

# JOURNAL

de Théorie des Nombres  
de BORDEAUX

*anciennement Séminaire de Théorie des Nombres de Bordeaux*

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Tome 31, n° 2 (2019), p. 343-360.

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## Generating functions for multiple zeta star values

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RÉSUMÉ. Nous étudions les fonctions génératrices des valeurs des fonctions polyzêta  $\zeta^*(s_1, \dots, s_m)$  dans le cadre général. Ces fonctions génératrices établissent un lien entre les nombres polyzêta et les sommes d'Euler multiples, ce qui nous permet d'exprimer chaque valeur polyzêta en termes de sommes d'Euler multiples alternées, et notamment réduire la longueur des blocs de deux dans les sommes résultantes.

ABSTRACT. We study generating functions for multiple zeta star values in general form. These generating functions provide a connection between multiple zeta star values and multiple Euler sums, which allows us to express each multiple zeta star value in terms of multiple alternating Euler sums, and specifically, reduce the length of blocks of twos in the resulting sums.

### 1. Introduction

Multiple Euler sums and multiple zeta values (MZVs) have been of interest for mathematicians and physicists for more than two decades. The systematic study of MZVs has started from works of Hoffman and Zagier in the 1990s, although some partial historical results are dated back to the work of Euler. In this paper, we will study generating functions for multiple zeta star values in general form. These generating functions provide a connection between multiple zeta star values (MZSVs) and multiple Euler sums, which allows us to express each MZSV in terms of multiple Euler sums, and specifically, reduce the length of blocks of twos in the resulting sums.

To begin with precise definitions, let  $\mathbb{N}$  be the set of positive integers,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\overline{\mathbb{N}} = \{\overline{s} : s \in \mathbb{N}\}$ , and  $\mathbb{D} = \mathbb{N} \cup \overline{\mathbb{N}}$  be the set of signed positive integers. For all  $s \in \mathbb{N}$ , the absolute value function  $|\cdot|$  on  $\mathbb{D}$  is defined by  $|s| = |\overline{s}| = s$  and the sign function is given by  $\text{sgn}(s) = 1$ ,  $\text{sgn}(\overline{s}) = -1$ .

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Manuscrit reçu le 9 septembre 2018, accepté le 11 mai 2019.

2010 *Mathematics Subject Classification.* 11M32, 11M35, 05A15, 30B10, 30D05, 39B32.

*Mots-clefs.* Multiple zeta star value, multiple zeta value, generating function, Euler sum.

For  $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{D}^m$ , we define the (alternating) Euler sums by nested sums with strict and non-strict inequalities,

$$(1.1) \quad \zeta(\mathbf{s}) = \sum_{k_1 > \dots > k_m \geq 1} \prod_{j=1}^m \frac{(\text{sgn}(s_j))^{k_j}}{k_j^{|s_j|}}, \quad \zeta^*(\mathbf{s}) = \sum_{k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{(\text{sgn}(s_j))^{k_j}}{k_j^{|s_j|}},$$

respectively, where  $s_1 \neq 1$  in order for the series to converge. If  $\mathbf{s} \in \mathbb{N}^m$ , then  $\zeta(\mathbf{s})$  is called a *multiple zeta value* (MZV) and  $\zeta^*(\mathbf{s})$  a *multiple zeta star value* (MZSV). We assign two characteristics to each of the sums above: the length (or depth)  $\ell(\mathbf{s}) := m$  and the weight  $|\mathbf{s}| := |s_1| + \dots + |s_m|$ . By convention, we set  $\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ . By  $\{s\}^m$  we denote the sequence formed by repeating the symbol  $\{s\}$   $m$  times.

In particular, for  $s \in \mathbb{N}$  we have

$$\zeta(\overline{s}) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = \begin{cases} -\log 2, & \text{if } s = 1; \\ (2^{1-s} - 1)\zeta(s), & \text{if } s \geq 2, \end{cases}$$

where  $\zeta(s)$  is a value of the Riemann zeta function. The simplest explicit evaluations of multiple zeta values include

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m + 1)!}, \quad \zeta^*(\{2\}^m) = -2\zeta(\overline{2m}).$$

These formulas follow from the Laurent series expansions of two functions  $\sin(\pi z)/\pi z$  and  $\pi z/\sin(\pi z)$  (see [4] for more details).

In 2012, Zagier [11] found explicit formulas for the values  $\zeta(\{2\}^a, 3, \{2\}^b)$  and  $\zeta^*(\{2\}^a, 3, \{2\}^b)$ ,  $a, b \in \mathbb{N}_0$ , in terms of rational linear combinations of products  $\zeta(m)\pi^{2n}$  with  $m + 2n = 2a + 2b + 3$ . These formulas played an important role in Brown’s proof [2] of the Hoffman conjecture [8] that every multiple zeta value is a  $\mathbb{Q}$ -linear combination of values  $\zeta(s_1, \dots, s_r)$  for which each  $s_i$  is either 2 or 3.

In [4], we gave another proof of the theorem of Zagier via a new representation of the generating function for  $\zeta^*(\{2\}^a, 3, \{2\}^b)$  in terms of the double series

$$(1.2) \quad \sum_{a,b \geq 0} \zeta^*(\{2\}^a, 3, \{2\}^b) x^{2a} y^{2b} = -2 \sum_{j=1}^{\infty} \frac{(-1)^j j}{(j^2 - x^2)(j^2 - y^2)} - 4 \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2 - x^2} \sum_{k=1}^{j-1} \frac{k}{k^2 - y^2}$$

by using a hypergeometric identity of Andrews and then evaluating the right-hand side in terms of the digamma function with the help of complex integration and the residue theorem. Formula (1.2) also implies that  $\zeta^*(\{2\}^a, 3, \{2\}^b)$  is expressible in terms of double Euler sums

