Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 30, nº 1 (2018), p. 195-202.

<http://jtnb.cedram.org/item?id=JTNB_2018__30_1_195_0>

Journal de Théorie des Nombres
de Bordeaux **30** (2018), 195–202

Note on the Stern-Brocot sequence, some relatives, and their generating power series

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Résumé. Trois variantes de la suite de Stern-Brocot sont liées à la célèbre suite de Thue-Morse. Dans la présente note, les fonctions génératrices de ces quatre suites sont considérées. Tandis que l'une d'entre elles est connue comme étant rationnelle, l'indépendance algébrique sur $\mathbb{C}(z)$ des trois autres est démontrée ici. Puis, ce théorème est généralisé de sorte que les fonctions $\Phi(z), \Phi(-z), \Psi(z), \Psi(z^2)$ sont aussi considérées, où Φ et Ψ indiquent les fonctions génératrices des suites de Rudin-Shapiro et de Baum-Sweet, respectivement. Quelques applications arithmétiques sont également données.

Abstract. Three variations on the Stern-Brocot sequence are related to the celebrated Thue-Morse sequence. In the present note, the generating power series of these four sequences are considered. Whereas one of these was known to define a rational function, the other three are proved here to be algebraically independent over $\mathbb{C}(z)$. Then this statement is fairly generalized by including the functions $\Phi(z), \Phi(-z), \Psi(z), \Psi(z^2)$, where Φ and Ψ denote the generating power series of the Rudin-Shapiro and Baum-Sweet sequence, respectively. Moreover, some arithmetical applications are given.

1. Introduction and first results

In their Remark 1.3, Allouche and Mendès France [1] defined a map $\mathcal{C}$ associating with two sequences $\mathbf{a} = (a_n)_{n\geq 0}$ and $\mathbf{b} = (b_n)_{n\geq 0}$ the sequence

$$\mathcal{C}(\mathbf{a}, \mathbf{b}) := \left(\sum_{0\leq 2\nu \leq n} \binom{n-\nu}{\nu}_2 a_\nu b_{n-\nu} \right)_{n\geq 0}.$$

Here, for integers u, v with $v \geq 0$, the symbol $\binom{u}{v}_2$ equals to 0 or 1 depending on whether the corresponding binomial coefficient $\binom{u}{v}$ is even or odd. In particular, $\mathcal{C}(\mathbf{1}, \mathbf{1})$ is the Stern-Brocot sequence ($\mathbf{1}$ the obvious constant

Manuscrit reçu le 25 avril 2016, révisé le 4 septembre 2016, accepté le 12 septembre 2016.
2010 *Mathematics Subject Classification.* 11J81, 11J91, 11B37.
Mots-clefs. Stern-Brocot sequence, transcendence, algebraic independence, Mahler's method.

sequence), which is identical with the sequence $(a_{n+1})_{n\geq 0}$ considered in [2, (1)]. The authors of [1] recalled that the ± 1 Thue-Morse sequence $\mathbf{t} = (t_n)_{n\geq 0}$ is given by $t_0 = 1$ and, for all $n \geq 0$, by $t_{2n} = t_n$ and $t_{2n+1} = -t_n$. Then they introduced the sequences $\alpha = (\alpha_n)_{n\geq 0}, \beta = (\beta_n)_{n\geq 0}, \gamma = (\gamma_n)_{n\geq 0}$ by

$$\alpha := \mathcal{C}(\mathbf{t}, \mathbf{1}),\ \beta := \mathcal{C}(\mathbf{1}, \mathbf{t}),\ \gamma := \mathcal{C}(\mathbf{t}, \mathbf{t})$$

to which $\sigma := \mathcal{C}(\mathbf{1}, \mathbf{1})$ is added, σ for **S**tern, and noted that these four sequences satisfy the recurrences

