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Résumé. Ce travail fait suite à l'article [4] de Satoshi Fujii et l'auteur. Soient k un corps de nombres, p un nombre premier, et k^c/k la $\mathbb{Z}_p$-extension cyclotomique. Pour un ensemble fini S de nombres premiers qui ne contient pas p, le module d'Iwasawa (par rapport à la pro-p extension abélienne maximale non ramifiée en dehors de S) a été étudié dans plusieurs articles. Nous donnons des exemples non-triviaux où $X_S(k^c)$ a un sous-module fini non-nul avec k totalement réel. Nous donnons également un exemple similaire dans le cas de la $\mathbb{Z}_p^{\oplus 2}$-extension d'un corps quadratique imaginaire. De plus, nous discutons en appendice des analogues faibles de la conjecture de Greenberg pour $X_S(k^c)$.

Abstract. The present paper is a sequel to the previous paper [4] (by Satoshi Fujii and the author). Let k be an algebraic number field, p a prime number, and k^c/k the cyclotomic $\mathbb{Z}_p$-extension. For a finite set S of prime numbers which does not contain p, the Iwasawa module $X_S(k^c)$ (with respect to the maximal pro-p abelian extension unramified outside S) has been studied in several papers. We will give some non-trivial examples such that $X_S(k^c)$ has no non-trivial finite submodules even when k is totally real. We also give a similar example for the case of the $\mathbb{Z}_p^{\oplus 2}$-extension of an imaginary quadratic field. Moreover, weak analogs of Greenberg's conjecture for $X_S(k^c)$ are also discussed in the appendix.

1. Introduction and results

Let p be a prime number, S a finite set of prime numbers which does not contain p. For an algebraic extension $\mathcal{K}/\mathbb{Q}$, let $L_S(\mathcal{K})/\mathcal{K}$ be the maximal abelian (pro-)p-extension unramified outside S. We put $X_S(\mathcal{K}) = \mathrm{Gal}(L_S(\mathcal{K})/\mathcal{K})$. When $\mathcal{K}$ is an algebraic number field (i.e., $\mathcal{K}/\mathbb{Q}$ is finite), $L_S(\mathcal{K})/\mathcal{K}$ is finite because all ramified primes are tamely ramified.

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Let k be an algebraic number field, and K/k a $\mathbb{Z}_p$-extension. We put $\Lambda_{K/k} = \mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$. Then, we can show that $X_S(K)$ is a finitely generated torsion module over $\Lambda_{K/k}$ (this is often called a tamely ramified Iwasawa module). We will consider the existence of a non-trivial pseudo-null $\Lambda_{K/k}$-submodule of $X_S(K)$. (For the definition of pseudo-nullity, see, e.g., [18]. In this case, a pseudo-null $\Lambda_{K/k}$-module is just a finite $\Lambda_{K/k}$-module.) We denote by k^c/k the cyclotomic $\mathbb{Z}_p$-extension.

In [4], it was shown that if p is odd and $X_S(\mathbb{Q}^c) \neq 0$, then $X_S(\mathbb{Q}^c)$ always contains a non-trivial finite submodule. On the other hand, when $p = 2$, Mizusawa's result [16, Theorem 7.3] implies the existence of the case that $X_S(\mathbb{Q}^c) \cong \mathbb{Z}_2$ as a $\mathbb{Z}_2$-module. Hence, the case when $p = 2$ is more complicated. Our first result is a determination of the set S of odd prime numbers such that $X_S(\mathbb{Q}^c)$ does not contain a non-trivial finite submodule for $p = 2$ (the proof will be given in Section 3).

Theorem 1.1. *Assume that $p = 2$. Let S be a non-empty finite set of odd prime numbers. For an odd prime number q, we denote by $P(q)$ the number of primes in $\mathbb{Q}^c$ lying above q. (Note that q is finitely decomposed in $\mathbb{Q}^c$.) Then $X_S(\mathbb{Q}^c)$ does not have a non-trivial finite $\Lambda_{\mathbb{Q}^c/\mathbb{Q}}$-submodule if and only if $S = \{q_1, \ldots, q_r\}$ satisfies $q_1 \equiv \cdots \equiv q_r \equiv 3 \pmod 4$ and $P(q_1) = \cdots = P(q_r)$ (where $q_1, \ldots, q_r$ are distinct prime numbers).*

When p is an odd prime number, we can also find an example of a totally real number field k such that $X_S(k^c)$ does not contain a non-trivial finite submodule. Our second result is a simple criterion whether $X_S(k^c)$ has no non-trivial finite submodules for a real quadratic field k and certain p and S. We denote by $|A|$ the number of elements of a finite set A.

Theorem 1.2. *Let p be an odd prime number, and k a real quadratic field. Assume that p is inert in k and p does not divide the class number of k. Take distinct prime numbers $q_1, \ldots, q_r$ such that $q_i \equiv -1 \pmod p$ and q_i is inert in k for $i = 1, \ldots, r$. We put $S = \{q_1, \ldots, q_r\}$. We denote by $P(q_i)$ the number of primes of $\mathbb{Q}^c$ lying above q_i, and by P' the largest number of $P(q_i)$ for $i = 1, \ldots, r$. Then $X_S(k^c)$ does not have a non-trivial finite $\Lambda_{k^c/k}$-submodule if and only if*

$$|X_S(k)| = p^{r-1} \cdot \left(\prod_{i=1}^{r} P(q_i) \right) / P'.$$

(Note that $p^{r-1} \cdot \left(\prod_{i=1}^{r} P(q_i)\right) / P' \leq |X_S(k)| \leq p^r \cdot \prod_{i=1}^{r} P(q_i)$ in this case. We also see that $X_S(k^c)$ is infinite if $|S| \geq 2$.)