$$(1.3) \quad \zeta^*(\{2\}^a, 3, \{2\}^b) = -2\zeta(\overline{2a + 2b + 3}) - 4\zeta(\overline{2a + 2}, 2b + 1),$$

which by [3, Theorem 7.2] can be reduced to linear combinations of products of single zeta values, giving one more proof of Zagier’s theorem (see [6, Remark 2.7]). Note that formula (1.3) as well as the similar one

$$\zeta^*(\{2\}^a, 1, \{2\}^b) = -2\zeta(\overline{2a + 2b + 1}) - 4\zeta(2a + 1, \overline{2b}), \quad a, b \in \mathbb{N},$$

were proved even earlier in [6] by another method using finite binomial identities. These formulas were generalized by Zhao [12] and Linebarger and Zhao [9] that led to the proof of the Two-one formula conjectured by Ohno and Zudilin [10], which states that

$$\zeta^*(\{2\}^{a_1}, 1, \{2\}^{a_2}, \dots, 1, \{2\}^{a_d}, 1) = \sum_{\mathbf{p}=(2a_1+1)\circ\cdots\circ(2a_d+1)} 2^{\ell(\mathbf{p})} \zeta(\mathbf{p}),$$

where  $\mathbf{p}$  runs through all indices of the form  $(2a_1 + 1) \circ \cdots \circ (2a_d + 1)$  with “ $\circ$ ” being either the symbol “,” or the sign “+”. These results were extended further by the present authors and Zhao [7, Theorem 1.4], to get formulas for arbitrary multiple zeta star values  $\zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d})$  in terms of multiple Euler sums.

In this paper, we generalize the formula for generating function (1.2) to include generating functions of multiple zeta star values with an arbitrary number of blocks of twos.

**Theorem 1.1.** *For any integer  $d \geq 1$  and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have*

$$\begin{aligned} \sum_{a_0, \dots, a_d \geq 0} \zeta^*(\{2\}^{a_0}, 3, \{2\}^{a_1}, \dots, 3, \{2\}^{a_d}) z_0^{2a_0} \dots z_d^{2a_d} \\ = -2 \sum_{k_0 \geq \dots \geq k_d \geq 1} \frac{(-1)^{k_0} k_d^2}{k_0^2 - z_0^2} \prod_{i=1}^d \frac{2^{\Delta(k_{i-1}, k_i)}}{k_i(k_i^2 - z_i^2)}, \end{aligned}$$

where

$$\Delta(a, b) = \begin{cases} 0, & \text{if } a = b; \\ 1, & \text{else.} \end{cases}$$

In particular,

$$\sum_{a_0 \geq 0} \zeta^*(\{2\}^{a_0}) z_0^{2a_0} = 1 - 2z_0^2 \sum_{k_0 \geq 1} \frac{(-1)^{k_0}}{k_0^2 - z_0^2}.$$

In the same vein we give the formulas for the Two-one and Two-three-two-one generating functions.

**Theorem 1.2.** For any integer  $d \geq 0$  and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$\sum_{a_0, \dots, a_d \geq 0} \zeta^* (\{2\}^{a_0+1}, 1, \{2\}^{a_1}, \dots, 1, \{2\}^{a_d}) z_0^{2a_0} \dots z_d^{2a_d} = - \sum_{k_0 \geq \dots \geq k_d \geq 1} \frac{(-1)^{k_d} k_d}{k_0^2} \prod_{i=0}^d \frac{k_i \cdot 2^{\Delta(k_{i-1}, k_i)}}{k_i^2 - z_i^2}$$

and

$$\sum_{a_0, \dots, a_d \geq 0} \zeta^* (\{2\}^{a_0+1}, 1, \{2\}^{a_1}, \dots, 1, \{2\}^{a_d}, 1) z_0^{2a_0} \dots z_d^{2a_d} = \sum_{k_0 \geq \dots \geq k_d \geq 1} \frac{1}{k_0^2} \prod_{i=0}^d \frac{k_i \cdot 2^{\Delta(k_{i-1}, k_i)}}{k_i^2 - z_i^2},$$

where  $k_{-1} = 0$ .

**Theorem 1.3.** For any integer  $d \geq 1$  and any complex numbers  $z_0, \dots, z_{2d}$  with  $|z_j| < 1, j = 0, 1, \dots, 2d$ , we have

$$\sum_{a_0, \dots, a_{2d} \geq 0} \zeta^* (\{2\}^{a_0}, 3, \{2\}^{a_1}, 1, \dots, 3, \{2\}^{a_{2d-1}}, 1, \{2\}^{a_{2d}}) z_0^{2a_0} \dots z_{2d}^{2a_{2d}} = - \sum_{k_0 \geq \dots \geq k_{2d} \geq 1} k_{2d}^2 \prod_{i=0}^{2d} \frac{(-1)^{k_i} \cdot 2^{\Delta(k_{i-1}, k_i)}}{k_i^2 - z_i^2},$$

where  $k_{-1} = 0$ , and

$$\sum_{a_1, \dots, a_{2d} \geq 0} \zeta^* (\{2\}^{a_1}, 3, \{2\}^{a_2}, 1, \dots, 3, \{2\}^{a_{2d}}, 1) z_1^{2a_1} \dots z_{2d}^{2a_{2d}} = \sum_{k_1 \geq \dots \geq k_{2d} \geq 1} \prod_{i=1}^{2d} \frac{(-1)^{k_i} \cdot 2^{\Delta(k_{i-1}, k_i)}}{k_i^2 - z_i^2},$$

where  $k_0 = 0$ .

