} \frac{1}{d} \leq x(\log xV)^2.$$

If $q > 1$ and χ is a primitive character mod q, we use the Pólya–Vinogradov inequality (see [5, Theorem 2] for explicit results): for all x, y we have

$$\Bigg| \sum_{x \leq n \leq y} \chi(n) \Bigg| < \sqrt{q} \log q. \tag{2.3}$$

Then

$$|\sigma_1| < (\log y) \sum_{d \leq V} \max_{1 \leq u \leq y} q^{\frac{1}{2}} \log q < q^{\frac{1}{2}} V (\log xV)^2.$$

Further, for σ_2 we put the absolute value inside the integral to get

$$|\sigma_2| \leq \frac{\log y}{\log(V_0/V)} \sum_{V \leq d \leq V_0} \log \frac{V_0}{d} \max_{1 \leq u \leq y} \Bigg| \sum_{u \leq h \leq y/d} \chi(h) \Bigg|$$

and work with the internal sum in the same way as in σ_1 to get $|\sigma_2| \leq x(\log xV_0)^2$ for $q = 1$ and $|\sigma_2| \leq q^{\frac{1}{2}} V_0 (\log xV_0)^2$ for $q > 1$.

On inserting the estimate for s_1 in (2.1) we obtain

$$\begin{aligned} S_1 &\leq \left(\max_{y\leq x} |s_1| + \sum_{1<q\leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}{}^{*} \max_{y\leq x} |s_1| \right) \\ &\leq \left(x(\log xV_0)^2 + V_0(\log xV_0)^2 \sum_{1<q\leq Q} \frac{q^{\frac{3}{2}}}{\varphi(q)} \sum_{\chi(q)}{}^{*} 1 \right) \\ &\leq (x + Q^{\frac{5}{2}} V_0)(\log xV_0)^2. \end{aligned} \tag{2.4}$$

Type II sums. Now we work with s_2 and want to use dyadic decomposition. Write

$$\begin{aligned} s_2 &= \sum_{\substack{cdr\leq y\\ c\leq U}} \Lambda(c)\mu(d)\eta(d)\chi(cdr) = \sum_{\substack{ct\leq y\\ c\leq U}} \sum_{d|t} \Lambda(c)\mu(d)\eta(d)\chi(ct) \\ &= \sum_{c\leq w} + \sum_{w<c\leq U} = s_2' + s_2'', \end{aligned}$$

where we introduced a new parameter w, that should be smaller than U and will be chosen later. We deal first with the small range of the parameter c in s_2, namely with s_2'. Write

$$s_2' = \sum_{c\leq w} \Lambda(c)\chi(c) \sum_{t\leq y/c} \sum_{d|t} \mu(d)\eta(d)\chi(t).$$

Since we have the bound

$$\left| \sum_{\substack{cd=t\\ c\leq w}} \Lambda(c)\mu(d)\eta(d)\chi(t) \right| \leq \sum_{c|t} \Lambda(c) = \log t,$$

then proceeding as for s_1 via the Pólya–Vinogradov inequality and using the fact that $cd = t \leq wV_0$ we get

$$|S_2'| \leq (x + Q^{\frac{5}{2}} wV_0)(\log(xwV_0))^2, \tag{2.5}$$

where the x term comes from the contribution of $q = 1$ and $Q^{\frac{5}{2}} wV_0$ from the remaining $q \neq 1$.

Next consider s_2''. Writing s_2'' as a dyadic sum we have

$$s_2'' = \sum_{\substack{M=2^{\alpha}\\ w/2<M\leq U}} \sum_{\substack{w<c\leq U\\ M<c\leq 2M}} \sum_{t\leq y/c} \sum_{d|t} \Lambda(c)\mu(d)\eta(d)\chi(ct).$$

Denote the contributions of s_2' and s_2'' to S_2 by S_2' and S_2'' respectively. Inserting the above into (2.1) and using the triangle inequality we get

$$S_2'' \le \sum_{\substack{M=2^\alpha \\ w/2<M\le U}} \sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \max_{y\le x} \left| \sum_{\substack{w<c\le U \\ M<c\le 2M}} \sum_{t\le y/c} \sum_{d|t} \Lambda(c)\mu(d)\eta(d)\chi(ct) \right|.$$

