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Résumé. Soit F un corps CM et p un nombre premier. Soit $X^-_{F_\infty}$ le quotient "moins" du groupe de Galois de la pro-p-extension abélienne non ramifiée maximale de la $\mathbb{Z}_p$-extension cyclotomique de F. Si p ne vaut pas 2, il est bien connu que $X^-_{F_\infty}$ n'a pas de sous-module fini non-trivial. Mais pour $p = 2$, il peut arriver que $X^-_{F_\infty}$ contient un sous-module fini non-trivial. Dans cet article, nous étudions le sous-module fini maximal de $X^-_{F_\infty}$ pour $p = 2$, et nous déterminons ce module sous certaines légères hypothèses.

Abstract. Let p be a prime, $X^-_{F_\infty}$ the minus quotient of the Iwasawa module, which we define to be the Galois group of the maximal unramified abelian pro-p-extension over the cyclotomic $\mathbb{Z}_p$-extension over a CM field F. If p is an odd prime, it is well known that $X^-_{F_\infty}$ has no non-trivial finite $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule. But $X^-_{F_\infty}$ has non-trivial finite $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule in some cases for $p = 2$. In this paper, we study the maximal finite $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule of $X^-_{F_\infty}$ for $p = 2$. We determine the size of the maximal finite $\mathbb{Z}_2[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule of $X^-_{F_\infty}$ under some mild assumptions.

1. Introduction

Let p be a prime, F a CM field, and F_∞ the cyclotomic $\mathbb{Z}_p$-extension of F. We denote by X_{F_∞} the Galois group of the maximal unramified abelian pro-p-extension of F_∞. Iwasawa proved that X_{F_∞} is a finitely generated $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-module ([6, Theorem 5]). If p is an odd prime, Iwasawa also proved that the minus part of X_{F_∞} has no non-trivial finite $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule (see, for example [9, Proposition 13.28]). But for $p = 2$, Ferrero proved that if F is an imaginary quadratic field that is not $F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$ and the prime above 2 ramifies in $F_\infty/\mathbb{Q}_\infty$, the maximal finite $\mathbb{Z}_2[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule of X_{F_∞} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and this submodule is generated by the prime above 2 ([1, Theorem 5]). One of our purposes is to generalize Ferrero's result to an arbitrary CM field.

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In this paper, we study the maximal finite $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-submodule of the minus quotient of X_{F_∞} of a CM field for $p=2$.

For any number field K, we denote by K_∞/K the cyclotomic $\mathbb{Z}_2$-extension of K, K_n the n-th layer of K_∞/K, and $\mathrm{CL}(K)$ the ideal class group of K. We denote by $S_2(K)$, $S_\infty(K)$ the set of primes of K lying above 2, ∞, respectively.

Let F be a CM field and F^+ the maximal real subfield of F. Put $\Lambda := \mathbb{Z}_2[\![\mathrm{Gal}(F_\infty/F)]\!]$. We define the subset $\mathscr{S}(F^+)$ of $S_2(F^+)\cup S_\infty(F^+)$ by

$$\mathscr{S}(F^+) = \{v \in S_2(F^+) \mid v \text{ ramifies in } F_\infty/F_\infty^+\} \cup S_\infty(F^+).$$

For any extension K/F^+, we denote by $\mathscr{S}(K)$ the set of primes of K lying above $\mathscr{S}(F^+)$. We put

$$d = \sharp\bigl(S_2(F_\infty)\cap \mathscr{S}(F_\infty)\bigr).$$

Using this particular $\mathscr{S}(K)$, we define $\mathrm{CL}_{\mathscr{S}}(K)$ by the $\mathscr{S}(K)$-ideal class group of K, i.e

$$\mathrm{CL}_{\mathscr{S}}(K) = \mathrm{coker}(K^\times \overset{\oplus \mathrm{ord}_v}{\longrightarrow} \textstyle\bigoplus_{v\notin \mathscr{S}(K)} \mathbb{Z}).$$

We denote by A_K (resp. $A_{K,\mathscr{S}}$) the 2-Sylow subgroup of the ideal class group $\mathrm{CL}(K)$ (resp. $\mathrm{CL}_{\mathscr{S}}(K)$). By class field theory, we have $X_{F_\infty} \cong \varprojlim A_{F_n}$. There are several ways to define the minus quotient, but we adopt the following. Let J be the complex conjugation. We define the minus quotient $X_{F_\infty}^-$ by

$$X_{F_\infty}^- = X_{F_\infty}/(1+J)X_{F_\infty}.$$

We denote by $F_\Lambda(X_{F_\infty}^-)$ the maximal finite Λ-submodule of $X_{F_\infty}^-$. We define

$$D_{n,\mathscr{S}} = \ker(A_{F_n} \longrightarrow A_{F_n,\mathscr{S}}),\;\; D_{n,\mathscr{S}}^+ = \ker(A_{F_n^+} \longrightarrow A_{F_n^+,\mathscr{S}}),$$
$$\delta_1 = \mathrm{rank}_2\bigl(\varprojlim\,((\mathcal{O}_{F_n,\mathscr{S}}^\times)^{1-J}/(\mathcal{O}_{F_n}^\times)^{1-J})\bigr),$$
$$\delta_2 = \mathrm{rank}_2\bigl(\varprojlim \ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}})\bigr),$$

where $\mathcal{O}_{F_n,\mathscr{S}}^\times$ is the $\mathscr{S}(F_n)$-unit group of F_n, $\mathcal{O}_{F_n}^\times$ the unit group of F_n, both projective limits are taken with respect to the norm maps, and $\mathrm{rank}_2(A)$ is the 2-rank, namely the dimension of $A/2A$ as an $\mathbb{F}_2$-vector space. We note that $0 \le \delta_2 \le \delta_1 \le 1$ and the 2-rank of $\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$ is d or $d-1$, where d is the number of certain 2-adic prime defined above. (see Remark 2.4 in this paper). Our main result is the following.

Theorem 1.1. *Assume that Leopoldt's conjecture is valid for F^+ and the lifting maps $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ are injective for all sufficiently large $n \gg 0$. Then we have*

$$F_\Lambda(X_{F_\infty}^-) = \varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus d} & (\textit{if } \mu_{2^\infty} \not\subset F_\infty) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus d-\delta_1+\delta_2} & (\textit{if } \mu_{2^\infty} \subset F_\infty), \end{cases}$$

where d is the number of primes of F_∞ above 2 which ramify in F_∞/F_∞^+ and μ_{2^∞} is the group of all 2 power roots of unity.

This is a generalization of the Ferrero's result(see Example 2.7 in this paper). We prove Theorem 1.1 in Section 2. Concerning the injectivity of the lifting map $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ for an imaginary abelian field F, we get Lemma 2.6 in Section 2. Theorem 1.1 and Lemma 2.6 imply the following result.

Corollary 1.2. *Assume that F is an imaginary abelian field and all primes above 2 ramify in F_∞/F_∞^+. If F_∞ contains μ_{2^∞} or Hasse's unit index $[\mathcal{O}_{F_n}^\times : \mu(F_n)\mathcal{O}_{F_n^+}^\times] = 2$ for all sufficiently large $n \gg 0$, we have*

$$F_\Lambda(X_{F_\infty}^-) = \varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus d} & (\textit{if } \mu_{2^\infty} \not\subset F_\infty) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus d-\delta_1+\delta_2} & (\textit{if } \mu_{2^\infty} \subset F_\infty), \end{cases}$$

where d is the number of primes of F_∞ above 2, $\mu(F_n)$ is the group of roots of unity contained in F_n.

For example, let F^+ be a real abelian field which is unramified at 2, and $F = F^+(\sqrt{-1})$. Then, we have

$$F_\Lambda(X_{F_\infty}^-) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus d-1},$$

where g is the number of primes of F lying above 2 (see Example 2.8 in this paper).

For any extension K/F^+, let $\mathscr{S}(K)$ be the set we defined on page 1018. We define $X_{F_\infty,\mathscr{S}}, X_{F_\infty,\mathscr{S}}^-$ by

$$X_{F_\infty,\mathscr{S}} = \varprojlim A_{F_n,\mathscr{S}}, \quad X_{F_\infty,\mathscr{S}}^- = X_{F_\infty,\mathscr{S}}/(1+J)X_{F_\infty,\mathscr{S}},$$

where the projective limit is taken with respect to the norm maps. We also prove the following result which plays an important rule in the proof of Theorem 1.1.

