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Transcendence of the Hodge–Tate filtration

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Résumé. Soit C une extension algébriquement close et complète de $\mathbb{Q}_p$. Nous démontrons qu'un groupe p-divisible $G/\mathcal{O}_C$ de dimension 1 est défini sur un sous-corps $L \subset C$ complet pour une valuation discrète et contenant les ratios entre les périodes de Hodge–Tate si et seulement si G est de type CM et si et seulement si les ratios entre les périodes engendrent une extension de $\mathbb{Q}_p$ de degré égal à la hauteur de la composante connexe neutre de G. C'est un analogue p-adique du résultat classique de transcendance de Schneider qui dit que, pour τ dans le demi-plan complexe supérieur, τ et $j(\tau)$ sont tous les deux algèbriques sur $\mathbb{Q}$ si et seulement si τ appartient à une extension quadratique de $\mathbb{Q}$. Nous discutons aussi brièvement d'une conjecture qui généralise ce résultat aux shtukas à une patte.

Abstract. For C a complete algebraically closed extension of $\mathbb{Q}_p$, we show that a one-dimensional p-divisible group $G/\mathcal{O}_C$ can be defined over a complete discretely valued subfield $L \subset C$ with Hodge–Tate period ratios contained in L if and only if G has CM, if and only if the period ratios generate an extension of $\mathbb{Q}_p$ of degree equal to the height of the connected part of G. This is a p-adic analog of a classical transcendence result of Schneider which states that for τ in the complex upper half plane, τ and $j(\tau)$ are simultaneously algebraic over $\mathbb{Q}$ if and only if τ is contained in a quadratic extension of $\mathbb{Q}$. We also briefly discuss a conjectural generalization to shtukas with one paw.

1. Introduction

1.1. Transcendence of τ. An elliptic curve E over the complex numbers $\mathbb{C}$ equipped with a basis for its integral Betti homology gives rise to a point

$$[\tau : 1] \in \mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \backslash \mathbb{P}^1(\mathbb{R})$$

describing the position of the Hodge filtration on $H_1(E(\mathbb{C}), \mathbb{C})$ with respect to the fixed basis (thus, τ is the ratio of two periods of a non-zero holomorphic differential on E). The $GL_2(\mathbb{Z})$-orbits on this set are in bijection with the isomorphism classes of elliptic curves over $\mathbb{C}$. These isomorphism classes

are also parameterized by an algebraic modulus, the j-invariant, which has the property that the elliptic curve E_τ corresponding to τ has a model over the field $\mathbb{Q}(j(\tau))$. A classical transcendence result of Schneider [6] gives:

Theorem 1.1 (Schneider [6]). *For $\tau \in \mathbb{H}^\pm$, the following are equivalent:*

(1) τ and $j(\tau)$ are both in $\overline{\mathbb{Q}}$.
(2) $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.
(3) E_τ has CM.

1.2. An analog for p-divisible groups. In this note, we prove a p-adic analog of Theorem 1.1, where the role of elliptic curves is taken up by one-dimensional p-divisible groups, $\mathbb{C}$ is replaced by a complete algebraically closed $C/\mathbb{Q}_p$, and $\mathbb{Q}$ is replaced by any complete discretely valued subfield of C (e.g., $\mathbb{Q}_p$ or $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$, the completion of the maximal unramified extension of $\mathbb{Q}_p$).