By the large sieve inequality given in Lemma 1.8 we get

$$S_2'' \ll \sum_{\substack{M=2^\alpha \\ w/2<M\le U}} (M'+Q^2)^{\frac12}(K'+Q^2)^{\frac12} \times \sigma_1(M)^{\frac12}\sigma_2(M)^{\frac12} \log\left(\frac{2x}{M}\min(U,2M)\right),$$

where M' and K' are the number of terms in sums over c and t respectively and

$$\sigma_1(M) = \sum_{\substack{w<c<U \\ M<c\le 2M}} \Lambda(c)^2,$$

$$\sigma_2(M) = \sum_{t\le y/M} \left| \sum_{d|t} \mu(d)\eta(d) \right|^2,$$

and $\log(2x/M\min(U,2M)) \ll \log x$. By Chebyshev's estimate we have $\sigma_1(M) \ll M\log U$ and using the bounds $M'\le M$, $K'\le x/M$ we obtain

$$S_2'' \ll (\log x)(\log U)^{\frac12} \sum_{\substack{M=2^\alpha \\ w/2<M\le U}} (M+Q^2)^{\frac12}\left(\frac{x}{M}+Q^2\right)^{\frac12} M^{\frac12}\sigma_2(M)^{\frac12}.$$

To bound $\sigma_2(M)$ we apply Corollary 1.7 and get

$$\sigma_2(M) \ll \frac{y}{M\log(V_0/V)}.$$

On putting it together we obtain

$$\begin{aligned} S_2'' &\ll (\log x)(\log U)^{\frac12}x^{\frac12} \sum_{\substack{M=2^\alpha \\ \frac12 w<M\le U}} (M+Q^2)^{\frac12}\left(\frac{x}{M}+Q^2\right)^{\frac12}\left(\log\frac{V_0}{V}\right)^{-\frac12} \\ &\ll (\log x)\frac{(\log U)^{\frac12}}{(\log(V_0/V))^{\frac12}}(\log(Uw))\left(x+Qxw^{-\frac12}+Q^2x^{\frac12}+U^{\frac12}Qx^{\frac12}\right), \end{aligned} \tag{2.6}$$

where we applied the bound

$$\sum_{\substack{M=2^\alpha \\ w/2<M\le U}} 1 \le \frac{\log(2U/w)}{\log 2}.$$

We continue with an estimate for S_3 that uses the large sieve inequality (Lemma 1.8) and properties of $\eta(\,\cdot\,)$ from Lemma 1.6. Writing s_3 as a dyadic sum we have

$$s_3 = \sum_{\substack{M=2^\alpha \\ U/2<M\le x/V}} \sum_{\substack{U<m\le x/V \\ M<m\le 2M}} \sum_{\substack{V<k\le x/M \\ mk\le y}} a_m c_k \chi(mk),$$

where $a_m = \Lambda(m)$ and $c_k = \sum_{d|k,\, d>V} \mu(d)(1-\eta(d))$. Inserting the above into (2.1) and using the triangle inequality we get

$$S_3 \le \sum_{\substack{M=2^\alpha \\ U/2<M\le x/V}} \sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \max_{y\le x} \left| \sum_{\substack{U<m\le x/V \\ M<m\le 2M}} \sum_{\substack{V<k\le x/M \\ mk\le y}} a_m c_k \chi(mk) \right|.$$

Now we apply the large sieve inequality given in Lemma 1.8 to get

$$S_3 \ll \sum_{\substack{M=2^\alpha \\ U/2<M\le x/V}} (M'+Q^2)^{\frac12}(K'+Q^2)^{\frac12}\sigma_1'(M)^{\frac12} \times \sigma_2'(M)^{\frac12} \log\left(\frac{2x}{M}\min\left(\frac{x}{V}, 2M\right)\right),$$

where

$$\sigma_1'(M) = \sum_{V<k\le x/M} |c_k|^2, \qquad \sigma_2'(M) = \sum_{\substack{U<m\le x/V \\ M<m\le 2M}} |a_m|^2.$$

and M' and K' denote the number of terms in the sums over m and k, respectively. Clearly, $\log(2x/M\min(x/V,2M)) \ll \log x$. From the definition of M' and N' we conclude

$$\begin{aligned} M' &= \min(2M, x/V) - \max(M+1, U+1) \le M, \\ K' &= x/M - (V+1) + 1 \le x/M. \end{aligned}$$