Theorem 1.3. *$X_{F_\infty,\mathscr{S}}^-$ has no non-trivial finite Λ-submodule.*

To prove Theorem 1.3, we use a result of Greenberg in [4]. We prove Theorem 1.3 in Section 3. A key point to prove this theorem is to choose some appropriate local conditions.

If all primes above 2 are unramified in F_∞/F_∞^+, the set $\mathscr{S}(F^+)$ defined on page 1018 coincides with $S_\infty(F^+)$ by definition. Therefore we have $A_{F_n,\mathscr{S}} = A_{F_n}$, and $X_{F_\infty,\mathscr{S}}^- = X_{F_\infty}^-$. Thus Theorem 1.3 implies the following result.

Corollary 1.4. *Assume that all primes above 2 are unramified in F_∞/F_∞^+. Then, $X_{F_\infty}^-$ has no non-trivial finite Λ-submodule.*

Remark 1.5. Put $S(F_n) = S_2(F_n) \cup S_\infty(F_n)$. We denote by $A_{F_n,S}$ is the 2-Sylow subgroup of the $S(F_n)$-ideal class group. We define $X_{F_\infty,S} = \varprojlim A_{F_n,S}$ and $X^-_{F_\infty,S} = X_{F_\infty,S}/(1+J)X_{F_\infty,S}$. We can also prove that $X^-_{F_\infty,S}$ has no non-trivial finite Λ-submodule, using the result of Greenberg in [4].

2. The maximal finite Λ-submodule of $X^-_{F_\infty}$

In this section, we prove Theorem 1.1 assuming Theorem 1.3. We use the same notation as in the previous section.

Lemma 2.1. *We have an exact sequence*

$$\begin{aligned} 0 &\longrightarrow \ker(D^+_{n,\mathscr{S}} \to D_{n,\mathscr{S}}) \longrightarrow (\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^\times_{F_n})^{1-J} \\ &\longrightarrow \bigoplus_{w\in\mathscr{S}(F_n)\cap S_2(F_n)} \mathbb{Z}/2\mathbb{Z} \longrightarrow D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \longrightarrow 0 \end{aligned}$$

of $\mathbb{F}_2$-vector spaces for all sufficiently large $n \gg 0$, where J is the complex conjugation and $(\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J} = \{(1-J)x \mid x \in \mathcal{O}^\times_{F_n,\mathscr{S}}\}$.

Proof. For any extension K/F^+, put $\mathscr{S}_f(K) = \mathscr{S}(K) \cap S_2(K)$. We take n sufficiently large such that the primes above 2 are totally ramified in F_∞/F_n and F^+_∞/F^+_n. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^\times_{F_n^+,\mathscr{S}}/\mathcal{O}^\times_{F_n^+}\otimes\mathbb{Z}_2 & \longrightarrow & \bigoplus_{v\in\mathscr{S}_f(F_n^+)}\mathbb{Z}_2 & \longrightarrow & D^+_{n,\mathscr{S}} & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow \times 2 & & \downarrow f_2 & & \\ 0 & \longrightarrow & \mathcal{O}^\times_{F_n,\mathscr{S}}/\mathcal{O}^\times_{F_n}\otimes\mathbb{Z}_2 & \longrightarrow & \bigoplus_{w\in\mathscr{S}_f(F_n)}\mathbb{Z}_2 & \longrightarrow & D_{n,\mathscr{S}} & \longrightarrow & 0, \end{array}$$

where f_1, f_2 are homomorphisms induced by the natural maps $\mathcal{O}^\times_{F_n^+,\mathscr{S}} \longrightarrow \mathcal{O}^\times_{F_n,\mathscr{S}}$ and $\mathrm{CL}_\mathscr{S}(F_n^+) \longrightarrow \mathrm{CL}_\mathscr{S}(F_n)$. By the snake lemma, we get an exact sequence

$$0 \longrightarrow \ker f_2 \longrightarrow \operatorname{coker} f_1 \longrightarrow \bigoplus_{w\in\mathscr{S}_f(F_n)} \mathbb{Z}/2\mathbb{Z} \longrightarrow \operatorname{coker} f_2 \longrightarrow 0.$$

Therefore, it suffices to show that $\operatorname{coker} f_1 \cong (\mathcal{O}^{\times}_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^{\times}_{F_n})^{1-J}$ and $\operatorname{coker} f_2 \cong D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$. We consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{\times}_{F_n^+} & \longrightarrow & \mathcal{O}^{\times}_{F_n^+,\mathscr{S}} & \longrightarrow & \mathcal{O}^{\times}_{F_n^+,\mathscr{S}}/\mathcal{O}^{\times}_{F_n^+} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f1'} & & \\
0 & \longrightarrow & \mathcal{O}^{\times}_{F_n} & \longrightarrow & \mathcal{O}^{\times}_{F_n,\mathscr{S}} & \longrightarrow & \mathcal{O}^{\times}_{F_n,\mathscr{S}}/\mathcal{O}^{\times}_{F_n} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 1-J} & & \big\downarrow{\scriptstyle 1-J} & & & & \\
 & & (\mathcal{O}^{\times}_{F_n})^{1-J} & \longrightarrow & (\mathcal{O}^{\times}_{F_n,\mathscr{S}})^{1-J}. & & & &
\end{array}$$

Since the map f_1' is injective and $(\mathcal{O}^{\times}_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^{\times}_{F_n})^{1-J}$ is a 2 group (see Remark 1.4 in this paper), we have

$$\operatorname{coker} f_1 = \operatorname{coker} f_1' \otimes \mathbb{Z}_2 \cong (\mathcal{O}^{\times}_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^{\times}_{F_n})^{1-J}$$

by the snake lemma.

Next we show that $\operatorname{coker} f_2 \cong D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$. Since $D_{n,\mathscr{S}}$ is equal to $\ker\big(\mathrm{CL}(F_n) \to \mathrm{CL}_{\mathscr{S}}(F_n)\big) \otimes \mathbb{Z}_2$, we have $\operatorname{coker} f_2 \cong D_{n,\mathscr{S}}/D^+_{n,\mathscr{S}}$. We consider the following diagram,

$$\begin{array}{ccc}
D_{n,\mathscr{S}} & \xrightarrow{1+J} & D_{n,\mathscr{S}} \\
{\scriptstyle N_{F_n/F_n^+}}\big\downarrow & & \big\uparrow{\scriptstyle f_2} \\
D^+_{n,\mathscr{S}} & = & D^+_{n,\mathscr{S}}.
\end{array}$$

Since all primes above 2 which are contained in $\mathscr{S}(F_n)$ ramify in F_n/F_n^+, the norm map $N_{F_n/F_n^+} : D_{n,\mathscr{S}} \longrightarrow D^+_{n,\mathscr{S}}$ is surjective. This implies that $\operatorname{coker} f_2 \cong D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$. □

Lemma 2.2 ([2, Corollary])**.** *Assume that Leopoldt's conjecture is valid for F^+. Then the order of $D^+_{n,\mathscr{S}}$ remains bounded as $n \to \infty$.*

Proof. Put $\Gamma_n = \mathrm{Gal}(F_n^+/F^+)$. If Leopoldt's conjecture is valid for F^+, the order of the Galois invariant $A_{F_n^+}^{\Gamma_n}$ remains bounded as $n \to \infty$ (see [2, Proposition 1]). This implies that the order of $D^+_{n,\mathscr{S}}$ remains bounded as $n \to \infty$. □

Proposition 2.3. *Assume that Leopoldt's conjecture is valid for F^+. Then,*

$$\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus d} & (\mu_{2^\infty} \not\subset F_\infty) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus d-\delta_1+\delta_2} & (\mu_{2^\infty} \subset F_\infty), \end{cases}$$

where d is the number of primes of F_∞ above 2 which ramify in F_∞/F_∞^+, and δ_1, δ_2 are defined just before Theorem 1.1.

