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Lukas SPIEGELHOFER

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par Lukas SPIEGELHOFER

Résumé. Pour un nombre irrationnel $\alpha \in (0,1)$, nous étudions la fonction somme des chiffres d'Ostrowski σ_α. Étant donné un nombre α à quotients partiels bornés et un nombre $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$, nous montrons que la fonction $g : n \mapsto \mathrm{e}(\vartheta\sigma_\alpha(n))$, où $\mathrm{e}(x) = \mathrm{e}^{2\pi i x}$, est pseudo-aléatoire dans le sens suivant : pour tout $r \in \mathbb{N}$ la limite

$$\gamma_r = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} g(n+r)\overline{g(n)}$$

existe et on a

$$\lim_{R\to\infty} \frac{1}{R} \sum_{0\le r<R} \left|\gamma_r\right|^2 = 0.$$

Abstract. For an irrational $\alpha \in (0,1)$, we investigate the Ostrowski sum-of-digits function σ_α. For α having bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$, we prove that the function $g : n \mapsto \mathrm{e}(\vartheta\sigma_\alpha(n))$, where $\mathrm{e}(x) = \mathrm{e}^{2\pi i x}$, is pseudorandom in the following sense: for all $r \in \mathbb{N}$ the limit

$$\gamma_r = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} g(n+r)\overline{g(n)}$$

exists and we have

$$\lim_{R\to\infty} \frac{1}{R} \sum_{0\le r<R} \left|\gamma_r\right|^2 = 0.$$

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1. Introduction and main results

Let g be an arithmetical function. The set of $\beta \in [0,1)$ satisfying

$$\limsup_{N\to\infty} \frac{1}{N} \left| \sum_{0\le n<N} g(n)\,\mathrm{e}(-n\beta) \right| > 0$$

is called the *Fourier–Bohr spectrum* of g.

The function g is called *pseudorandom in the sense of Bertrandias* [4] or simply *pseudorandom* if the limit

$$\gamma_r = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} g(n+r)\overline{g(n)}$$

exists for all $r \ge 0$ and is zero in quadratic mean, that is,

$$\lim_{R\to\infty} \frac{1}{R} \sum_{0\le r<R} \left|\gamma_r\right|^2 = 0.$$

Pseudorandomness can be understood as a property of the *spectral measure* associated to g: Assume that the correlation γ_r of g exists for all $r \ge 0$. By the Bochner representation theorem there exists a unique measure μ on the Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ such that

$$\gamma_r = \int_{\mathbb{T}} \mathrm{e}(rx)\,\mathrm{d}\mu(x)$$

for all r. Then g is pseudorandom if and only if the discrete component of μ vanishes. We refer to [9] for more details.

It is known that pseudorandomness of a bounded arithmetic function g implies that the spectrum of g is empty, which can be proved using van der Corput's inequality. For the convenience of the reader, we give a proof of this fact in Section 2.

The converse of this statement does not always hold. However, it is true for q-multiplicative functions $g : \mathbb{N} \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which has been proved by Coquet [5, 6, 7]. Here a function $g : \mathbb{N} \to \mathbb{C}$ is called *q-multiplicative* if $f(q^k n + b) = f(q^k n) f(b)$ for all integers $k, n > 0$ and $0 \le b < q^k$.

The purpose of this paper is to prove an analogous statement for the Ostrowski numeration system, that is, for *α-multiplicative* functions. Assume that $\alpha \in (0,1)$ is irrational. The Ostrowski numeration system has as its scale of numeration the sequence of denominators of the convergents of the regular continued fraction expansion of α. More precisely, let $\alpha = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of α and $p_i/q_i =$

$[0; a_1, \ldots, a_i]$ the i-th convergent to α, where $i \geq 0$. By the greedy algorithm, every nonnegative integer n has a representation

$$n = \sum_{k \geq 0} \varepsilon_k q_k \tag{1.1}$$

such that

$$\sum_{0 \leq k < K} \varepsilon_k q_k < q_K$$

for all $K \geq 0$. This algorithm yields the unique expansion of the form (1.1) having the properties that $0 \leq \varepsilon_0 < a_1$, $0 \leq \varepsilon_k \leq a_{k+1}$ and $\varepsilon_k = a_{k+1} \Rightarrow \varepsilon_{k-1} = 0$ for $k \geq 1$, the *Ostrowski expansion of* n.