By Chebyshev's estimate we have an upper bound

$$\sigma_2'(M) \le \sum_{m\le 2M} \Lambda(m)^2 \le \psi(2M)\log 2M \ll M\log M.$$

Thus by the above we conclude

$$S_3 \ll (\log x) \sum_{\substack{M=2^\alpha \\ U/2<M\le x/V}} (M+Q^2)^{\frac12}\left(\frac{x}{M}+Q^2\right)^{\frac12}(M\log M)^{\frac12}\sigma_1'(M)^{\frac12}.$$

Further, we use

$$M(M+Q^2)(x/M+Q^2) = Mx + Q^2x + M^2Q^2 + MQ^4$$

and $\log M \le \log(x/V)$. to get

$$S_3 \ll (\log x)\left(\log\frac{x}{V}\right)^{\frac12}\sum_{\substack{M=2^\alpha\\ U/2<M\le x/V}}(Mx+Q^2x+M^2Q^2+MQ^4)^{\frac12}\sigma_1'(M)^{\frac12}.$$

By the definition of η given in (2.2) we have

$$1-\eta(d)=1-\frac{\log(V_0/d)}{\log(V_0/V)}=\frac{\log(d/V)}{\log(V_0/V)},\quad V\le d\le V_0.$$

On applying Corollary 1.6 we obtain

$$\sigma_1(M)=\sum_{V<k\le x/M}\left(\sum_{d|k}\mu(d)-\sum_{d|k}\mu(d)\eta(d)\right)^2\ll\frac{x}{M\log(V_0/V)}$$

that implies

$$S_3\ll\frac{\log x}{(\log(V_0/V))^{\frac12}}(\log(x/V))^{\frac12}\sum_{\substack{M=2^\alpha\\ U/2<M\le x/V}}\left(x^2+\frac{Q^2x^2}{M}+MQ^2x+Q^4x\right)^{\frac12}.$$

Since

$$\sum_{\substack{M=2^\alpha\\ U/2<M\le x/V}}1\le\frac{\log(2x/V)}{\log 2},$$

then

$$S_3\ll\frac{\log x}{(\log(V_0/V))^{\frac12}}(\log(x/V))^{\frac32}\left(x+QxU^{-\frac12}+QxV^{-\frac12}+Q^2x^{\frac12}\right). \tag{2.7}$$

Finally we have to adjust the parameters U, V, V_0, w using all our previous estimates $S_0 \ll UQ^2$, (2.4), (2.5), (2.6) and (2.7) We take $U=V$ and get

$$\begin{aligned}S=\sum_{q\le Q}\frac{q}{\varphi(q)}{\sum}^*_{\chi(q)}\max_{y\le x}|\psi(y,\chi)|\\ \ll \mathrm{R}(x,Q,w,V,V_0)\mathrm{G}(x,w,V,V_0),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&\mathrm{R}(x,Q,w,V,V_0)\\ &\quad=x+Q^{\frac52}(V+wV_0)+Qxw^{-\frac12}+QxV^{-\frac12}+Q^2x^{\frac12}+V^{\frac12}Qx^{\frac12},\end{aligned} \tag{2.9}$$

$$\mathrm{G}(x,Q,w,V,V_0) = \max\Big\{(\log xV_0)^2, (\log(xwV_0))^2, (\log(Vw))\frac{(\log V)^{\frac12}}{(\log(V_0/V))^{\frac12}}\log x, (\log(x/V))^{\frac32}\,\frac{\log x}{(\log(V_0/V))^{1/2}}\Big\}. \tag{2.10}$$

Now we specify V and V_0. We introduce a parameter $0 < \alpha < 1/2$ to be chosen later and subdivide into two ranges

(1) $x^{\alpha} \le Q \le x^{\frac12}$,
(2) $Q \le x^{\alpha}$.