We fix some notation: for a p-divisible group $G/\mathcal{O}_C$, we denote by $\dim G$ the dimension of G and by $\mathrm{ht}G$ the height of G. The Tate module of G,

$$T_p G := \varprojlim G[p^n](C)$$

is a free $\mathbb{Z}_p$-module of rank $\mathrm{ht}G$. We denote by G° the connected part of G, and by $\mathrm{Lie}\,G$ the tangent space to the identity element of G° (thinking of G° as a formal Lie group), which is a free $\mathcal{O}_C$-module of rank $\dim G$. We denote by ω_G the cotangent space to the identity element of G°; it is dual to $\mathrm{Lie}\,G$. We denote by $G^\vee$ the dual p-divisible group (defined using Cartier duality), which satisfies $\mathrm{ht}G^\vee = \mathrm{ht}G$ and $\dim G^\vee = \mathrm{ht}G - \dim G$. Finally, we let $\mathbb{Z}_p(1) := T_p\mu_{p^\infty}$ be the Tate $\mathbb{Z}_p$-module, and for any $\mathbb{Z}_p$-module M and $n \in \mathbb{Z}$, we define Tate twists

$$M(n) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}.$$

After tensoring with C, the Tate module of a p-divisible group G is equipped with a one-step Hodge–Tate filtration

(1.1) $0 \to \mathrm{Lie}\,G(1) \otimes C \to T_p G \otimes C \to \omega_{G^\vee} \otimes C \to 0.$

If G is a one-dimensional p-divisible group of height n over $\mathcal{O}_C$, $\mathrm{Lie}(G)(1)\otimes C$ is one-dimensional, and, after twisting by $\mathbb{Z}_p(-1)$, the Hodge–Tate filtration (1.1) gives a point

$$\mathrm{HT}(G) \in \mathbb{P}(T_p G \otimes C(-1)).$$

Following Tate [9], we can describe $\mathrm{HT}(G)$ explicitly: there is a map

$$\mathrm{HT} : T_p G^\vee \to \omega_G$$

given by viewing an element of $T_p G^\vee$ as a map from G to μ_{p^∞} and pulling back the invariant differential $\frac{dt}{t}$. If we fix a basis $e_1, \ldots, e_n$ for $T_p G(-1)$

giving an identification

$$\mathbb{P}(T_p G \otimes C(-1)) \cong \mathbb{P}^{n-1}(C)$$

and a basis ∂ for $\operatorname{Lie} G$, we have

$$\operatorname{HT}(G) = [(\operatorname{HT}(e_1^*), \partial) : (\operatorname{HT}(e_2^*), \partial) : \cdots : (\operatorname{HT}(e_n^*), \partial)]$$

where e_i^* is the dual basis for $T_p G^\vee$ under the natural duality $T_p G^\vee \cong T_p G(-1)^*$. Thus, in this presentation the homogeneous coordinates of $\operatorname{HT}(G)$ are the Hodge–Tate periods of ∂. The field of definition, $\mathbb{Q}_p(\operatorname{HT}(G))$, which depends only on G, is generated by the ratios of these periods.

We say that a p-divisible group $G/\mathcal{O}_C$ has CM if $\operatorname{End}(G) \otimes \mathbb{Q}_p$ contains a commutative semisimple algebra over $\mathbb{Q}_p$ of rank equal to $\operatorname{ht} G$. Our main result is

Theorem 1.2. *Let $G/\mathcal{O}_C$ be a one-dimensional p-divisible group. The following are equivalent:*

(1) *There is a complete discretely valued field $L \subset C$ such that G can be defined over $\mathcal{O}_L$ and $\mathbb{Q}_p(\operatorname{HT}(G)) \subset L$.*
(2) $[\mathbb{Q}_p(\operatorname{HT}(G)) : \mathbb{Q}_p] = \operatorname{ht} G^\circ$.
(3) *G has CM.*

Remark 1.3. When the conditions of Theorem 1.2 hold, one can always find a complete discretely valued field $L \subset C$ and a p-divisible group $G'/\mathcal{O}_L$ such that $G'_{\mathcal{O}_C} \cong G$ and G' has CM over $\mathcal{O}_L$. It is not typically true, however, that *every* $G'/\mathcal{O}_L$ such that $G'_{\mathcal{O}_C} \cong G$ has CM, even after allowing a finite extension of L. For example, over the ring of integers of a finite extension of $\mathbb{Q}_p$, the only extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} with CM will be those with Serre–Tate coordinate a root of unity, however, after passing to $\mathcal{O}_C$ all will have CM, as over $\mathcal{O}_C$ the extensions split.