For a nonnegative integer n let $(\varepsilon_k(n))_{k \geq 0}$ be its Ostrowski expansion. An arithmetic function f is *α-additive* resp. *α-multiplicative* if

$$f(n) = \sum_{k \geq 0} f(\varepsilon_k(n) q_k) \quad \text{resp.} \quad f(n) = \prod_{k \geq 0} f(\varepsilon_k(n) q_k)$$

for all n. Examples of α-additive functions are the functions $n \mapsto \beta n$ (for $\beta \in \mathbb{R}$) and the *α-sum of digits* of n [8]:

$$\sigma_\alpha(n) = \sum_{i \geq 0} \varepsilon_i(n).$$

We refer the reader to [3] for a survey on the Ostrowski numeration system. In particular, we want to note that the Ostrowski numeration system is a useful tool for studying the discrepancy modulo 1 of $n\alpha$-sequences, see for example the references contained in the aforementioned paper.

Moreover, see [2] for a dynamical viewpoint of the Ostrowski numeration system, see also [1, 12] for more general numeration systems.

Our main theorem establishes a connection between the Fourier–Bohr spectrum and pseudorandomness for α-multiplicative functions.

Theorem 1.1. *Assume that g is a bounded α-multiplicative function. The Fourier–Bohr spectrum of g is empty if and only if g is pseudorandom.*

Using a theorem by Coquet, Rhin and Toffin [11, Theorem 2], we obtain the following corollary.

Corollary 1.2. *Assume that $\alpha \in (0, 1)$ is irrational and has bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$. Then $n \mapsto \mathrm{e}(\vartheta \sigma_\alpha(n))$ is pseudorandom.*

In particular, this holds for the *Zeckendorf sum-of-digits function*, which corresponds to the case $\alpha = (\sqrt{5} - 1)/2 = [0; 1, 1, \ldots]$. This special case can be found in the author's thesis [14].

We first present a series of auxiliary results, and proceed to the proof of Theorem 1.1 in Section 3.

2. Lemmas

We begin with the well-known inequality of van der Corput.

Lemma 2.1 (Van der Corput's inequality)**.** *Let I be a finite interval in $\mathbb{Z}$ and let $a_n \in \mathbb{C}$ for $n \in I$. Then*

$$\left|\sum_{n\in I} a_n\right|^2 \le \frac{|I|-1+R}{R} \sum_{0\le |r|<R} \left(1-\frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+r\in I}} a_{n+r}\overline{a_n}$$

for all integers $R \ge 1$.

In the definition of pseudorandomness for bounded arithmetic functions g, we do not actually need the square.

Lemma 2.2. *Let g be a bounded arithmetic function such that the correlation γ_r of g exists for all $r \ge 0$. The function g is pseudorandom if and only if*

$$\lim_{R\to\infty} \frac{1}{R} \sum_{0\le r<R} |\gamma_r| = 0.$$

For the proof of sufficiency we note that we may without loss of generality assume that $|g| \le 1$. The other direction is an application of the Cauchy-Schwarz inequality.

As we noted before, pseudorandomness of g implies that the spectrum of g is empty.

Lemma 2.3. *Let g be a bounded arithmetic function. If g is pseudorandom, then the Fourier–Bohr spectrum of g is empty.*

Proof. The proof is an application of van der Corput's inequality (Lemma 2.1). We have for all $R \in \{1,\ldots,N\}$

$$\begin{aligned}\left|\frac{1}{N}\sum_{0\le n<N} g(n)\,\mathrm{e}(n\beta)\right|^2 &\le \frac{N-1+R}{RN^2}\sum_{0\le|r|<R}\left(1-\frac{|r|}{R}\right)\mathrm{e}(r\beta)\sum_{0\le n,n+r<N} g(n+r)\overline{g(n)}\\ &\ll \frac{1}{R}\sum_{0\le r<R}\left|\frac{1}{N}\sum_{0\le n<N} g(n+r)\overline{g(n)}\right| + O\left(\frac{R}{N}\right).\end{aligned}$$