In what follows we let $\beta_i, \gamma_i, \delta_i \in (0,1)$, $i = 1,2$ be chosen later. If $x^{\alpha} \le Q \le x^{\frac12}$, then we set $V = x^{\beta_1}/Q$. We choose $V_0 = x^{\delta_1}/Q$ and $w = x^{\gamma_1}/Q$. Then putting that into (2.9) we get

$$\mathrm{R}(x,Q) \ll x + Q^{\frac32}x^{\beta_1} + Q^{\frac12}x^{\gamma_1+\delta_1} + Q^{\frac32}x^{1-\frac{\gamma_1}{2}} + Q^{\frac32}x^{1-\frac{\beta_1}{2}} + Q^2x^{\frac12} + Q^{\frac12}x^{\frac{1+\beta_1}{2}}. \tag{2.11}$$

If $Q \le x^{\alpha}$, we let $V = x^{\beta_2}$, $V_0 = x^{\delta_2}$, $w = x^{\gamma_2}$. On inserting it into (2.9) we obtain

$$\mathrm{R}(x,Q) \ll x + Q^2x^{\beta_2} + Q^{\frac52}x^{\beta_2} + Q^{\frac52}x^{\gamma_2+\delta_2} + Qx^{1-\frac{\gamma_2}{2}} + Qx^{1-\frac{\beta_2}{2}} + Q^2x^{\frac12} + Qx^{\frac{1+\beta_2}{2}}. \tag{2.12}$$

Let $0 < \varepsilon < 1/14$. We keep in mind conditions $\alpha < 1/2$, $\gamma_1 < \beta_1$, $\delta_1 > \beta_1$ and put

$$\alpha = 3/7 + \varepsilon, \quad \beta_1 = 4/7, \quad \gamma_1 = 4/7 - \varepsilon, \quad \delta_1 = 4/7 + 5\varepsilon/2.$$

Then, substitution into (2.11) gives

$$\begin{aligned}\mathrm{R}(x,Q) &\ll x + Q^{\frac32}x^{\frac47} + Q^{\frac12}x^{\frac87+\frac{3\varepsilon}{2}} + Q^{\frac32}x^{\frac57+\frac{\varepsilon}{2}} + Q^{\frac32}x^{\frac57} + Q^2x^{\frac12} + Q^{\frac12}x^{\frac{11}{14}}\\ &\ll x + Q^2x^{\frac12},\end{aligned}$$

where we used

$$\begin{aligned}Q^{\frac32}x^{\frac47} &\le Q^2x^{\frac47-\frac{3}{14}-\frac{\varepsilon}{2}} < Q^2x^{\frac12},\\ Q^{\frac12}x^{\frac87+\frac{3\varepsilon}{2}} &\le Q^2x^{\frac87+\frac{3\varepsilon}{2}-\frac32\left(\frac37+\varepsilon\right)} = Q^2x^{\frac12},\\ Q^{\frac32}x^{\frac57+\frac{\varepsilon}{2}} &\le Q^2x^{\frac57+\frac{\varepsilon}{2}-\frac{3}{14}-\frac{\varepsilon}{2}} = Q^2x^{\frac12},\\ Q^{\frac32}x^{\frac57} &\le Q^2x^{\frac57-\frac12\left(\frac37+\varepsilon\right)} = Q^2x^{\frac12-\frac{\varepsilon}{2}} < Q^2x^{\frac12},\\ Q^{\frac12}x^{\frac{9}{14}} &\le Q^2x^{\frac{9}{14}-\frac32\left(\frac37+\varepsilon\right)} < Q^2x^{\frac12}.\end{aligned}$$