Remark 1.4. Theorem 1.2 can be used to produce transcendental elements of C as follows: if L is a complete discretely valued extension of $\mathbb{Q}_p$ and G is a formal group over $\mathcal{O}_L$ without CM, then we find that at least one of the period ratios of G is contained in $\widehat{\overline{L}} \backslash \overline{L}$ (since any algebraic extension of L is also discretely valued). The CM lifts of formal groups are well understood (cf. e.g. [3] and [12]). In particular, using the Gross–Hopkins period map it is easy to see that "most" formal groups over complete discretely valued extensions of $\mathbb{Q}_p$ do not have CM.

The main tools used in the proof of Theorem 1.2 are the Scholze–Weinstein [7] classification of p-divisible groups over $\mathcal{O}_C$ by the Hodge–Tate filtration, and Tate's [9] theorem on full-faithfulness of the Tate module of a p-divisible group over a complete discretely valued field. We recall both results in Section 2.

Remark 1.5. The Scholze–Weinstein classification identifies the set of isomorphism classes of one-dimensional height n p-divisible groups over $\mathcal{O}_C$ with the set of $\mathrm{GL}_n(\mathbb{Z}_p)$-orbits in $\mathbb{P}^{n-1}(C)$. If we restrict to formal groups, then the Hodge–Tate filtration lies in the Drinfeld upper half space Ω_{n-1} (constructed by removing from $\mathbb{P}^{n-1}$ all lines that are contained in a proper $\mathbb{Q}_p$-rational subspace). In particular, for $n = 2$, we have

$$\Omega_1(C) = \mathbb{P}^1(C) - \mathbb{P}^1(\mathbb{Q}_p)$$

which we can write with coordinate $[\tau_{\mathrm{HT}} : 1]$. Thus, for height 2 formal groups Theorem 1.2 becomes a precise analog of Theorem 1.1. The role of the j-invariant is played by the explicit statement that G can be defined over $\mathcal{O}_L$.

1.3. Period mappings. No function playing the role of the j-invariant appears in Theorem 1.1 because $\mathbb{P}^n$ is not a well-behaved moduli space for the p-divisible groups we consider. After restricting to formal groups we can remedy this: let G_0 be the unique one-dimensional height n formal group over $\overline{\mathbb{F}_p}$, and let LT_n^∞ denote the infinite level Lubin–Tate space parameterizing deformations of G_0 with a basis of the Tate module, which is a preperfectoid space over $\mathrm{Frac} W(\overline{\mathbb{F}_p})$ by work of Weinstein [11]. It admits two period maps (cf. [7]): the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathrm{LT}_n^\infty \to \Omega_{n-1},$$

which is described on C-points using the Hodge–Tate filtration (1.1) as above, and the Gross–Hopkins [4] period map

$$\pi_{\mathrm{GH}} : \mathrm{LT}_n^\infty \to \mathbb{P}^{n-1}.$$

The Gross–Hopkins period map factors through level zero Lubin–Tate space, and remembers the field of definition of a deformation of G_0 up to a finite extension. Fixing an embedding $W(\overline{\mathbb{F}_p}) \to C$, we obtain

Corollary 1.6. *For $x \in \mathrm{LT}_n^\infty(C)$ with corresponding formal group $G_x/\mathcal{O}_C$, the following are equivalent:*

1. *There is a complete discretely valued $L \subset C$ such that $\pi_{\mathrm{GH}}(x) \in \mathbb{P}^{n-1}(L)$ and $\pi_{\mathrm{HT}}(x) \in \Omega_{n-1}(L)$.*
2. $[\mathbb{Q}_p(\pi_{\mathrm{HT}}(x)) : \mathbb{Q}_p] = n$.
3. G_x *has CM.*

1.4. Outline. In Section 2 we recall some results on p-divisible groups. In Section 3 we prove Theorem 1.2 and Corollary 1.6. In Section 4 we briefly discuss a conjectural generalization to higher dimensional p-divisible groups and shtukas with one paw.