$$\begin{array}{lllll}
\sigma_0 = 1, & \sigma_1 = 1, & \sigma_{2n} = \sigma_n + \sigma_{n-1}, & \sigma_{2n+1} = \sigma_n & \text{for any } n \geq 1, \\
\alpha_0 = 1, & \alpha_1 = 1, & \alpha_{2n} = \alpha_n - \alpha_{n-1}, & \alpha_{2n+1} = \alpha_n & \text{for any } n \geq 1, \\
\beta_0 = 1, & \beta_1 = -1, & \beta_{2n} = \beta_n - \beta_{n-1}, & \beta_{2n+1} = -\beta_n & \text{for any } n \geq 1, \\
\gamma_0 = 1, & \gamma_1 = -1, & \gamma_{2n} = \gamma_n + \gamma_{n-1}, & \gamma_{2n+1} = -\gamma_n & \text{for any } n \geq 1.
\end{array}$$

Let τ be any of these sequences $\sigma, \alpha, \beta, \gamma$ and write $\tau = (\tau_n)_{n\geq 0}$ with $\tau_0 = 1$. Then one concludes inductively $\tau_{2^m-1} = \tau_1^m$ for any $m = 0, 1, \ldots$ and, moreover, $|\tau_n| \leq n$ for $n = 1, 2, \ldots$ Thus, the generating power series

$$\Gamma_\tau(z) := \sum_{n=0}^{\infty} \tau_n z^n \in \mathbb{Z}[\![z]\!]$$

of τ has convergence radius 1. In this situation, a classical result of Carlson [5] says that Γ_τ either represents a rational function or cannot be analytically continued beyond the unit circle.

At the beginning of Section 2, for each of the functions $\Gamma_\tau, \tau \in \{\sigma, \alpha, \beta, \gamma\}$, a functional equation of so-called Mahler-type

$$\Gamma_\tau(z) = p_\tau(z)\Gamma_\tau(z^2) \tag{1.1}$$

will be specified with a polynomial $p_\tau(z)$ of degree 2 satisfying $p_\tau(0) = 1$ and the other coefficients being 1 or -1. Although these equations look rather similar, the analytic behaviour of the four functions is quite different. While $\Gamma_\gamma(z)$ turns out to be rational, the other three are hypertranscendental, hence transcendental over $\mathbb{C}(z)$ and therefore, by Carlson's above-mentioned theorem, have the unit circle as natural boundary. Remember here that an analytic function is called hypertranscendental if it satisfies no algebraic differential equation, that is, no finite collection of derivatives of the function is algebraically dependent over $\mathbb{C}(z)$. By the way, the hypertranscendence of Γ_σ was proved in [2, Theorem 1]. In the case of Γ_α, the hypertranscendence is shown in the proof of [3, Corollary 3], and the case of Γ_β is similar. All these results are consequences of the general hypertranscendence criterion for infinite products $F(z) = \prod_{j=0}^{\infty} P(z^{2^j})$ with a polynomial $P(z)$, see [3, Theorem 1], namely $\Gamma_\tau(0) = 1$ and iteration of (1.1) immediately leads to the product representation

$$\Gamma_\tau(z) = \prod_{j=0}^{\infty} p_\tau(z^{2^j}) \tag{1.2}$$

of $\Gamma_\tau(z)$ valid in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Note also that in [4] we proved the algebraic independence of $\Gamma_\sigma(z)$ and its twist.

The sole transcendence over $\mathbb{C}(z)$ of each of the functions $\Gamma_\sigma, \Gamma_\alpha, \Gamma_\beta$ will now be generalized in a different direction. Namely, our first main result reads as follows.

Theorem 1.1. *The functions* $\Gamma_\sigma(z), \Gamma_\alpha(z), \Gamma_\beta(z)$ *are algebraically independent over* $\mathbb{C}(z)$.

Using [7, Corollary 2] or [8, Theorem 4.2.1] we deduce from Theorem 1.1 the following arithmetical consequence.