For a string of positive integers  $\mathbf{r} = (r_1, \dots, r_c) \in \mathbb{N}^c$  and positive integers  $k, m$ , define the multiple sharp sum

$$S_{k,m}^\#(r_1, \dots, r_c) = \begin{cases} \sum_{k \geq l_1 \geq \dots \geq l_c \geq m} \frac{2^{\Delta(k, l_1) + \Delta(l_1, l_2) + \dots + \Delta(l_{c-1}, l_c) + \Delta(l_c, m)}}{l_1^{r_1} l_2^{r_2} \dots l_c^{r_c}}, & \text{if } \mathbf{r} \neq \emptyset \text{ and } k \geq m; \\ 2^{\Delta(k, m)}, & \text{if } \mathbf{r} = \emptyset \text{ or } k < m. \end{cases}$$

Actually, our main result is more general than the above three theorems, since it provides formulas for generating functions of arbitrary multiple zeta star values with an arbitrary number of blocks of twos.

**Theorem 1.4.** For any integers  $d \geq 1$ ,  $c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ ,  $c_1 \geq 3$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$(1.4) \quad \sum_{a_0, a_1, \dots, a_d \geq 0} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ = - \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\# (\{1\}^{c_i - 3}),$$

where

$$c_0 = 1, \quad c_{d+1} = 0, \quad k_{-1} = 0, \quad \delta_i = \delta(c_i) + \delta(c_{i+1}), \quad \text{and } \delta(c) = \begin{cases} 2, & \text{if } c = 0; \\ 1, & \text{if } c = 1; \\ 0, & \text{if } c \geq 3. \end{cases}$$

If all  $c_i$  take only values 1 or 3, the formula can be simplified.

**Corollary 1.5.** For any integers  $d \geq 1$ ,  $c_1, \dots, c_d \in \{1, 3\}$ ,  $c_1 \geq 3$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$\sum_{a_0, a_1, \dots, a_d \geq 0} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ = - \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} 2^{\Delta(k_{i-1}, k_i)}$$

with the same notation as in Theorem 1.4.

The case  $z_d = 0$  and  $c_d = 1$  for  $d \geq 2$  also leads to the simplification of the right hand side of (1.4). Note that after substitution  $z_d = 0$  and  $c_d = 1$  in (1.4), we replace  $d - 1$  by  $d$  to get the next corollary.

**Corollary 1.6.** For any integers  $d \geq 1$ ,  $c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ ,  $c_1 \geq 3$ , and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$\sum_{a_0, a_1, \dots, a_d \geq 0} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ = \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\# (\{1\}^{c_i - 3}),$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 1.4.

**Theorem 1.7.** For any integers  $d \geq 0, 0 \leq m \leq d, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$  such that  $c_1 \geq 3$  if  $m \geq 1$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with

$|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$(1.5) \quad \sum_{\substack{a_0, a_1, \dots, a_d \geq 0 \\ a_m \geq 1}} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ = -z_m^2 \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{k_m^2} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$$

with the same notation as in Theorem 1.4.

**Corollary 1.8.** For any integers  $d \geq 0, 0 \leq m \leq d, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$  such that  $c_1 \geq 3$  if  $m \geq 1$ , and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$\sum_{\substack{a_0, a_1, \dots, a_d \geq 0 \\ a_m \geq 1}} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ = z_m^2 \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{k_m^2} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 1.4.

The next theorem provides a formula for arbitrary MZSV in terms of Euler sums. One can consider this type of relations as duality relations between multiple zeta star values and multiple Euler sharp sums. For different types of duality relations known so far for multiple Euler sums, see [1, Section 6].

**Theorem 1.9.** For any integers  $d \geq 0, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}, a_0, \dots, a_d \in \mathbb{N}_0$ , and  $c_1 \geq 3$  if  $a_0 = 0$  and  $d \geq 1$ , and  $a_0 \geq 1$  if  $d = 0$ , we have

$$(1.6) \quad \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) \\ = - \sum_{k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i}}{k_i^{2a_i + 3 - \delta_i}} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$$

with the same notation as in Theorem 1.4; and

$$(1.7) \quad \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) \\ = \sum_{k_0 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i}}{k_i^{2a_i + 3 - \delta_i}} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}),$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 1.4. Moreover, if  $c_i \in \{1, 3\}$ , then  $S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$  is replaced by  $2^{\Delta(k_{i-1}, k_i)}$  in the above formulas.

Note that Theorem 1.9 implies [7, Theorem 1.4] if we expand and regroup the inner sharp sums in powers of 2.

Summarizing, it is worth mentioning that generating functions (1.4) and (1.5) in their generality and simplicity may be very useful in applications. For example, in [5], we show how to apply generating functions obtained in this paper for evaluation of certain explicit formulas as well as sum formulas for multiple zeta star values on 3-2-1 indices.

### 2. Auxiliary Statements

In this section, we prove several lemmas that will be needed in the sequel. From [6, (2.1), (2.2), and (2.5)] we have the following statement.

**Lemma 2.1.** *For any positive integer  $n$  and a non-negative integer  $l$ , we have*

$$(2.1) \quad 2 \sum_{k=l+1}^n (-1)^k \frac{\binom{n}{k}}{\binom{n+k}{k}} = \frac{(-1)^{l+1}}{n} \cdot \frac{(n-l)\binom{n}{l}}{\binom{n+l}{l}},$$

$$(2.2) \quad 2 \sum_{k=l+1}^n \frac{k\binom{n}{k}}{\binom{n+k}{k}} = \frac{(n-l)\binom{n}{l}}{\binom{n+l}{l}}.$$

If  $n \geq 2$ , then

$$(2.3) \quad \sum_{k=1}^n \frac{(-1)^k k^2 \binom{n}{k}}{\binom{n+k}{k}} = 0.$$

**Lemma 2.2.** *For any positive integers  $n, l$  and a non-negative integer  $c$ , we have*

$$(2.4) \quad \sum_{k=l}^n \frac{k\binom{n}{k} 2^{\Delta(k,l)}}{\binom{n+k}{k}} = \frac{n\binom{n}{l}}{\binom{n+l}{l}},$$

$$(2.5) \quad \sum_{k=l}^n \frac{(-1)^k \binom{n}{k}}{\binom{n+k}{k}} S_{k,l}^{\sharp}(\{1\}^c) = \frac{(-1)^l}{n^{c+1}} \cdot \frac{l\binom{n}{l}}{\binom{n+l}{l}}.$$

