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# Newton–Okounkov bodies: an approach of function field arithmetic

par HUAYI CHEN

RÉSUMÉ. En utilisant les méthodes de la géométrie d’Arakelov dans le cadre de corps de fonctions, on associe, à chaque système linéaire gradué birationnel et de type sous-fini, un corps convexe dont la mesure de Lebesgue s’identifie au volume du système linéaire gradué. Comparé à d’autres approches dans la littérature, cette nouvelle approche demande moins de paramètres non intrinsèques de la variété projective. En outre, cette méthode n’exige pas l’existence d’un point rationnel régulier sur la variété projective, ce qui est supposé, par exemple, dans la construction de Lazarsfeld et Mustață.

ABSTRACT. By using the method of Arakelov geometry in the function field setting, we associate, to each graded linear series which is birational and of sub-finite type, a convex body whose Lebesgue measure identifies with the volume of the graded linear series. Compared to other constructions in the literature, less non-intrinsic parameters of the projective variety are involved in this new approach. Moreover, this method does not require the existence of a regular rational point in the projective variety, which was assumed for example in the construction of Lazarsfeld and Mustață.

## 1. Introduction

The theory of Newton–Okounkov bodies, initialised by Okounkov [19, 20] and then developed by Lazarsfeld and Mustață [16], and Kaveh and Khovanskii [11, 12] respectively, is an efficient tool to study the asymptotic behaviour of graded linear series on projective varieties. Let  $X$  be an integral projective scheme of Krull dimension  $d$  over a field  $k$  and  $L$  be an invertible  $\mathcal{O}_X$ -module. This theory associates, to each graded linear series of  $L$  (satisfying some mild conditions), a convex body of  $\mathbb{R}^d$  (called *Newton–Okounkov body* of the graded linear series) whose Lebesgue measure is equal to the volume of the graded linear series divided by  $d!$ . The construction relies on a valuation of the rational function field  $k(X)$  valued in the group  $\mathbb{Z}^d$  equipped with a monomial order, such that the residue field of the valuation coincides with  $k$ . This valuation can be constructed through the choice of a flag of smooth subschemes of  $X$  containing a regular rational

point (see [16]), or through the choice of a regular sequence of the local ring of a regular rational point (see [12]). Moreover, in [8, Theorem 2.9] it is shown that any valuation of maximal rank of the field  $k(X)$  is equivalent to a valuation associated with a flag in a birational modification of  $X$ . From the point of view of birational geometry, the dependence of the Newton–Okounkov body on the choice of the parameters (regular point, flag of smooth closed subvarieties, monomial order on  $\mathbb{R}^d$ ) is a subtle problem. We refer the readers to [14] for a detailed discussion on the convex bodies appearing as Newton–Okounkov bodies of divisors, see also [16, §5]. Moreover, the existence of a regular rational point actually implies that the  $k$ -scheme is geometrically irreducible. Although interesting consequences of the Okounkov convex body approach, such as Fujita approximation and the volume as a limite, can still be obtained by using fine technics of local multiplicities (see [9]), the construction of convex bodies becomes more complicated and relies on even more non-intrinsic choices.

The purpose of this article is to introduce a new approach of relating graded linear series to convex bodies in Euclidean spaces, which is based on function field arithmetics. The construction only depends on the choice of a flag of successive field extensions

$$k = K_0 \subset K_1 \subset \cdots \subset K_d = k(X),$$

such that each extension  $K_i/K_{i-1}$  is of transcendence degree 1 (the extensions  $K_i/K_{i-1}$  are necessarily finitely generated, see [3, Chapter V, §14, no. 7 Corollary 3]). Thus we can consider  $K_i$  as the field of rational functions of a regular projective curve  $C_i$  over  $\text{Spec}(K_{i-1})$ . Given a graded linear series  $V_\bullet$  of a Cartier divisor on  $X$ , we let  $V_{\bullet, K_i}$  be the graded sub- $K_i$ -algebra of  $\bigoplus_{n \in \mathbb{N}} k(X)$  generated by  $V_\bullet$ . It generates a graded  $\mathcal{O}_{C_i}$ -algebra of vector bundles  $E_\bullet^i$  on the curve  $C_i$ . In the case where the graded linear series  $V_\bullet$  is birational (see Definition 3.5), we show that the volume of  $V_{\bullet, K_i}$  is equal to the arithmetic volume of the graded- $\mathcal{O}_{C_i}$ -algebra  $E_\bullet^i$ . We also construct by induction a family of convex bodies  $(\Delta^i)_{i=0}^d$ , where  $\Delta^i$  is a convex body in  $\mathbb{R}^i$ , which is delimited by the graph of a non-negative concave function on  $\Delta^{i-1}$  constructed from the function field arithmetic of  $E_\bullet^i$  by using the method of Fujita approximation for arithmetic graded linear series (see [1, 5]). Note that Katz and Urbinati [10] have studied the construction of Newton–Okounkov bodies on schemes over a discrete valuation ring, where the idea of arithmetic Newton–Okounkov body [1, 22] also appears.

From the point of view of birational geometry, the above procedure holds for the general setting of graded linear series of a finitely generated extension  $K$  of the base field  $k$  and it is not necessary to fix a projective model of  $K$ . For any  $d \in \mathbb{N}$ ,  $d \geq 1$ , we denote by  $\mathcal{C}_d$  the set of  $d$ -uplets  $(\Delta^i)_{i=0}^d$

where each  $\Delta^i$  is a convex body in  $\mathbb{R}^i$  and such that the projection of  $\Delta^i$  on  $\mathbb{R}^{i-1}$  by the  $i - 1$  first coordinates is contained in  $\Delta^{i-1}$ .

**Theorem 1.1.** *Let  $k$  be a field and  $K/k$  be a finitely generated extension of transcendence degree  $d \geq 1$ . We fix a flag*

$$(1.1) \quad k = K_0 \subset K_1 \subset \cdots \subset K_d = K$$

*of field extensions such that  $K_i/K_{i-1}$  is of transcendence degree 1. Let  $\mathcal{A}(K/k)$  be the set of all birational graded linear series of sub-finite type of  $K/k$  (see Definition 3.5). There exists a map  $(\Delta^i)_{i=0}^d$  from  $\mathcal{A}(K/k)$  to  $\mathcal{C}_d$ , only depending on the flag (1.1), which satisfies the following conditions:*

- (1) *For all graded linear series  $V_\bullet$  and  $V'_\bullet$  in  $\mathcal{A}(K/k)$  such that  $V_n \subset V'_n$  for sufficiently positive  $n$ , one has  $\Delta^i(V_\bullet) \subset \Delta^i(V'_\bullet)$  for any  $i \in \{0, \dots, d\}$ .*
- (2) *For any graded linear series  $V_\bullet$  in  $\mathcal{A}(K/k)$ , any integer  $m \geq 1$  and any  $i \in \{0, \dots, d\}$ , one has  $\Delta^i(V_\bullet^{(m)}) = m\Delta^i(V_\bullet)$ , where  $V_\bullet^{(m)} := \bigoplus_{n \in \mathbb{N}} V_{mn}$ .*
- (3) *For all graded linear series  $V_\bullet$  and  $W_\bullet$  in  $\mathcal{A}(K/k)$ , one has*

$$\Delta^i(V_\bullet) + \Delta^i(W_\bullet) \subset \Delta^i(V_\bullet \cdot W_\bullet),$$