Corollary 1.2. *If* ξ *is an algebraic number with* $0 < |\xi| < 1$ *such that every* ξ^{2^j} $(j = 0, 1, \dots)$ *is different from* $\pm(1-\sqrt{5})/2$, *then* $\Gamma_\sigma(\xi), \Gamma_\alpha(\xi), \Gamma_\beta(\xi)$ *are algebraically independent over* $\mathbb{Q}$.

Note that here the exceptional conditions on ξ are necessary. Namely, if ξ^{2^j} equals to $(1-\sqrt{5})/2$ or to $(\sqrt{5}-1)/2$ for some $j \geq 0$, then the product representation (1.2) of Γ_α or Γ_β implies $\Gamma_\alpha(\xi) = 0$ or $\Gamma_\beta(\xi) = 0$, respectively.

The question of what happens in the direction of Corollary 1.2 for transcendental ξ can be answered as follows using [11, Théorème 4].

Corollary 1.3. *For transcendental* $\xi \in \mathbb{D}$, *one has the estimate*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\xi, \Gamma_\sigma(\xi), \Gamma_\alpha(\xi), \Gamma_\beta(\xi)) \geq 3$$

and this is best possible.

2. Proof of Theorem 1.1

To begin with, we use the recurrences for $\tau \in \{\sigma, \alpha, \beta, \gamma\}$ as given above to easily demonstrate the functional equation (1.1) for Γ_τ with

$$p_\sigma(z) = 1 + z + z^2, \tag{2.1}$$

$$p_\alpha(z) = 1 + z - z^2, \tag{2.2}$$

$$p_\beta(z) = 1 - z - z^2 \ \ (= p_\alpha(-z)), \tag{2.3}$$

$$p_\gamma(z) = 1 - z + z^2 \ \ (= p_\sigma(-z)). \tag{2.4}$$

Note that the first case, $\Gamma_\sigma(z) = (1 + z + z^2)\Gamma_\sigma(z^2)$, was explicitly checked in [2, (7)].

To realize the rationality of $\Gamma_\gamma(z)$, we simply note $p_\gamma(z)p_\sigma(z) = p_\sigma(z^2)$ and $p_\sigma(z) \neq 0$ in $\mathbb{D}$, by (2.1) and (2.4). Using (1.2) with $\tau = \gamma$ we obtain

$$\Gamma_\gamma(z) = \frac{1}{p_\sigma(z)} = \frac{1-z}{1-z^3} = (1-z)\sum_{m\geq 0} z^{3m},$$

and this even shows that γ_n equals $1, -1, 0$ if $n \equiv 0, 1, 2 \pmod 3$, respectively.

The main tool in our proof of Theorem 1.1 is the subsequent criterion of Kubota [6, Proposition 3] (see also [8, Theorem 3.5]) to be quoted here in a very particular version, which suffices for our purposes.

Lemma 2.1. *Let $d \geq 2$ be an integer. Suppose that the series $f_1, \dots, f_k \in \mathbb{C}[\![z]\!] \setminus \{0\}$ converge on $\mathbb{D}$ and satisfy the functional equations*

$$f_i(z^d) = b_i(z) f_i(z) \qquad (i = 1, \dots, k)$$

with all $b_i \in \mathbb{C}(z) \setminus \{0\}$ fulfilling the condition that, for no $(n_1, \dots, n_k) \in \mathbb{Z}^k \setminus \{\underline{0}\}$, the functional equation

$$r(z^d) = r(z)\prod_{i=1}^{k} b_i(z)^{n_i}$$

has a solution $r \in \mathbb{C}(z) \setminus \{0\}$. Then the functions $f_1, \dots, f_k$ are algebraically independent over $\mathbb{C}(z)$.