Proof. Put $B_n = (\mathcal{O}_{F_n,\mathscr{S}}^{\times})^{1-J}/(\mathcal{O}_{F_n}^{\times})^{1-J}$ for any $n \in \mathbb{Z}_{\geq 0}$. We consider the following commutative diagram which is obtained by Lemma 1.1 for $n \geq m \gg 0$,

$$\begin{array}{ccccccc} B_n & \longrightarrow & \bigoplus_{v \in \mathscr{S}(F_n) \cap S_2(F_n)} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} & \longrightarrow & 0 \\ \downarrow N_{F_n/F_m} & & \| & & \downarrow N_{F_n/F_m} & & \\ B_m & \longrightarrow & \bigoplus_{w \in \mathscr{S}(F_m) \cap S_2(F_m)} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & D_{m,\mathscr{S}}/(1+J)D_{m,\mathscr{S}} & \longrightarrow & 0 \end{array}$$

Since the action of J on $\mathscr{S}(F_n)$ is trivial, we have $(\mathcal{O}_{F_n,\mathscr{S}}^{\times})^{1-J} \subset \mu(F_n)$ for all $n \geq 0$, where $\mu(F_n)$ is the set of root of unity which contains in F_n (see [9, Lemma 1.6]).

If F_∞ does not contain μ_{2^∞}, the 2-Sylow subgroup of $\mu(F_n)$ is $\{\pm 1\}$ for all $n \geq 0$. Therefore the norm map $\mu(F_n) \otimes \mathbb{Z}_2 \to \mu(F_m) \otimes \mathbb{Z}_2$ is the 0-map. This fact and $(\mathcal{O}_{F_n,\mathscr{S}}^{\times})^{1-J} \subset \mu(F_n)$ imply that $B_n \to B_m$ is the 0-map for all $n \geq m \geq 0$. Therefore we have $\bigoplus_{v \in \mathscr{S}(F_n) \cap S_2(F_n)} \mathbb{Z}/2\mathbb{Z} \cong D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$ for all sufficiently large $n \gg 0$. Taking the projective limit, we have

$$\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus d}.$$

If F_∞ contains μ_{2^∞}, we have $(\mathcal{O}_{F_n,\mathscr{S}}^{\times})^{1-J} = \mu(F_n)$ or $\mu(F_n)^2$ (see [9, Theorem 4.12]). Since the norm map $\mu(F_n) \to \mu(F_m)$ is surjective, the norm map $B_n \to B_m$ is surjective for all sufficiently large $n \geq m \gg 0$.

We claim that the norm map

$$N_{F_n^+/F_m^+} : \ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}}) \longrightarrow \ker(D_{m.\mathscr{S}}^+ \to D_{m,\mathscr{S}})$$

is also surjective for all sufficiently large $n \geq m \gg 0$. We consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}}) & \longrightarrow & D_{n,\mathscr{S}}^+ \\ & & \downarrow N_{F_n^+/F_m^+} & & \downarrow N_{F_n^+/F_m^+} \\ 0 & \longrightarrow & \ker(D_{m,\mathscr{S}}^+ \to D_{m,\mathscr{S}}) & \longrightarrow & D_{m,\mathscr{S}}^+. \end{array}$$

Lemma 2.2 implies that the norm map $N_{F_n^+/F_m^+} : D_{n,\mathscr{S}}^+ \to D_{m,\mathscr{S}}^+$ is an isomorphism for all sufficiently large $n \geq m \gg 0$. Therefore the norm map $N_{F_n^+/F_m^+} : \ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}}) \longrightarrow \ker(D_{m,\mathscr{S}}^+ \to D_{m,\mathscr{S}})$ is injective for all sufficiently large $n \geq m \gg 0$. Since the order of $\ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}})$ is 1 or 2 for all $n \geq 0$ (see [9, Theorem 10.3]), the norm map $N_{F_n^+/F_m^+}$: $\ker(D_{n,\mathscr{S}}^+ \to D_{n,\mathscr{S}}) \longrightarrow \ker(D_{m,\mathscr{S}}^+ \to D_{m,\mathscr{S}})$ is surjective for all sufficiently large $n \geq m \gg 0$. Therefore, taking the projective limit of the exact sequences obtained from in Lemma 2.1, we get an exact sequence

$$0 \longrightarrow \varprojlim \left(\ker(D^+_{n,\mathscr{S}} \to D_{n,\mathscr{S}}) \right) \longrightarrow \varprojlim \left((\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^\times_{F_n})^{1-J} \right) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\oplus d} \longrightarrow \varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \longrightarrow 0$$

of $\mathbb{F}_2$-vector spaces. Proposition 2.3 is obtained by considering the 2-rank of this exact sequence. □

Remark 2.4. Since $(\mathcal{O}^\times_{F_n})^{1-J} = \mu(F_n)$ or $\mu(F_n)^2$, $(\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J}/(\mathcal{O}^\times_{F_n})^{1-J}$ is isomorphic to 0 or $\mathbb{Z}/2\mathbb{Z}$ for all $n \geq 0$. Thus we have $0 \leq \delta_2 \leq \delta_1 \leq 1$ and the 2-rank of $\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}}$ is d or $d-1$.

Lemma 2.5. *We have*

$$\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong (\varprojlim D_{n,\mathscr{S}})/(1+J)(\varprojlim D_{n,\mathscr{S}}).$$

Proof. Put $D'_{n,\mathscr{S}} := \ker(D_{n,\mathscr{S}} \xrightarrow{1-J} D_{n,\mathscr{S}})$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D'_{n,\mathscr{S}} & \longrightarrow & D_{n,\mathscr{S}} & \xrightarrow{1+J} & (1+J)D_{n,\mathscr{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N_{F_n/F_m}} & & \downarrow{\scriptstyle N_{F_n/F_m}} & & \downarrow{\scriptstyle N_{F_n/F_m}} & & \\ 0 & \longrightarrow & D'_{m,\mathscr{S}} & \longrightarrow & D_{m,\mathscr{S}} & \xrightarrow{1+J} & (1+J)D_{m,\mathscr{S}} & \longrightarrow & 0. \end{array}$$

Since $D'_{n,\mathscr{S}}$ is finite for any $n \geq 0$, the system $(D'_{n,\mathscr{S}}, N_{F_n/F_{n-1}})$ satisfies the Mittag-Leffler property (see [8, Chapter 2, §7]). Therefore taking projective limits, we get an exact sequence

$$0 \longrightarrow \varprojlim D'_{n,\mathscr{S}} \longrightarrow \varprojlim D_{n,\mathscr{S}} \xrightarrow{1+J} \varprojlim (1+J)D_{n,\mathscr{S}} \longrightarrow 0.$$

Thus we have $\varprojlim(1+J)D_{n,\mathscr{S}} \cong (1+J)\varprojlim D_{n,\mathscr{S}}$. This implies that

$$\varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong (\varprojlim D_{n,\mathscr{S}})/(1+J)(\varprojlim D_{n,\mathscr{S}}).$$ □

Now we proceed to the proof of Theorem 1.1, assuming Theorem 1.3.

Proof of Theorem 1.1. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim D_{n,\mathscr{S}} & \longrightarrow & X_{F_\infty} & \longrightarrow & X_{F_\infty,\mathscr{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & \varprojlim D_{n,\mathscr{S}} & \longrightarrow & X_{F_\infty} & \longrightarrow & X_{F_\infty,\mathscr{S}} & \longrightarrow & 0, \end{array}$$

where f_1, f_2, f_3 are induced by $1+J$, and $X_{F_\infty,\mathscr{S}} = \varprojlim A_{F_n,\mathscr{S}}$. By the snake lemma, we get an exact sequence

$$\text{(2.1)}\quad \ker f_2 \longrightarrow \ker f_3 \longrightarrow (\varprojlim D_{n,\mathscr{S}})/(1+J)(\varprojlim D_{n,\mathscr{S}}) \longrightarrow X^-_{F_\infty} \longrightarrow X^-_{F_\infty,\mathscr{S}} \longrightarrow 0$$

where $X^-_{F_\infty,\mathscr{S}} = X_{F_\infty,\mathscr{S}}/(1+J)X_{F_\infty,\mathscr{S}}$. We claim that the map $\ker f_2 \longrightarrow \ker f_3$ is surjective if $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ is injective for sufficiently large $n \gg 0$. We define

$$A'_{F_n} = \ker(A_{F_n} \xrightarrow{1+J} A_{F_n}),$$
$$A'_{F_n,\mathscr{S}} = \ker(A_{F_n,\mathscr{S}} \xrightarrow{1+J} A_{F_n,\mathscr{S}}),$$
$$D'_{n,\mathscr{S}} = \ker(D_{n,\mathscr{S}} \xrightarrow{1+J} D_{n,\mathscr{S}}).$$