This theorem will be shown in Section 4. As a consequence, one can find an explicit example such that $X_S(k^c)$ does not have a non-trivial finite submodule (see Remark 4.3). We note that when p splits in a real quadratic field k, the same type result does not hold (see Appendix A). We also give another (non-trivial) example of a totally real field k such that $X_S(k^c)$ does not have a non-trivial finite submodule (Proposition 4.5).

Next, we will consider the case of the $\mathbb{Z}_p^{\oplus 2}$-extension of an imaginary quadratic field k. Concerning this paragraph, see also [4] for the details. Let $\widetilde{k}/k$ be the unique $\mathbb{Z}_p^{\oplus 2}$-extension. We put $\Lambda_{\widetilde{k}/k} = \mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/k)]\!]$. Then $X_S(\widetilde{k})$ is a finitely generated torsion $\Lambda_{\widetilde{k}/k}$-submodule. In [4], some sufficient conditions such that $X_S(\widetilde{k})$ has a non-trivial pseudo-null $\Lambda_{\widetilde{k}/k}$-submodule were given. However, there is a non-trivial example such that $X_S(\widetilde{k})$ does not contain a non-trivial pseudo-null submodule. In Section 5, we will prove the following:

Theorem 1.3. *We put $k = \mathbb{Q}(\sqrt{-3})$ and $p = 3$. Let $\widetilde{k}/k$ be the unique $\mathbb{Z}_3^{\oplus 2}$-extension. Take a set $S = \{q_1, q_2\}$ of distinct prime numbers which satisfy $q_i \equiv 2 \pmod 3$ and q_i is not decomposed in $\mathbb{Q}^c$ for $i = 1, 2$. Then $X_S(\widetilde{k})$ is not pseudo-null, and it does not contain a non-trivial pseudo-null submodule.*

In Appendix A, we consider analogs of weak forms of Greenberg's conjecture in the sense of Nguyen Quang Do.

2. Preliminaries

We shall define some notations. Let $|\cdot|_p$ be the multiplicative p-adic absolute value normalized as $|p|_p = p^{-1}$. In the following of this section, k denotes an arbitrary algebraic number field. We denote by O_k the ring of integers of k, and by $E(k)$ the group of units of k. For a non-zero integral ideal $\mathfrak{m}$ of k, we put $R(k, \mathfrak{m}) = (O_k/\mathfrak{m})^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider every algebraic extension field over $\mathbb{Q}$ as a subfield of $\mathbb{C}$, and put $\zeta_n = e^{2\pi i/n}$ for a positive integer n.

Let K/k be a $\mathbb{Z}_p$-extension. Put $\Gamma = \mathrm{Gal}(K/k)$, and $\Lambda_{K/k} = \mathbb{Z}_p[\![\Gamma]\!]$. We also note that $\Lambda_{K/k}$ is isomorphic to the power series ring $\mathbb{Z}_p[\![T]\!]$ (and we fix an isomorphism). Let S be a finite set of prime numbers not containing p. In this case, there is a pseudo-isomorphism

$$X_S(K) \to \bigoplus_{i=1}^{m} \mathbb{Z}_p[\![T]\!]/p^{c_i}\mathbb{Z}_p[\![T]\!] \oplus \bigoplus_{j=1}^{n} \mathbb{Z}_p[\![T]\!]/g_j^{d_j}\mathbb{Z}_p[\![T]\!],$$

where c_i, d_j are positive integers and g_j is an irreducible distinguished polynomial for each i, j (see, e.g., [18], [26]). We put

$$\mu_{K/k,S} = \sum_{i=1}^{m} c_i \qquad \text{and} \qquad F_{K/k,S}(T) = \prod_{j=1}^{n} g_j^{d_j}.$$

$F_{K/k,S}(T)$ is called the characteristic polynomial of $X_S(K)$ (in the sense of [18, (5.3.9) Definition]). Note that $\mu_{K/k,S} = 0$ if and only if $X_S(K)$ is finitely generated as a $\mathbb{Z}_p$-module. In particular, if $k/\mathbb{Q}$ is an abelian extension and $K = k^c$ (the cyclotomic $\mathbb{Z}_p$-extension), then $\mu_{k^c/k,S} = 0$ (see, e.g., [10, p. 1494]). Note also that if $X_S(K)$ is finite, then $\mu_{K/k,S} = 0$ and $F_{K/k,S}(T) = 1$. Moreover, we denote by $X_S(K)^\Gamma$ the Γ-invariant submodule of $X_S(K)$, and by $X_S(K)_\Gamma$ the Γ-coinvariant quotient of $X_S(K)$ (similar notations will be used in Section 5 under a slightly different setting).

The following is our main criterion. (This type result seems well known, however, we will give a brief proof.)

Proposition 2.1. *Assume that there is only one prime $\mathfrak{p}$ of k lying above p, and $\mathfrak{p}$ is totally ramified in K/k. Then $X_S(K)$ does not have a non-trivial finite $\Lambda_{K/k}$-submodule if and only if*

$$|X_S(k)| \cdot |F_{K/k,S}(0)|_p = p^{\mu_{K/k,S}}.$$

Proof. By our assumptions, we can show that $X_S(K)_\Gamma \cong X_S(k)$ (see also, e.g., [13, Proposition 2.2.2] for a more general result), and hence $X_S(K)_\Gamma$ is finite. This implies that $X_S(K)^\Gamma$ is also finite, and $F_{K/k,S}(0) \neq 0$ (see, e.g., [18, p. 300, Exercise 3]). We can show that $X_S(K)$ does not have a non-trivial finite $\Lambda_{K/k}$-submodule if and only if $X_S(K)^\Gamma$ is trivial (see, e.g., [18, (5.3.19) Proposition] or the argument given in the proof of [22, Proposition 2]). It is known that

$$|X_S(K)_\Gamma| \cdot |F_{K/k,S}(0)|_p = |X_S(K)^\Gamma| \cdot p^{\mu_{K/k,S}}$$

(see, e.g., [18, p. 300, Exercise 3]). The assertion follows from this. □

As a corollary, we can obtain the following simpler criterion. (This type result also seems well known. See, e.g., the proof of [23, Theorem 2].)

