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Heights and representations of split tori

par Valerio TALAMANCA

Résumé. Soit $\mathbb{G}_m^d$ le tore déployé de dimension d, défini sur un corps de nombres k. À chaque $\mathbb{G}_m^d$-module E nous associons une fonction hauteur h_E définie en utilisant la hauteur spectrale sur $\mathrm{GL}(E)$. Cela donne lieu à un accouplement de hauteur entre le monoïde des $\mathbb{G}_m^d$-modules irréductible de $\mathbb{G}_m^d$ et le groupe $\mathbb{G}_m^d(\overline{k})$. Nos résultats principaux sont une caractérisation de ces $\mathbb{G}_m^d$-modules E pour lequel h_E satisfait le théorème de finitude de Northcott, la détermination des noyaux des accouplements de hauteur, ainsi que, pour quelques classes de $\mathbb{G}_m^d$-modules E, le calcul du groupe des automorphismes qui préservent h_E.

Abstract. Let $\mathbb{G}_m^d$ denote the d-dimensional split torus defined over a number field k. To each $\mathbb{G}_m^d$-module E we associate a height function h_E defined by means of the spectral height on $\mathrm{GL}(E)$. This gives rise to a height pairing between the monoid of irreducible $\mathbb{G}_m^d$-modules of $\mathbb{G}_m^d$ and the group $\mathbb{G}_m^d(\overline{k})$. Our main results are a characterization of those $\mathbb{G}_m^d$-modules E for which h_E satisfeis Northcott's finiteness theorem, the determination of the kernels of the height pairing, as well as, for a few special classes of $\mathbb{G}_m^d$-modules, of the group of automorphisms that preserve h_E.

Introduction

Let A be an abelian variety defined over a number field k and $\widehat{A}$ be its dual. Let $\overline{k}$ be a fixed algebraic closure of k. In [7] Néron constructed a height pairing $\langle\,\cdot\,,\cdot\,\rangle_N$ between $A(\overline{k})$ and $\widehat{A}(\overline{k})$ and used it to define height functions attached to divisors on A. Moreover, by means of properties of the height pairing, Néron established the quadraticity of height functions associated to symmetric ample divisors.

The present work stems from the attempt to construct a height pairing for algebraic tori. It extends and refines results obtained by the author in his Ph. D. thesis ([11]) many years ago. In this paper we treat only the case

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of k-split tori, even though part of the construction can be carried over to affine algebraic groups.

Let $\mathbb{G}_m^d$ denote the standard k-split torus. The role of $\widehat{A}$ is played by $\mathrm{Rep}(\mathbb{G}_m^d)$, the monoid of isomorphism classes of $\mathbb{G}_m^d$-modules. The construction of the height pairing is done by attaching to each $\mathbb{G}_m^d$-module E a height function h_E which only depends on the isomorphism class of E. We define h_E as follows: let $\rho_E : \mathbb{G}_m^d \to \mathrm{GL}(E)$ denote the homomorphism associated to the $\mathbb{G}_m^d$-module E. We set $h_E := h_s \circ \rho_E$ where h_s denotes the spectral height on $\mathrm{GL}(E)$. For convenience of the reader we recall the definition and the main properties of the spectral height in Section 1; here we just want to stress that the spectral height is invariant under conjugation, and so h_E only depends on the isomorphism class of E. We thus obtain a height pairing between $\mathrm{Rep}(\mathbb{G}_m^d)$ and $\mathbb{G}_m^d(\overline{k})$, whose main properties are listed in Proposition 1.4. In particular we explicitly determine the kernels of the pairing. One main feature that is typically expected to be possessed by a height function is the finiteness of the set of points of bounded height. Clearly not all heights associated to $\mathbb{G}_m^d$-modules can enjoy this property, but in Theorem 1.3 we determine those who do. Let us go back for a moment to the abelian variety setting. Let $\mathcal{L}$ be a symmetric line bundle on A. Consider the associated isogeny $\varphi_{\mathcal{L}} : A \to \hat{A}$, mapping a to $\mathrm{t}_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where t_a is the translation by a map on A. Then one defines the height pairing on $A(\overline{k})$ associated to $\mathcal{L}$ by setting $\langle a, b\rangle_{\mathcal{L}} := \langle a, \varphi_{\mathcal{L}}(b)\rangle_N$. It was proven in [12, Corollary of §2] that an endomorphism $f : A \to A$ preserves the height pairing $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{L}}$ if and only if $\varphi_{\mathcal{L}} = \hat{f} \circ \varphi_{\mathcal{L}} \circ f$.