*Proof.* The proof of (2.4) easily follows from Lemma 2.1, identity (2.2),

$$\sum_{k=l}^n \frac{k\binom{n}{k} 2^{\Delta(k,l)}}{\binom{n+k}{k}} = \frac{l\binom{n}{l}}{\binom{n+l}{l}} + 2 \sum_{k=l+1}^n \frac{k\binom{n}{k}}{\binom{n+k}{k}} = \frac{l\binom{n}{l}}{\binom{n+l}{l}} + \frac{(n-l)\binom{n}{l}}{\binom{n+l}{l}} = \frac{n\binom{n}{l}}{\binom{n+l}{l}}.$$

To prove (2.5), we apply induction on  $c$ . For  $c = 0$ ,  $S_{k,l}^{\sharp}(\{1\}^c) = 2^{\Delta(k,l)}$  and we have by (2.1),

$$(2.6) \quad \sum_{k=l}^n \frac{(-1)^k \binom{n}{k} 2^{\Delta(k,l)}}{\binom{n+k}{k}} = \frac{(-1)^l \binom{n}{l}}{\binom{n+l}{l}} + 2 \sum_{k=l+1}^n \frac{(-1)^k \binom{n}{k}}{\binom{n+k}{k}} = \frac{(-1)^l}{n} \cdot \frac{l\binom{n}{l}}{\binom{n+l}{l}}.$$



If  $c \geq 1$ , then changing the order of summation and applying identity (2.6), we obtain

$$\begin{aligned}
 & \sum_{k=l}^n \frac{(-1)^k \binom{n}{k}}{\binom{n+k}{k}} S_{k,l}^\#(\{1\}^c) \\
 &= \sum_{k=l}^n \frac{(-1)^k \binom{n}{k}}{\binom{n+k}{k}} \sum_{k \geq l_1 \geq \dots \geq l_c \geq l} \frac{2^{\Delta(k,l_1) + \Delta(l_1,l_2) + \dots + \Delta(l_c,l)}}{l_1 l_2 \dots l_c} \\
 &= \sum_{n \geq l_1 \geq \dots \geq l_c \geq l} \left( \sum_{k=l_1}^n \frac{(-1)^k \binom{n}{k} 2^{\Delta(k,l_1)}}{\binom{n+k}{k}} \right) \frac{2^{\Delta(l_1,l_2) + \dots + \Delta(l_c,l)}}{l_1 l_2 \dots l_c} \\
 &= \sum_{n \geq l_1 \geq \dots \geq l_c \geq l} \frac{(-1)^{l_1}}{n} \cdot \frac{l_1 \binom{n}{l_1}}{\binom{n+l_1}{l_1}} \cdot \frac{2^{\Delta(l_1,l_2) + \dots + \Delta(l_c,l)}}{l_1 l_2 \dots l_c} \\
 &= \frac{1}{n} \sum_{l_1=l}^n \frac{(-1)^{l_1} \binom{n}{l_1}}{\binom{n+l_1}{l_1}} S_{l_1,l}^\#(\{1\}^{c-1}).
 \end{aligned}$$

Now formula (2.5) easily follows by induction on  $c$ . □

The proof of the next lemma was essentially given in detail in [9, Lemma 4.2]. We slightly modified the formulation to embrace a more general class of series.

**Lemma 2.3.** *Let  $M, c, a \in \mathbb{R}, M > 0, a > 1$ , and let  $R_k$  be a sequence of real numbers satisfying  $|R_k| < \frac{M(\log k+1)^c}{k^a}$  for all  $k = 1, 2, \dots$ . Then*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n |R_k| \left( 1 - \frac{\binom{n}{k}}{\binom{n+k}{k}} \right) = 0.$$

### 3. Generating Functions For Multiple Harmonic Sums

In this section, we prove a finite version of identity (1.4), from which Theorem 1.4 will follow by limit transition. For any  $n, m \in \mathbb{N}$  and  $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{D}^m$ , we define the (alternating) multiple harmonic sums by

$$H_n(\mathbf{s}) = \sum_{n \geq k_1 > \dots > k_m \geq 1} \prod_{j=1}^m \frac{(\text{sgn}(s_j))^{k_j}}{k_j^{|s_j|}}$$

and

$$H_n^*(\mathbf{s}) = \sum_{n \geq k_1 \geq \dots \geq k_m \geq 1} \prod_{j=1}^m \frac{(\text{sgn}(s_j))^{k_j}}{k_j^{|s_j|}}.$$

By convention, we put  $H_n(\mathbf{s}) = 0$  if  $n < m$ , and  $H_n(\emptyset) = H_n^*(\emptyset) = 1$ .