*for any  $i \in \{0, \dots, d\}$ , where “+” denotes the Minkowski sum.*

- (4) *For any graded linear series  $V_\bullet$  in  $\mathcal{A}(K/k)$  and any  $i \in \{0, \dots, d\}$ , the mass of  $\Delta^i(V_\bullet)$  with respect to the Lebesgue measure is equal to the volume of the graded linear series  $V_{\bullet, K_{d-i}}$  of  $K/K_{d-i}$  divided by  $i!$ .*

*Moreover, the graded linear series  $V_\bullet$  satisfies the Fujita approximation property (see Definition 3.7).*

As explained above, for any  $V_\bullet \in \mathcal{A}(K/k)$ , the family of convex bodies  $\Delta^i(V_\bullet)$  are constructed in a recursive way. More precisely, it can be shown that the graded linear series  $V_{\bullet, K_1}$  belongs to  $\mathcal{A}(K/K_1)$  (see Remark 3.6). We let  $(\Delta^0(V_\bullet), \dots, \Delta^{d-1}(V_\bullet))$  be the family  $(\Delta^0(V_{\bullet, K_1}), \dots, \Delta^{d-1}(V_{\bullet, K_1}))$ , where by abuse of notation we use the expression  $(\Delta^i)_{i=0}^{d-1}$  to denote the map  $\mathcal{A}(K/K_1) \rightarrow \mathcal{C}_{d-1}$  predicted by the theorem as an induction hypothesis. The extension  $K_1/k$  corresponds to a regular projective curve  $C$  over  $\text{Spec } k$  whose rational function field is  $K_1$ . The convex body  $\Delta^d(V_\bullet)$  is then constructed as the arithmetic Newton–Okounkov body of the graded  $\mathcal{O}_C$ -algebra on vector bundles generated by  $V_\bullet$ .

Compared to the classic approach of Newton–Okounkov bodies, the above construction relies on fewer parameters, which are closely related to the birational geometry of the extension  $K/k$ . Moreover, the convex bodies  $\Delta^i(V_\bullet)$ ,  $i \in \{0, \dots, d\}$  are related by linear projections. This fact reflects interesting geometric information of the graded linear series  $V_{\bullet, K_i}$ .

The rest of the article is organised as follows. In the second section, we recall some classic results on function field arithmetic of vector bundles. In the third section, we discuss several properties of graded linear series. Finally, in the fourth section, we explain the construction of convex bodies associated with graded linear series and prove the main theorem.

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### 2. Function field arithmetic of vector bundles

In this section, we let  $k$  be a field and  $C$  be a regular projective curve over  $\text{Spec } k$  (namely an integral regular projective scheme of dimension 1 over  $\text{Spec } k$ ). By vector bundle on  $C$ , we mean a locally free  $\mathcal{O}_C$ -module  $E$  of finite rank.

**2.1. Degree function.** Recall that Riemann–Roch formula shows that, for any vector bundle  $E$  on  $C$ , one has

$$(2.1) \quad h^0(E) - h^1(E) = \text{deg}(E) + \text{rk}(E)(1 - g(C/k)),$$

where  $h^0(E)$  and  $h^1(E)$  are respectively the ranks of  $H^0(C, E)$  and  $H^1(C, E)$  over  $k$ ,  $\text{deg}(E) = \text{deg}(c_1(E) \cap [C])$  is the degree of  $E$ , and  $g(C/k)$  is the genus of the curve  $C$  relatively to  $k$ , which is equal to the rank of  $H^0(C, \omega_{C/k})$  over  $k$ ,  $\omega_{C/k}$  being the relative dualising sheaf. Note that by Serre duality one has  $h^1(E) = h^0(E^\vee \otimes \omega_{C/k})$ .

Let  $K = k(C)$  be the field of rational functions on  $C$ . Recall that any closed point  $x$  of  $C$  determines a discrete valuation  $\text{ord}_x(\cdot)$  on the field  $K$ . We denote by  $|\cdot|_x$  the absolute value on  $K$  defined as  $|\cdot|_x := e^{-\text{ord}_x(\cdot)}$ , and by  $K_x$  the completion of  $K$  with respect to  $|\cdot|_x$ , on which the absolute value  $|\cdot|_x$  extends by continuity. If  $E$  is a vector bundle on  $C$ , the  $\mathcal{O}_C$ -module structure on  $E$  determines, for each closed point  $x \in C$ , a norm  $\|\cdot\|_x$  on  $E_{K_x}$  defined as

$$\|s\|_x := \inf\{|a| : a \in K_x^\times, a^{-1}s \in E \otimes_{\mathcal{O}_C} \mathfrak{o}_x\},$$

where  $\mathfrak{o}_x$  is the valuation ring of  $K_x$ . Recall that the degree of  $E$  can also be computed as

$$(2.2) \quad \text{deg}(E) = - \sum_{\text{closed point } x \in C} [k(x) : k] \log \|s_1 \wedge \cdots \wedge s_r\|_{x, \det},$$

where  $(s_1, \dots, s_r)$  is an arbitrary basis of  $E_K$  over  $K$ ,  $k(x)$  is the residue field of the closed point  $x$ , and  $\|\cdot\|_{x, \det}$  is the determinant norm on  $\det(E_{K_x})$

induced by  $\|\cdot\|_x$ . Similarly, for any non-zero vector  $s$  in  $E_K$ , we define

$$(2.3) \quad \deg(s) = - \sum_{\text{closed point } x \in C} [k(x) : k] \log \|s\|_x.$$

Note that, if  $E$  is an invertible  $\mathcal{O}_C$ -module, then for any non-zero vector  $s \in E_K$ , one has  $\deg(s) = \deg(E)$ .

**Proposition 2.1.** *Let  $C$  be a regular projective curve over a field  $k$  and  $E$  be a vector bundle on  $C$ . For any non-zero section  $s \in H^0(C, E)$  viewed as a vector in  $E_K$ , one has  $\deg(s) \geq 0$ .*

*Proof.* Since  $s$  is a global section of  $E$ , for any closed point  $x \in C$ , one has

$$-\log \|s\|_x = \text{ord}_x(s) \geq 0.$$

Hence we obtain from (2.3) the relation  $\deg(s) \geq 0$ . □

**Proposition 2.2.** *Let  $C$  be a regular projective curve over a field  $k$  and  $L$  be an invertible  $\mathcal{O}_C$ -module. If  $\deg(L) \geq 0$ , then  $h^0(L) \leq \deg(L) + 1$ .*

*Proof.* The inequality is trivial if  $h^0(L) = 0$ . In the following, we assume that  $L$  admits a non-zero global section  $s$ , which defines an injective homomorphism from  $L^\vee$  to  $\mathcal{O}_C$ . Therefore one has

$$h^0(L^\vee \otimes \omega_{C/k}) \leq h^0(\omega_{C/k}) = g(C/k),$$

where  $g(C/k)$  is the genus of  $C$ . By Riemann–Roch formula (2.1) we obtain

$$h^0(L) = h^0(L^\vee \otimes \omega_{C/k}) + \deg(L) + 1 - g(C/k) \leq \deg(L) + 1. \quad \square$$

**2.2. Successive minima.** Let  $k$  be a field,  $C$  be a regular projective curve over  $\text{Spec } k$  and  $E$  be a vector bundle on  $C$ . For any  $i \in \{1, \dots, \text{rk}(E)\}$ , let  $\lambda_i(E)$  be the supremum of  $\lambda \in \mathbb{R}$  such that the  $K$ -vector subspace of  $E_K$  generated by the vectors  $s \in E_K \setminus \{0\}$  satisfying  $\deg(s) \geq \lambda$  has rank  $\geq i$ . By definition one has

$$\lambda_1(E) \geq \dots \geq \lambda_r(E), \quad \text{with } r = \text{rk}(E).$$

These invariants are similar to (the minus logarithmic version of) successive minima in geometry of numbers.