We apply Lemma 2.1 with $d = 2, k = 3$ and $\Gamma_\sigma, \Gamma_\alpha, \Gamma_\beta$ for the f_1, f_2, f_3. According to their functional equations (1.1) with $p_\tau(z)$ as given by (2.1)–(2.3), we have $b_1(z) = 1/p_\sigma(z), b_2(z) = 1/p_\alpha(z), b_3(z) = 1/p_\beta(z)$ and we must show, for any $(n_\sigma, n_\alpha, n_\beta) \in \mathbb{Z}^3 \setminus \{\underline{0}\}$, that the equation

$$r(z) = r(z^2)p_\sigma(z)^{n_\sigma} p_\alpha(z)^{n_\alpha} p_\beta(z)^{n_\beta} \tag{2.5}$$

has no non-trivial rational solution r.

To consider first the case $n_\sigma \neq 0, n_\alpha = n_\beta = 0$, we iterate (2.5) j times and obtain

$$r(z) = r(z^{2^j})\left(\prod_{i=0}^{j-1} p_\sigma(z^{2^i})\right)^{n_\sigma}.$$

Letting $j \to \infty$ and noting $r(0) \neq 0, \infty$, this formula leads to $r(z) = r(0)\Gamma_\sigma(z)^{n_\sigma}$, by (2.5) with $\tau = \sigma$, a contradiction. Namely, for rational $r(z)$, the function $\Gamma_\sigma(z)$ would be algebraic.

To consider the second case $(n_\alpha, n_\beta) \neq (0,0)$ of (2.5), we apply Lemma 2.2 below with $d = 2$ and

$$\begin{aligned} R(z) &:= p_\sigma(z)^{n_\sigma} p_\alpha(z)^{n_\alpha} p_\beta(z)^{n_\beta} \\ &= \left((\zeta - z)\left(\frac{1}{\zeta} - z\right)\right)^{n_\sigma}\left((G-z)\left(\frac{1}{G}+z\right)\right)^{n_\alpha}\left((G+z)\left(\frac{1}{G}-z\right)\right)^{n_\beta} \end{aligned} \tag{2.6}$$

being the factor appearing in the right-hand side of (2.5). Here $\zeta := e^{2\pi i/3}$ and $G := (1+\sqrt{5})/2$ denotes the golden ratio. Clearly, we may choose $\Omega = G, \omega = -1/G$ if $n_\alpha \neq 0$, or $\Omega = -G, \omega = 1/G$ if $n_\alpha = 0$ but $n_\beta \neq 0$. To examine the last hypothesis of Lemma 2.2, let $j \geq 1$. All solutions of $z^{2^j} = \pm 1/G$ (or of $z^{2^j} = \pm G$) satisfy $|z| = G^{-1/2^j}$ (or $|z| = G^{1/2^j}$), and no such z can be a zero or pole of $R(z^{2^{j-1}}) \dots R(z)$. Namely otherwise, according to the right-hand side of (2.6), zeros or poles of $R(z^{2^k})$ at z inside or outside of $\mathbb{D}$ satisfy $z^{2^k} = \pm 1/G$ (or $z^{2^k} = \pm G$) with some $k \in \{0, \dots, j-1\}$ which is impossible. □

Lemma 2.2. *Let $d \geq 2$ be an integer. Let $R \in \mathbb{C}(z)$ have zeros or poles in $0 < |z| < 1$ or in $1 < |z| < +\infty$, where ω and Ω are zeros or poles of minimal and of maximal absolute value, respectively. Assume that, for each integer $j \geq 1$, not all solutions of $z^{d^j} = \omega$ (or of $z^{d^j} = \Omega$) are zeros or poles of the product $R(z^{d^{j-1}}) \dots R(z)$. Then the functional equation*

$$r(z) = r(z^d) R(z) \tag{2.7}$$

has no rational solution $r \neq 0$.