2. Recollections

In this section we recall two theorems on p-divisible groups:

Theorem 2.1 (Tate [9, Theorem 4])**.** *For L a complete discretely valued extension of $\mathbb{Q}_p$, the functor*

$$G \mapsto T_p G$$

is fully faithful from the category of p-divisible groups over $\mathcal{O}_L$ to the category of finite free $\mathbb{Z}_p$-representations of $\mathrm{Gal}(\overline{L}/L)$.

Theorem 2.2 (Scholze–Weinstein [7, Theorem B])**.** *For C a complete algebraically closed extension of $\mathbb{Q}_p$, the functor*

$$G \mapsto (T_p G, \mathrm{Lie}\, G \subset T_p G \otimes C(-1))$$

is an equivalence between the category of p-divisible groups over $\mathcal{O}_C$ and the category of pairs (M, Fil) consisting of a finite free $\mathbb{Z}_p$-module M and a C-vector subspace $\mathrm{Fil} \subset M \otimes C(-1)$.

Remark 2.3. We note that for 1-dimensional p-divisible formal groups, Theorem 2.2 is due to Fargues [2].

3. Proofs

In this section we prove Theorem 1.2 and then deduce Corollary 1.6 from it. The key observation in the proof of Theorem 1.2 is that the combination of Tate's theorem (Theorem 2.1 above) and the fact that the Hodge–Tate filtration is determined by the Galois representation put strong restrictions on the field of definition of the Hodge–Tate filtration for a p-divisible group over a complete discretely valued field.

To prove Theorem 1.2, we will need a linear algebra lemma. For F'/F an extension of fields, V a finite dimensional vector space over F, and $W \subset V \otimes_F F'$ a one-dimensional subspace, we will denote by $F(W)$ the field of definition of W inside of F'. It is the smallest subfield $L \subset F'$ such that there exists a subspace $W_L \subset V \otimes_F L$ with $W_L \otimes_L F = W$. Equivalently, it is the field generated over F by the ratios of the homogeneous coordinates of W as a point in $\mathbb{P}(V \otimes F')$ with respect to a fixed F-basis of V.

Lemma 3.1. *Let V be a finite dimensional vector space over a perfect field F, and let F'/F be an extension containing an algebraic closure of F. If*

$$W \subset V \otimes F'$$

is a line whose conjugates under $\mathrm{Aut}(F'/F)$ span $V \otimes F'$, then the stabilizer of W in $\mathrm{End}_F(V)$ is isomorphic to a subfield $K \subset F(W)$ such that $[K : F] \leq \dim V$. Furthermore, in this situation $K = F(W)$ if and only if $[F(W) : F] = \dim V$ if and only if $[K : F] = \dim V$.

Proof. We denote $d = [F(W) : F]$ (which could be ∞), and $n = \dim V$. We denote by K the stabilizer of W in $\operatorname{End}_F(V)$.

The action of K on the line

$$W_{F(W)} := W \cap V \otimes_F F(W)$$

induces a map from K to $F(W)$. It is an injection because any element of $\operatorname{End}_F(V)$ that acts as zero on W also acts as zero on all of its conjugates under $\operatorname{Aut}(F'/F)$, which span $V \otimes_F F'$. Thus we obtain $K \hookrightarrow F(W)$. In particular, K is a field. Then V is a K-vector space, so we find $[K : F] \leq n$.

We now show $[K : F] = n$ if and only if $K = F(W)$ if and only if $[F(W) : F] = n$.

Suppose $[K : F] = n$. Then, V is isomorphic to K as a K-module. Because F'/F contains $\overline{F}$, and K/F is separable (F is perfect), we find $V \otimes F'$ splits into n distinct characters of K, and thus these are the only lines stabilized by K. Each of these lines is defined over an extension of degree n of F, so that $[F(W) : F] \leq n$. Since $K \subset F(W)$, $[F(W) : F] = n$, and $K = F(W)$.