**Proposition 2.3.** *Let  $C$  be a regular projective curve over a field  $k$  and  $E$  be a vector bundle on  $C$ . If  $E$  is generated by global sections, then for any  $i \in \{1, \dots, \text{rk}(E)\}$  one has  $\lambda_i(E) \geq 0$ .*

*Proof.* Let  $K = k(C)$  be the field of rational functions on  $C$ . Since  $E$  is generated by global sections, there exist non-zero global sections  $s_1, \dots, s_r$  of  $E$  which form a basis of  $E_K$  over  $K$ . By Proposition 2.1, one has  $\deg(s_i) \geq 0$  for any  $i \in \{1, \dots, n\}$ , which implies the assertion of the proposition. □

By Hadamard’s inequality, if  $(s_1, \dots, s_r)$  is a basis of  $E_K$  over  $K$ , then for any closed point  $x \in C$  one has

$$\log \|s_1 \wedge \dots \wedge s_r\|_{x, \det} \leq \sum_{i=1}^r \log \|s_i\|_x.$$

Thus we obtain

$$(2.4) \quad \lambda_1(E) + \dots + \lambda_r(E) \leq \deg(E).$$

Roy and Thunder have proved the following converse inequality (see [21, Theorem 2.1]):

$$(2.5) \quad \lambda_1(E) + \dots + \lambda_r(E) \geq \deg(E) - \text{rk}(E)\ell_C(g(C/k)),$$

where  $\ell_C$  is a non-negative function on  $\mathbb{R}_+$  depending only on the curve  $C$ .

The successive minima are related to the  $\mathbb{R}$ -filtration by minima. Let  $E$  be a vector bundle on  $C$ . For any  $t \in \mathbb{R}$ , let  $\mathcal{F}^t(E_K)$  be the  $K$ -vector subspace of  $E_K$  generated by non-zero vectors  $s \in E_K$  such that  $\deg(s) \geq t$ . Then  $(\mathcal{F}^t(E_K))_{t \in \mathbb{R}}$  is a decreasing  $\mathbb{R}$ -filtration of  $E_K$ . Note that the function  $(t \in \mathbb{R}) \mapsto \dim_K(\mathcal{F}^t(E_K))$  is left continuous, and one has the following equality of distributions

$$\sum_{i=1}^{\text{rk}(E)} \delta_{\lambda_i(E)} = -\frac{d}{dt} \dim_K(\mathcal{F}^t(E_K)),$$

where  $\delta_{\lambda_i(E)}$  denotes the Dirac measure on  $\lambda_i(E)$ .

**2.3. Positive degree.** Let  $k$  be a field,  $C$  be a regular projective curve and  $E$  be a vector bundle on  $C$ . We denote by  $\deg_+(E)$  the supremum of the degrees of all vector subbundles of  $E$ , called the positive degree of  $E$ . Recall that one has (see [7, Theorem 2.4])

$$(2.6) \quad |h^0(E) - \deg_+(E)| \leq \text{rk}(E) \max(g(C/k) - 1, 1),$$

where  $h^0(E)$  is the rank of  $H^0(C, E)$  over  $k$ . Moreover, by Hadamard’s inequality, one has

$$(2.7) \quad \sum_{i=1}^{\text{rk}(E)} \max(\lambda_i(E), 0) \leq \deg_+(E),$$

which is similar to (2.4). Also we can deduce from (2.5) the following inequality (we refer the readers to [7, Proposition 8.1] for a proof)

$$(2.8) \quad \deg_+(E) \leq \sum_{i=1}^{\text{rk}(E)} \max(\lambda_i(E), 0) + \text{rk}(E)\ell_C(g(C/k)).$$

Combining the inequalities (2.6)–(2.8), we obtain the following result.

**Proposition 2.4.** *Let  $C$  be a regular projective curve over a field  $k$  and  $E$  be a vector bundle on  $C$ . One has*

$$(2.9) \quad \left| h^0(E) - \sum_{i=1}^{\text{rk}(E)} \max(\lambda_i(E), 0) \right| \leq \text{rk}(E) \tilde{\ell}_C(g(C/k)),$$

where  $\tilde{\ell}_C(x) = \ell_C(x) + \max(x - 1, 1)$  for any  $x \in \mathbb{R}_+$ .

### 3. Graded linear series

**3.1. Graded linear series of a divisor.** Let  $k$  be a field and  $\pi : X \rightarrow \text{Spec } k$  be an integral projective  $k$ -scheme. Let  $K$  be the field of rational functions on  $X$ . For any Cartier divisor  $D$  on  $X$ , we denote by  $H^0(D)$  the  $k$ -vector space

$$H^0(D) = \{f \in K : D + \text{div}(f) \geq 0\} \cup \{0\}.$$

Denote by  $\mathcal{O}_X(D)$  the sub- $\mathcal{O}_X$ -module of the constant sheaf  $\pi^*(K)$  generated by  $-D$ . It is an invertible  $\mathcal{O}_X$ -module. For any finite dimensional  $k$ -vector subspace  $V$  of  $K$  which is contained in  $H^0(D)$ , the canonical homomorphism  $\pi^*(V) \rightarrow \pi^*(K)$  factors through  $\mathcal{O}_X(D)$ . The locus where the homomorphism  $\pi^*(V) \rightarrow \mathcal{O}_X(D)$  is not surjective is called the *base locus* of  $V$  with respect to the Cartier divisor  $D$ , denoted by  $B_D(V)$ . It is a Zariski closed subset of  $X$ . If  $V$  is non-zero, then one has  $B_D(V) \subsetneq X$ . Moreover, the homomorphism  $\pi^*(V) \rightarrow \mathcal{O}_X(D)$  induces a  $k$ -morphism  $j_{V,D} : X \setminus B_D(V) \rightarrow \mathbb{P}(V)$  such that  $j_{V,D}^*(\mathcal{O}_V(1)) \cong \mathcal{O}_X(D)|_{X \setminus B_D(V)}$ , where  $\mathcal{O}_V(1)$  denotes the universal invertible sheaf of  $\mathbb{P}(V)$ . Note that the rational morphism  $X \dashrightarrow \mathbb{P}(V)$  determined by  $j_{V,D}$  does not depend on the choice of the Cartier divisor  $D$  (such that  $H^0(D) \supset V$ ). We denote by  $j_V$  this rational morphism.

**Definition 3.1.** Let  $V$  be a finite dimensional  $k$ -vector subspace of  $K$ . We say that  $V$  is *birational* if the rational morphism  $j_V : X \dashrightarrow \mathbb{P}(V)$  maps  $X$  birationally to its image. Note that this condition is equivalent to the condition  $K = k(V)$ , where  $k(V)$  denotes the sub-extension of  $K/k$  generated by elements of the form  $a/b$  in  $K$ ,  $a$  and  $b$  being elements in  $V$ ,  $b \neq 0$ . By definition, if  $V$  is birational and if  $f$  is a non-zero element of  $K$ , then  $fV := \{fg \mid g \in V\}$  is also birational.

**Remark 3.2.** Let  $V$  be a finite dimensional  $k$ -vector subspace of  $K$ . If there exists a very ample Cartier divisor  $A$  on  $X$  such that  $H^0(A) \subset V$ , then the rational morphism  $j_V : X \dashrightarrow \mathbb{P}(V)$  maps  $X$  birationally to its image, namely  $V$  is birational. More generally, if  $V$  is a finite dimensional  $k$ -vector subspace of  $K$  which is birational and if  $W$  is another finite dimensional  $k$ -vector subspace of  $K$  containing  $V$ , then  $W$  is also birational.