Corollary 2.2. *Let the assumptions be as in Proposition 2.1. If $|X_S(k)| = p$ and $X_S(K)$ is infinite, then $X_S(K)$ does not have a non-trivial finite $\Lambda_{K/k}$-submodule.*

We will prove Theorems 1.1 and 1.2 by using Proposition 2.1 directly. (Note that a similar idea is already used in [4] to show the existence of a non-trivial finite submodule of $X_S(\mathbb{Q}^c)$ when p is odd.)

3. Proof of Theorem 1.1

In this section, we will only treat the case of $\mathbb{Q}^c/\mathbb{Q}$ when $p = 2$. Let S be a non-empty finite set of odd prime numbers. In this section, we write $F(T) = F_{\mathbb{Q}^c/\mathbb{Q},S}(T)$ for simplicity (note that $\mu_{\mathbb{Q}^c/\mathbb{Q},S} = 0$). We can compute $F(T)$ from the results given in [10]. To state this, we need some preparations.

We define a topological generator of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$ similar to [10]. That is, let γ be the topological generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_4)^c/\mathbb{Q}(\zeta_4))$ satisfying $\zeta_{2^n}^{\gamma} = \zeta_{2^n}^{5}$, and let γ_1 be the restriction of γ to $\mathbb{Q}^c$. Then γ_1 is a topological generator of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$. We fix an isomorphism $\Lambda_{\mathbb{Q}^c/\mathbb{Q}} \to \mathbb{Z}_2[\![T]\!]$ satisfying $\gamma_1 \mapsto 1+T$.

We define the subsets S° and $S^\bullet$ of S by

$$S^\circ = \{q \in S \mid q \equiv 1 \pmod 4\}, \qquad S^\bullet = \{q \in S \mid q \equiv 3 \pmod 4\}.$$

For $q \in S$, we put $P(q)$ the number of primes of $\mathbb{Q}^c$ lying above q. Let P° be the largest number of $P(q)$ for $q \in S^\circ$ (if S° is empty, we put $P^\circ = 0$). Moreover, let $\mathcal{P}^\bullet$ be the set of (distinct) numbers $P(q)$ for $q \in S^\bullet$, and put $\mathcal{P}^{\bullet\bullet} = \{P \in \mathcal{P}^\bullet \mid P \geq P^\circ\}$ (if $S^\bullet$ is empty, then both $\mathcal{P}^\bullet$ and $\mathcal{P}^{\bullet\bullet}$ are also empty). We define the following polynomials

$$F^\circ(T) = \begin{cases} \left(\prod_{q\in S^\circ}((1+T)^{P(q)} - 5^{P(q)})\right) / ((1+T)^{P^\circ} - 5^{P^\circ}) & \text{if } S^\circ \neq \emptyset, \\ 1 & \text{if } S^\circ = \emptyset, \end{cases}$$

$$F^\bullet(T) = \begin{cases} \left(\prod_{q\in S^\bullet}((1+T)^{P(q)} + 5^{P(q)})\right) / \left(\prod_{P\in\mathcal{P}^{\bullet\bullet}}((1+T)^{P} + 5^{P})\right) & \text{if } S^\bullet \neq \emptyset, \\ 1 & \text{if } S^\bullet = \emptyset. \end{cases}$$

Then, from the arguments and results given in [10] (especially, see the proof of Lemma 2.3 of [10]), we see that

$$F(T) = F^\circ(T) \cdot F^\bullet(T).$$

By using this formula, the value $|F(0)|_2$ can be obtained. (Note that $|1-5^{2^a}|_2 = 2^{-(a+2)}$ and $|1+5^{2^a}|_2 = 2^{-1}$.) We can also compute $|X_S(\mathbb{Q})|$ from the following exact sequence

$$0 \to E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \bigoplus_{q\in S} R(\mathbb{Q}, q\mathbb{Z}) \to X_S(\mathbb{Q}) \to 0$$

and the fact that $E(\mathbb{Q}) = \{\pm 1\}$.

At first, we assume that $S^\circ = \emptyset$. In this case, we see that

$$F(T) = \left(\prod_{q\in S^\bullet}((1+T)^{P(q)} + 5^{P(q)})\right) / \left(\prod_{P\in\mathcal{P}^{\bullet}}((1+T)^{P} + 5^{P})\right).$$

From this,

$$|F(0)|_2 = 2^{|\mathcal{P}^\bullet| - |S|}.$$

We also see that $|X_S(\mathbb{Q})| = 2^{|S|-1}$, and hence

$$|X_S(\mathbb{Q})| \cdot |F(0)|_2 = 2^{|\mathcal{P}^\bullet|-1}.$$

This implies that $|X_S(\mathbb{Q})| \cdot |F(0)|_2 = 1$ if and only if $|\mathcal{P}^\bullet| = 1$. When $S = \{q_1, \dots, q_r\}$, $|\mathcal{P}^\bullet| = 1$ if and only if $P(q_1) = \cdots = P(q_r)$. Hence, by using Proposition 2.1, the assertion of Theorem 1.1 has been shown for this case.