Let $\varepsilon \in (0,1)$. By hypothesis and Lemma 2.2 we may choose R so large that

$$\frac{1}{R}\sum_{0\le r<R} |\gamma_r| < \varepsilon^2.$$

Moreover, we choose N_0 in such a way that $R/N_0 < \varepsilon^2$ and

$$\left| \frac{1}{N} \sum_{0 \le n < N} g(n+r)\overline{g(n)} - \gamma_r \right| < \varepsilon^2$$

for all $r < R$ and $N \ge N_0$. Then for $N \ge N_0$ we have

$$\begin{aligned} &\left| \frac{1}{N} \sum_{0 \le n < N} g(n)\,\mathrm{e}(n\beta) \right|^2 \\ &\quad \ll \frac{1}{R} \sum_{0 \le r < R} \left| \frac{1}{N} \sum_{0 \le n < N} g(n+r)\overline{g(n)} - \gamma_r \right| + \frac{1}{R} \sum_{0 \le r < R} |\gamma_r| + O\left(\frac{R}{N_0}\right) \\ &\quad \ll \varepsilon^2. \end{aligned}$$

□

The following lemma is a generalization of Dini's Theorem.

Lemma 2.4. *Assume that $(f_i)_{i \ge 0}$ is a sequence of nonnegative continuous functions on $[0,1]$ converging pointwise to the zero function. Assume that $|f_{i+1}(x)| \le \max\{|f_i(x)|, |f_{i-1}(x)|\}$ for $x \in [0,1]$. Then the convergence is uniform in x.*

Proof. For $\varepsilon > 0$ and nonnegative N we set

$$A_N(\varepsilon) = \{x \in [0,1] : f_N(x) < \varepsilon \text{ and } f_{N+1}(x) < \varepsilon\}.$$

Note that this is an open set. By induction, using the property $|f_{i+1}(x)| \le \max\{|f_i(x)|, |f_{i-1}(x)|\}$, we obtain

$$A_N(\varepsilon) = \{x \in [0,1] : f_n(x) < \varepsilon \text{ for all } n \ge N\}.$$

Trivially, we have $A_N(\varepsilon) \subseteq A_{N+1}(\varepsilon)$. For each $x \in [0,1]$ there is an $N(x)$ such that $f_n(x) < \varepsilon$ for all $n \ge N(x)$. Then $x \in A_{N(x)}(\varepsilon)$, therefore $\left(A_{N(x)}(\varepsilon)\right)_{x \in [0,1]}$ is an open cover of the compact set $[0,1]$. Choose $x_1, \ldots, x_k$ such that $A_{N(x_1)}(\varepsilon) \cup \cdots \cup A_{N(x_k)}(\varepsilon) \supseteq [0,1]$ and set $N = \max\{N(x_1), \ldots, N(x_k)\}$. By monotonicity of the sets $A_N(\varepsilon)$, we obtain $A_N(\varepsilon) \supseteq [0,1]$, in other words, $f_n(x) < \varepsilon$ for all $x \in [0,1]$ and all $n \ge N$. □

Lemma 2.5. *Assume that $\lambda \ge 1$ and let $(w_i)_i$ be the increasing enumeration of the integers n such that $\varepsilon_0(n) = \cdots = \varepsilon_{\lambda-1}(n) = 0$. The intervals $[w_i, w_{i+1})$ constitute a partition of the set $\mathbb{N}$ into intervals of length q_λ and $q_{\lambda-1}$, where $w_{i+1} - w_i = q_{\lambda-1}$ if and only if $\varepsilon_\lambda(w_i) = a_{\lambda+1}$.*

Proof. Assume first that $\varepsilon_\lambda(w_i) = a_{\lambda+1}$. We want to show that $w_{i+1} = w_i + q_{\lambda-1}$. Assume that $\lambda \ge 2$ and let $w_i \le n < w_i + q_{\lambda-1}$. Then the Ostrowski expansion of n is obtained by superposition of the expansions of w_i and of $n - w_i$. In particular, for $w_i < n < w_i + q_{\lambda-1}$ we have $\varepsilon_j(n) \ne 0$

for some $j < \lambda - 1$. (Trivially, this also holds for $\lambda = 1$.) Moreover, in the addition $w_i + q_{\lambda-1}$ a carry occurs, producing $\varepsilon_j(w_i + q_{\lambda-1}) = 0$ for $j \le \lambda$, therefore $w_{i+1} = w_i + q_{\lambda-1}$. The case $\varepsilon_\lambda(w_i) < a_{\lambda+1}$ is similar, in which case $w_{i+1} = w_i + q_\lambda$. □

For an α-multiplicative function g and an integer $\lambda \ge 0$ we define a function g_λ by truncating the digital expansion: we define $\psi_\lambda(n) = \sum_{i<\lambda} \varepsilon_i(n) q_i$ and

$$g_\lambda(n) = g(\psi_\lambda(n)).$$

We will need the following carry propagation lemma for the Ostrowski numeration system.