Let  $F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d)$  denote the generating function of the multiple harmonic star sum,

$$F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = \sum_{a_0, a_1, \dots, a_d \geq 0} H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}.$$

**Theorem 3.1.** For any integers  $n \geq 1, d \geq 0, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$(3.1) \quad F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = - \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}),$$

where

$$c_0 = 1, c_{d+1} = 0, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1}), \text{ and } \delta(c) = \begin{cases} 2, & \text{if } c = 0; \\ 1, & \text{if } c = 1; \\ 0, & \text{if } c \geq 3. \end{cases}$$

*Proof.* If  $n = 1$ , the theorem is obviously true. We have

$$F_1(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = \sum_{a_0, a_1, \dots, a_d \geq 0} z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} = \prod_{j=0}^d \frac{1}{1 - z_j^2},$$

and the right-hand side of (3.1) is

$$-\frac{\binom{1}{1}}{\binom{2}{1}} \left( \prod_{i=0}^d \frac{(-1)^{\delta_i}}{1 - z_i^2} \right) 2^{\Delta(0,1)} = (-1)^{1+\delta_0+\delta_1+\dots+\delta_d} \prod_{i=0}^d \frac{1}{1 - z_i^2} = \prod_{i=0}^d \frac{1}{1 - z_i^2}.$$

If  $d = 0$ , the formula becomes

$$(3.2) \quad F_n(; z_0) = \sum_{a_0 \geq 0} H_n^*(\{2\}^{a_0}) z_0^{2a_0} = -2 \sum_{n \geq k_0 \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \cdot \frac{(-1)^{k_0} k_0^2}{k_0^2 - z_0^2}.$$

Then it is easy to see that

$$\begin{aligned} F_n(; z_0) &= \sum_{a_0 \geq 0} H_n^*(\{2\}^{a_0}) z_0^{2a_0} = \sum_{a_0=0}^{\infty} z_0^{2a_0} \sum_{k=0}^{a_0} \frac{1}{n^{2(a_0-k)}} H_{n-1}^*(\{2\}^k) \\ &= \sum_{k=0}^{\infty} H_{n-1}^*(\{2\}^k) z_0^{2k} \sum_{a_0=k}^{\infty} \frac{z_0^{2(a_0-k)}}{n^{2(a_0-k)}} = \frac{n^2}{n^2 - z_0^2} F_{n-1} (; z_0) \end{aligned}$$

and (3.2) follows immediately by induction on  $n$ . Indeed, we have

$$\begin{aligned} F_n(; z_0) &= \frac{n^2}{n^2 - z_0^2} F_{n-1} (; z_0) = \frac{-2n^2}{n^2 - z_0^2} \sum_{k=1}^{n-1} \frac{\binom{n-1}{k}}{\binom{n-1+k}{k}} \frac{(-1)^k k^2}{k^2 - z_0^2} \\ &= 2 \sum_{k=1}^n \frac{(-1)^k k^2}{k^2 - z_0^2} \frac{\binom{n}{k}}{\binom{n+k}{k}} \left( \frac{k^2 - z_0^2}{n^2 - z_0^2} - 1 \right) = -2 \sum_{k=1}^n \frac{(-1)^k k^2}{k^2 - z_0^2} \frac{\binom{n}{k}}{\binom{n+k}{k}}, \end{aligned}$$

where in the last equality we used identity (2.3).

For the general case  $n > 1$  and  $d > 0$ , we proceed by induction on  $n + d$ . When  $n + d = 1$  or  $2$  the formula is true by the above. Let  $N$  be a positive integer. Suppose (3.1) is true for all  $n + d \leq N$ . To prove it for  $n + d = N + 1$  with  $n > 1$  and  $d > 0$ , we consider the expansion of the finite sum

$$\begin{aligned} &H_n^* (\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) \\ &= \frac{1}{n^{2a_0+c_1}} H_n^* (\{2\}^{a_1}, c_2, \{2\}^{a_2}, \dots, c_d, \{2\}^{a_d}) \\ &\quad + \sum_{k=0}^{a_0} \frac{1}{n^{2(a_0-k)}} H_{n-1}^* (\{2\}^k, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}), \end{aligned}$$

which leads to the following reduction of the generating function:

$$\begin{aligned} F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) &= \sum_{a_0 \geq 0} \frac{z_0^{2a_0}}{n^{2a_0+c_1}} \cdot F_n(c_2, \dots, c_d; z_1, \dots, z_d) \\ &+ \sum_{a_0, \dots, a_d \geq 0} \sum_{k=0}^{a_0} \frac{1}{n^{2(a_0-k)}} H_{n-1}^* (\{2\}^k, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} \dots z_d^{2a_d}. \end{aligned}$$

Changing the order of summation in the second sum and simplifying, we get

$$\begin{aligned} &\sum_{a_0, \dots, a_d \geq 0} \sum_{k=0}^{a_0} \frac{1}{n^{2(a_0-k)}} H_{n-1}^* (\{2\}^k, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} \dots z_d^{2a_d} \\ &= \sum_{k=0}^{\infty} \sum_{a_1, \dots, a_d \geq 0} H_{n-1}^* (\{2\}^k, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2k} \dots z_d^{2a_d} \sum_{a_0=k}^{\infty} \frac{z_0^{2(a_0-k)}}{n^{2(a_0-k)}} \\ &= \frac{n^2}{n^2 - z_0^2} \cdot F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) \end{aligned}$$

and therefore,

$$\begin{aligned} (3.3) \quad F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) &= \frac{n^{2-c_1}}{n^2 - z_0^2} \cdot F_n(c_2, \dots, c_d; z_1, \dots, z_d) \\ &+ \frac{n^2}{n^2 - z_0^2} \cdot F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d). \end{aligned}$$

Let  $\Sigma_0$  denote the right-hand side of (3.1). Consider the difference

$$F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0$$

and evaluate  $\frac{n^2}{n^2 - z_0^2} F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0$ . By the induction hypothesis for  $(n - 1) + d = n + (d - 1) = N$ , we have

$$\begin{aligned} & \frac{n^2}{n^2 - z_0^2} F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0 \\ &= \frac{-n^2}{n^2 - z_0^2} \sum_{n-1 \geq k_0 \geq \dots \geq k_d \geq 1} \frac{\binom{n-1}{k_0}}{\binom{n-1+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \\ & \quad + \sum_{n \geq k_0 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \\ &= \sum_{n \geq k_0 \geq \dots \geq k_d \geq 1} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \\ & \quad \times \left( \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} - \frac{n^2}{n^2 - z_0^2} \cdot \frac{\binom{n-1}{k_0}}{\binom{n-1+k_0}{k_0}} \right). \end{aligned}$$