We shall show the remaining case. It is sufficient to show that $X_S(\mathbb{Q}^c)$ has a non-trivial finite submodule when $S^\circ \neq \emptyset$. In this case, we see that

$$|F^\circ(0)|_2 = 2^2 \cdot P^\circ \prod_{q \in S^\circ} (2^{-2} P(q)^{-1}), \quad |F^\bullet(0)|_2 = 2^{|\mathcal{P}^{\bullet\bullet}| - |S^\bullet|}.$$

On the other hand, we can show that

$$|X_S(\mathbb{Q})| = \left(2^{|S^\bullet|} \cdot \prod_{q \in S^\circ} (2^2 P(q)) \right) /2.$$

Hence

$$|X_S(\mathbb{Q})| \cdot |F(0)|_2 = 2^{|\mathcal{P}^{\bullet\bullet}|+1} \cdot P^\circ > 1.$$

By using Proposition 2.1, we see that $X_S(\mathbb{Q}^c)$ has a non-trivial finite submodule in this case. Thus we have completed the proof of Theorem 1.1. □

4. Totally real fields

We shall show Theorem 1.2, however, we will give a simple remark before this.

Remark 4.1. Let k be a real quadratic field, and p an odd prime number. Let S be a non-empty finite set of prime numbers not containing p. For the structure of $X_S(k^c)$, it is sufficient to consider the case that every $q \in S$ satisfies either

(a) $q \equiv 1 \pmod p$, or
(b) $q \equiv -1 \pmod p$ and q is inert in k

(see [8]). We put

$$S_1 = \{q \in S \mid q \equiv 1 \pmod p\}.$$

We note that if $S_1 \neq \emptyset$, then $X_S(k^c)$ always contains a non-trivial finite $\Lambda_{k^c/k}$-submodule. Indeed, since $\mathrm{Gal}(k^c/\mathbb{Q}^c)$ acts on $X_S(k^c)$, the plus and minus parts

$$X_S(k^c)^\pm = \{x \in X_S(k^c) \mid \sigma(x) = \pm x \text{ for the generator } \sigma \text{ of } \mathrm{Gal}(k^c/\mathbb{Q}^c)\}$$

can be defined, and we see that $X_S(k^c) \cong X_S(k^c)^+ \oplus X_S(k^c)^-$. We can show that $X_S(k^c)^+$ is isomorphic to $X_{S_1}(\mathbb{Q}^c)$, and this is not trivial because $S_1 \neq \emptyset$ (see also [10]). Hence, the assertion follows from the fact (which is shown in [4]) that $X_{S_1}(\mathbb{Q}^c)$ contains a non-trivial finite submodule. (The same type result for imaginary quadratic fields is given in [4].)

Proof of Theorem 1.2. Let k be a real quadratic field. Assume that p is inert in k, and p (>2) does not divide the class number of k. Let k_n^c be the nth layer of k^c/k. Take a topological generator γ of $\mathrm{Gal}(k(\zeta_p)^c/k(\zeta_p))$ which satisfies $\zeta_{p^n}^{\gamma} = \zeta_{p^n}^{1+p}$ for all n. Let γ_1 be the restriction of γ to k^c, then γ_1 is a topological generator of $\mathrm{Gal}(k^c/k)$. We fix an isomorphism from $\Lambda_{k^c/k}$ to $\mathbb{Z}_p[\![T]\!]$ satisfying $\gamma_1 \mapsto 1+T$.

Let $q_1, \ldots, q_r$ be distinct prime numbers satisfying the assumption of this theorem. For each i, we see that $R(k, q_i O_k)$ is a cyclic group of order $p \cdot P(q_i)$. Since $X_\emptyset(k) = 0$, we obtain the following exact sequence

$$E(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus_{i=1}^{r} R(k, q_i O_k) \to X_S(k) \to 0.$$

We note that $E(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a cyclic $\mathbb{Z}_p$-module. Hence, we have the inequalities

$$p^{r-1} \cdot \left(\prod_{i=1}^{r} P(q_i) \right) / P' \leq |X_S(k)| \leq p^r \cdot \prod_{i=1}^{r} P(q_i).$$

We will compute the characteristic polynomial of $X_S(k^c)$. The following argument is essentially given in [8, Section 6], however, we shall reconstruct it for our situation. We denote by k_n^c the nth layer of k^c/k. We put $R_i = \varprojlim_n R(k_n^c, q_i O_{k_n^c})$ (for each i) and $\mathcal{E} = \varprojlim_n E(k_n^c) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where the projective limits are taken with respect to the norm mappings. By using class field theory, we obtain the following exact sequence

$$\mathcal{E} \to \bigoplus_{i=1}^{r} R_i \to X_S(k^c) \to 0$$

of $\mathbb{Z}_p[\![\mathrm{Gal}(k^c/\mathbb{Q})]\!]$-modules (note that $X_\emptyset(k^c) = 0$ by Iwasawa's result [11]). For each term of the above exact sequence, we can consider its plus and minus parts with respect to the action of $\mathrm{Gal}(k^c/\mathbb{Q}^c)$. We note that $X_S(k^c)^+ \cong X_S(\mathbb{Q}^c) = 0$ because $q_i \equiv -1 \pmod{p}$ for all i (see also Remark 4.1). Hence we will consider the structure of $X_S(k^c)^- (\cong X_S(k^c))$ as a module over $\Lambda_{k^c/k} \cong \mathbb{Z}_p[\![T]\!]$. We can show that

$$R_i^- \cong R_i \cong \mathbb{Z}_p[\![T]\!]/((1+T)^{P(q_i)} - (1+p)^{P(q_i)})$$