Lemma 2.6. *Let $\lambda \ge 1$ be an integer and $N, r \ge 0$. Assume that $\alpha \in (0,1)$ is irrational and let g be an α-multiplicative function. Then*

$$\Bigl|\bigl\{n < N : g(n+r)\overline{g(n)} \ne g_\lambda(n+r)\overline{g_\lambda(n)}\bigr\}\Bigr| \le N \frac{r}{q_{\lambda-1}}. \tag{2.1}$$

Proof. The statement we want to prove is trivial for $r \ge q_{\lambda-1}$, we assume therefore that $r < q_{\lambda-1}$. Let w be the family from Lemma 2.5. For $w_i \le n < w_{i+1} - r$, we have $\varepsilon_j(n+r) = \varepsilon_j(n)$ for $j \ge \lambda$. It follows that

$$\Bigl|\bigl\{n \in \{w_i, \ldots, w_{i+1} - 1\} : g(n+r)\overline{g(n)} \ne g_\lambda(n+r)\overline{g_\lambda(n)}\bigr\}\Bigr| \le r.$$

By concatenating blocks, the statement follows therefore for the case that $N = w_i$ for some i. It remains to treat the case that $w_i < N < w_{i+1}$ for some i. To this end, we denote by $L(N)$ resp. $R(N)$ the left hand side resp. the right hand side of (2.1). For $w_i \le N \le w_{i+1}$ we have

$$L(N) = \begin{cases} L(w_i), & N \le w_{i+1} - r; \\ L(w_i) + N - (w_{i+1} - r), & N \ge w_{i+1} - r. \end{cases}$$

Note that L is a polygonal line that lies below $R(N)$ for $N \in \{w_i, w_{i+1} - r, w_{i+1}\}$ and therefore for all $N \in [w_i, w_{i+1}]$. By concatenating blocks, the full statement follows. □

We define Fourier coefficients for g:

$$G_\lambda(h) = \frac{1}{q_\lambda} \sum_{0 \le u < q_\lambda} g(u)\, \mathrm{e}(-huq_\lambda^{-1}).$$

Lemma 2.7. *Assume that i is such that $w_{i+1} - w_i = q_\lambda$ and let $r \ge 0$. We have*

$$\sum_{h<q_\lambda} \bigl|G_\lambda(h)\bigr|^2 \mathrm{e}(hrq_\lambda^{-1}) = \frac{1}{q_\lambda} \sum_{w_i \le v < w_{i+1}} g_\lambda(v+r)\overline{g_\lambda(v)} + O\Bigl(\frac{r}{q_\lambda}\Bigr). \tag{2.2}$$

Proof.

$$\begin{aligned}
\sum_{0\le h<q_\lambda} \left|G_\lambda(h)\right|^2 \mathrm{e}\left(hrq_\lambda^{-1}\right) \\
&= q_\lambda^{-1} \sum_{0\le u,v<q_\lambda} g_\lambda(u)\overline{g_\lambda(v)}\, q_\lambda^{-1} \sum_{0\le h<q_\lambda} \mathrm{e}\left(\frac{h}{q_\lambda}(v+r-u)\right)\\
&= \frac{1}{q_\lambda} \sum_{0\le u,v<q_\lambda} [\![v+r\equiv u \bmod q_\lambda]\!]\, g_\lambda(u)\overline{g_\lambda(v)}\\
&= \frac{1}{q_\lambda} \sum_{w_i\le u,v<w_{i+1}} [\![v+r\equiv u \bmod q_\lambda]\!]\, g_\lambda(u)\overline{g_\lambda(v)}\\
&= \frac{1}{q_\lambda} \sum_{w_i\le u<w_{i+1}-r} g_\lambda(v+r)\overline{g_\lambda(v)} + O\left(\frac{r}{q_\lambda}\right)\\
&= \frac{1}{q_\lambda} \sum_{w_i\le u<w_{i+1}} g_\lambda(v+r)\overline{g_\lambda(v)} + O\left(\frac{r}{q_\lambda}\right). \qquad \square
\end{aligned}$$