Similarly to satisfy $\gamma_2 < \beta_2$, $\delta_2 > \beta_2$ we put

$$\beta_2 = 1/7, \quad \gamma_2 = 1/7 - \varepsilon, \quad \delta_2 = 1/7 + \varepsilon/2$$

in (2.12) to obtain

$$\begin{aligned}\mathrm{R}(x,Q) &\ll x + Q^{\frac{5}{2}}x^{\frac{1}{7}} + Q^{\frac{5}{2}}x^{\frac{2}{7}-\frac{\varepsilon}{2}} + Qx^{\frac{13}{14}+\frac{\varepsilon}{2}} + Qx^{\frac{13}{14}} + Q^2x^{\frac{1}{2}} + Qx^{\frac{4}{7}} \\ &\ll x + Q^2x^{\frac{1}{2}} + Qx^{\frac{13}{14}+\frac{\varepsilon}{2}},\end{aligned}$$

where we used

$$\begin{aligned}Q^{\frac{5}{2}}x^{\frac{1}{7}} &\le Q^2x^{\frac{1}{7}+\frac{3}{14}+\frac{\varepsilon}{2}} = Q^2x^{\frac{5}{14}+\frac{\varepsilon}{2}} < Q^2x^{\frac{1}{2}}, \\ Q^{\frac{5}{2}}x^{\frac{2}{7}-\frac{\varepsilon}{2}} &\le Q^2x^{\frac{2}{7}-\frac{\varepsilon}{2}+\frac{3}{14}+\frac{\varepsilon}{2}} = Q^2x^{\frac{1}{2}}.\end{aligned}$$

Now we bound $\mathrm{G}(x,Q,w,V,V_0)$ given by (2.10). We notice that with our choice of parameters above $\log(V_0/V) \gg_\varepsilon \log x$, where the implied constant depends on β_i, δ_i. Thus $\log \ll_\varepsilon (\log x)^2$. Finally, we have

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\max_{y\le x}|\psi(y,\chi)| \ll_\varepsilon (x + Q^2x^{\frac{1}{2}} + Qx^{\frac{13}{14}+\frac{\varepsilon}{2}})(\log x)^2.$$

The power $13/14+\varepsilon/2$ is optimal here. Indeed, let us show first that $\alpha > 3/7$. The system

$$\begin{cases} Q^{\frac{1}{2}}x^{\gamma_1+\delta_1} \le Q^2x^{\frac{1}{2}}, \\ Q^{\frac{3}{2}}x^{1-\frac{\gamma_1}{2}} \le Q^2x^{\frac{1}{2}}, \end{cases}$$

brings us to

$$\begin{cases} \gamma_1+\delta_1-\frac{3\alpha}{2} \le \frac{1}{2}, \\ 1-\frac{\gamma_1}{2}-\frac{\alpha}{2} \le \frac{1}{2}. \end{cases}$$

Solving this we obtain $\delta_1 \le 5\alpha/2 - 1/2$. Further since $Q^{\frac{3}{2}}x^{1-\frac{\beta_1}{2}} \le Q^2x^{\frac{1}{2}}$, we get

$$1-\alpha \le \beta_1 < \delta_1 \le 5\alpha/2 - 1/2.$$

Thus $\alpha > 3/7$. We use that to obtain the fact that the term Qx^A has $A > 13/14$. Since $Q^{\frac{5}{2}}x^{\gamma_2+\delta_2} \le Q^2x^{\frac{1}{2}}$, we get $\gamma_2+\delta_2 \le 1/2-\alpha/2 < 2/7$. The inequality $Qx^{1-\frac{\beta_2}{2}} \le Qx^A$ gives us $\delta_2 > \beta_2 \ge 2(1-A)$. Similarly for γ_2 we obtain $\gamma_2 \ge 2(1-A)$ because of the term $Qx^{1-\frac{\gamma_2}{2}} \le Qx^A$. Combining all of this we get $4(1-A) < \gamma_2+\delta_2 < 2/7$ and thus $A > 13/14$.

3. Proofs of Theorem 1.2 and Corollary 1.4

The proof is essentially the same and differs only in the last step. It is classical and almost identical to the one in [1] and [13], hence we only sketch it here. Let $y \ge 2$,$(a,q) = 1$ and χ_0 be the principal character modulo q.