A possibile analogue in the case of k-split tori concerns the determination of $\mathcal{H}_E$, the *group of symmetries* of h_E, which is defined as:

$$\mathcal{H}_E := \left\{ \varphi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, h_E(\varphi(g)) = h_E(g), \forall\, g \in \mathbb{G}_m^d(k) \right\}.$$

Ideally one would like to pin down $\mathcal{H}_E$ for every $\mathbb{G}_m^d$-module E, but at the present time this does not seem to be achievable. In this paper we compute $\mathcal{H}_E$ in a few basic examples. Firstly we determine $\mathcal{H}_E$ for all E such that $\rho_E(\mathbb{G}_m^d)$ is a maximal torus of $\mathrm{GL}(E)$. In particular we prove that if $E = k^d$ endowed with the standard action of $\mathbb{G}_m^d$ as diagonal matrices then $\mathcal{H}_E$ is the group consisting of *twisted permutation matrices*, see Section 2 for definitions. Next suppose ρ_E gives the standard realization of $\mathbb{G}_m^d$ as maximal torus of diagonal matrices of one of the following classical groups SL_{d+1}, Sp_{2d}, SO_{2d+1}, and SO_{2d} (cf. Section 3 for the definitions). Let $\mathrm{N}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d))$ (respectively $\mathrm{C}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d)))$ denote the normalizer (respectively the centralizer) of $\rho_E(\mathbb{G}_m^d)$ in $\mathrm{GL}(E)$. The Weyl group of $\mathrm{GL}(E)$ relative to $\rho_E(\mathbb{G}_m^d))$ is (cf. [5, 24.1])

$$\mathcal{W}(\rho_E(\mathbb{G}_m^d), \mathrm{GL}(E)) = \mathrm{N}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d))/\, \mathrm{C}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d))$$

and acts on $\rho_E(\mathbb{G}_m^d)$ by conjugation, and so leaves invariant the spectral height. It follows that the pull back of the relative Weyl group to $\mathrm{Aut}(\mathbb{G}_m^d)$ is contained in $\mathcal{H}_E$. Set

$$\mathcal{W}_E = \left\{ \varphi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, \varphi = \rho_E^{-1} \circ \omega \circ \rho_E, \text{ for some } \omega \in \mathcal{W}\big(\rho_E(\mathbb{G}_m^d), \mathrm{GL}(E)\big) \right\}$$

Our main result on symmetries can be stated as follows: (cf. Theorem 3.2 for the precise statement)

Theorem. *Let E be one of the above $\mathbb{G}_m^d$-modules. Then $\mathcal{H}_E = \mathcal{W}_E$.*

As the referee pointed out, our h_E is a special case of a height function on toric varieties introduced by V. Malliot in [6] and later generalized by J. I. Burgos Gil, P. Philippon, and M. Sombra in [3]. This is evident if one compares our expression for h_E given in formula (1.2), with the formula computing Malliot's height [6, Corollary 8.3.2]. Our approach is rather more elementary than the sophisticated Arakelov geometry techniques employed by Maillot and later by Burgos Gil, Philippon, and M. Sombra. Also the questions addressed here are of a different nature than those taken up in [6] and [3] as we are mostly interested in computing the group $\mathcal{H}_E$.