Notice that the expression in the parenthesis simplifies to

$$\frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} - \frac{n^2}{n^2 - z_0^2} \cdot \frac{\binom{n-1}{k_0}}{\binom{n-1+k_0}{k_0}} = \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \cdot \frac{k_0^2 - z_0^2}{n^2 - z_0^2}$$

and therefore,

$$\begin{aligned} (3.4) \quad & \frac{n^2}{n^2 - z_0^2} F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0 \\ &= \frac{2}{n^2 - z_0^2} \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0} (-1)^{k_0(1+\delta(c_1))} k_0^{\delta(c_1)}}{\binom{n+k_0}{k_0}} \\ & \quad \times \prod_{i=1}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}). \end{aligned}$$

Now let us evaluate the inner single sum over  $k_0$  in (3.4), which is

$$\Sigma_{k_0} := \sum_{k_0=k_1}^n \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} (-1)^{k_0(1+\delta(c_1))} \cdot k_0^{\delta(c_1)} \cdot S_{k_0, k_1}^\#(\{1\}^{c_1 - 3}).$$

If  $c_1 = 1$ , then  $\delta(c_1) = 1$  and we get by Lemma 2.2, (2.4),

$$\Sigma_{k_0} = \sum_{k_0=k_1}^n \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \cdot k_0 \cdot 2^{\Delta(k_0, k_1)} = \frac{n \binom{n}{k_1}}{\binom{n+k_1}{k_1}},$$

which implies

$$\begin{aligned} & \frac{n^2}{n^2 - z_0^2} F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0 \\ &= \frac{2n}{n^2 - z_0^2} \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}} \frac{(-1)^{k_1(1+\delta(c_2))} k_1^{\delta(c_2)}}{k_1^2 - z_1^2} \\ & \quad \times \prod_{i=2}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \\ &= \frac{-n}{n^2 - z_0^2} \cdot F_n(c_2, \dots, c_d; z_1, \dots, z_d), \end{aligned}$$

where in the last equality we applied the induction hypothesis for  $n + (d - 1) = N$ . Now from the above and recurrence (3.3) we conclude that  $F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = \Sigma_0$  and therefore, the theorem is proved in this case.

If  $c_1 \geq 3$ , then  $\delta(c_1) = 0$  and by Lemma 2.2, (2.5), we obtain

$$\Sigma_{k_0} = \sum_{k_0=k_1}^n \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} (-1)^{k_0} \cdot S_{k_0, k_1}^\#(\{1\}^{c_1 - 3}) = \frac{(-1)^{k_1} \cdot k_1}{n^{c_1 - 2}} \cdot \frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}},$$

which implies

$$\begin{aligned} & \frac{n^2}{n^2 - z_0^2} F_{n-1}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) - \Sigma_0 \\ &= \frac{2}{n^{c_1 - 2} (n^2 - z_0^2)} \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}} \frac{(-1)^{k_1(1+\delta(c_2))} k_1^{\delta(c_2)}}{k_1^2 - z_1^2} \\ & \quad \times \prod_{i=2}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \\ &= \frac{-n^{2-c_1}}{n^2 - z_0^2} \cdot F_n(c_2, \dots, c_d; z_1, \dots, z_d), \end{aligned}$$

and therefore, the proof is complete. □

**Corollary 3.2.** *For any integers  $n \geq 1, d \geq 0, c_1, \dots, c_d \in \{1, 3\}$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have*

$$F_n(c_1, \dots, c_d; z_0, \dots, z_d) = - \sum_{n \geq k_0 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} 2^{\Delta(k_{i-1}, k_i)}$$

with the same notation as in Theorem 3.1.

**Corollary 3.3.** For any integers  $n \geq 1, d \geq 0, 0 \leq m \leq d, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and any complex numbers  $z_0, z_1, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$\sum_{\substack{a_0, a_1, \dots, a_d \geq 0 \\ a_m \geq 1}} H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}$$

$$= -z_m^2 \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \frac{1}{k_m^2} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$$

with the same notation as in Theorem 3.1.

*Proof.* The formula easily follows from Theorem 3.1 and the relation

$$(3.5) \quad \sum_{\substack{a_0, a_1, \dots, a_d \geq 0 \\ a_m \geq 1}} H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}$$

$$= F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d)$$

$$- F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_{m-1}, 0, z_{m+1}, \dots, z_d). \quad \square$$

**Corollary 3.4.** For any integers  $n \geq 1, d \geq 0, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have

$$(3.6) \quad \sum_{a_0, a_1, \dots, a_d \geq 0} H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}$$

$$= \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}),$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 3.1.

*Proof.* Note that the generating function on the left of (3.6) is exactly

$$F_n(c_1, \dots, c_d, 1; z_0, z_1, \dots, z_d, 0).$$

Therefore, by Theorem 3.1 with  $d$  replaced by  $d + 1$ , the inner sum over  $k_{d+1}$  is reduced to

$$\sum_{k_{d+1}=1}^{k_d} \frac{(-1)^{k_{d+1}}}{k_{d+1}^2 - 0^2} \cdot k_{d+1}^2 \cdot 2^{\Delta(k_d, k_{d+1})} = \sum_{k_{d+1}=1}^{k_d} (-1)^{k_{d+1}} \cdot 2^{\Delta(k_d, k_{d+1})}$$

$$= 2 \sum_{k_{d+1}=1}^{k_d-1} (-1)^{k_{d+1}} + (-1)^{k_d} = -1,$$

since

$$\sum_{k_{d+1}=1}^{k_d-1} (-1)^{k_{d+1}} = \begin{cases} -1, & \text{if } k_d \text{ is even;} \\ 0, & \text{if } k_d \text{ is odd,} \end{cases}$$

and the formula follows. □

**Corollary 3.5.** *For any integers  $n \geq 1, d \geq 0, 0 \leq m \leq d, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and any complex numbers  $z_0, \dots, z_d$  with  $|z_j| < 1, j = 0, 1, \dots, d$ , we have*

$$\begin{aligned} & \sum_{\substack{a_0, a_1, \dots, a_d \geq 0 \\ a_m \geq 1}} H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d} \\ &= z_m^2 \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \frac{1}{k_m^2} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i} k_i^{\delta_i - 1}}{k_i^2 - z_i^2} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}), \end{aligned}$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 3.1.