Lemma 2.8. *Let $H\ge 1$ be an integer and R a real number. For all real numbers t we have*

$$\sum_{h<H}\left|\frac{1}{R}\sum_{r<R}\mathrm{e}\big(r(t+h/H)\big)\right|^2 \le \frac{H+R-1}{R}.$$

This lemma is an immediate consequence of the analytic form of the large sieve, see [13, Theorem 3]. This form of the theorem, featuring the optimal constant $N-1+\delta^{-1}$, is due to Selberg.

Lemma 2.9 (Selberg)**.** *Let $N\ge 1, R\ge 1, M$ be integers, $\alpha_1,\ldots,\alpha_R\in\mathbb{R}$ and $a_{M+1},\ldots,a_{M+N}\in\mathbb{C}$. Assume that $\|\alpha_r-\alpha_s\|\ge\delta$ for $r\ne s$. Then*

$$\sum_{r=1}^{R}\left|\sum_{n=M+1}^{M+N} a_n\,\mathrm{e}(n\alpha_r)\right|^2 \le \left(N-1+\delta^{-1}\right)\sum_{n=M+1}^{M+N}\left|a_n\right|^2.$$

As an important first step in the proof of Theorem 1.1, we show that for the functions in question we have the following uniformity property.

Proposition 2.10. *Let g be a bounded α-multiplicative function. Assume that the Fourier–Bohr spectrum of g is empty, that is,*

$$\left|\sum_{0\le n<N} g(n)\,\mathrm{e}(-n\beta)\right| = o(N)$$

as $N\to\infty$ for all $\beta\in\mathbb{R}$. Then

$$\sup_{\beta\in\mathbb{R}}\left|\sum_{0\le n<N} g(n)\,\mathrm{e}(-n\beta)\right| = o(N).$$

Proof of Proposition 2.10. Without loss of generality we may assume that $|g| \le 1$, since the full statement follows by scaling. We first prove the special case

$$\lim_{i\to\infty} \sup_{\beta\in\mathbb{R}} \frac{1}{q_i} \left| \sum_{0\le n<q_i} g(n)\,\mathrm{e}(-n\beta) \right| = 0.$$

We set $h(n) = g(n)\,\mathrm{e}(-n\beta)$ and

$$S_i = S_i(\beta) = \frac{1}{q_i} \sum_{0\le n<q_i} h(n).$$

For all $i \ge 1$ we have

$$\begin{aligned} S_{i+1} &= \frac{1}{q_{i+1}} \sum_{0\le b<a_{i+1}} \sum_{0\le u<q_i} h(u+bq_i) + \frac{1}{q_{i+1}} \sum_{0\le u<q_{i-1}} h(u+a_{i+1}q_i) \\ &= \frac{q_i}{q_{i+1}} \left(\sum_{0\le b<a_{i+1}} h(bq_i) \right) \cdot S_i + \frac{q_{i-1}}{q_{i+1}} h(a_{i+1}q_i) S_{i-1}. \end{aligned}$$

Using the recurrence for q_i, it follows that $|S_{i+1}| \le \max\{|S_i|, |S_{i-1}|\}$. By Lemma 2.4 we obtain the statement.

We proceed to the general case. We consider partial sums of $g(n)\,\mathrm{e}(n\beta)$ up to N. Assume that $w_i \le N < w_{i+1}$. We have

$$\left| \sum_{0\le n<N} g(n)\,\mathrm{e}(n\beta) \right|^2 \le \left| \sum_{0\le n<w_i} g(n)\,\mathrm{e}(n\beta) \right|^2 + q_\lambda^2 + 2Nq_\lambda.$$

We apply the inequality of van der Corput (Lemma 2.1) to obtain

$$\left| \sum_{0\le n<w_i} g(n)\,\mathrm{e}(n\beta) \right|^2 \le \frac{N+R-1}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \mathrm{e}(r\beta) \sum_{0\le n,n+r<w_i} g(n+r)\overline{g(n)}.$$