By definition, $\ker f_2 = \varprojlim A'_{F_n}$ and $\ker f_3 = \varprojlim A'_{F_n,\mathscr{S}}$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_{n,\mathscr{S}} & \longrightarrow & A_{F_n} & \longrightarrow & A_{F_n,\mathscr{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N_{F_n/F_n^+}} & & \downarrow{\scriptstyle N_{F_n/F_n^+}} & & \downarrow{\scriptstyle N_{F_n/F_n^+}} & & \\ 0 & \longrightarrow & D^+_{n,\mathscr{S}} & \longrightarrow & A_{F_n^+} & \longrightarrow & A_{F_n^+,\mathscr{S}} & \longrightarrow & 0. \end{array}$$

Since infinite primes ramify in F_n/F_n^+, all norm maps N_{F_n/F_n^+} are surjective by class field theory. Since we assumed that $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ is injective,

$$A'_{F_n,\mathscr{S}} = \ker\left(A_{F_n,\mathscr{S}} \xrightarrow{N_{F_n/F_n^+}} A_{F_n^+,\mathscr{S}}\right).$$

Put

$$D''_{n,\mathscr{S}} = \ker\left(D_{n,\mathscr{S}} \xrightarrow{N_{F_n/F_n^+}} D^+_{n,\mathscr{S}}\right),$$
$$A''_{F_n} = \ker\left(A_{F_n} \xrightarrow{N_{F_n/F_n^+}} A_{F_n^+}\right).$$

By the snake lemma, we get an exact sequence,

$$0 \longrightarrow D''_{n,\mathscr{S}} \longrightarrow A''_{F_n} \longrightarrow A'_{F_n,\mathscr{S}} \longrightarrow 0$$

for all sufficiently large $n \gg 0$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D''_{n,\mathscr{S}} & \longrightarrow & D'_{n,\mathscr{S}} & \longrightarrow & \ker(D^+_{n,\mathscr{S}} \to D_{n,\mathscr{S}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_4} & & \\ 0 & \longrightarrow & A''_{F_n} & \longrightarrow & A'_{F_n} & \longrightarrow & \ker(A_{F_n^+} \to A_{F_n}) & \longrightarrow & 0. \end{array}$$

Since the map $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ is injective, the map f_4 is an isomorphism. Therefore we get an exact sequence,

$$0 \longrightarrow D'_{n,\mathscr{S}} \longrightarrow A'_{F_n} \longrightarrow A'_{F_n,\mathscr{S}} \longrightarrow 0$$

for all sufficiently large $n \gg 0$. Lemma 1.2 implies that the map

$$D'_{n,\mathscr{S}} \overset{N_{F_n/F_{n-1}}}{\longrightarrow} D'_{n-1,\mathscr{S}}$$

is surjective for sufficiently $n \gg 0$. Therefore, taking the projective limit, we get an exact sequence

$$0 \longrightarrow \varprojlim D'_{n,\mathscr{S}} \longrightarrow \varprojlim A'_{F_n} \longrightarrow \varprojlim A'_{F_n,\mathscr{S}} \longrightarrow 0.$$

This implies that $\ker f_2 \longrightarrow \ker f_3$ is surjective. Therefore, it follows from (2.1) that we have an exact sequence

$$0 \longrightarrow (\varprojlim D_{n,\mathscr{S}})/(1+J)(\varprojlim D_{n,\mathscr{S}}) \longrightarrow X^-_{F_\infty} \longrightarrow X^-_{F_\infty,\mathscr{S}} \longrightarrow 0.$$

If $X^-_{F_\infty,\mathscr{S}}$ has no non-trivial finite Λ-submodule, we have

$$F_\Lambda(X^-_{F_\infty}) = (\varprojlim D_{n,\mathscr{S}})/(1+J)(\varprojlim D_{n,\mathscr{S}}).$$

Proposition 2.3, Lemma 2.5 and the above equality imply that

$$F_\Lambda(X^-_{F_\infty}) = \varprojlim D_{n,\mathscr{S}}/(1+J)D_{n,\mathscr{S}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus d} & (\text{if } \mu_{2^\infty} \not\subset F_\infty) \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus d-\delta_1+\delta_2} & (\text{if } \mu_{2^\infty} \subset F_\infty), \end{cases}$$

if $X^-_{F_\infty,\mathscr{S}}$ has no non-trivial finite Λ-submodule. This completes the proof of Theorem 1.1. □

Next we study certain conditions on the injectivity of the map $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ for an imaginary abelian field F. Leopoldt's conjecture is valid for a real abelian field. Hence the following result implies Corollary 1.2.

Lemma 2.6. *Assume that F is an imaginary abelian field and all primes above 2 ramify in F_∞/F_∞^+. If F_∞ contains μ_{2^∞} or Hasse's unit index $[\mathcal{O}_{F_n}^\times : \mu(F_n)\mathcal{O}_{F_n^+}^\times] = 2$ for all sufficiently large $n \gg 0$, the lifting map $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ is injective for all sufficiently large $n \gg 0$.*

Proof. It is well known that the kernel of the map $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ coincides with the kernel of the map

$$H^1(F_n/F_n^+, \mathcal{O}_{F_n,\mathscr{S}}^\times) \longrightarrow H^1\left(F_n/F_n^+, \prod_{v \notin \mathscr{S}(F_n)} \mathcal{O}_{F_n,v}^\times\right),$$

where $\mathcal{O}_{F_n,v}^\times$ is the unit group of the completion of F_n at v. Therefore it suffices to show that $H^1(F_n/F_n^+, \mathcal{O}_{F_n,\mathscr{S}}^\times) = 0$ for all sufficiently large $n \gg 0$.

If F_∞ contains μ_{2^∞}, since all primes above 2 are contained in $\mathscr{S}(F_n)$, $\mathcal{O}^\times_{F_n,\mathscr{S}}$ contains $1-\zeta_{2^m}$ for all 2^mth roots of unity ζ_{2^m} in $\mu(F_n)$. This implies that $(\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J} = \mu(F_n)$ for all sufficiently large $n \gg 0$.

If Hasse's unit index $[\mathcal{O}^\times_{F_n} : \mu(F_n)\mathcal{O}^\times_{F_n^+}] = 2$, we also have $(\mathcal{O}^\times_{F_n,\mathscr{S}})^{1-J} = \mu(F_n)$ (see [5, Satz 14]).

Therefore, we get an exact sequence for all sufficiently large $n \gg 0$,

$$0 \longrightarrow \mathcal{O}^\times_{F_n^+,\mathscr{S}} \longrightarrow \mathcal{O}^\times_{F_n,\mathscr{S}} \xrightarrow{1-J} \mu(F_n) \to 0.$$

Put $G = \mathrm{Gal}(F_n/F_n^+)$. Taking Galois cohomology, we get an exact sequence

$$0 \longrightarrow \mathcal{O}^\times_{F_n^+,\mathscr{S}} \xrightarrow{f_1} \mathcal{O}^\times_{F_n^+,\mathscr{S}} \xrightarrow{f_2} \{\pm 1\} \xrightarrow{f_3} H^1(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) \xrightarrow{f_4} H^1(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}})$$
$$\xrightarrow{f_5} H^1(G,\ \mu(F_n)) \xrightarrow{f_6} H^2(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) \xrightarrow{f_7} H^2(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}}).$$

Since f_1 is an isomorphism, f_2 is the 0-map. Therefore f_3 is injective. Since F_n/F_n^+ is a cyclic extension,

$$H^1(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) = \frac{\ker\left(1+J : \mathcal{O}^\times_{F_n^+,\mathscr{S}} \to \mathcal{O}^\times_{F_n^+,\mathscr{S}}\right)}{\left(\mathcal{O}^\times_{F_n^+,\mathscr{S}}\right)^{1-J}} = \{\pm 1\}.$$

Thus f_3 is also an isomorphism and f_4 is the 0-map. Therefore f_5 is injective. Since

$$H^1(G,\ \mu(F_n)) = \frac{\ker(1+J : \mu(F_n) \to \mu(F_n))}{\mu(F_n)^{1-J}} = \frac{\mu(F_n)}{\mu(F_n)^2} = \{\pm 1\},$$

we get an exact sequence

$$0 \longrightarrow H^1(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}}) \xrightarrow{f_5} \{\pm 1\} \xrightarrow{f_6} H^2(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) \xrightarrow{f_7} H^2(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}}).$$

If f_7 is not injective, f_6 is not the 0-map. This implies that $H^1(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}}) = 0$. We show that f_7 is not injective. Since F_n/F_n^+ is a cyclic extension,

$$H^2(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) = \hat{H}^0(G,\ \mathcal{O}^\times_{F_n^+,\mathscr{S}}) = \frac{(\mathcal{O}^\times_{F_n^+,\mathscr{S}})^G}{N_{F_n/F_n^+}(\mathcal{O}^\times_{F_n^+,\mathscr{S}})} = \frac{\mathcal{O}^\times_{F_n^+,\mathscr{S}}}{(\mathcal{O}^\times_{F_n^+,\mathscr{S}})^2},$$
$$H^2(G,\ \mathcal{O}^\times_{F_n,\mathscr{S}}) = \frac{\mathcal{O}^\times_{F_n^+,\mathscr{S}}}{N_{F_n/F_n^+}(\mathcal{O}^\times_{F_n,\mathscr{S}})}.$$

If Hasse's unit index $[\mathcal{O}^\times_{F_n} : \mu(F_n)\mathcal{O}^\times_{F_n^+}] = 2$, Satz 14 in [5] shows that $[(\mathcal{O}^\times_{F_n})^{1+J} : (\mathcal{O}^\times_{F_n^+})^2] = 2$. This implies that f_7 is not injective.