The paper is organized as follows: in Section 1, after recalling some facts about the spectral height, we set up our notation about representations of $\mathbb{G}_m^d$ and we prove a characterization of the $\mathbb{G}_m^d$-modules whose associated height satisfies Northcott's property. The section is ended by the construction of the height pairing between $\mathrm{Rep}(\mathbb{G}_m^d)$ and $\mathbb{G}_m^d(\overline{k})$. Sections 2 and 3 are devoted to the computation of $\mathcal{H}_E$ in the particular cases mention above.

Notations and conventions. Let k be a number field. We denote by $\mathcal{M}_k$ the set of places of k. We normalize absolute values as follows: if v is archimedean we require that $|\cdot|_v$ restricted to $\mathbb{Q}$ is the standard archimedean absolute value, while if v is a finite place, say $v|p$, then we require that $|p|_v = p^{-1}$. Let k_v be the completion of k with respect to $|\cdot|_v$. We denote by n_v the local degree, i.e. $n_v = [k_v : \mathbb{Q}_p]$, and set $d_v = n_v/d$, where d is the degree of k over $\mathbb{Q}$. With this normalization the product formula reads $\prod_{v\in\mathcal{M}_k} |\lambda|_v^{n_v} = 1$.

1. Heights and representations

In this section we describe in detail the construction of the height associated to a $\mathbb{G}_m^d$-module and prove some basic results about it. Let us start by recalling the definition of the Northcott–Weil height. Let $\overline{k}$ be an algebraic closure of k which we fix once and for all. The *(absolute logarithmic) Northcott–Weil* height on $\overline{k}^n$, is defined by setting $h_{NW}(0, \dots, 0) = 0$ and

$$h_{NW}(x_1, \dots, x_n) = \sum_{v\in\mathcal{M}_l} d_v \log(\max\{|x_1|_v, \dots, |x_n|_v\})$$

where $(0,\dots,0) \neq (x_1,\dots,x_n) \in \overline{k}^n$ and $l \supset k$ is any number field containing all the x_i's. Our normalization of absolute values implies that h_{NW} does not depend on the choice of the field l (see e.g. [10, Chapter VIII, Proposition 5.4]). By the product formula it descends to a function, which also is denoted by h_{NW}, on $\mathbb{P}_n(\overline{k})$.

Next we recall the definition of the spectral height of linear operators on a k-vector space. Let E be a finite dimensional k-vector space. Given $T \in \mathrm{GL}(E)$ and $v \in \mathcal{M}_k$ we also denote by T, by an abuse of notation, the linear transformation induced by T on $E_v := E \otimes_k k_v$. The *v-adic spectral radius of T* is

$$\rho_v(T) = \sup_{\lambda \in \mathrm{sp}(T)} |\lambda|_{k_v(\lambda)},$$

where $\mathrm{sp}(T)$ denotes the set of characteristic roots of T in an algebraic closure of k_v, and $|\cdot|_{k_v(\lambda)}$ is the unique extension of $|\cdot|_v$ to $k_v(\lambda)$. The *(logarithmic) spectral height*[1] of T is defined as

$$h_s(T) = \sum_{v \in \mathcal{M}_k} d_v \log(\rho_v(T)),$$

for T not nilpotent. If T is nilpotent we set $h_s(T) = 0$. Let us recall the properties of spectral height that we need in the following [13, Section 2]:

- (S1) h_s is invariant under field extensions.
- (S2) $h_s(T) \geq 0$.
- (S3) $h_s(T^m) = m h_s(T)$.
- (S4) If T and T' commute, then $h_s(TT') \leq h_s(T) + h_s(T')$.
- (S5) h_s is invariant under conjugation.
- (S6) If F is a k-vector space and $S \in \mathrm{GL}(F)$, then $h_s(T \otimes S) = h_s(T) + h_s(S)$
- (S7) If $\lambda \in k$, then $h_s(\lambda T) = h_s(T)$.
- (S8) Let $\mathrm{sp}(T) = \{\lambda_1,\dots,\lambda_r\}$, set $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_r) \in \overline{k}^r$, then $h_s(T) = h_{NW}(\boldsymbol{\lambda})$.