as $\mathbb{Z}_p[\![T]\!]$-modules for each i (see, e.g., the argument given in the proof of [10, Lemma 2.1]). Note also that $\mathcal{E}^- \cong \mathbb{Z}_p[\![T]\!]$ as a $\mathbb{Z}_p[\![T]\!]$-module. (For example, by using [18, (11.3.11) Theorem (iii)], we see that

$$\mathcal{E} \cong \mathbb{Z}_p[\![\mathrm{Gal}(k^c/\mathbb{Q})]\!] \cong \mathbb{Z}_p[\mathrm{Gal}(k/\mathbb{Q})][\![T]\!],$$

and hence the fact follows.) From [8, Theorem 1.1], we see that $X_{\{q_i\}}(k^c)$ is finite for each i. Hence, by using the same type argument given in [10], we obtain the following exact sequence

$$0 \to \mathcal{E}^-/((1+T)^{P'} - (1+p)^{P'}) \to \bigoplus_{i=1}^{r} R_i \to X_S(k^c) \to 0$$

as $\mathbb{Z}_p[\![T]\!]$-modules. From the above results, we can see that

$$F_{k^c/k,S}(T) = \left(\prod_{i=1}^{r}((1+T)^{P(q_i)} - (1+p)^{P(q_i)})\right)/((1+T)^{P'} - (1+p)^{P'})$$

(and $\mu_{k^c/k,S} = 0$). Then, we can obtain the formula

$$|F_{k^c/k,S}(0)|_p = p^{1-r} \cdot P'/\left(\prod_{i=1}^{r} P(q_i)\right).$$

Consequently, we see that $|X_S(k)| \cdot |F_{k^c/k,S}(0)|_p = 1$ if and only if

$$|X_S(k)| = p^{r-1} \cdot \left(\prod_{i=1}^{r} P(q_i)\right)/P'.$$

The assertion follows from Proposition 2.1. □

As a special case of Theorem 1.2, we obtain the following:

Corollary 4.2. *Let the assumptions be as in Theorem 1.2, and suppose also that $P(q_i) = 1$ for $i = 1, \dots, r$. Then, $X_S(k^c)$ does not have a non-trivial finite submodule if and only if $|X_S(k)| = p^{r-1}$.*

Remark 4.3. Let the assumptions be as in the above corollary (that is, $P(q_i) = 1$ for all i). In this case, we can show that if $|X_{\{q_i\}}(k)| = 1$ for some q_i, then $|X_S(k)| = p^{r-1}$, and hence $X_S(k^c)$ does not have a non-trivial finite submodule. We will give an example. In the case when $k = \mathbb{Q}(\sqrt{2})$ and $p = 3$, it can be shown that $|X_{\{q\}}(k)| = 1$ for $q = 5, 11, 83$. (The author used PARI/GP [25] (versions 2.9.1 and 2.9.3) to check these examples.) As a consequence, at least for $k = \mathbb{Q}(\sqrt{2})$ and $p = 3$, we can take a set S such that $X_S(k^c) \cong \mathbb{Z}_3^{\oplus c}$ (as a $\mathbb{Z}_3$-module) for any given positive integer c (e.g., $X_{\{5,11,29\}}(k^c) \cong \mathbb{Z}_3^{\oplus 2}$).

Remark 4.4. Let the assumptions be as in Theorem 1.2, however, we remove the assumption that p does not divide the class number of k. We also assume that $X_\emptyset(k^c)$ is non-trivial and finite. Under these assumptions, we can see that the characteristic polynomial $F_{k^c/k,S}(T)$ of $X_S(k^c)$ is the same as in the proof of Theorem 1.2. In this case, we see that

$$|X_S(k)| \cdot |F_{k^c/k,S}(0)|_p \geq |X_\emptyset(k)| > 1$$

(recall also that $\mu_{k^c/k,S} = 0$), and hence $X_S(k^c)$ contains a non-trivial finite submodule.

We will give another example. The method of construction is different from Theorem 1.2.

Proposition 4.5. *Let p be an odd prime number. There is a finite set S of prime numbers (not containing p), and a finite p-extension k of $\mathbb{Q}$ such that $X_S(k^c) \cong \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module.*

Proof. We use the result given in [17] (see also [9]). Let $S = \{q_1, q_2\}$ be a set of distinct prime numbers satisfying the condition of [17, Theorem 1]. (We will not use this condition directly in this proof. For the existence of such a set S, see also [17, Remark 1, Remark 2].) Let $\mathcal{L}_S(\mathbb{Q}^c)/\mathbb{Q}^c$ be the maximal pro-p extension unramified outside S. From [17, Theorem 1], we see that $\mathrm{Gal}(\mathcal{L}_S(\mathbb{Q}^c)/\mathbb{Q}^c)$ is isomorphic to an infinite metacyclic pro-p group G topologically generated by a, b which satisfy

$$a^{p^2} = 1, \ \ b^{-1}ab = a^{1+p}.$$

(In the following, we will identify $\mathrm{Gal}(\mathcal{L}_S(\mathbb{Q}^c)/\mathbb{Q}^c)$ with G.) For a positive integer n, let H_n be the open subgroup of G which is topologically generated by b^{p^n}. Then, we can take n such that the fixed field L of $\mathcal{L}_S(\mathbb{Q}^c)$ by H_n is a Galois extension over $\mathbb{Q}$. Since $L/\mathbb{Q}^c$ is finite, there is a finite p-extension $k/\mathbb{Q}$ such that $L = k^c$. We also note that k is totally real.