*Proof.* The formula follows from identity (3.5) with  $d$  replaced by  $d + 1$  and then setting  $c_{d+1} = 1, z_{d+1} = 0$ , and applying Corollary 3.4. □

**Corollary 3.6.** *For any integers  $n \geq 1, d \geq 0, c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and any  $a_0, a_1, \dots, a_d \in \mathbb{N}_0$ , we have*

$$\begin{aligned} & H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) \\ &= - \sum_{n \geq k_0 \geq k_1 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i}}{k_i^{2a_i + 3 - \delta_i}} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}) \end{aligned}$$

with the same notation as in Theorem 3.1; and

$$\begin{aligned} & H_n^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}, 1) \\ &= \sum_{n \geq k_0 \geq \dots \geq k_d \geq 1} \frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}} \prod_{i=0}^d \frac{(-1)^{k_i \delta_i}}{k_i^{2a_i + 3 - \delta_i}} S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3}), \end{aligned}$$

where  $c_0 = c_{d+1} = 1, k_{-1} = 0, \delta_i = \delta(c_i) + \delta(c_{i+1})$ , and  $\delta(c)$  is defined in Theorem 3.1. Moreover, if  $c_i \in \{1, 3\}$ , then  $S_{k_{i-1}, k_i}^\#(\{1\}^{c_i - 3})$  is replaced by  $2^{\Delta(k_{i-1}, k_i)}$  in the above formulas.

*Proof.* Expanding the fractions  $\frac{1}{k_i^2 - z_i^2}$  in powers of  $z_i$  and comparing coefficients of  $z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}$  on both sides of equation (3.1), we get the first formula. The second formula follows similarly from (3.6). □

### 4. Limit transfer to multiple zeta star values

The purpose of this section is to justify the possibility of limit transfer from generating functions of multiple harmonic star sums to generating functions of multiple zeta star values.

Let

$$F(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = \sum_{a_0, a_1, \dots, a_d \geq 0} \zeta^*(\{2\}^{a_0}, c_1, \{2\}^{a_1}, \dots, c_d, \{2\}^{a_d}) z_0^{2a_0} z_1^{2a_1} \dots z_d^{2a_d}.$$

**Lemma 4.1.** *Let  $d \in \mathbb{N}_0$ ,  $z_0, z_1, \dots, z_d \in \mathbb{C}$ ,  $|z_j| < 1$ ,  $j = 0, 1, \dots, d$ ,  $c_1, \dots, c_d \in \mathbb{N} \setminus \{2\}$ , and  $c_1 \geq 3$  if  $d \geq 1$ . Then*

$$\lim_{n \rightarrow \infty} F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = F(c_1, \dots, c_d; z_0, z_1, \dots, z_d).$$

Moreover, the convergence is uniform in any closed region  $D : |z_0| \leq q_0 < 1, |z_1| \leq q_1 < 1, \dots, |z_d| \leq q_d < 1$ .

*Proof.* For positive integers  $n, m, s_1, \dots, s_r$  and  $n \geq m$ , let

$$H_{n,m}^*(s_1, \dots, s_r) = \sum_{n \geq k_1 \geq \dots \geq k_r \geq m} \frac{1}{k_1^{s_1} \dots k_r^{s_r}}.$$

Notice that we allow also the case  $n = \infty$  in the above definition if  $s_1 > 1$ . Then observing that

$$\prod_{k=m}^n \left(1 - \frac{z^2}{k^2}\right)^{-1} = \sum_{l=0}^{\infty} z^{2l} \cdot H_{n,m}^*(\{2\}^l), \quad |z| < 1,$$

we have the following representations:

$$(4.1) \quad F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{\prod_{k=k_1}^n \left(1 - \frac{z_0^2}{k^2}\right) k_1^{c_1} \prod_{k=k_2}^{k_1} \left(1 - \frac{z_1^2}{k^2}\right) k_2^{c_2} \dots k_d^{c_d} \prod_{k=1}^{k_d} \left(1 - \frac{z_d^2}{k^2}\right)},$$

if  $d \geq 1$ , and

$$(4.2) \quad F_n(; z_0) = \prod_{k=1}^n \left(1 - \frac{z_0^2}{k^2}\right)^{-1}.$$

Note that the case  $n = \infty$  in (4.1) and (4.2) corresponds to the generating function  $F(c_1, \dots, c_d; z_0, z_1, \dots, z_d)$ , i.e.,

$$(4.3) \quad F_{\infty}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) = F(c_1, \dots, c_d; z_0, z_1, \dots, z_d).$$

Then by (4.2) and (4.3), we easily conclude the theorem for  $d = 0$ .



Now consider the case  $d \geq 1$ . From (4.1) we have the following upper bound:

$$\begin{aligned}
 (4.4) \quad & |F_n(c_1, \dots, c_d; z_0, z_1, \dots, z_d)| \\
 & \leq \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{\prod_{k=k_1}^n \left(1 - \frac{|z_0|^2}{k^2}\right) k_1^{c_1} \dots k_d^{c_d} \prod_{k=1}^{k_d} \left(1 - \frac{|z_d|^2}{k^2}\right)} \\
 & < \frac{\pi|z_0|}{\sin(\pi|z_0|)} \dots \frac{\pi|z_d|}{\sin(\pi|z_d|)} \cdot H_n^*(c_1, \dots, c_d) \\
 & < \prod_{j=0}^d \frac{\pi q_j}{\sin(\pi q_j)} \cdot \zeta^*(c_1, \dots, c_d)
 \end{aligned}$$

for all  $z_0, \dots, z_d \in \mathbb{C}$  such that  $|z_0| \leq q_0 < 1, \dots, |z_d| \leq q_d < 1$ .