By a $\mathbb{G}_m^d$*-module* we mean a pair (E, ρ_E), where E is finite dimensional $\overline{k}$-vector space and $\rho_E : \mathbb{G}_m^d \longrightarrow \mathrm{GL}(E)$ is a homomorphism of algebraic groups. We often drop ρ_E and use only E to denote a G-module.

Definition 1.1. Let (E, ρ_E) be a $\mathbb{G}_m^d$-module. The (logarithmic) *height* associated to (E, ρ_E) is the function $h_E : \mathbb{G}_m^d(\overline{k}) \to \mathbb{R}$, defined by setting $h_E = h_s \circ \rho_E$.

[1]The logarithmic spectral height can be regarded as the normalised or minimal height for all reasonable (logarithmic) height functions on $\mathrm{End}(E)$ (i.e. twisted heights [8, Section 1] or heights defined via adelic vector bundles [4, définition 2.1]). Namely let h be such a height. Set $h^{\mathrm{op}}(T) = \sup_{e \in E}\{h(T(e)) - h(e)\}$. Then it can be proved that (see [13, Theorem A]) $\lim_{m\to\infty} \frac{1}{m} h^{\mathrm{op}}_{E}(T^m) = h_s(T)$ for all $T \in \mathrm{End}(E)$. See also [2, Section 2.2]

The following properties of h_E follow from the corresponding ones of h_s:

(h1) h_E is invariant under field extension.
(h2) $h_E(g) \geq 0$.
(h3) $h_E(g^m) = mh_E(g)$.
(h4) $h_E(gg') \leq h_E(g) + h_E(g')$.
(h5) $h_E = h_F$ if E and F are isomorphic as $\mathbb{G}_m^d$-modules
(h6) If F is another $\mathbb{G}_m^d$-module, then $h_{E\otimes F}(g) = h_E(g) + h_F(g)$

The celebrated Northcott's finiteness theorem (see [9, Section 2.4]) states that the set

$$\left\{P \in \mathbb{P}_n(\overline{k}) \,\middle|\, h_{NW}(P) \leq B \,\text{and}\, P \in \mathbb{P}_n(k_0) \,\text{with}\, [k_0 : k] \leq C\right\}$$

is finite for every $B, C > 0$. We want to determine for which $\mathbb{G}_m^d$-module Northcott's theorem holds. We start with the following:

Definition 1.2. We say that a $\mathbb{G}_m^d$-module E *satisfy property (N)* if the set

$$\begin{aligned}&\Omega_E(\mathbb{G}_m^d, B, C)\\ &\qquad = \left\{g \in \mathbb{G}_m^d(\overline{k}) \,\middle|\, h_E(g) \leq B \text{ and } g \in \mathbb{G}_m^d(k_0) \text{ with } [k_0 : k] \leq C\right\}\end{aligned}$$

is finite for every $B, C > 0$.

By (h5) E satisfy property (N) if and only if every $\mathbb{G}_m^d$-module isomorphic to E does. Before characterising which $\mathbb{G}_m^d$-modules do satisfy property (N) we need to recall a few facts about $\mathrm{Rep}(\mathbb{G}_m^d)$, the monoid of isomorphism classes of $\mathbb{G}_m^d$-modules, and at the same time set our notation. There is a simple description of $\mathrm{Rep}(\mathbb{G}_m^d)$: let $\mathbf{\Gamma}_d := \widehat{\mathbb{G}_m^d} \simeq \mathbb{Z}^d$ denote the group of characters of $\mathbb{G}_m^d$, then $\mathrm{Rep}(\mathbb{G}_m^d)$ is isomorphic to the group semiring $\mathbb{N}[\mathbf{\Gamma}_d]$. The isomorphism is defined as follows: given a $\mathbb{G}_m^d$-module E we can always decompose it as a direct sum of subspaces on which $\mathbb{G}_m^d$ acts by scalar multiplication via a character. Namely given $\chi \in \mathbf{\Gamma}_d$ the set $E_\chi := \{\mathbf{e} \in E \mid \rho_E(g)\mathbf{e} = \chi(g)\mathbf{e}\, \forall\, g \in \mathbb{G}_m^d\}$ is called the χ-*isotypical component* of E. Clearly $E_\chi \neq 0$ only for finitely many χ's. Moreover, since $\mathbb{G}_m^d$ is abelian, we have