Let  $D$  be a Cartier divisor on  $X$ . We denote by  $V_\bullet(D) := \bigoplus_{n \in \mathbb{N}} H^0(nD)$ . This is a graded  $k$ -algebra. We call *graded linear series* of  $D$  any graded sub- $k$ -algebra of  $V_\bullet(D)$ . If  $V_\bullet$  is a graded linear series of  $D$ , its *volume* is defined as

$$\text{vol}(V_\bullet) := \limsup_{n \rightarrow \infty} \frac{\dim_k(V_n)}{n^d/d!},$$

where  $d$  is the dimension of  $X$ . In particular, if  $V_\bullet$  is the total graded linear series  $V_\bullet(D)$ , its volume is also called the *volume of  $D$* , denoted by  $\text{vol}(D)$ . The divisor  $D$  is said to be *big* if  $\text{vol}(D) > 0$ . Note that if  $D$  is an ample divisor, then its volume is positive, and can be written in terms of the self-intersection number  $(D^d)$  (see [15, §2.2.C] for more details).

**Definition 3.3.** Let  $D$  be a Cartier divisor on  $X$  and  $V_\bullet$  be a graded linear series of  $D$ . Following [16, Definition 2.5], we say that the graded linear series  $V_\bullet$  is *birational* if for sufficiently positive integer  $n$ , the rational map  $j_{V_n} : X \dashrightarrow \mathbb{P}(V_n)$  maps  $X$  birationally to its image.

**Remark 3.4.**

- (1) Let  $D$  be a Cartier divisor on  $X$  and  $V_\bullet$  be a birational graded linear series of  $D$ . If  $\nu : X' \rightarrow X$  is a birational projective  $k$ -morphism from an integral projective  $k$ -scheme  $X'$  to  $X$ , then  $V_\bullet$  is also a birational graded linear series of  $\nu^*(D)$ .
- (2) Let  $D$  be a Cartier divisor on  $X$  and  $V_\bullet$  be a graded linear series. We assume that  $V_\bullet$  contains an ample divisor, namely  $V_n \neq \{0\}$  for sufficiently positive integer  $n$ , and there exist an ample Cartier divisor  $A$  on  $X$  and an integer  $p \geq 1$  such that  $V_n(A) \subset V_{np}$  for  $n \in \mathbb{N}_{\geq 1}$ . Then  $V_\bullet$  is a birational graded linear series of  $D$ .

**3.2. Graded linear series of a finitely generated extension.** Let  $k$  be a field and  $K$  be a field extension of  $k$  which is finitely generated over  $k$ .

**Definition 3.5.** By *linear series* of  $K/k$  we mean any finite dimensional  $k$ -vector subspace  $V$  of  $K$ . We say that a linear series  $V$  of  $K/k$  is *birational* if  $K = k(V)$ . Recall that  $k(V)$  denotes the sub-extension of  $K/k$  generated by elements of the form  $a/b$  with  $a, b \in V, b \neq 0$ .

We call *graded linear series* of  $K/k$  any graded sub- $k$ -algebra  $V_\bullet$  of  $\bigoplus_{n \in \mathbb{N}} K$  (equipped with the polynomial graded ring structure) such that each homogeneous component  $V_n$  is a linear series of  $K/k$  for any  $n \in \mathbb{N}$ . If  $V_\bullet$  is a graded linear series of  $K$ , its *volume* is defined as

$$\text{vol}(V_\bullet) := \limsup_{n \rightarrow \infty} \frac{\dim_k(V_n)}{n^d/d!} \in [0, +\infty],$$

where  $d$  is the transcendence degree of  $K$  over  $k$ .

Let  $V_\bullet$  be a graded linear series of  $K/k$ . We say that  $V_\bullet$  is of *finite type* if it is finitely generated as a  $k$ -algebra. We say that  $V_\bullet$  is of *sub-finite type*

if it is contained in a graded linear series of finite type. We say that  $V_\bullet$  *birational* if  $k(V_n) = K$  for sufficiently positive  $n$ .

We denote by  $\mathcal{A}(K/k)$  the set of all birational graded linear series of sub-finite type of  $K/k$ .

**Remark 3.6.**

- (1) Let  $V_\bullet$  and  $V'_\bullet$  be two graded linear series of  $K/k$ . Denote by  $V_\bullet \cdot V'_\bullet$  the graded linear series  $\bigoplus_{n \in \mathbb{N}} (V_n \cdot V'_n)$  of  $K/k$ , where  $V_n \cdot V'_n$  is the  $k$ -vector space generated by  $\{ff' \mid f \in V_n, f' \in V'_n\}$ . If both graded linear series  $V_\bullet$  and  $V'_\bullet$  are of finite type (resp. of sub-finite type, birational), then also is  $V_\bullet \cdot V'_\bullet$ .
- (2) Let  $V_\bullet$  be a graded linear series of  $K/k$ . If  $k'/k$  is a field extension such that  $k' \subset K$ , we denote by  $V_{\bullet, k'}$  the graded linear series  $\bigoplus_{n \in \mathbb{N}} V_{n, k'}$  of  $K/k'$ , where  $V_{n, k'}$  is the  $k'$ -vector subspace of  $K$  generated by  $V_n$ . If  $V_\bullet$  is of finite type (resp. of sub-finite type, birational), then also is  $V_{\bullet, k'}$ .
- (3) Let  $V_\bullet$  be a graded linear series of  $K/k$  which is of sub-finite type. There then exists a birational graded linear series of finite type  $W_\bullet$  of  $K/k$  such that  $V_n \subset W_n$  for any  $n \in \mathbb{N}$ . Without loss of generality we may assume that  $1 \in W_1$  and that  $W_\bullet$  is generated as  $W_0$ -algebra by  $W_1$ . Then the scheme  $X = \text{Proj}(W_\bullet)$  is a projective model of the field  $K$  over  $k$  (namely an integral projective  $k$ -scheme such that  $k(X) = K$ ). Moreover, the element  $1 \in W_1$  defines an ample Cartier divisor  $A$  on  $X$  such that  $H^0(nA) = W_n$  for sufficiently positive integer  $n$ . Therefore, in the asymptotic study of the behaviour of  $V_n$  when  $n \rightarrow \infty$ , we may assume that  $V_\bullet$  is a graded linear series of a Cartier divisor on an integral projective scheme over  $k$  which is a projective model of  $K$  over  $k$ .

**Definition 3.7.** Let  $V_\bullet$  be a graded linear series of  $K/k$ . We say that  $V_\bullet$  satisfies the *Fujita approximation property* if the relation

$$\sup_{W_\bullet \subset V_\bullet} \text{vol}(W_\bullet) = V_\bullet$$

holds, where  $W_\bullet$  runs over the set of graded sub- $k$ -algebra of finite type of  $V_\bullet$ .

**Remark 3.8.** If a graded linear series  $V_\bullet$  satisfies the Fujita approximation property, then the “limsup” in the definition of its volume is actually a limit, provided that  $V_n \neq \{0\}$  for sufficiently positive  $n$ . Let  $W_\bullet$  be a graded sub- $k$ -algebra of finite type of  $V_\bullet$ . For sufficiently divisible integer  $m \geq 1$ , the graded  $k$ -algebra

$$W_\bullet^{(m)} := k \oplus \bigoplus_{n \in \mathbb{N}, n \geq 1} W_{nm}$$

is generated by  $W_m$  (see [2, III.§1, no. 3, Lemma 2]) and, by the classic theory of Hilbert–Samuel functions (see [4, VIII.§4]), the sequence

$$(d! \dim_k(W_{nm}) / (nm)^d)_{n \geq 1}$$

converges to the volume of  $W_\bullet$ . By the assumption that  $V_n \neq \{0\}$  for sufficiently positive  $n$ , we obtain that

$$\liminf_{n \rightarrow \infty} \frac{\dim_k(V_n)}{n^d/d!} \geq \text{vol}(W_\bullet),$$

which implies the convergence of the sequence  $(d! \dim_k(V_n) / n^d)_{n \geq 1}$  if  $V_\bullet$  satisfies the Fujita approximation property.