By the above results, $\mathrm{Gal}(\mathcal{L}_S(\mathbb{Q}^c)/k^c) \cong \mathbb{Z}_p$. Note that $\mathcal{L}_S(\mathbb{Q}^c)$ is also the maximal pro-p extension of k^c unramified outside S. This implies that $X_S(k^c) \cong \mathrm{Gal}(\mathcal{L}_S(\mathbb{Q}^c)/k^c)$. The assertion follows. □

5. Proof of Theorem 1.3

In this section, we put $k = \mathbb{Q}(\sqrt{-3})$ and $p = 3$. Note that there is only one prime $\mathfrak{p}$ of k lying above 3. Suppose that q_1, q_2 satisfy the assumptions of Theorem 1.3, and put $S = \{q_1, q_2\}$. We see that q_i is inert in k and $|R(k, q_iO_k)| = 3$ for $i = 1, 2$. Note also that $|X_S(k)| = 3$ because the image of ζ_3 in $R(k, q_1O_k) \oplus R(k, q_2O_k)$ is not trivial.

First, we will show that $X_S(\widetilde{k})$ is not pseudo-null. To see this, we need some preparations. Let k^a/k be the anti-cyclotomic $\mathbb{Z}_3$-extension, and k^a_m its mth layer.

Lemma 5.1. *Let the assumptions be as in Theorem 1.3. We put $F = k^a_m$. Then $\dim_{\mathbb{Q}_3} X_S(F^c) \otimes_{\mathbb{Z}_3} \mathbb{Q}_3 \geq 3^m - 1$.*

Proof. Our proof of this lemma uses a method given in [8], [10], [17], etc.

Take a topological generator γ of $\mathrm{Gal}(F^c/F)$ satisfying $\zeta_{3^n}^{\gamma} = \zeta_{3^n}^4$ for all n, and fix an isomorphism from $\Lambda_{F^c/F}$ to $\mathbb{Z}_p[\![T]\!]$ satisfying $\gamma \mapsto 1 + T$.

We remark that the prime of k lying above q_i splits completely in F (see, e.g., [12]), and hence there are 3^m primes in F lying above q_i (for $i = 1, 2$). Put $r = 3^m$. We denote by $\mathfrak{q}_{1,1}, \ldots, \mathfrak{q}_{r,1}$ (resp. $\mathfrak{q}_{1,2}, \ldots, \mathfrak{q}_{r,2}$) the primes of F lying above q_1 (resp. q_2). Note that each $\mathfrak{q}_{i,j}$ is not decomposed in F^c/F

by the assumptions. Let F_n^c be the nth layer of F^c/F. We denote by $i_n(\mathfrak{q}_{i,j})$ the extension of $\mathfrak{q}_{i,j}$ in F_n^c. Note that we can see that

$$\varprojlim_n R(F_n^c, i_n(\mathfrak{q}_{i,j})) \cong \mathbb{Z}_3[\![T]\!]/(T-3)$$

as a $\mathbb{Z}_3[\![T]\!]$-module (see also the proof of Theorem 1.2 in Section 4).

Note that 3 does not divide the class number of k, and only $\mathfrak{p}$ is ramified in F_n^c/k. Hence $X_\emptyset(F^c)$ is trivial (by Iwasawa's result [11]). We put $\mathcal{E} = \varprojlim_n E(F_n^c) \otimes_{\mathbb{Z}} \mathbb{Z}_3$ where the projective limit is taken with respect to the norm mappings. We can also regard $\mathcal{E}$ as a $\mathbb{Z}_3[\![T]\!]$-module. By using class field theory, we obtain the following exact sequence

$$\mathcal{E}/(T-3) \to \varprojlim_n \bigoplus_{j=1}^{2} \bigoplus_{i=1}^{r} R(F_n^c, i_n(\mathfrak{q}_{i,j})) \to X_S(F^c) \to 0$$

(cf. e.g., [17], [10]). Note that the second term is isomorphic to $(\mathbb{Z}_p[\![T]\!]/(T-3))^{\oplus 2r}$, and hence it is free of rank $2r$ as a $\mathbb{Z}_3$-module.

By using [18, (11.3.11) Theorem (iii)] (F^c/F satisfies the assumption of this theorem), we see that

$$\mathcal{E} \cong \mathbb{Z}_3[\![T]\!]^{\oplus r} \oplus \mathbb{Z}_3(1),$$

here $\mathbb{Z}_3(1)$ is the first Tate twist of $\mathbb{Z}_3$. Hence $\dim_{\mathbb{Q}_3} \mathcal{E}/(T-3)\otimes_{\mathbb{Z}_3}\mathbb{Q}_3 = r+1$.

The assertion follows from these facts. □

Lemma 5.2. *Under the assumptions of Theorem 1.3, $X_S(\widetilde{k})$ is not pseudo-null as a $\Lambda_{\widetilde{k}/k}$-module.*

Proof. Let the notations be as in Lemma 5.1. Note that F^c is an intermediate field of $\widetilde{k}/k^c$, and F^c/k^c is a cyclic extension of degree 3^m. Since $\mathfrak{p}$ is totally ramified in $\widetilde{k}/k$, we see that the $\mathrm{Gal}(\widetilde{k}/F^c)$-coinvariant quotient $X_S(\widetilde{k})_{\mathrm{Gal}(\widetilde{k}/F^c)}$ is isomorphic to $X_S(F^c)$. We can also show that $X_S(\widetilde{k})$ is a finitely generated $\mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k^c)]\!]$-module because $X_S(\widetilde{k})_{\mathrm{Gal}(\widetilde{k}/k^c)} \cong X_S(k^c)$ is finitely generated over $\mathbb{Z}_3$ (see, e.g., [10]). However, Lemma 5.1 implies that $X_S(\widetilde{k})$ is not a torsion $\mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k^c)]\!]$-module. From this, we can deduce that $X_S(\widetilde{k})$ is not pseudo-null as a $\Lambda_{\widetilde{k}/k}$-module (see Greenberg [7], or Lemma 2.3 of Fujii [3]). □

Remark 5.3. Concerning the non-pseudo-nullity of $X_S(\tilde{k})$ (for a general imaginary quadratic field k), see also Kataoka [14]. However, Kataoka's result does not cover our case ($p = 3$ and $k = \mathbb{Q}(\sqrt{-3})$).