Suppose $K = F(W)$. Because W has n distinct conjugates, we have $d \geq n$. Because $[K : F] \leq n$, we have $d \leq n$. Thus $d = n$, and $[K : F] = [F(W) : F] = n$.

Suppose $[F(W) : F] = n$. It suffices to show $F(W) \hookrightarrow K$, since that implies $[K : F] \geq n$, and we obtain $[K : F] = n$ and $K = F(W)$. We observe

$$V^* \cong (V^* \otimes F(W))/W^{\perp}_{F(W)}$$

as F vector spaces, since both have dimension n, and the map is injective (if it were not, there would be a non-zero linear form on V defined over F and vanishing on W, thus the conjugates of W would not span $V \otimes F'$). The right hand side is a $F(W)$-vector space, so this isomorphism equips V^* with an action of $F(W)$ preserving $W^{\perp} \subset V^* \otimes F'$, and thus dually equips V with an action of $F(W)$ preserving $W \subset V \otimes F'$. This gives the desired inclusion $F(W) \hookrightarrow K$. $\qquad\square$

Remark 3.2. We will use Lemma 3.1 in the proof of Theorem 1.2 to force the image of the Galois representation on the Tate module of a formal group to be abelian. For higher dimensional formal groups this step breaks down because for a higher dimensional filtration the stabilizers in Lemma 3.1 are no longer necessarily commutative.

We will also need the following structural lemma:

Lemma 3.3. *If $G/\mathcal{O}_C$ is a p-divisible group, then*

$$G \cong G^{\circ} \times (\mathbb{Q}_p/\mathbb{Z}_p)^{\operatorname{ht}G - \dim G},$$

and the conjugates of $\operatorname{Lie} G$ *under* $\operatorname{Aut}(C/\mathbb{Q}_p)$ *span* $T_p G^{\circ}(-1) \otimes C$.

Proof. Let W' be the span of the conjugates of $W = \operatorname{Lie} G$ in $T_p G(-1) \otimes C$, and let $W_0' = (W' \cap T_p G(-1))(1)$, which we identify with a submodule of $T_p G$. W_0' is a free $\mathbb{Z}_p$-module of rank $\dim W'$, saturated in $T_p G$ (here we use some standard descent results for vector spaces to deduce that $W' \cap T_p G \otimes \mathbb{Q}_p$ has $\mathbb{Q}_p$-dimension equal to $\dim W'$ and spans W', cf., e.g., [5, Propositions 16.1 and 16.7]).

By the Scholze–Weinstein classification, Theorem 2.2, the pair $(W_0', \operatorname{Lie} G)$ defines a p-divisible group H over $\mathcal{O}_C$. Furthermore, any map from H to $\mathbb{Q}_p/\mathbb{Z}_p$ comes from a map from $(W_0', \operatorname{Lie} G)$ to $(\mathbb{Z}_p, \{0\})$, and thus must be zero since it sends $\operatorname{Lie} G$ and all of its conjugates to zero. Thus, H is connected. If we choose a free $\mathbb{Z}_p$-module M of rank m complementary to W in $T_p G$ (which exists because W_0' is saturated in $T_p G$), and a trivialization $\mathbb{Z}_p^m \cong M$, then the resulting isomorphism

$$(W_0', \operatorname{Lie} G) \times (\mathbb{Z}_p^m, \{0\}) \cong (T_p G, \operatorname{Lie} G)$$

gives an isomorphism

(3.1) $$H \times (\mathbb{Q}_p/\mathbb{Z}_p)^m \to G.$$

Since H is connected, we deduce $H \cong G^\circ$, and $m = \dim G - \operatorname{ht} G$. Thus (3.1) is the desired decomposition, and the statement about the conjugates of $\operatorname{Lie} G$ follows from the construction of W'. $\qquad\qquad\square$