$$E \simeq \bigoplus_{\chi\in\mathbf{\Gamma}_d} E_\chi\,. \tag{1.1}$$

In the above isomorphism $[E]$ is mapped to $\mathbf{f}_E := \sum_{\chi\in\mathbf{\Gamma}_d} f_E^\chi \chi$, where $f_E^\chi = \dim_k E_\chi$. It is then natural to say that E is *isotypical* if $E_\chi \neq 0$ for exactly one $\chi \in \mathbf{\Gamma}_d$. Given $\mathbf{f} = \sum_{\chi\in\mathbf{\Gamma}_d} f^\chi \chi \in \mathbb{N}[\mathbf{\Gamma}_d]$, set $\mathrm{supp}(\mathbf{f}) = \{\chi \in \mathbf{\Gamma}_d \mid f^\chi \neq 0\} \subset \mathbf{\Gamma}_d$, then from (1.1) we get the following alternative description of h_E:

$$h_E(g) = \sum_{v\in\mathcal{M}_k} d_v \max_{\chi\in\operatorname{supp}(\mathbf{f}_E)} \log(|\chi(g)|_v)\,. \tag{1.2}$$

The isomorphism class of G-modules associated to any given

$$\mathbf{f} = \sum_{\chi\in\mathbf{\Gamma}_d} f^\chi \chi \in \mathbb{N}[\mathbf{\Gamma}_d]$$

is the one containing

$$E_{\mathbf{f}} := \bigoplus_{\chi\in\operatorname{supp}(\mathbf{f})} k^{f_\chi} \tag{1.3}$$

where the action of $\mathbb{G}_m^d$ on the direct summand k^{f_χ} is given by scalar multiplication by χ.

Note that the convolution product in $\mathbb{N}[\mathbf{\Gamma}_d]$ corresponds to the operation induced by the tensor product of $\mathbb{G}_m^d$-modules, while the sum in $\mathbb{N}[\mathbf{\Gamma}_d]$ corresponds to the operation induced by the direct sum of $\mathbb{G}_m^d$-modules.

Recall that there is a natural action of $\mathbf{\Gamma}_d$ on $\mathbb{N}[\mathbf{\Gamma}_d]$, given by

$$\mathbf{f}^\eta = \left(\sum_{\chi\in\mathbf{\Gamma}_d} f^\chi \chi\right)^\eta = \sum_{\chi\in\mathbf{\Gamma}_d} f^\chi (\eta\chi)\,.$$

Given $\mathbf{f} \in \mathbb{N}[\mathbf{\Gamma}_d]$ we denote by $\mathcal{O}_{\mathbf{\Gamma}_d}(\mathbf{f})$ its orbit under $\mathbf{\Gamma}_d$. It follows from (1.2) and the product formula that (isomorphism classes of) $\mathbb{G}_m^d$-modules in the same orbit under $\mathbf{\Gamma}_d$ give rise to the same height function. We say that two $\mathbb{G}_m^d$-modules are in the same orbit under $\mathbf{\Gamma}_d$ if their isomorphism classes are. Thus $\mathbb{G}_m^d$-modules in the same orbit either all satisfy property (N) or none does. A $\mathbb{G}_m^d$-module (E, ρ_E) is said to have finite kernel if ρ_E does.