Lemma 5.4. *Under the assumptions of Theorem 1.3, $X_S(K)$ does not have a non-trivial finite $\Lambda_{K/k}$-submodule for every $\mathbb{Z}_3$-extension K/k.*

Proof. Let K/k be an arbitrary $\mathbb{Z}_3$-extension, and put $H = \mathrm{Gal}(\widetilde{k}/K)$. We recall that $\mathfrak{p}$ is totally ramified in K/k, and $|X_S(k)| = 3$. Hence, by Corollary 2.2, it is sufficient to show that $X_S(K)$ is infinite. Assume that $X_S(K)$ is finite. In our situation, the H-coinvariant quotient $X_S(\widetilde{k})_H$ is isomorphic to $X_S(K)$, and hence $X_S(\widetilde{k})_H$ is also finite. Then we can see that $X_S(\widetilde{k})$ is pseudo-null by using Perrin-Riou's result [24, Lemme 4] (see also Minardi [15]). However, this contradicts to Lemma 5.2. □

The remaining part of our proof of Theorem 1.3 is heavily relied on Greenberg's results given in [6]. Assume that the maximal pseudo-null submodule Z of $X_S(\widetilde{k})$ is not trivial. Let I be the augmentation ideal of $\mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k)]\!]$. We claim that Z/I is finite. To show this, we use a similar argument which is given in the paragraph before Lemma 5 of [6]. Let K/k be a $\mathbb{Z}_3$-extension. We put $H = \mathrm{Gal}(\widetilde{k}/K)$ and $\Gamma = \mathrm{Gal}(K/k)$. Recall that $\mathfrak{p}$ is totally ramified in $\widetilde{k}/k$. Then $X_S(\widetilde{k})_H \cong X_S(K)$, and it is a finitely generated torsion $\mathbb{Z}_3[\![\mathrm{Gal}(K/k)]\!]$-module. From this, we can see that $(X_S(\widetilde{k})/Z)^H$ is trivial, and hence the natural $\mathbb{Z}_3[\![\mathrm{Gal}(K/k)]\!]$-module homomorphism $Z_H \to X_S(\widetilde{k})_H (\cong X_S(K))$ is injective. Moreover, since $X_S(K)_\Gamma \cong X_S(k)$ is finite, we can show that $(Z_H)_\Gamma$ is finite. Then the claim follows. From this, we can apply [6, Lemma 5]. In this case, there must be a $\mathbb{Z}_3$-extension $K^\dagger/k$ such that $Z_{\mathrm{Gal}(K^\dagger/k)}$ has a non-trivial finite submodule, and then $X_S(K^\dagger)$ has a non-trivial finite submodule from the above argument. This contradicts Lemma 5.4. Hence, Theorem 1.3 completely follows. □

Remark 5.5. There is another method to deduce Theorem 1.3 from the lemmas. We shall state this briefly. We take the isomorphism $\mathbb{Z}_3[\![\mathrm{Gal}(k^c/k)]\!] \cong \mathbb{Z}_3[\![T]\!]$ given in the proof of Lemma 5.1, then we see that $\mu_{k^c/k,S} = 0$ and $F_{k^c/k,S}(T) = T - 3$ (see, e.g., [10]). Moreover, by Lemma 5.4, we see that $X_S(k^c) \cong \mathbb{Z}_3$ as a $\mathbb{Z}_3$-module. Recall also that $X_S(\widetilde{k})_{\mathrm{Gal}(\widetilde{k}/k^c)} \cong X_S(k^c)$. Moreover, it can be shown that $X_S(\widetilde{k})$ is a cyclic $\mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k)]\!]$-module. Hence, $X_S(\widetilde{k}) \cong \mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k)]\!]/\mathfrak{A}$, where $\mathfrak{A}$ is the annihilator ideal of $\mathbb{Z}_3[\![\mathrm{Gal}(\widetilde{k}/k)]\!]$ for $X_S(\widetilde{k})$. By Lemma 5.2, $X_S(\widetilde{k})$ is not pseudo-null. By a similar argument given in the proof of [2, Proposition 3.1] (or [15, Section 3.D]), we can show that $\mathfrak{A}$ is a principal ideal generated by an irreducible element. (The fact that $F_{k^c/k,S}(T) = T - 3$ is crucial.) Hence, $X_S(\widetilde{k})$ does not contain a non-trivial pseudo-null submodule.

Appendix A. Weak analogs of Greenberg's conjecture

Let k be a totally real field and p an odd prime number. We denote by S a finite set of prime numbers which does not contain p. To consider the structure of $X_S(k^c)$, it is sufficient to treat S satisfying the following condition (see also, e.g., [10]):

(R) For each $q \in S$, there is a prime $\mathfrak{q}$ of k lying above q such that $R(k, \mathfrak{q})$ is not trivial.

When k is a real quadratic field, the condition (R) is equivalent to the condition that every $q \in S$ satisfies (a) or (b) in Remark 4.1.

Let $M_{S,p}(k^c)$ be the maximal abelian pro-p extension of k^c unramified outside $S \cup \{p\}$. In this case, we see that $\mathrm{Gal}(M_{S,p}(k^c)/k^c)$ is a finitely generated torsion $\Lambda_{k^c/k}$-module. We also see that $\mathrm{Gal}(M_{S,p}(k^c)/k^c)$ does not contain a non-trivial finite $\Lambda_{k^c/k}$-submodule. (For these results, see, e.g., [6], [18].)