Let

$$\begin{aligned}
 & F_{n,\infty}(c_1, \dots, c_d; z_0, z_1, \dots, z_d) \\
 & = \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{\prod_{k=k_1}^\infty \left(1 - \frac{z_0^2}{k^2}\right) k_1^{c_1} \prod_{k=k_2}^{k_1} \left(1 - \frac{z_1^2}{k^2}\right) k_2^{c_2} \dots k_d^{c_d} \prod_{k=1}^{k_d} \left(1 - \frac{z_d^2}{k^2}\right)}.
 \end{aligned}$$

Then

$$\begin{aligned}
 (4.5) \quad & |F_{n,\infty}(c_1, \dots, c_d; z_0, \dots, z_d) - F_n(c_1, \dots, c_d; z_0, \dots, z_d)| \\
 & \leq \sum_{n \geq k_1 \geq \dots \geq k_d \geq 1} \frac{1}{\prod_{k=k_1}^n \left(1 - \frac{|z_0|^2}{k^2}\right) k_1^{c_1} \dots k_d^{c_d} \prod_{k=1}^{k_d} \left(1 - \frac{|z_d|^2}{k^2}\right)} \\
 & \qquad \qquad \qquad \times \left| \frac{1}{\prod_{k=n+1}^\infty \left(1 - \frac{z_0^2}{k^2}\right)} - 1 \right|.
 \end{aligned}$$

Since

$$\begin{aligned}
 \left| \frac{1}{\prod_{k=n+1}^\infty \left(1 - \frac{z_0^2}{k^2}\right)} - 1 \right| & = \left| \sum_{l=1}^\infty H_{\infty, n+1}^* (\{2\}^l) z_0^{2l} \right| \\
 & \leq \sum_{l=1}^\infty H_{\infty, n+1}^* (\{2\}^l) |z_0|^{2l} < \sum_{l=1}^\infty H_{\infty, n+1}^* (\{2\}^l)
 \end{aligned}$$

and

$$H_{\infty, n+1}^* (\{2\}^l) = \sum_{k_1 \geq \dots \geq k_l \geq n+1} \frac{1}{k_1^2 \dots k_l^2} < \left( \sum_{k=n+1}^\infty \frac{1}{k^2} \right)^l < \left( \int_n^\infty \frac{dx}{x^2} \right)^l = \frac{1}{n^l}$$

we get

$$(4.6) \quad \left| \frac{1}{\prod_{k=n+1}^{\infty} \left(1 - \frac{z_0^2}{k^2}\right)} - 1 \right| < \sum_{l=1}^{\infty} \frac{1}{n^l} = \frac{1/n}{1 - 1/n} = \frac{1}{n - 1}$$

and therefore, by (4.4)–(4.6),

$$(4.7) \quad |F_{n,\infty}(c_1, \dots, c_d; z_0, \dots, z_d) - F_n(c_1, \dots, c_d; z_0, \dots, z_d)| < \frac{1}{n - 1} \prod_{j=0}^d \frac{\pi q_j}{\sin(\pi q_j)} \cdot \zeta^*(c_1, \dots, c_d) \rightarrow 0$$

as  $n \rightarrow \infty$  for all  $z_0, \dots, z_d \in \mathbb{C}$  such that  $|z_0| \leq q_0 < 1, \dots, |z_d| \leq q_d < 1$ .

From the other side,

$$(4.8) \quad |F(c_1, \dots, c_d; z_0, \dots, z_d) - F_{n,\infty}(c_1, \dots, c_d; z_0, \dots, z_d)| \leq \sum_{\substack{k_1 \geq \dots \geq k_d \geq 1 \\ k_1 > n}} \frac{1}{\prod_{k=k_1}^{\infty} \left(1 - \frac{|z_0|^2}{k^2}\right) k_1^{c_1} \dots k_d^{c_d} \prod_{k=1}^{k_d} \left(1 - \frac{|z_d|^2}{k^2}\right)} < \prod_{j=0}^d \frac{\pi q_j}{\sin(\pi q_j)} \cdot \left(\zeta^*(c_1, \dots, c_d) - H_n^*(c_1, \dots, c_d)\right) \rightarrow 0 \quad (n \rightarrow \infty)$$

uniformly on  $D$ . Now combining (4.7) and (4.8), we conclude the proof.  $\square$

Taking the limit as  $n \rightarrow \infty$  in Theorem 3.1 by Lemmas 2.3 and 4.1, we get Theorem 1.4. Theorem 1.9 follows from Corollary 3.6 by taking the limit  $n \rightarrow \infty$  and applying Lemma 2.3. Theorem 1.7 for  $m \geq 1$  follows from (3.5), Lemma 4.1, and Lemma 2.3 by taking the limit  $n \rightarrow \infty$ . Theorem 1.7 for  $m = 0$  follows from Theorem 1.9 by summing identity (1.6) over the corresponding set of integers  $a_0, a_1, \dots, a_d$ . Corollary 1.8 follows from Theorem 1.7 by setting  $c_d = 1, z_d = 0$ , and applying the same argument as in the proof of Corollary 3.4. Theorems 1.1–1.3 are simple consequences of Theorem 1.4 and Theorem 1.7. Note also that Theorem 1.4 can be obtained from Theorem 1.9 by summation over  $a_0, a_1, \dots, a_d \geq 0$ .

### Acknowledgement

The authors would like to thank the anonymous referee for the careful reading of the manuscript and helpful comments and suggestions.

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