If F_∞ contains μ_{2^∞}, $\mathcal{O}^\times_{F_n,\mathscr{S}}$ contains $1+\zeta_{2^l}$ where 2^l is the order of the 2-Sylow subgroup of $\mu(F_n)$. Thus we have

$$N_{F_n/F_n^+}(1+\zeta_{2^l}) = 2 + \zeta_{2^l} + \zeta_{2^l}^{-1} \in N_{F_n/F_n^+}(\mathcal{O}^\times_{F_n,\mathscr{S}}).$$

Since $\sqrt{2+\zeta_{2^l}+\zeta_{2^l}^{-1}} = \pm(\zeta_{2^{l+1}}+\zeta_{2^{l+1}}^{-1}) \notin F_n^\times$, we have

$$2+\zeta_{2^{l+1}}+\zeta_{2^{l+1}}^{-1} \notin (\mathcal{O}_{F_n^+,\mathscr{S}}^\times)^2.$$

This implies that f_7 is not injective. This completes the proof of Lemma 2.6. □

We give some examples here.

Example 2.7. Let F be an imaginary quadratic field that is not $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and the prime above 2 ramifies in F_∞/F_∞^+. Then, Leopoldt's conjecture is valid for $F^+ = \mathbb{Q}$. Since the class numbers of F_n^+ are odd for all $n \geq 0$ (see [5, Satz 6]), the lifting maps $A_{F_n^+,\mathscr{S}} \longrightarrow A_{F_n,\mathscr{S}}$ are injective for all $n \geq 0$. Since F_∞ does not contain all 2^nth roots of unity for $n \geq 1$, Theorem 0.1 implies that

$$F_\Lambda(X_{F_\infty}^-) \cong \mathbb{Z}/2\mathbb{Z}.$$

This is the result which was proved by Ferrero in [1].

Example 2.8. Let F^+ be a real abelian field which is unramified at 2, and $F = F^+(\sqrt{-1})$. Then, we have

$$F_\Lambda(X_{F_\infty}^-) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus d-1},$$

where d is the number of primes of F lying above 2. In fact, Theorem 1 in [7] implies that $(\mathcal{O}_{F_n}^\times)^{1-J} = \mu(F_n)^2$ for all $n \geq 0$. Since $1-\zeta_{2^{n+2}} \in \mathcal{O}_{F_n,\mathscr{S}}^\times$, we have $(\mathcal{O}_{F_n,\mathscr{S}}^\times)^{1-J} = \mu(F_n)$ for all $n \geq 0$. Therefore we have $(\mathcal{O}_{F_n,\mathscr{S}}^\times)^{1-J}/(\mathcal{O}_{F_n}^\times)^{1-J} \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 0$ and $\delta_1 = 1$. Theorem 1 in [7] also implies that $\delta_2 = 0$.

3. $\mathscr{S}$-modified Iwasawa module $X_{F_\infty,\mathscr{S}}$

We use the same notation as in the Introduction. We defined $X_{F_\infty,\mathscr{S}}$, $X_{F_\infty,\mathscr{S}}^-$ by

$$X_{F_\infty,\mathscr{S}} = \varprojlim A_{F_n,\mathscr{S}}, \quad X_{F_\infty,\mathscr{S}}^- = X_{F_\infty,\mathscr{S}}/(1+J)X_{F_\infty,\mathscr{S}}.$$

In this section, we prove Theorem 1.3, using a result of Greenberg in [4]. At first, we introduce the result of Greenberg. Greenberg describes his theorems in a much more general setting in [4]. However, we describe it in a restricted setting here.

Let p be a prime. Suppose that K is a finite extension of $\mathbb{Q}$ and that Σ is a finite set of primes of K. Let K_Σ be the maximal extension of K unramified outside Σ. We assume that Σ contains all archimedean primes and all primes lying above p. Put $\Lambda := \mathbb{Z}_p[[T]]$ and let $\mathcal{T}$ be a $\mathrm{Gal}(K_\Sigma/K)$-module such that $\mathcal{T} \cong \Lambda$ as a group and $\mathrm{Gal}(K_\Sigma/K)$ acts on $\mathcal{T}$ continuously. We define $\mathcal{D} = \mathcal{T} \otimes_\Lambda \hat{\Lambda}$, where $\hat{\Lambda} = \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual of Λ. The Galois group $\mathrm{Gal}(K_\Sigma/K)$ acts on $\mathcal{D}$ through its action on the first factor $\mathcal{T}$. We note that $\mathcal{D}$ is a discrete abelian group and the Galois cohomology group $H^1(K_\Sigma/K,\mathcal{D})$ is a discrete Λ-module. Let $L(K_v,\mathcal{D})$ be a Λ-submodule of $H^1(K_v,\mathcal{D})$ for each $v\in\Sigma$, where K_v is the completion of K at v. Put $Q(K,\mathcal{D}) := \prod_{v\in\Sigma} H^1(K_v,\mathcal{D})/L(K_v,\mathcal{D})$. The natural global-to-local maps induce a map

$$\phi : H^1(K_\Sigma/K,\mathcal{D}) \longrightarrow Q(K,\mathcal{D}).$$

The kernel of ϕ is denoted by $S(K,\mathcal{D})$. We define $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D},\mu_{p^\infty})$, and

$$Ш^2(K,\Sigma,\mathcal{D}) = \ker\Big(H^2(K_\Sigma/K,\mathcal{D}) \longrightarrow \prod_{v\in\Sigma} H^2(K_v,\mathcal{D})\Big).$$

We say that a finitely generated Λ-module M is reflexive if the map

$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_\Lambda\left(\mathrm{Hom}_\Lambda(M,\Lambda),\Lambda\right) \\ m &\longmapsto [\alpha\mapsto \alpha(m)]. \end{aligned}$$

is an isomorphism. Suppose that N is a discrete Λ-module and that its Pontryagin dual is finitely generated. We say that N is almost Λ-divisible if there is a nonzero element $f(T)\in\Lambda$ such that $g(T)N = N$ for all irreducible elements $g(T)\in\Lambda$ not dividing $f(T)$.

Theorem 3.1 (Greenberg [4, Proposition 4.1.1])**.** *Suppose that the following assumptions are satisfied,*

(a) *The Λ-module $Ш^2(K,\Sigma,\mathcal{D})$ is Λ-cotorsion,*
(b) *The Λ-module $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive for all $v\in\Sigma$,*
(c) *There exists a non-archimedean prime $v\in\Sigma$ such that $(\mathcal{T}^*)^{G_{K_v}}=0$,*
(d) *$\prod_{v\in\Sigma} L(K_v,\mathcal{D})$ is almost Λ-divisible,*
(e) $\mathrm{corank}_\Lambda\left(H^1(K_\Sigma/K,\mathcal{D})\right) = \mathrm{corank}_\Lambda\left(S(K,\mathcal{D})\right) + \mathrm{corank}_\Lambda\left(Q(K,\mathcal{D})\right)$*,*
(f) *At least one of the following additional assumptions is satisfied.*
   - *$\mathcal{D}[\mathfrak{m}]$ has no subquotient isomorphic to μ_p for the action of $G_K = \mathrm{Gal}(\bar{K}/K)$.*
   - *$\mathcal{D}$ is a cofree Λ-module and $\mathcal{D}[\mathfrak{m}]$ has no quotient isomorphic to μ_p for the action of G_K.*
   - *There is a prime $v\in\Sigma$ which satisfies* (c) *and such that $H^1(K_v,\mathcal{D})/L(K_v,\mathcal{D})$ is coreflexive as a Λ-module.*

Then $S(K,\mathcal{D})$ is almost Λ-divisible.