First, we consider the case when $S = \emptyset$. It is conjectured that $X_\emptyset(k^c)$ is finite (Greenberg's conjecture [5]). Moreover, some weak forms of this conjecture are also proposed (see Nguyen Quang Do [19], [20]).

(Conj1) $X_\emptyset(k^c)$ is trivial or contains a non-trivial finite $\Lambda_{k^c/k}$-submodule.
(Conj2) $\mathrm{Gal}(M_{\emptyset,p}(k^c)/k^c)$ is trivial or $\mathrm{Gal}(M_{\emptyset,p}(k^c)/L_\emptyset(k^c))$ is not trivial.

Remark A.1. Note that (Conj1) implies (Conj2). If p splits completely in k and Leopoldt's conjecture holds for k and p, then (Conj1) and (Conj2) are equivalent (see [19], [22]). In [19] and [20], Nguyen Quang Do considered these conjectures for the case when k is a real abelian field and p splits completely in k (see also [21]).

Next, we shall consider the "S-ramified" version of these assertions:

(Conj1S) $X_S(k^c)$ is trivial or contains a non-trivial finite $\Lambda_{k^c/k}$-submodule.
(Conj2S) $\mathrm{Gal}(M_{S,p}(k^c)/k^c)$ is trivial or $\mathrm{Gal}(M_{S,p}(k^c)/L_S(k^c))$ is not trivial.

However, the results given in Section 4 imply that the assertion (Conj1S) does not hold in general. For the assertion (Conj2S), we can obtain the following:

Theorem A.2. *Let k be a totally real field, p an odd prime number, and S a non-empty finite set of prime numbers which does not contain p. Assume that S satisfies* (R). *If* $\mathrm{Gal}(L_{\{q\}}(k^c)/L_\emptyset(k^c))$ *is finite for some $q \in S$, then* $\mathrm{Gal}(M_{S,p}(k^c)/L_S(k^c))$ *is not trivial.*

Proof. When $k = \mathbb{Q}$, this assertion is already mentioned in [4] (and the proof of our case is almost similar). Take a prime number $q \in S$ such that $\mathrm{Gal}(L_{\{q\}}(k^c)/L_\emptyset(k^c))$ is finite. Since S satisfies (R), we see that $\mathrm{Gal}(M_{\{q\},p}(k^c)/M_{\emptyset,p}(k^c))$ is infinite (see, e.g., [18, (11.3.5) Theorem and (11.3.6) Corollary]). Hence $\mathrm{Gal}(M_{\{q\},p}(k^c)/L_\emptyset(k^c))$ is also infinite. Thus, we conclude that $\mathrm{Gal}(M_{\{q\},p}(k^c)/L_{\{q\}}(k^c))$ is infinite. Since $L_S(k^c) \subseteq M_{\{q\},p}(k^c)L_S(k^c) \subseteq M_{S,p}(k^c)$, we can show that $\mathrm{Gal}(M_{S,p}(k^c)/L_S(k^c))$ is not trivial. (Note that we do not need the validity of Leopoldt's conjecture in this proof.) $\square$

Moreover, we can also see the following:

Proposition A.3. *Let k be a totally real field, p an odd prime number, and S a non-empty finite set of prime numbers which does not contain p. Assume that Leopoldt's conjecture holds for k and p. Assume also that p splits completely in k. Then $X_S(k^c)$ contains a non-trivial finite $\Lambda_{k^c/k}$-submodule if and only if* $\mathrm{Gal}(M_{S,p}(k^c)/L_S(k^c))$ *is not trivial.*

Proof. The proof is quite similar to the case when $S = \emptyset$. See [22] or [19] (see also [4] for the case when $k = \mathbb{Q}$ and $S \neq \emptyset$). Note that we need the finiteness of $\mathrm{Gal}(M_{S,p}(k^c)/k^c)_\Gamma$ (where $\Gamma = \mathrm{Gal}(k^c/k)$) to show the triviality of $\mathrm{Gal}(M_{S,p}(k^c)/k^c)^\Gamma$, however, this follows from class field theory and the validity of Leopoldt's conjecture for k and p (see, e.g., [18]). ☐

Hence we obtained the following:

Corollary A.4. *Let k be a totally real field, p an odd prime number, and S a non-empty finite set of prime numbers which does not contain p. Assume that S satisfies* (R). *Assume also that p splits completely in k, and Leopoldt's conjecture holds for k and p. If* $\mathrm{Gal}(L_{\{q\}}(k^c)/L_\emptyset(k^c))$ *is finite for some $q \in S$, then $X_S(k^c)$ contains a non-trivial finite submodule.*

When k is a real abelian field and p is an odd prime, it was shown that $\mathrm{Gal}(L_{\{q\}}(k^c)/L_\emptyset(k^c))$ is finite for every prime number $q (\neq p)$ (see [8, Theorem 1.1]). Hence, by combining the validity of Leopoldt's conjecture (see [1]), we can also obtain the following:

Corollary A.5. *Let k be a real abelian field, p an odd prime number, and S a non-empty finite set of prime numbers which does not contain p. Assume that S satisfies* (R)*. Then the following statements hold.*

(1) $\mathrm{Gal}(M_{S,p}(k^c)/L_S(k^c))$ *is not trivial.*
(2) *Moreover, if p splits completely in k, then $X_S(k^c)$ contains a non-trivial finite submodule.*

Note that the assertions of the above corollary were already shown in [4] for the case when $k = \mathbb{Q}$.

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