Define $\psi'(y,\chi)=\psi(y,\chi)$ if $\chi\neq\chi_0$ and $\psi'(y,\chi)=\psi(y,\chi)-\psi(y)$ otherwise. Then by orthogonality of characters modulo q, we have

$$\psi(y,q,a)-\frac{\psi(y)}{\varphi(q)}=\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\psi'(y,\chi).$$

For a character $\chi\pmod q$, we let χ^* be the primitive character modulo q^* inducing χ. Thus

$$\psi'(y,\chi^*)-\psi'(y,\chi)=\psi(y,\chi^*)-\psi(y,\chi)=\sum_{p^k\le y}(\log p)(\chi^*(p^k)-\chi(p^k)).$$

If $p|q$ then $(p^k,q^*)=1$, and hence $\chi^*(p^k)=\chi(p^k)$. If $p|q$ then $\chi(p^k)=0$. Therefore

$$|\psi'(y,\chi^*)-\psi'(y,\chi)|\le\sum_{\substack{p^k\le y\\ p|q}}(\log p)\le(\log y)\sum_{p|q}1\le(\log qy)^2.$$

Thus the quantity we want to average can be bounded as

$$\left|\psi(y,q,a)-\frac{\psi(y)}{\varphi(q)}\right|\le\frac{1}{\varphi(q)}\sum_{\chi}|\psi'(y,\chi)|\le(\log qy)^2+\frac{1}{\varphi(q)}\sum_{\chi}|\psi'(y,\chi^*)|.$$

We have to take care just of the second term in the inequality above, since the first one is smaller than the desired bound. A primitive character $\chi^*\bmod q^*$ induces characters of moduli dq^* and $\psi'(y,\chi^*)=0$ for χ principal, we observe

$$\sum_{\substack{q\le Q\\ l(q)>Q_1}}\frac{1}{\varphi(q)}\sum_{\substack{q^*|q\\ q^*\neq 1}}\sideset{}{^*}\sum_{\chi(q^*)}\max_{2\le y\le x}|\psi'(y,\chi)| \le\sum_{\substack{q^*\le Q\\ l(q^*)>Q_1}}\sideset{}{^*}\sum_{\chi(q^*)}\max_{2\le y\le x}|\psi'(y,\chi)|\sum_{k\le Q/q^*}\frac{1}{\varphi(kq^*)}. \tag{3.1}$$

Since $q^*\le Q\le x^{1/2}$ and $\varphi(k)\varphi(q^*)\le\varphi(kq^*)$ we have

$$\sum_{k\le Q/q^*}\frac{1}{\varphi(kq^*)}\ll\frac{\log x}{\varphi(q^*)}.$$

For $q>1$ and χ primitive character $(\bmod\, q)$, we know that χ is non-principal and $\psi(y,\chi)=\psi'(y,\chi)$. Since we assumed $Q_1\ge 1$ then we can replace $\psi'(y,\chi)$ by $\psi(y,\chi)$ inside the internal sum in (3.1). Hence

$$\sum_{\substack{q\le Q\\ l(q)>Q_1}}\frac{1}{\varphi(q)}\sum_{\substack{q^*|q\\ q^*\ne 1}}\sideset{}{^*}\sum_{\chi(q^*)}\max_{2\le y\le x}|\psi'(y,\chi)| \\ \ll (\log x)\sum_{\substack{q\le Q\\ l(q)>Q_1}}\frac{1}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\max_{2\le y\le x}|\psi(y,\chi)|.$$

For the right-hand side of the above apply the partial summation

$$\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\max_{2\le y\le x}|\psi(y,\chi)| \\ =\frac{1}{Q}\sum_{q\le Q}f(q)-\frac{1}{Q_1}\sum_{q\le Q_1}f(q)+\int_{Q_1}^{Q}\left(\sum_{q\le t}f(q)\right)\frac{\mathrm{d}t}{t},$$

where

$$f(q)=\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi(q)}\max_{2\le y\le x}|\psi(y,\chi)|$$

and use Proposition 1.3 to get

$$\sum_{q\le Q}f(q)\ll_\varepsilon\left(x+Q^2x^{\frac12}+Qx^{\frac{13}{14}+\varepsilon}\right)(\log x)^2.$$

This finishes the proof of Corollary 1.4. We apply Siegel–Walfisz theorem for the remaining moduli to get Theorem 1.2.

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