We adjust the summation range by omitting the condition $0 \le n+r < w_i$. This introduces an error term $O(NR)$. Moreover, α-multiplicative functions satisfy Lemma 2.6, therefore we may replace g by g_λ for the price of another error term, $O(N^2 R q_{\lambda-1}^{-1})$. Using (2.2) we get

$$\begin{aligned} &\left| \sum_{0\le n<w_i} g(n)\,\mathrm{e}(n\beta) \right|^2 \\ &\quad \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \mathrm{e}(r\beta) \left(\sum_{0\le n<w_i} g_\lambda(n+r)\overline{g_\lambda(n)} + O\left(R + \frac{NR}{q_{\lambda-1}}\right) \right) \\ &\quad \ll NR + \frac{N^2 R}{q_{\lambda-1}} + \frac{N}{R} w_i \sum_{h<q_\lambda} \left|G_\lambda(h)\right|^2 \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \mathrm{e}\left(r\left(\beta + \frac{h}{q_\lambda}\right)\right). \end{aligned}$$

Note that the sum over r is a nonnegative real number. This follows from the identity

$$\sum_{|r|<R} (R-|r|)\,\mathrm{e}(rx) = \left|\sum_{0\le r<R} \mathrm{e}(rx)\right|^2,$$

which can be proved by an elementary combinatorial argument. We use this equation and collect the error terms to get

$$(2.3)\quad \left|\frac{1}{N}\sum_{0\le n<N} g(n)\,\mathrm{e}(n\beta)\right|^2 \ll \frac{q_\lambda^2}{N^2}+\frac{q_\lambda}{N}+\frac{R}{N}+\frac{R}{q_{\lambda-1}} + \sum_{0\le h<q_\lambda} |G_\lambda(h)|^2 \left|\frac{1}{R}\sum_{0\le r<R} \mathrm{e}\left(r\left(\beta+\frac{h}{q_\lambda}\right)\right)\right|^2.$$

Next, using Lemma 2.8 we get

$$(2.4)\quad \sum_{0\le h<q_\lambda} |G_\lambda(h)|^2 \left|\frac{1}{R}\sum_{0\le r<R} \mathrm{e}\left(r\left(\beta+\frac{h}{q_\lambda}\right)\right)\right|^2 \le \sup_{0\le h<q_\lambda} |G_\lambda(h)|^2 \frac{q_\lambda+R-1}{R}.$$

Using the special case proved before and choosing R and λ appropriately, we obtain the statement. □

In order to establish the existence of the correlation γ_r of g for all $r\ge 0$, we use the following theorem [10, Théorème 4]. (Note that we defined $\psi_\lambda(n)=\sum_{0\le i<\lambda}\varepsilon_i(n)q_i$.)

Lemma 2.11 (Coquet–Rhin–Toffin)**.** *Let* $\lambda \ge 1$ *and* $a<q_\lambda$. *The set* $\mathcal{E}(\lambda,a)=\{n\in\mathbb{N} : \psi_\lambda(n)=a\}$ *possesses an asymptotic density given by*

$$\delta = (q_\lambda+q_{\lambda-1}[0;a_{\lambda+1},\ldots])^{-1} \qquad \text{if } a\ge q_{\lambda-1};$$
$$\delta' = \delta(1+[0;a_{\lambda+1},\ldots]) \qquad \text{if } a< q_{\lambda-1}.$$

Lemma 2.12. *Let g be a bounded α-multiplicative function. Then for every* $r\ge 0$ *the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N} g(n+r)\overline{g(n)}$$

exists.

We note that the existence of the correlation was established in [10] for the special case that $g(n)=\mathrm{e}(y\sigma_\alpha(n))$, where $\mathrm{e}(x)=\mathrm{e}^{2\pi i x}$ and $y\in\mathbb{R}$.