**3.3. Construction of vector bundles from linear series.** Let  $C$  be a regular projective curve over a field  $k$  and  $k(C)$  be the field of rational functions on  $C$ . Denote by  $\eta$  the generic point of  $C$ .

**Definition 3.9.** Let  $M$  be a vector space over  $k(C)$  and  $V$  be a finite dimensional  $k$ -vector subspace of  $M$ . For any affine open subset  $U$  of  $C$ , we let  $E(U)$  be the sub- $\mathcal{O}_C(U)$ -module of  $M$  generated by  $V$ . These modules define a torsion-free coherent sheaf  $E$  on  $C$  which is a vector bundle since  $C$  is a regular curve. We say that  $E$  is the *vector bundle on  $C$  generated by the couple  $(M, V)$* .

**Remark 3.10.** By definition any element  $s \in V$  defines a global section of  $E$  over  $C$ , which is non-zero when  $s \neq 0$ . Hence we can consider  $V$  as a  $k$ -vector subspace of  $H^0(C, E)$ . Moreover, the vector bundle  $E$  is generated by global sections, and hence  $\lambda_i(E) \geq 0$  for any  $i \in \{1, \dots, \text{rk}(E)\}$  (see Proposition 2.3).

We now consider the particular case where  $V$  is a linear series. Let  $K/k$  be a finitely generated extension of fields such that  $K$  contains  $k(C)$ . Suppose given a projective model  $X$  of the field  $K$  (namely  $X$  is an integral projective  $k$ -scheme such that  $k(X) \cong K$ ) equipped with a projective surjective  $k$ -morphism  $\pi : X \rightarrow C$  and a Cartier divisor  $D$  on  $X$ . Let  $V$  be a finite dimensional  $k$ -vector subspace of  $H^0(D)$ . Note that the vector bundle  $E$  generated by  $(K, V)$  identifies with the vector subbundle of  $\pi_*(\mathcal{O}_X(D))$  generated by  $V$ . Moreover, the generic fibre  $E_\eta$  of  $E$  is the  $k(C)$ -vector subspace of  $K$  generated by  $V$  (with the notation of Remark 3.6(2),  $E_\eta = V_{k(C)}$ ). In particular, if  $V$  is a birational linear series of  $K/k$ , then  $E_\eta$  is a birational linear series of  $K/k(C)$ .

### 4. Construction of convex bodies

In this section, we prove the main theorem (Theorem 1.1) of the article. Throughout the section, we fix a field  $k$  and a finitely generated extension  $K/k$  and we let  $d$  be the transcendence degree of  $K$  over  $k$ . In the case where

$d = 0$ , we assume that  $K = k$ , and the Lebesgue measure on  $\mathbb{R}^0 = \{0\}$  is assumed to be the Dirac measure on 0 by convention. We fix also a flag

$$(4.1) \quad k = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_d = K$$

of sub-extensions of  $K/k$  such that each extension  $K_i/K_{i-1}$  is of transcendence degree 1. We denote by  $\mathcal{A}(K/k)$  the set of graded linear series  $V_\bullet$  of  $K/k$  which are of sub-finite type and birational (see Definition 3.5). In the following, we will construct by induction a map  $\Delta^d$  from  $\mathcal{A}(K/k)$  to the set of convex bodies in  $\mathbb{R}^d$  which only depends on the choice of the flag (4.1) and satisfies the following properties:

- (a) if  $V_\bullet$  and  $V'_\bullet$  are two elements of  $\mathcal{A}(K/k)$  such that  $V_n \subset V'_n$  for sufficiently positive  $n$ , then one has  $\Delta^d(V_\bullet) \subset \Delta^d(V'_\bullet)$ ;
- (b) for any  $V_\bullet \in \mathcal{A}(K/k)$  and any  $m \in \mathbb{N}$ ,  $m \geq 1$ , one has

$$\Delta^d(V_\bullet^{(m)}) = m\Delta^d(V_\bullet);$$

- (c) if  $V_\bullet$  and  $V'_\bullet$  are elements in  $\mathcal{A}(K/k)$ , one has

$$\Delta^d(V_\bullet \cdot V'_\bullet) \supset \Delta^d(V_\bullet) + \Delta^d(V'_\bullet).$$

- (d) for any graded linear series  $V_\bullet \in \mathcal{A}(K/k)$ , the Lebesgue measure of  $\Delta^d(V_\bullet)$  is equal to  $\text{vol}(V_\bullet)/d!$ .

We also prove the following assertion by induction on  $d$ .

**Proposition 4.1.** *Any graded linear series  $V_\bullet \in \mathcal{A}(K/k)$  satisfies the Fujita approximation property (see Definition 3.7). In particular, the sequence*

$$\frac{\dim_k(V_n)}{n^d/d!}, \quad n \in \mathbb{N}, n \geq 1$$

*converges to  $\text{vol}(V_\bullet)$  when  $n$  tends to the infinity.*

**4.1. Case of  $d = 0$ .** In the case where  $d = 0$ , one has  $V_n = k$  for sufficiently positive  $n$  since  $K$  is assumed to be  $k$ . Moreover, the graded linear series  $V_\bullet$  is of finite type and one has  $\text{vol}(V_\bullet) = 1$ . Clearly  $V_\bullet$  satisfies the Fujita approximation property. We let  $\Delta^0(V_\bullet)$  be  $\{0\} = \mathbb{R}^0$ . Its mass with respect to the Lebesgue measure (which is the Dirac measure on 0 by convention) is 1, which is equal to  $\text{vol}(V_\bullet)/0!$ .

**4.2. Induction hypothesis.** We assume that  $d \geq 1$  and that the construction of convex bodies associated to birational graded linear series of finite type has been defined and that Proposition 4.1 has been established for field extensions of transcendence degree  $d - 1$ . In particular, we assume that a map  $\Delta^{d-1}$  from  $\mathcal{A}(K/K_1)$  to the set of convex bodies in  $\mathbb{R}^{d-1}$  has been constructed, which satisfies the conditions (a)–(d) above.

**4.3. Construction of convex bodies.** Since the extension  $K_1/k$  (which is of finite type) is of transcendence degree 1, there exists a regular projective curve  $C$  over  $\text{Spec } k$  such that  $k(C) = K_1$ . We denote by  $\eta$  the generic point of the curve  $C$ . In the following, we consider a graded linear series  $V_\bullet$  in  $\mathcal{A}(K/k)$ . Note that  $V_{\bullet, K_1}$  is a graded linear series in  $\mathcal{A}(K/K_1)$  (see Remark 3.6(2)). For any  $n \in \mathbb{N}$ , we let  $E_n$  be the vector bundle on  $C$  generated by  $(K, V_n)$  (see Definition 3.9). Note that one has  $E_{n, \eta} = V_{n, K_1}$ . Moreover, the direct sum  $E_\bullet = \bigoplus_{n \in \mathbb{N}} E_n$  forms a graded  $\mathcal{O}_C$ -algebra, and one has  $V_{\bullet, K_1} \cong E_\bullet \otimes_{\mathcal{O}_C} K_1$  as graded- $K_1$ -algebras. We call  $E_\bullet$  the *graded system of vector bundles on  $C$  generated by  $V_\bullet$* .