In Sections 3.4 and 3.5 in [4], Greenberg discuss the case that the assumption (f) is not satisfied. We can replace the assumption (f) to (f*) as following.

Theorem 3.2 (Greenberg [4, Proposition 4.1.1 and Section 3.4, 3.5])**.** *Suppose that the following assumptions are satisfied,*

(a) *The Λ-module $Ш^2(K,\Sigma,\mathcal{D})$ is Λ-cotorsion,*

(b) *The Λ-module $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive for all $v \in \Sigma$,*
(c) *There exists a non-archimedean prime $v \in \Sigma$ such that $(\mathcal{T}^*)^{G_{K_v}} = 0$,*
(d) *$\prod_{v\in\Sigma} L(K_v, \mathcal{D})$ is almost Λ-divisible,*
(e) $\mathrm{corank}_\Lambda(H^1(K_\Sigma/K, \mathcal{D})) = \mathrm{corank}_\Lambda(S(K, \mathcal{D})) + \mathrm{corank}_\Lambda(Q(K, \mathcal{D}))$,
(f*) *$L(K_v, \mathcal{D}) \subset H^1(K_v, \mathcal{D})_{\Lambda-div}$ for all $v \in \Sigma$.*

Then $S(K, \mathcal{D})$ is almost Λ-divisible.

Remark 3.3. Let M be a finitely generated Λ-module, and N the Pontryagin dual of M (i.e., $N = \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$). Then, the following two statements are equivalent:

- M has no non-trivial finite Λ-submodule.
- N is almost Λ-divisible.

The proof of this fact can be found in Proposition 2.4 in Greenberg [3]

We prove Theorem 1.3 using Theorem 3.2, taking $K = F^+, p = 2$. We may assume that all primes above 2 are totally ramified in F_∞/F and F_∞^+/F^+. We define

$$\Sigma = S_{\mathrm{ram}}(F/F^+) \cup S_\infty(F^+) \cup S_2(F^+),$$

where $S_{\mathrm{ram}}(F/F^+)$ is the set of primes of F^+ which ramify in F/F^+. Let F_Σ^+ be the maximal extension of F^+ unramified outside Σ. By definition, $F_\infty \subset F_\Sigma^+$. Put $\Gamma := \mathrm{Gal}(F_\infty^+/F^+)$, and $\Lambda := \mathbb{Z}_2[\![\Gamma]\!] \cong \mathbb{Z}_2[\![T]\!]$. Let J be the complex conjugation. By definition, $\mathrm{Gal}(F/F^+) = \{1, J\}$. We take $\mathcal{T}$ to be a $\mathrm{Gal}(F_\Sigma^+/F^+)$-module such that $\mathcal{T} \cong \Lambda$ as a Λ-module, for which J acts as -1, and the group $\mathrm{Gal}(F_\Sigma^+/F^+)$ acts on $\mathcal{T}$ through the natural map $\mathrm{Gal}(F_\Sigma^+/F^+) \twoheadrightarrow \mathrm{Gal}(F_\infty/F^+) \cong \mathrm{Gal}(F/F^+) \times \mathrm{Gal}(F_\infty^+/F^+)$. We define

$$\mathcal{D} = \mathcal{T} \otimes_\Lambda \hat{\Lambda}, \quad \mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{2^\infty}),$$

where $\hat{\Lambda} = \mathrm{Hom}(\Lambda, \mathbb{Q}_2/\mathbb{Z}_2)$ is the Pontryagin dual of Λ. We define the Λ-submodule $L(F_v^+, \mathcal{D})$ of $H^1(F_v^+, \mathcal{D})$ for each $v \in \Sigma$

$$L(F_v^+, \mathcal{D}) = \begin{cases} \ker\left(H^1(F_v^+, \mathcal{D}) \to H^1(F_v^{+\,\mathrm{unr}}, \mathcal{D})\right), & \text{if } v \notin S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+) \\ 0, & \text{if } v \in S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+), \end{cases}$$

where $F_v^{+\,\mathrm{unr}}$ is the maximal unramified extension of F_v^+ and $S_{\mathrm{ram}}(F_\infty/F_\infty^+)$ is the set of primes of F^+ which ramify in F_∞/F_∞^+. Put $Q(F^+, \mathcal{D}) := \prod_{v\in\Sigma} H^1(F_v^+, \mathcal{D})/L(F_v^+, \mathcal{D})$. The natural global-to-local maps induce a map

$$\phi : H^1(F_\Sigma^+/F^+, \mathcal{D}) \longrightarrow Q(F^+, \mathcal{D}).$$

The kernel of ϕ is denoted by $S(F^+, \mathcal{D})$. In this situation, we check the assumptions (a), (b), (c), (d), (e), (f*) in Theorem 3.2.

Proof of Theorem 1.3. (c) $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{2^\infty}) = \mathrm{Hom}(\mathcal{D}, \mathbb{Q}_2/\mathbb{Z}_2) \otimes \mathbb{Z}_2(1) = \Lambda(1)$ as Λ-modules. Since no prime splits completely in the cyclotomic $\mathbb{Z}_2$-extension in F_∞^+/F^+, G_{K_v} acts on $\Lambda(1)$ nontrivially for each v. Therefore, $(\mathcal{T}^*)^{G_{F_v^+}} = 0$ for any non-archimedean prime $v \in \Sigma$.

(b) If v is non-archimedean, $\mathcal{T}^*/(\mathcal{T}^*)^{G_{F_v^+}} \cong \Lambda(1)$. This Λ-module $\Lambda(1)$ is reflexive. If v is archimedean, $G_{F_v^+} = \{1, J\}$. Since

$$(Jf)(x) = J\big(f(J^{-1}x)\big) = J\big(f(-x)\big) = J\big(f(x)^{-1}\big) = f(x)$$

for any $f \in \mathcal{T}^*$ and $x \in \mathcal{D}$, so J acts trivially on $\mathcal{T}^*$. Thus, $\mathcal{T}^*/(\mathcal{T}^*)^{G_{F_v^+}} = 0$.

(d) We claim that

$$L(F_v^+, \mathcal{D}) \cong \begin{cases} \mathbb{Q}_2/\mathbb{Z}_2 & (\text{if } v \in S_2(F^+) \text{ and } v \text{ splits in } F_\infty/F_\infty^+) \\ 0 & (\text{otherwise}) \end{cases}$$

for each $v \in \Sigma$. This fact implies that $\prod_{v\in\Sigma} L(K_v, \mathcal{D})$ is almost Λ-divisible.

If $v \in S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$, this is trivial by definition. Thus, we consider the case $v \notin S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$. The inflation-restriction sequence shows that

$$L(F_v^+, \mathcal{D}) \cong H^1(F_v^{+\mathrm{unr}}/F_v^+, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}).$$

If v is archimedean, $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+) = 1$ implies $L(F_v^+, \mathcal{D}) = 0$.