Proof of Theorem 1.2. We first reduce to the case of G connected (i.e. to G a formal group). Observe that passing to the connected component G° does not change $\mathbb{Q}_p(\operatorname{HT}(G))$, so that condition (2) of the theorem is unchanged. Furthermore, if G can be defined over $\mathcal{O}_L$ as in (1), then so can G°, and vice versa by Lemma 3.3. Thus, if we have (1) $\iff$ (2) for G connected, we obtain (1) $\iff$ (2) for all G. Furthermore, the decomposition of Lemma 3.3 and the Scholze–Weinstein classfication (Theorem 2.2) imply that

$$\operatorname{End}(G) = \begin{pmatrix} \operatorname{End}(G^\circ) & \operatorname{Hom}((\mathbb{Q}_p/\mathbb{Z}_p)^{n-\operatorname{ht} G^\circ}, G^\circ) \\ 0 & M_{n-\operatorname{ht} G^\circ}(\mathbb{Q}_p) \end{pmatrix}.$$

Here, the lower left is 0 because any map from a connected finite group scheme to an étale finite group scheme is zero (alternatively, this can be seen from the Scholze–Weinstein classification). From this we see that (2) $\iff$ (3) for G connected implies (2) $\iff$ (3) for all G.

From now on we will assume G is connected. Denote by W the Hodge–Tate filtration in $T_p G(-1) \otimes C$. By Lemma 3.3, the conjugates of W span $T_p G(-1) \otimes C$. Note $\mathbb{Q}_p(W)$ is the field $\mathbb{Q}_p(\operatorname{HT}(G))$ in the statement of the theorem. Now, as a consequence of the Scholze–Weinstein classification (Theorem 2.2), $\operatorname{End}(G) \otimes \mathbb{Q}_p$ is equal to the stabilizer of W in $\operatorname{End}(T_p G \otimes \mathbb{Q}_p)$. We then obtain (2) $\iff$ (3) from Lemma 3.1. In general, we denote this stabilizer, which is equal to $\operatorname{End} G \otimes \mathbb{Q}_p$, by K. By Lemma 3.1, it is a field with $[K : \mathbb{Q}_p] \leq n$.

We now show (1) $\implies$ (2). We assume (1), and by abuse of notation write $G/\mathcal{O}_L$ for some p-divisible group over $\mathcal{O}_L$ with base change to $\mathcal{O}_C$ equal to G. Because the induced L-semilinear action of $\mathrm{Gal}(\overline{L}/L)$ on $T_pG(-1) \otimes L$ preserves the Hodge–Tate filtration and is in fact L-linear, the line W is preserved by the linear action of $\mathrm{Gal}(\overline{L}/L)$. On the other hand, the image of $\mathrm{Gal}(\overline{L}/L)$ is also contained in $\mathrm{End}_{\mathbb{Q}_p}(T_pG(-1) \otimes \mathbb{Q}_p)$. Thus, Galois acts through K, and Tate's theorem (Theorem 2.1) implies that the centralizer of K also acts by isogenies. So, we must have that K contains its centralizer in $\mathrm{End}_{\mathbb{Q}_p}(T_pG(-1) \otimes \mathbb{Q}_p)$, which implies that $[K : \mathbb{Q}_p] = n$. Thus, by Lemma 3.1, $[\mathbb{Q}_p(\mathrm{HT}(G)) : \mathbb{Q}_p] = n$.