By Remark 3.6(3), there exists a birational projective model  $X$  of the field  $K$  over  $k$  and ample Cartier divisor  $D$  on  $X$  such that  $V_\bullet$  identifies with a graded linear series of  $D$ . The inclusion  $K_1 \rightarrow K$  defines a rational  $k$ -morphism from  $X$  to  $C$ . By replacing  $X$  by a birational modification of  $X$ , we may assume that the rational  $k$ -morphism  $X \dashrightarrow C$  extends to a flat projective  $k$ -morphism  $\pi : X \rightarrow C$ . In particular, we can consider  $E_\bullet$  as a graded  $\mathcal{O}_C$ -sub-algebra of  $\bigoplus_{n \in \mathbb{N}} \pi_*(\mathcal{O}(nD))$ , see Section 3.3 for more details.

For each integer  $n \in \mathbb{N}$ , the vector space  $V_{n, K_1}$  is equipped with the  $\mathbb{R}$ -filtration by minima as follows

$$\forall t \in \mathbb{R}, \quad \mathcal{F}^t(V_{n, K_1}) = \text{Vect}_{K_1}(\{s \in V_{n, K_1} \mid s \neq 0, \deg(s) \geq t\}),$$

where in the computation of degree, we consider the vector bundle structure of  $E_n$ . In particular, the filtration is multiplicative. In other words, for all  $(n, m) \in \mathbb{N}^2$  and  $(t_1, t_2) \in \mathbb{R}^2$ , one has

$$(4.2) \quad \mathcal{F}^{t_1}(V_{n, K_1})\mathcal{F}^{t_2}(V_{m, K_1}) \subset \mathcal{F}^{t_1+t_2}(V_{n+m, K_1}).$$

For any  $t \in \mathbb{R}$ , let  $V_{\bullet, K_1}^t$  be the graded sub- $K_1$ -algebra of  $V_{\bullet, K_1}$  defined as

$$\bigoplus_{n \in \mathbb{N}} \mathcal{F}^{nt}(V_{n, K_1}).$$

Clearly, the graded linear series  $V_{\bullet, K_1}^t$  is of sub-finite type. Let

$$\lambda_{\max}^{\text{asy}}(E_\bullet) := \sup_{n \in \mathbb{N}, n \geq 1} \frac{\lambda_1(E_n)}{n} = \lim_{n \rightarrow +\infty} \frac{\lambda_1(E_n)}{n}.$$

The second equality comes from the fact that the sequence  $(\lambda_1(E_n))_{n \geq 1}$  is super-additive. Since  $E_n$  is a vector subbundle of  $\pi_*(\mathcal{O}(D))$ , we obtain that

$$\lambda_1(E_n) \leq \mu_{\max}(\pi_*(\mathcal{O}(nD))),$$

where for any vector bundle  $F$  on  $C$ ,  $\mu_{\max}(F)$  denote the maximal slope of  $F$ , defined as

$$\sup_{F' \subset F} \deg(F') / \text{rk}(F'),$$

with  $F'$  running over the set of all non-zero vector subbundles of  $F$ . By [6, Theorem 4.3.6], we obtain that  $\lambda_{\max}^{\text{asy}}(E_{\bullet})$  is finite. The following lemma is similar to [1, Lemma 1.6].

**Lemma 4.2.** *For any real number  $t < \lambda_{\max}^{\text{asy}}(E_{\bullet})$ , the graded linear series  $V_{\bullet, K_1}^t$  belongs to  $\mathcal{A}(K/K_1)$ .*

*Proof.* Since  $V_{\bullet, K_1}^t$  is a graded sub- $K_1$ -algebra of  $V_{\bullet, K_1}$ , it is of sub-finite type (see Remark 3.6(2)). It remains to verify that  $V_{\bullet, K_1}^t$  is birational. By Remark 3.6(2), the graded linear series  $V_{\bullet, K_1}$  is birational. Let  $N_0 \in \mathbb{N}$  such that the linear series  $V_{N_0, K_1}$  is birational. Let  $\varepsilon > 0$  such that  $t + \varepsilon < \lambda_{\max}^{\text{asy}}(E_{\bullet})$ . for sufficiently positive integer  $n$ , there exists an element  $s_n \in V_{n, K_1} \setminus \{0\}$  such that  $\deg(s_n) > (t + \varepsilon)n$ . For any section  $s \in V_{N_0, K_1}$  such that  $\deg(s) \geq 0$ , one has

$$(4.3) \quad \deg(ss_n) \geq \deg(s) + \deg(s_n) > (t + \varepsilon)n$$

which is bounded from below by  $t(n + N_0)$  when  $n \geq tN_0\varepsilon^{-1}$ , where the first inequality (4.3) comes from the fact that (since  $E_{\bullet}$  is a graded  $\mathcal{O}_C$ -algebra)

$$\forall \text{ closed point } x \in C, \quad \|ss_n\|_x \leq \|s\|_x \cdot \|s_n\|_x.$$

We then deduce that the linear series  $V_{n, K_1}^t$  is birational for sufficiently positive  $n$ , namely the graded linear series  $V_{\bullet}^t$  is birational.  $\square$

By the induction hypothesis, for any real number  $t < \lambda_{\max}^{\text{asy}}(E_{\bullet})$  the convex body  $\Delta^{d-1}(V_{\bullet, K_1}^t)$  is well defined. Since  $E_n$  is generated by global sections, one has  $V_{\bullet, K_1}^t = V_{\bullet, K_1}$  for  $t \leq 0$ . Moreover, for any couple  $(t_1, t_2)$  of real numbers such that  $t_1 \leq t_2 \leq \lambda_{\max}^{\text{asy}}(E_{\bullet})$  one has  $\Delta^{d-1}(V_{\bullet, K_1}^{t_1}) \supset \Delta^{d-1}(V_{\bullet, K_1}^{t_2})$ . We define a function  $G_{E_{\bullet}} : \Delta^{d-1}(V_{\bullet, K_1}) \rightarrow [0, \lambda_{\max}^{\text{asy}}(E_{\bullet})]$  such that (see [1, Definition 1.8])

$$G_{E_{\bullet}}(x) = \sup\{t \mid x \in \Delta^{d-1}(V_{\bullet, K_1}^t)\}.$$

By convention, if  $\{t \mid x \in \Delta^{d-1}(V_{\bullet, K_1}^t)\}$  is empty, then  $G_{E_{\bullet}}(x)$  is defined as  $\lambda_{\max}^{\text{asy}}(E_{\bullet})$ . By definition, for any  $t \in [0, \lambda_{\max}^{\text{asy}}(E_{\bullet})]$ , one has

$$\{x \in \Delta^{d-1}(V_{\bullet, K_1}) \mid G_{E_{\bullet}}(x) \geq t\} = \bigcap_{\varepsilon > 0} \Delta^{d-1}(V_{\bullet, K_1}^t).$$

Therefore, the function  $G_{E_{\bullet}}$  is upper semi-continuous. Moreover, by the condition (4.2) and the induction hypothesis (notably the conditions (b) and (c)), we obtain that, for any rational number  $\lambda \in [0, 1]$  and all real numbers  $t_1$  and  $t_2$  bounded from above by  $\lambda_{\max}^{\text{asy}}(E_{\bullet})$ , one has

$$\Delta(V_{\bullet, K_1}^{\lambda t_1 + (1-\lambda)t_2}) \supset \lambda \Delta(V_{\bullet, K_1}^{t_1}) + (1 - \lambda) \Delta(V_{\bullet, K_1}^{t_2}).$$

By Sierpiński’s theorem (see for example [13, Theorem 9.4.2]), we obtain that the function  $G_{E_\bullet}$  is concave and is continuous on  $\Delta(V_{\bullet, K_1})^\circ$ , the interior of  $\Delta(V_{\bullet, K_1})$ .