If v is non-archimedean and $v \notin S_2(F^+)$, then v is unramified in F_∞^+/F^+ and hence $F_{v,\infty}^+ \subset F_v^{+\mathrm{unr}}$, where $F_{v,\infty}^+$ is the cyclotomic $\mathbb{Z}_2$-extension of F_v^+. Thus, $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$ contains the unique subgroup P_v which is isomorphic to $\mathbb{Z}_2$ and the restriction map $P_v \to \Gamma_v = \mathrm{Gal}(F_{v,\infty}^+/F_v^+)$ is an isomorphism. The inflation-restriction sequence shows that the restriction map

$$H^1(F_v^{+\mathrm{unr}}/F_v^+, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) \longrightarrow H^1(P_v, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}})$$

is injective. Hence, it suffices to show that $H^1(P_v, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) = 0$. The action of $G_{F_v^{+\mathrm{unr}}}$ on $\mathcal{D}$ factors through $G_{F_v^{+\mathrm{unr}}} \twoheadrightarrow \mathrm{Gal}(F_w/F_v^+) = \{1, J\}$, where w is a prime of F lying above v. Since J acts on $\mathcal{D}$ as -1 and $\mathcal{D}$ is a divisible group, we get an exact sequence

$$0 \longrightarrow \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}} \longrightarrow \mathcal{D} \xrightarrow{1-J} \mathcal{D} \longrightarrow 0.$$

Taking Galois cohomology, we get an exact sequence

$$\mathcal{D}^{P_v} \xrightarrow{\times 2} \mathcal{D}^{P_v} \longrightarrow H^1(P_v, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) \longrightarrow H^1(P_v, \mathcal{D}).$$

Let γ_v be a topological generator of Γ_v. Then,

$$\mathcal{D}^{P_v} \cong \mathrm{Hom}_{\Gamma_v}(\Lambda, \mathbb{Q}_2/\mathbb{Z}_2) \cong \mathrm{Hom}(\Lambda/(1-\gamma_v), \mathbb{Q}_2/\mathbb{Z}_2) \cong (\mathbb{Q}_2/\mathbb{Z}_2)^{\oplus n}$$

where, $n = [\Gamma : \Gamma_v]$. Thus, $\mathcal{D}^{P_v}$ is a divisible group and the map $\mathcal{D}^{P_v} \xrightarrow{\times 2} \mathcal{D}^{P_v}$ is surjective. Therefore, the map $H^1(P_v, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) \longrightarrow H^1(P_v, \mathcal{D})$ is injective. Here, $H^1(P_v, \mathcal{D}) \cong \mathcal{D}/(1-\gamma_v)\mathcal{D} = 0$ because $1-\gamma_v$ acts on $\mathcal{D}$ as the multiplication by a nonzero element of Λ and $\mathcal{D}$ is Λ-divisible. Thus, $H^1(P_v, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) = 0$ for each non-archimedean prime $v \notin S_2(F^+)$.

We consider the case that $v \in S_2(F^+)$ and v is inert in F_∞/F_∞^+. Let P_v be the maximal subgroup of $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$ which is isomorphic to $\mathbb{Z}_2$, and γ_v a topological generator of P_v. The action of $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$ on $\mathcal{D}$ factors through $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+) \twoheadrightarrow \mathrm{Gal}(F_w/F_v^+) = \{1, J\}$, where w is the prime of F lying above v. Therefore, γ_v acts on $\mathcal{D}$ as -1. Thus,

$$\begin{aligned} H^1(F_v^{+\mathrm{unr}}/F_v^+, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) &= H^1(F_v^{+\mathrm{unr}}/F_v^+, A) \\ &= H^1(P_v, A) \\ &= A/(1-\gamma_v)A = 0, \end{aligned}$$

where A is a $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$-module such that A is isomorphic to $\mathbb{Q}_2/\mathbb{Z}_2$ as a group, for which J acts as -1, and $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$ acts via $\mathrm{Gal}(F_w/F_v^+) = \{1, J\}$.

If $v \in S_2(F^+)$ and v splits in F_∞/F_∞^+, the action of $\mathrm{Gal}(F_v^{+\mathrm{unr}}/F_v^+)$ on $\mathcal{D}$ is trivial since we assumed that all primes above 2 are totally ramified in F_∞^+/F^+. Therefore,

$$\begin{aligned} H^1(F_v^{+\mathrm{unr}}/F_v^+, \mathcal{D}^{G_{F_v^{+\mathrm{unr}}}}) &= H^1(F_v^{+\mathrm{unr}}/F_v^+, \mathbb{Q}_2/\mathbb{Z}_2) \\ &= \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Q}_2/\mathbb{Z}_2) \\ &\cong \mathbb{Q}_2/\mathbb{Z}_2. \end{aligned}$$

(f*) Let γ be a topological generator of Γ. $L(F_v^+, \mathcal{D})$ is a divisible group and annihilated by the ideal $(1-\gamma)$ for each $v \in \Sigma$. By Remark 3.5.2 in [4], we have the inclusion $L(F_v^+, \mathcal{D}) \subset H^1(F_v^+, \mathcal{D})_{\Lambda-div}$.

Before we check the assumptions (a) and (e), we prove the following lemma.

Lemma 3.4. *We have*

$$S(F^+, \mathcal{D}) = \mathrm{Hom}\left(X^-_{F_\infty, \mathscr{S}}, \mathbb{Q}_2/\mathbb{Z}_2\right),$$

where $X^-_{F_\infty, \mathscr{S}}$ is the Λ-module defined on page 1018.

Proof. For each w lying above $v \in \Sigma$, we define $L(F_w, \mathcal{D})$ by

$$L(F_w, \mathcal{D}) = \ker\left(H^1(F_w, \mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)} \to H^1({F_w}^{\mathrm{unr}}, \mathcal{D})\right)$$

if $v \notin S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$ and

$$L(F_w, \mathcal{D}) = 0$$

if $v \in S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$.

At first, we claim that the map

$$\frac{H^1(F_v^+,\mathcal{D})}{L(F_v^+,\mathcal{D})} \longrightarrow \frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})} \tag{3.1}$$

induced by the restriction map is injective for each w lying above $v \in \Sigma$.

If $v \notin S_2(F^+)$ or $v \in S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$, we have $L(F_v^+,\mathcal{D}) = 0$. Similarly, we have $L(F_w,\mathcal{D}) = 0$. Since $\mathcal{D}^{G_{F_w}}$ is a divisible group and J acts on $\mathcal{D}^{G_{F_w}}$ as -1, we have

$$H^1(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) = \frac{\ker(\mathcal{D}^{G_{F_w}} \overset{1+J}{\to} \mathcal{D}^{G_{F_w}})}{(\mathcal{D}^{G_{F_w}})^{1-J}} = 0.$$

Therefore the inflation-restriction sequence

$$0 \longrightarrow H^1(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) \longrightarrow H^1(F_v^+,\mathcal{D}) \longrightarrow H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}$$

implies the above map (3.1) is injective.

If $v \in S_2(F^+)$ and v is unramified in F_∞/F_∞^+, then $F_w^{\mathrm{unr}} = F_v^{+\mathrm{unr}}$. We consider the commutative diagram

$$
\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & H^1(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) & = & H^1(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) \\
 & & \downarrow \text{inf} & & \downarrow \text{inf} \\
0 & \longrightarrow & H^1(F_v^{+\mathrm{unr}}/F_v^+,\mathcal{D}^{G_{F_w^{\mathrm{unr}}}}) & \xrightarrow{\text{inf}} & H^1(F_v^+,\mathcal{D}) \\
 & & \downarrow \text{res} & & \downarrow \text{res} \\
0 & \longrightarrow & H^1(F_v^{+\mathrm{unr}}/F_w,\mathcal{D}^{G_{F_w^{\mathrm{unr}}}})^{\mathrm{Gal}(F_w/F_v^+)} & \xrightarrow{\text{inf}} & H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)} \\
 & & \downarrow & & \downarrow \\
 & & H^2(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) & = & H^2(F_w/F_v^+,\mathcal{D}^{G_{F_w}}).
\end{array}
$$

By the snake lemma, the map $\frac{H^1(F_v^+,\mathcal{D})}{L(F_v^+,\mathcal{D})} \longrightarrow \frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})}$ is injective. This completes the proof of the injectivity of (3.1).

Put

$$Q(F,\mathcal{D}) = \prod_{w \in \Sigma_F} \frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})},$$

where Σ_F is the set of primes of F lying above Σ. We consider the commutative diagram

$$\begin{array}{ccccccc}
 & & & & H^1(F/F^+, \mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)}) & & \\
 & & & & \big\downarrow{\scriptstyle \mathrm{inf}} & & \\
0 \longrightarrow & S(F^+,\mathcal{D}) & \longrightarrow & & H^1(F_\Sigma^+/F^+,\mathcal{D}) & \xrightarrow{\ \phi\ } & Q(F^+,\mathcal{D}) \\
 & \big\downarrow & & & {\scriptstyle \mathrm{res}}\big\downarrow{\scriptstyle g_1} & & \big\downarrow{\scriptstyle g_2} \\
0 \longrightarrow & \ker f & \longrightarrow & & H^1(F_\Sigma^+/F,\mathcal{D})^{\mathrm{Gal}(F/F^+)} & \xrightarrow{\ f\ } & Q(F,\mathcal{D}) \\
 & & & & \big\downarrow & & \big\downarrow \\
 & & & & \operatorname{coker} g_1 & \longrightarrow & \operatorname{coker} g_2
\end{array}$$

Here, we defined g_1 to be the restriction map

$$H^1(F_\Sigma^+/F^+,\mathcal{D}) \longrightarrow H^1(F_\Sigma^+/F,\mathcal{D})^{\mathrm{Gal}(F/F^+)}$$

and g_2 the map $Q(F^+,\mathcal{D}) \to Q(F,\mathcal{D})$ induced by the restriction map. Next, we show that the map $\operatorname{coker} g_1 \to \operatorname{coker} g_2$ is injective.