Finally we show (3) $\implies$ (1). For a fixed $[K : \mathbb{Q}_p] = n$, our computations so far along with the Scholze–Weinstein classification (Theorem 2.2) show that isogeny classes of G with CM by K correspond to orbits of $GL_n(\mathbb{Q}_p)$ on $\Omega_{n-1}(K)$. There is a unique such orbit, corresponding to bases for $K/\mathbb{Q}_p$ up to $\mathbb{Q}_p^\times$ homothety. Any Lubin–Tate formal group $G/\mathcal{O}_K$ for K is contained in this isogeny class and any group isogenous to it is defined over the ring of a integers of a finite extension of K, thus we conclude any formal group with CM by K can in fact be defined over the ring of integers of a finite extension of $\mathbb{Q}_p$. $\qquad\square$

Proof of Corollary 1.6. For a point $x \in \mathrm{LT}_n^\infty$, we will show Corollary 1.6 (1) is equivalent to Theorem 1.2 (1) for G_x. It suffices to show that if $\pi_{\mathrm{GH}}(x)$ is in a discretely valued extension L of $\mathrm{Frac}W(\overline{k})$ then G_x can be defined over a finite extension of L. As in [4], Corollary 23.21, π_{GH} is surjective (at level 0) on $\overline{L}$ points. The fibers of π_{GH} contain isogenous groups, and thus there is some G' isogenous to G_x defined over a finite extension of L. But then G_x is also defined over a finite extension of L, since the kernel of an isogeny from G' to G_x is defined over a finite extension. $\qquad\square$

4. Generalizations

In the archimedean setting, an analog of Theorem 1.1 holds for abelian varieties of higher dimension (cf. [1] and [8]). Our method of proof in the p-adic setting does not obviously generalize to p-divisible groups of dimension ≥ 1 (cf. Remark 3.2), however, it is natural to conjecture the analogous result.

We might even go further: p-divisible groups give the simplest examples of shtukas with one paw over $\mathrm{Spa}\,C^\flat$ as in [10]. By a result of Fargues (cf. [10, Theorem 12.4.4]), a shtuka with one paw is equivalent to a pair consisting of a finite free $\mathbb{Z}_p$-module M and a $\mathbb{B}_{\mathrm{dR}}^+$-lattice $\mathcal{L} \subset M \otimes \mathbb{B}_{\mathrm{dR}}$. The lattice $\mathcal{L}$ induces a filtration on $M \otimes \mathbb{B}_{\mathrm{dR}}$, thus by restriction on $M \otimes \mathbb{B}_{\mathrm{dR}}^+$, and then, via specialization along the natural map $\theta : \mathbb{B}_{\mathrm{dR}}^+ \to C$, on $M \otimes C$. The resulting filtration on $M \otimes C$ is called the Hodge–Tate filtration. This generalizes the Hodge–Tate filtration on a p-divisible group, in which case

$M = T_p G$ and $\mathcal{L}$ is uniquely determined by the Hodge–Tate filtration and the requirement that it lie in the natural minuscule Schubert cell relative to $M \otimes \mathbb{B}_{\mathrm{dR}}^+$. We note that, in general, the Hodge–Tate filtration can have multiple steps!

We say a shtuka $(M, \mathcal{L})$ has CM if $(M \otimes \mathbb{Q}_p, \mathcal{L})$ admits endomorphisms by a semi-simple commutative algebra over $\mathbb{Q}_p$ of dimension equal to the rank of M.

There is also a natural analog of being defined over a complete discretely valued subfield in this setting: Given a complete discretely valued field $L \subset C$ and a lattice M in a deRham representation of $\mathrm{Gal}(\overline{L}/L)$, we obtain a shtuka with one paw from the pair

$$(M, (M \otimes \mathbb{B}_{\mathrm{dR}})^{\mathrm{Gal}(\overline{L}/L)} \otimes_L \mathbb{B}_{\mathrm{dR}}^+).$$

We will say a shtuka with one paw is *arithmetic* if it is isomorphic to one of this form. We formulate the optimistic

Conjecture 4.1. *If $(M, \mathcal{L})$ is an arithmetic shtuka with one paw with Hodge–Tate filtration defined over a complete discretely valued subfield of C then $(M, \mathcal{L})$ has CM.*

Remark 4.2. For a CM shtuka, the Hodge–Tate filtration is algebraic since any sub-space preserved by the CM algebra K is a direct sum of the 1-dimensional character spaces for K.

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