Proof. Note first that $0 < |\omega| < 1$ or $|\Omega| > 1$ holds by hypothesis. Then suppose, on the contrary, that (2.7) has a rational solution $r(z) \neq 0$. Clearly, this $r(z)$ has no zero or pole in $|z| < |\omega|$ (or in $|z| > |\Omega|$). Since $|\omega^d| < |\omega|$ (or $|\Omega^d| > |\Omega|$) we know that ω^d (or Ω^d) is not a zero or pole of $r(z)$, whence ω (or Ω) is a zero or pole of $r(z)$, by (2.7) and the hypothesis on R.

On iterating (2.7) we conclude that r satisfies

$$r(z) = r(z^{d^j}) R(z^{d^{j-1}}) \dots R(z)$$

for any $j = 1, 2, \dots$. For any such j, we consider all d^j solutions of $z^{d^j} = \omega$ (or of $z^{d^j} = \Omega$), among which at least one is not a zero or pole of $R(z^{d^{j-1}}) \dots R(z)$. Thus, on every circle $|z| = |\omega|^{1/d^j}$ (or $|z| = |\Omega|^{1/d^j}$), we have a zero or a pole of r contradicting the rationality of r. □

3. Interactions of the Γ_τ's with other generating power series

Beside the Γ_τ's studied before, we recall two more integer sequences and their generating power series. First, let $\varphi = (\varphi_n)_{n \geq 0}$ denote the Rudin-Shapiro sequence, which can be recursively defined by $\varphi_0 = 1$, and $\varphi_{2n} = \varphi_n$, $\varphi_{2n+1} = (-1)^n \varphi_n$ for any $n \geq 0$. Its generating power series

$$\Phi(z) := \sum_{n=0}^{\infty} \varphi_n z^n$$

satisfies a system of two functional equations of Mahler-type, in matrix notation written as

$$\begin{pmatrix}\Phi(z)\\ \Phi(-z)\end{pmatrix} = \begin{pmatrix}1 & z\\ 1 & -z\end{pmatrix}\begin{pmatrix}\Phi(z^2)\\ \Phi(-z^2)\end{pmatrix} \tag{3.1}$$

(see [7]). Secondly, let $\psi = (\psi_n)_{n\geq 0}$ denote the Baum-Sweet sequence, which is given by $\psi_0 = 1, \psi_n = 1$ if the binary representation of $n \geq 1$ contains no block of consecutive 0's of odd length, and $\psi_n = 0$ otherwise. The corresponding generating power series

$$\Psi(z) := \sum_{n=0}^{\infty} \psi_n z^n$$

fulfills a system of functional equations similar to (3.1) (see [7]), namely

$$\begin{pmatrix}\Psi(z)\\ \Psi(z^2)\end{pmatrix} = \begin{pmatrix}z & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi(z^2)\\ \Psi(z^4)\end{pmatrix}. \tag{3.2}$$

K. Nishioka [7] (see also [8, pp. 158–160]) proved that both pairs $\{\Phi(z), \Phi(-z)\}$ and $\{\Psi(z), \Psi(z^2)\}$ consist of functions which are algebraically independent over $\mathbb{C}(z)$. Using this and one of her results cited before our Corollary 1.2, she obtained also the algebraic independence over $\mathbb{Q}$ of the numbers $\Phi(\xi), \Phi(-\xi)$ (and of $\Psi(\xi), \Psi(\xi^2)$) for any non-zero algebraic $\xi \in \mathbb{D}$.

In a recent paper, K. Nishioka and S. Nishioka [9] proved much stronger results on Φ and Ψ. Namely, they obtained the algebraic independence of all four functions $\Phi(z), \Phi(-z), \Psi(z), \Psi(z^2)$ not only over $\mathbb{C}(z)$ but over any *difference field extension of valuation ring type* over $\mathbb{C}(z)$ under the transformation $z \mapsto z^2$ (compare [9, Theorem 9]). This notion, too combersome to be fully quoted here, has been introduced by S. Nishioka in his deep work [10, Definition 1] on solvability of certain classes of difference equations. His Theorem 2 and Proposition 5 are the key tools of the proofs in [9].