**Definition 4.3.** We let  $\Delta^d(V_\bullet)$  be the convex body in  $\mathbb{R}^d$  delimited by the concave function  $G_{E_\bullet}$ , namely  $\Delta^d(V_\bullet)$  is by definition the closure of the set

$$\{(x, t) \mid x \in \Delta^{d-1}(V_{\bullet, K_1}), 0 \leq t \leq G_{E_\bullet}(x)\}.$$

By the same method of [1, Theorem 1.11 and Corollary 1.13], we deduce from the induction hypothesis (notably the limit property predicted in Proposition 4.1, which replaces the condition of containing an ample series in loc. cit.) on  $V_{\bullet, K_1}^t$  ( $t < \lambda_{\max}^{\text{asy}}(E_\bullet)$ ) that

$$\begin{aligned} \text{vol}(\Delta^d(V_\bullet)) &= \int_{t \in [0, \lambda_{\max}^{\text{asy}}(E_\bullet)]} \text{vol}(\Delta^{d-1}(V_{\bullet, K_1}^t)) dt \\ (4.4) \qquad &= \lim_{n \rightarrow +\infty} \frac{1}{n^d} \sum_{\substack{1 \leq i \leq \dim_{K_1}(V_{n, K_1}) \\ \lambda_i(E_n) \geq 0}} \lambda_i(E_n) = \text{vol}(E_\bullet), \end{aligned}$$

where by definition

$$\text{vol}(E_\bullet) := \lim_{n \rightarrow +\infty} \frac{\dim_k H^0(C, E_n)}{n^d},$$

and the last inequality of (4.4) comes from (2.9) and the fact that

$$\dim_{K_1}(V_{n, K_1}) = O(n^{d-1}), \quad n \rightarrow +\infty,$$

which is also a consequence of the induction hypothesis and the subfiniteness condition.

**Remark 4.4.** Similarly to [1, Theorem 1.14], the graded system  $E_\bullet$  of vector bundles on  $C$  satisfies the “arithmetic Fujita approximation property”, namely

$$(4.5) \qquad \sup_{p \geq 1} \frac{\text{vol}(E_\bullet^{[p]})}{p^d} = \text{vol}(E_\bullet),$$

where  $E_\bullet^{[p]}$  is the graded sub- $\mathcal{O}_C$ -algebra of  $E_\bullet^{(p)}$  generated by  $E_p$ .

**4.4. Verification of the properties.** In the previous subsection, we have constructed a map  $\Delta^d$  from  $\mathcal{A}(K/k)$  to the set of convex bodies in  $\mathbb{R}^d$ . It remains to verify that the map  $\Delta^d$  satisfies the properties (a)–(d) described in the beginning of the section and that any graded linear series in  $\mathcal{A}(K/k)$  satisfies the Fujita approximation property.

Let  $V_\bullet$  and  $V'_\bullet$  be two graded linear series in  $\mathcal{A}(K/k)$ . Let  $E_\bullet$  and  $E'_\bullet$  be graded systems of vector bundles on  $C$  generated by  $V_\bullet$  and  $V'_\bullet$  respectively. Assume that, for sufficiently positive integer  $n$ , one has  $V_n \subset V'_n$ , which implies that  $E_n$  is a vector subbundle of  $E'_n$ . In particular, for any  $t \in \mathbb{R}$ , one

has  $V_n^t \subset V_n'^t$  for sufficiently positive  $t$ . Thus by the induction hypothesis one has  $\Delta^{d-1}(V_n^t) \subset \Delta^{d-1}(V_n'^t)$  for any  $t < \lambda_{\max}^{\text{asy}}(E_\bullet)$ , which implies that  $\Delta^d(V_\bullet) \subset \Delta^d(V'_\bullet)$ . Therefore, the map  $\Delta^d$  satisfies the condition (a).

Let  $V_\bullet$  be a graded linear series in  $\mathcal{A}(K/k)$  and  $E_\bullet$  be the graded system of vector bundles on  $C$  generated by  $V_\bullet$ . For any  $m \in \mathbb{N}$ ,  $m \geq 1$ , let  $E_\bullet^{(m)} := \bigoplus_{n \in \mathbb{N}} E_{mn}$ , which is the graded system of vector bundles on  $C$  generated by  $V_\bullet^{(m)}$ . By definition one has  $\lambda_{\max}^{\text{asy}}(E_\bullet^{(m)}) = m\lambda_{\max}^{\text{asy}}(E_\bullet)$ . Moreover, for any  $t < \lambda_{\max}^{\text{asy}}(E_\bullet)$ , one has  $V_{\bullet, K_1}^{(m), mt} = V_{\bullet, K_1}^{t, (m)}$ . Therefore one has  $\Delta^d(V_\bullet^{(m)}) = m\Delta^d(V_\bullet)$ . In other words, the map  $\Delta^d$  satisfies the condition (b).

Let  $V_\bullet$  and  $V'_\bullet$  be two graded linear series in  $\mathcal{A}(K/k)$ , and  $E_\bullet$  and  $E'_\bullet$  be graded system of vector bundles on  $C$  generated by  $V_\bullet$  and  $V'_\bullet$ , respectively. Let  $F_\bullet$  be the graded system of vector bundles on  $C$  generated by  $V_\bullet \cdot V'_\bullet$ . For any  $n \in \mathbb{N}$ , one has a homomorphism of  $\mathcal{O}_C$ -modules  $E_n \otimes_{\mathcal{O}_C} E'_n \rightarrow F_n$  induced by the canonical  $k$ -linear map  $V_n \otimes_k V'_n \rightarrow V_n \cdot V'_n$ . In particular, for any  $(t, t') \in \mathbb{R}^2$  one has

$$\mathcal{F}^t(V_{n, K_1}) \cdot \mathcal{F}^{t'}(V'_{n, K_1}) \subset \mathcal{F}^{t+t'}(V_{n, K_1} \cdot V'_{n, K_1}).$$

This implies that, for any  $(x, y) \in \Delta^{d-1}(V_\bullet) \times \Delta^{d-1}(V'_\bullet)$ , one has

$$G_{F_\bullet}(x + y) \geq G_{E_\bullet}(x) + G_{E'_\bullet}(y).$$

Hence  $\Delta^d(V_\bullet \cdot V'_\bullet) \supset \Delta^d(V_\bullet) + \Delta^d(V'_\bullet)$ . Namely the map  $\Delta^d$  satisfies the condition (c).

The condition (d) and the Fujita approximation property of  $V_\bullet$  follow from (4.4) and the following lemma, which concludes the proof of Theorem 1.1.

**Lemma 4.5.** *Let  $V_\bullet$  be a graded linear series in  $\mathcal{A}(K/k)$  and  $E_\bullet$  be the graded system of vector bundles on  $C$  generated by  $V_\bullet$ . One has*

$$(4.6) \quad \text{vol}(V_\bullet) = \text{vol}(E_\bullet) := \lim_{n \rightarrow +\infty} \frac{\dim_k H^0(C, E_n)}{n^d/d!}.$$

*Moreover, the graded linear series  $V_\bullet$  satisfies the Fujita approximation property.*

*Proof.* For any  $n \in \mathbb{N}$ , one has (see Remark 3.10)

$$\dim_k(V_n) \leq \dim_k H^0(C, E_n).$$

Therefore  $\text{vol}(V_\bullet) \leq \text{vol}(E_\bullet)$ .