Proof. Let $\lambda, N \geq 0$ and $r \geq 1$ and set $k = \max\{j : w_j \leq N\}$. Moreover, let $a = a(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q_\lambda$ and $b = b(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q_{\lambda-1}$. By Lemma 2.11 $a(N)/N$ and $b(N)/N$ converge, say to A and B respectively. Let λ be so large that $r/q_{\lambda-1} < \varepsilon$. Moreover, choose N_0 so large that $|A - a(N)/N| < \varepsilon q_\lambda^{-1}$, $|B - b(N)/N| < \varepsilon q_{\lambda-1}^{-1}$ and $q_\lambda/N < \varepsilon$ for all $N \geq N_0$.

Then by Lemma 2.6 we get

$$\begin{aligned}\sum_{0\leq n<N} g(n+r)\overline{g(n)} &= \sum_{0\leq n<N} g_\lambda(n+r)\overline{g_\lambda(n)} + O\big(Nrq_{\lambda-1}^{-1}\big)\\ &= \sum_{0\leq n<w_k} g_\lambda(n+r)\overline{g_\lambda(n)} + O\big(q_\lambda + Nrq_{\lambda-1}^{-1}\big),\end{aligned}$$

therefore

$$\begin{aligned}&\left|\frac{1}{N}\sum_{0\leq n<N} g(n+r)\overline{g(n)} - A\sum_{0\leq n<q_\lambda} g_\lambda(n+r)\overline{g_\lambda(n)} - B\sum_{0\leq n<q^{\lambda-1}} g_\lambda(n+r)\overline{g_\lambda(n)}\right|\\ &\quad\ll \left|\frac{1}{N}\sum_{0\leq n<N} g(n+r)\overline{g(n)} - \frac{a}{N}\sum_{0\leq n<q_\lambda} g_\lambda(n+r)\overline{g_\lambda(n)}\right.\\ &\qquad\qquad\left. - \frac{b}{N}\sum_{0\leq n<q_{\lambda-1}} g_\lambda(n+r)\overline{g_\lambda(n)}\right| + 2\varepsilon\\ &\quad= \left|\frac{1}{N}\sum_{0\leq n<N} g(n+r)\overline{g(n)} - \frac{1}{N}\sum_{0\leq n<w_k} g_\lambda(n+r)\overline{g_\lambda(n)}\right| + 2\varepsilon\\ &\quad\ll \frac{q_\lambda}{N} + \frac{r}{q_{\lambda-1}} + 2\varepsilon.\end{aligned}$$

By the triangle inequality it follows that the values $\frac{1}{N}\sum_{n<N} g(n+r)\overline{g(n)}$ form a Cauchy sequence and therefore a convergent sequence, which proves the existence of the correlation of g. □

3. Proof of the theorem

Now we are prepared to prove Theorem 1.1. If g is pseudorandom, then by Lemma 2.3 its spectrum is empty. We are therefore concerned with the converse. Let $\ell \geq 0$. We denote by a the number of $i < \ell$ such that $w_{i+1} - w_i = q_\lambda$ and by b the number of $i < \ell$ such that $w_{i+1} - w_i = q_{\lambda-1}$.

Choose ε_r such that $|\varepsilon_r| = 1$ and

$$\varepsilon_r \sum_{0\leq h<q_\lambda} \big|G_\lambda(h)\big|^2 \operatorname{e}\big(hrq_\lambda^{-1}\big)$$

is a nonnegative real number. Similarly choose ε'_r for $\lambda - 1$. We have

$$
\begin{aligned}
&\frac{1}{R}\sum_{0\le r<R}\left|\frac{1}{w_\ell}\sum_{0\le n<w_\ell} g_\lambda(n+r)\overline{g_\lambda(n)}\right| \\
&= \left|\frac{aq_\lambda}{w_\ell}\frac{1}{R}\sum_{0\le r<R}\varepsilon_r\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^2 \mathrm{e}\left(\frac{hr}{q_\lambda}\right)\right. \\
&\qquad \left. + \frac{bq_{\lambda-1}}{w_\ell}\frac{1}{R}\sum_{0\le r<R}\varepsilon'_r\sum_{0\le h<q_{\lambda-1}}\left|G_{\lambda-1}(h)\right|^2 \mathrm{e}\left(\frac{hr}{q_{\lambda-1}}\right)\right| + O\left(\frac{ar}{w_\ell}+\frac{br}{w_\ell}\right) \\
&= \frac{1}{R}\left|\frac{aq_\lambda}{w_\ell}\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^2\sum_{0\le r<R}\varepsilon_r e\left(\frac{hr}{q_\lambda}\right)\right. \\
&\qquad \left. + \frac{bq_{\lambda-1}}{w_\ell}\sum_{0\le h<q_{\lambda-1}}\left|G_{\lambda-1}(h)\right|^2\sum_{0\le r<R}\varepsilon'_r e\left(\frac{hr}{q_{\lambda-1}}\right)\right| + O\left(\frac{r}{q_{\lambda-1}}\right) \\
&\le \frac{1}{R}\left|\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^2\sum_{0\le r<R}\varepsilon_r e\left(\frac{hr}{q_\lambda}\right)\right| \\
&\qquad + \frac{1}{R}\left|\sum_{0\le h<q_{\lambda-1}}\left|G_{\lambda-1}(h)\right|^2\sum_{0\le r<R}\varepsilon'_r e\left(\frac{hr}{q_{\lambda-1}}\right)\right| + O\left(\frac{r}{q_{\lambda-1}}\right).
\end{aligned}
$$