By definition, we have

$$\operatorname{coker} g_2 = \prod_{v\in\Sigma} \operatorname{coker}\left(\frac{H^1(F_v^+,\mathcal{D})}{L(F_v^+,\mathcal{D})} \to \bigoplus_{w|v} \frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})}\right).$$

For any prime $v\in\Sigma$, put

$$(\operatorname{coker} g_2)_v = \operatorname{coker}\left(\frac{H^1(F_v^+,\mathcal{D})}{L(F_v^+,\mathcal{D})} \to \bigoplus_{w|v} \frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})}\right).$$

It is sufficiently to show that the map $\operatorname{coker} g_1 \to (\operatorname{coker} g_2)_v$ is injective for any $v \in S_\infty(F^+)$. Since $L(F_v^+,\mathcal{D}) = 0, L(F_w,\mathcal{D}) = 0$ for any $v \in S_\infty(F^+)$, the inflation-restriction sequence implies the commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & \operatorname{coker} g_1 & \longrightarrow & H^2(F/F^+, \mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)}) & \\
 & \big\downarrow & & \big\downarrow{\scriptstyle g_3} & \\
0 \longrightarrow & (\operatorname{coker} g_2)_v & \longrightarrow & H^2(F_w/F_v^+, \mathcal{D}^{G_{F_w}}). &
\end{array}$$

We show that the map g_3 is injective. We know that $H^2(F_w/F_v^+,\mathcal{D}^{G_{F_w}}) = H^2(F/F^+,\mathcal{D})$. Put $\mathcal{D}' = \operatorname{coker}(\mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)} \to \mathcal{D})$. Since both $\mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)}$ and $\mathcal{D}$ are divisible groups, $\mathcal{D}'$ is also divisible. We consider the exact sequence

$$H^1(F/F^+,\mathcal{D}') \longrightarrow H^2(F/F^+,\mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)}) \xrightarrow{g_3} H^2(F/F^+,\mathcal{D}).$$

Since $\mathcal{D}'$ is divisible, we have

$$H^1(F/F^+,\mathcal{D}') = \frac{\ker(\mathcal{D}' \overset{1+J}{\to} \mathcal{D}')}{\mathcal{D}'^{1-J}} = \frac{\mathcal{D}'}{2\mathcal{D}'} = 0.$$

Therefore the map g_3 is injective. This implies that the map the map $\operatorname{coker} g_1 \to \operatorname{coker} g_2$ is injective.

The map g_2 is injective by the injectivity of (3.1). And we have $H^1(F/F^+,\mathcal{D}^{\mathrm{Gal}(F_\Sigma^+/F)}) = 0$. Thus, $S(F^+,\mathcal{D})$ is isomorphic to $\ker f$ by the snake lemma. We also have $\mathcal{D} = \mathrm{Ind}_{\mathrm{Gal}(F_\Sigma^+/F)}^{\mathrm{Gal}(F_\Sigma^+/F_\infty)}(A)$, where A is a $\mathrm{Gal}(F_\Sigma^+/F^+)$-module such that A is isomorphic to $\mathbb{Q}_2/\mathbb{Z}_2$ as a group, for which J acts as -1, and $\mathrm{Gal}(F_\Sigma^+/F^+)$ acts on A via $\mathrm{Gal}(F/F^+) = \{1, J\}$. Thus, we have

$$\begin{aligned} H^1(F_\Sigma^+/F,\mathcal{D})^{\mathrm{Gal}(F/F^+)} &\cong H^1(F_\Sigma^+/F_\infty, A)^{\mathrm{Gal}(F/F^+)} \\ &= \mathrm{Hom}_{\mathrm{Gal}(F/F^+)}(\mathrm{Gal}(F_\Sigma^{\mathrm{ab}}/F_\infty), A) \end{aligned}$$

by Shapiro's lemma, where F_Σ^{ab} is the maximal abelian pro-2-extension of F unramified outside Σ_F. We may assume that all primes in Σ_F does not split in F_∞/F. We denote by $F_{w,\infty}$ the cyclotomic $\mathbb{Z}_2$-extension of F_w. Similarly, we have

$$\begin{aligned} &\frac{H^1(F_w,\mathcal{D})^{\mathrm{Gal}(F_w/F_v^+)}}{L(F_w,\mathcal{D})} \\ &\quad = \begin{cases} \mathrm{Hom}_{\mathrm{Gal}(F_w/F_v^+)}(I_{F_{w,\infty}}, A) & (v \notin S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)) \\ \mathrm{Hom}_{\mathrm{Gal}(F_w/F_v^+)}(G_{F_{w,\infty}}, A) & (v \in S_2(F^+) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)), \end{cases} \end{aligned}$$

for each $w|v$, where $I_{F_{w,\infty}}$ is the inertia group in $G_{F_{w,\infty}}$. Therefore we have

$$\ker f \cong \mathrm{Hom}_{\mathrm{Gal}(F/F^+)}\big(\mathrm{Gal}(L'_\infty/F_\infty), A\big),$$

where L'_∞ is the maximal unramified abelian pro-2-extension of F_∞ in which the primes of F_∞ lying above $S_2(F) \cap S_{\mathrm{ram}}(F_\infty/F_\infty^+)$ split completely. By class field theory, $\mathrm{Gal}(L'_\infty/F_\infty)$ is isomorphic to $X_{F_\infty,\mathscr{S}}$. Thus, we have

$$S(F^+,\mathcal{D}) \cong \mathrm{Hom}_{\mathrm{Gal}(F/F^+)}\big(\mathrm{Gal}(L'_\infty/F_\infty), A\big) = \mathrm{Hom}\big(X_{F_\infty,\mathscr{S}}^-, \mathbb{Q}_2/\mathbb{Z}_2\big).$$

This completes the proof of Lemma 3.4. □

Finally, we check the assumptions (a) and (e) to complete the proof of Theorem 1.3.

(e) One has the following obvious inequality:

$$\mathrm{corank}_\Lambda\big(S(F^+,\mathcal{D})\big) \geq \mathrm{corank}_\Lambda\big(H^1(F_\Sigma^+/F^+,\mathcal{D})\big) - \mathrm{corank}_\Lambda\big(Q(F^+,\mathcal{D})\big).$$

The formulae in Section 2.3 in [4] show that the Λ-corank of $H^1(F_\Sigma^+/F^+,\mathcal{D})$ is at least $[F^+ : \mathbb{Q}]$ and the Λ-corank of $Q(F^+,\mathcal{D})$ is equal to $[F^+ : \mathbb{Q}]$.

Iwasawa proved that $S(F^+, \mathcal{D})$ is cotorsion Λ-module([6, Theorem 5]). This implies that the Λ-corank of $H^1(F_\Sigma^+/F^+, \mathcal{D})$ is equal to $[F^+ : \mathbb{Q}]$ and (e) is satisfied.

(a) The formulae in Section 2.3 in [4] also show that

$$\operatorname{corank}_\Lambda \left(H^1(F_\Sigma^+/F^+, \mathcal{D})\right) = \operatorname{corank}_\Lambda \left(H^2(F_\Sigma^+/F^+, \mathcal{D})\right) + [F^+ : \mathbb{Q}].$$

This implies that $H^2(F_\Sigma^+/F^+, \mathcal{D})$ is a cotorsion Λ-module and hence $Ш^2(K, \Sigma, \mathcal{D})$ is also Λ-cotorsion.

Thus Theorem 3.1 implies that $S(F^+, \mathcal{D})$ is almost Λ-divisible. Since $S(F^+, \mathcal{D}) = \operatorname{Hom}\left(X^-_{F_\infty, \mathscr{S}}, \mathbb{Q}_2/\mathbb{Z}_2\right)$ by Lemma 3.4, this is equivalent that $X^-_{F_\infty, \mathscr{S}}$ has no non-trivial finite Λ-submodule (Remark 3.3). This completes the proof of Theorem 1.3. □

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