The Nishiokas' just-quoted algebraic independence result implies the following, where we restrict ourselves to homogeneous systems of functional equations.

Lemma 3.1. *The four functions $\Phi(z), \Phi(-z), \Psi(z), \Psi(z^2)$ and power series $f_1(z), \ldots, f_k(z) \in \mathbb{C}[\![z]\!]$ satisfying a system*

$$^{\top}(f_1(z), \ldots, f_k(z)) = \mathcal{A}(z) \cdot {}^{\top}(f_1(z^2), \ldots, f_k(z^2)) \tag{3.3}$$

of functional equations with triangular $\mathcal{A} \in GL_k(\mathbb{C}(z))$, $^{\top}$ denoting matrix transposition, are algebraically independent over $\mathbb{C}(z)$ if this already applies to $f_1(z), \ldots, f_k(z)$.

Using this with $k = 3$, $\mathcal{A}(z) = \mathrm{diag}(p_\sigma(z), p_\alpha(z), p_\beta(z))$, $f_1 = \Gamma_\sigma$, $f_2 = \Gamma_\alpha, f_3 = \Gamma_\beta$, (see (2.1)–(2.3)), Theorem 1.1 leads us to the following.

Theorem 3.2. *The seven functions $\Phi(z), \Phi(-z), \Psi(z), \Psi(z^2), \Gamma_\sigma(z), \Gamma_\alpha(z), \Gamma_\beta(z)$ are algebraically independent over $\mathbb{C}(z)$. In addition, if $\xi \in \mathbb{D}$ is a non-zero algebraic number such that every $\xi^{2^j} (j = 0, 1, \dots)$ is different from $\pm(1 - \sqrt{5})/2$, then $\Phi(\xi), \Phi(-\xi), \Psi(\xi), \Psi(\xi^2), \Gamma_\sigma(\xi), \Gamma_\alpha(\xi), \Gamma_\beta(\xi)$ are algebraically independent over $\mathbb{Q}$.*

Note that the arithmetical part of this assertion is again an application of Nishioka's algebraic independence criterion cited before Corollary 1.2.

The hypertranscendence of the Γ_τ's for $\tau \in \{\sigma, \alpha, \beta\}$ we mentioned earlier can also be combined with Lemma 3.1 to get the following consequence.

Theorem 3.3. *For fixed $\tau \in \{\sigma, \alpha, \beta\}$, the functions $\Phi(z)$, $\Phi(-z)$, $\Psi(z)$, $\Psi(z^2)$, $\Gamma_\tau(z)$, $\Gamma'_\tau(z), \Gamma''_\tau(z), \dots$ are algebraically independent over $\mathbb{C}(z)$. Moreover, if $\xi \in \mathbb{D}$ is a non-zero algebraic number, then the values $\Phi(\xi)$, $\Phi(-\xi)$, $\Psi(\xi)$, $\Psi(\xi^2)$, $\Gamma_\tau(\xi), \Gamma'_\tau(\xi), \dots$ are algebraically independent over $\mathbb{Q}$ under the additional proviso that all ξ^{2^j} $(j = 0, 1, \dots)$ are different from $(1 - \sqrt{5})/2$ in case $\tau = \alpha$, and from $(\sqrt{5} - 1)/2$ in case $\tau = \beta$.*

Sketch of proof. We apply Lemma 3.1 with $f_1(z) = \Gamma_\tau(z), \dots, f_k(z) = \Gamma_\tau^{(k-1)}(z)$ for arbitrary but fixed integer $k \geq 1$. By taking ith derivatives $(i = 0, \dots, k-1)$ of the functional equation (1.1) with corresponding (2.1), (2.2), or (2.3), we obtain the validity of (3.3) with lower triangular matrix $\mathcal{A}(z)$ having $(2z)^i p_\tau(z)$, $i = 0, \dots, k-1$, on its main diagonal. □

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