By Cauchy-Schwarz we obtain

$$
\begin{aligned}
&\frac{1}{R^2}\left|\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^2\sum_{0\le r<R}\varepsilon_r e\left(\frac{hr}{q_\lambda}\right)\right|^2 \\
&\qquad\qquad \le \frac{1}{R^2}\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^4\sum_{0\le h<q_\lambda}\left|\sum_{0\le r<R}\varepsilon_r\,\mathrm{e}\left(\frac{hr}{q_\lambda}\right)\right|^2 \\
&\qquad\qquad \le \frac{1}{R^2}\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^4\sum_{0\le h<q_\lambda}\sum_{0\le r_1,r_2<R}\varepsilon_{r_1}\overline{\varepsilon_{r_2}}\,\mathrm{e}\left(h\frac{r_1-r_2}{q_\lambda}\right) \\
&\qquad\qquad = \frac{q_\lambda}{R^2}\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^4\sum_{0\le r_1,r_2<R}\varepsilon_{r_1}\overline{\varepsilon_{r_2}}\delta_{r_1,r_2} \\
&\qquad\qquad = \frac{q_\lambda}{R}\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^4,
\end{aligned}
$$

similarly for $\lambda - 1$. Using Lemma 2.6, we get

$$\begin{aligned}
&\frac{1}{R}\sum_{0\le r<R}|\gamma_r|\\
&= \lim_{\ell\to\infty}\frac{1}{R}\sum_{0\le r<R}\left|\frac{1}{w_\ell}\sum_{0\le n<w_\ell} g(n+r)\overline{g(n)}\right|\\
&= \lim_{k\to\infty}\frac{1}{R}\sum_{0\le r<R}\left|\frac{1}{w_\ell}\sum_{0\le n<w_\ell} g_\lambda(n+r)\overline{g_\lambda(n)}\right| + O\Big(\frac{R}{q_{\lambda-1}}\Big)\\
&\le \left[\Bigg(\sum_{0\le h<q_{\lambda-1}}\left|G_{\lambda-1}(h)\right|^4\Bigg)^{1/2} + \Bigg(\sum_{0\le h<q_\lambda}\left|G_\lambda(h)\right|^4\Bigg)^{1/2}\right]\Big(\frac{q_\lambda}{R}\Big)^{1/2} + O\Big(\frac{R}{q_{\lambda-1}}\Big).
\end{aligned}$$

Using the hypothesis and Proposition 2.10, we get $\sup_{h\in\mathbb{Z}}\left|G_\lambda(h)\right| = o(1)$ as $\lambda\to\infty$. By Parseval's identity this implies

$$\sum_{h<q_\lambda}|G_\lambda(h)|^4 = o(1).$$

By a straightforward argument we conclude that

$$\frac{1}{R}\sum_{0\le r<R}\left|\gamma_r\right| = o(1)$$

as $R\to\infty$. Since g is bounded, an application of Lemma 2.2 completes the proof of Theorem 1.1.

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Lukas Spiegelhofer
Institute of Discrete Mathematics and Geometry,
Vienna University of Technology
Wiedner Hauptstrasse 8–10
1040 Vienna, Austria
E-mail: lukas.spiegelhofer@tuwien.ac.at
URL: http://dmg.tuwien.ac.at/spiegelhofer/