In the following, we prove the converse inequality. By Remark 3.6 (iii), we may assume without loss of generality that  $K$  is the rational function field of a normal projective  $k$ -scheme  $X$  and that  $V_\bullet$  is a birational graded linear series of a very ample Cartier divisor  $D$  on  $X$ . Moreover, the inclusion  $k(C) \subset X$  defines a rational  $k$ -morphism from  $X$  to  $C$ . By replacing  $X$  by its blowing-up along the locus where the rational  $k$ -morphism is not



defined, we may suppose that the rational morphism  $X \dashrightarrow C$  extends to a flat projective  $k$ -morphism  $\pi : X \rightarrow C$ . We can thus identify each vector bundle  $E_n$  with the vector subbundle of  $\pi_*(\mathcal{O}_X(D))$  generated by  $V_n$ .

We denote by  $\varphi : C \rightarrow \text{Spec } k$  the structural morphism. Let  $p \geq 1$  be an integer such that the rational morphism  $j_p : X \dashrightarrow \mathbb{P}(V_p)$  defined by the linear series  $V_p$  maps  $X$  birationally to its image. Let  $u_p : X_p \rightarrow X$  be the blowing-up of  $X$  along the base locus of  $j_p$ , namely

$$X_p := \text{Proj} \left( \bigoplus_{m \geq 0} (\varphi\pi)^*(\text{Sym}^m(V_p)) \longrightarrow \mathcal{O}_X(pmD) \right).$$

Then the rational morphism  $j_p : X \dashrightarrow \mathbb{P}(V_p)$  gives rise to a projective  $k$ -morphism  $f_p : X_p \rightarrow \mathbb{P}(V_p)$  which maps  $X_p$  birationally to its image. Let  $L_p$  be the pull-back of the tautological invertible sheaf  $\mathcal{O}_{\mathbb{P}(V_p)}(1)$  by  $j_p$ . Hence one has

$$\text{vol}(L_p) = \text{vol}(\mathcal{O}_{V_p}(1)|_{f_p(X_p)})$$

since  $f_p$  maps birationally  $X_p$  to its image. Therefore the volume of  $L_p$  is equal to that of the graded linear series

$$V_{\bullet}^{[p]} := \bigoplus_{n \geq 0} \text{Im}(\text{Sym}_k^n(V_p) \longrightarrow V_{np}).$$

Let

$$E_{\bullet}^{[p]} = \bigoplus_{n \geq 0} \text{Im}(\text{Sym}_{\mathcal{O}_C}^n(E_p) \longrightarrow E_{np}).$$

For sufficiently positive integer  $n$ , one has  $E_n^{[p]} \subset (\pi u_p)_*(L_p^{\otimes n})$ . Hence

$$(4.7) \quad \text{vol}(V_{\bullet}) \geq \frac{\text{vol}(V_{\bullet}^{[p]})}{p^d} = \frac{\text{vol}(L_p)}{p^d} \geq \frac{\text{vol}(E_{\bullet}^{[p]})}{p^d}.$$

By (4.5) we obtain  $\text{vol}(V_{\bullet}) \geq \text{vol}(E_{\bullet})$ . The equality (4.6) is thus proved. Moreover, the inequality (4.7) and the equality (4.6) also imply that

$$\sup_{p \geq 1} \frac{\text{vol}(V_{\bullet}^{[p]})}{p^d} = \text{vol}(V_{\bullet}).$$

Hence the graded linear series  $V_{\bullet}$  satisfies the Fujita approximation property. □

**Remark 4.6.** The above result shows that any sub-finite and birational graded linear series verifies the Fujita approximation property. In recent works [17, 18], Maclean has constructed a graded algebra over a field, which satisfies the Fujita approximation property, but can not be realised as the graded linear series of a divisor over a projective model. It seems to be a natural question if the construction of Newton–Okounkov bodies can be realised for general graded linear series (non-necessarily of sub-finite type) which satisfy the Fujita approximation property.

## References

- [1] S. BOUCKSOM & H. CHEN, “Okounkov bodies of filtered linear series”, *Compos. Math.* **147** (2011), no. 4, p. 1205-1229.
- [2] N. BOURBAKI, *Éléments de mathématique. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.*, Actualités Scientifiques et Industrielles, vol. 1293, Hermann, 1961.
- [3] ———, *Éléments de mathématique. Algèbre. Chapitres 4 à 7*, Masson, 1981.
- [4] ———, *Éléments de mathématique. Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets*, Masson, 1983.
- [5] H. CHEN, “Arithmetic Fujita approximation”, *Ann. Sci. Éc. Norm. Supér.* **43** (2010), no. 4, p. 555-578.
- [6] ———, “Convergence des polygones de Harder-Narasimhan”, *Mém. Soc. Math. Fr., Nouv. Sér.* **120** (2010), p. 1-120.
- [7] ———, “Majorations explicites des fonctions de Hilbert-Samuel géométrique et arithmétique”, *Math. Z.* **279** (2015), no. 1, p. 99-137.
- [8] C. CILIBERTO, M. FARNIK, A. KÜRONYA, V. LOZOVANU, J. ROÉ & C. SHRAMOV, “Newton–Okounkov bodies sprouting on the valuative tree”, *Rend. Circ. Mat. Palermo* **66** (2017), no. 2, p. 161-194.
- [9] S. D. CUTKOSKY, “Asymptotic multiplicities of graded families of ideals and linear series”, *Adv. Math.* **264** (2014), p. 55-113.
- [10] E. KATZ & S. URBINATI, “Newton–Okounkov bodies over discrete valuation rings and linear systems on graphs”, preprint, 2016.
- [11] K. KAVEH & A. KHOVANSKII, “Algebraic equations and convex bodies”, in *Perspectives in analysis, geometry, and topology*, Progress in Mathematics, vol. 296, Birkhäuser, 2012, p. 263-282.
- [12] ———, “Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory”, *Ann. Math.* **176** (2012), no. 2, p. 925-978.
- [13] M. KUCZMA, *An introduction to the theory of functional equations and inequalities*, Birkhäuser, 2009, xiv+595 pages.
- [14] A. KÜRONYA, V. LOZOVANU & C. MACLEAN, “Convex bodies appearing as Okounkov bodies of divisors”, *Adv. Math.* **229** (2012), no. 5, p. 2622-2639.
- [15] R. LAZARSFELD, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48, Springer, 2004, xviii+387 pages.
- [16] R. LAZARSFELD & M. MUSTAŢĂ, “Convex bodies associated to linear series”, *Ann. Sci. Éc. Norm. Supér.* **42** (2009), no. 5, p. 783-835.
- [17] C. MACLEAN, “Approximable algebras and a question of Chen”, <https://arxiv.org/abs/1703.01801>, 2017.
- [18] ———, “Characterising approximable algebras”, preprint, 2017.
- [19] A. OKOUNKOV, “Brunn–Minkowski inequality for multiplicities”, *Invent. Math.* **125** (1996), no. 3, p. 405-411.
- [20] ———, “Why would multiplicities be log-concave?”, in *The orbit method in geometry and physics (Marseille, 2000)*, Progress in Mathematics, vol. 213, Birkhäuser, 2000, p. 329-347.
- [21] D. ROY & J. L. THUNDER, “An absolute Siegel’s lemma”, *J. Reine Angew. Math.* **476** (1996), p. 1-26.
- [22] X. YUAN, “On volumes of arithmetic line bundles”, *Compos. Math.* **145** (2009), no. 6, p. 1447-1464.

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