Theorem 1.3. *Let E be a $\mathbb{G}_m^d$-module. Then E satisfy property (N) if and only if there exists $\mathbf{f} \in \mathcal{O}_{\mathbf{\Gamma}_d}(\mathbf{f}_E)$ enjoying the following two properties:*

(1) $\operatorname{supp}(\mathbf{f})$ *contains the trivial character ε_0*
(2) *$E_{\mathbf{f}}$ has finite kernel.*

Proof. Suppose that $\mathbf{f} \in \mathcal{O}_{\mathbf{\Gamma}_d}(\mathbf{f}_E)$ enjoys (1) and (2). Let $\{\varepsilon_0, \chi_1, \dots, \chi_n\} = \operatorname{supp}(\mathbf{f})$. Then the map $\mathbb{G}_m^d(\overline{k}) \to \mathbb{P}_n(\overline{k})$, given by $g \mapsto [1 : \chi_1(g) : \cdots : \chi_n(g)]$ has finite fibers and it maps the set $\Omega_{E_{\mathbf{f}'}}(\mathbb{G}_m^d, B, C)$ to

$$\{P \in \mathbb{P}_n(\overline{k}) \mid h_{NW}(P) \le B \text{ and } [k(P) : k] \le C\}\,.$$

By Northcott's theorem the latter set is finite, and hence $E_{\mathbf{f}}$ satisfies property (N), which implies that E also does. Next we prove that if every $\mathbb{G}_m^d$-module in $\mathcal{O}_{\mathbf{\Gamma}_d}(\mathbf{f}_E)$ does not enjoy one between (1) and (2) then $E_{\mathbf{f}}$ does not satisfy property (N). Let $\{\chi_1, \dots, \chi_n\} = \operatorname{supp}(\mathbf{f}_E)$. Note that at least one of the χ_i is not the trivial character, otherwise we are done. We can assume without loss of generality that $\chi_1 \ne \varepsilon_0$. Let $\eta = (\chi_1)^{-1}$ and set

$F = E_{\mathbf{f}^\eta}$, so $\mathbf{f}_F = \mathbf{f}^\eta$. Since $\varepsilon_0 \in \operatorname{supp}(\mathbf{f}_F)$, the kernel of ρ_F has to be an infinite subgroup, otherwise $\mathbf{f}_F$ would satisfy both (1) and (2). Hence there exists a non-torsion $g \in \mathbb{G}_m^d(k)$ belonging to the kernel ρ_F. This amounts to say that $\chi_i(g^m) = \chi_1(g^m)$ for $i = 2, \ldots, n$ and for all $m \geq 1$, which in turn yields $h_E(g^m) = 0$ for all $m \geq 1$. But $g^m \neq g^n$ if $n \neq m$ and hence E does not satisfy property (N). □

We can finally define the *height pairing*, between $\operatorname{Rep}(\mathbb{G}_m^d)$ and $\mathbb{G}_m^d(\overline{k})$ by setting:

$$\begin{aligned} \langle\,\cdot\,,\cdot\,\rangle_h \colon \operatorname{Rep}(\mathbb{G}_m^d) \times \mathbb{G}_m^d(\overline{k}) &\longrightarrow \mathbb{R}_+ \\ ([E]\,,\,g) \qquad &\longmapsto h_E(g) \end{aligned}$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is also regarded as monoid under addiction.

Before proving a few properties of the height pairing we need one more notation. Given $\varphi \in \operatorname{Aut}(\mathbb{G}_m^d)$ we denote by $\widehat{\varphi}$ the automorphism induced by φ on both $\mathbf{\Gamma}_d$ and $\mathbb{N}[\mathbf{\Gamma}_d]$, the first one being defined by $\widehat{\varphi}(\chi) = \chi \circ \varphi$, the latter by $\widehat{\varphi}\left(\sum_{\chi \in \mathbf{\Gamma}_d} f^\chi \chi\right) = \sum_{\chi \in \mathbf{\Gamma}_d} f^\chi \widehat{\varphi}(\chi)$. The identification of $\operatorname{Rep}(\mathbb{G}_m^d)$ with $\mathbb{N}[\mathbf{\Gamma}_d]$ yields an automorphism $\widehat{\varphi} : \operatorname{Rep}(\mathbb{G}_m^d) \to \operatorname{Rep}(\mathbb{G}_m^d)$. On the other hand if $\varphi \in \operatorname{Aut}(\mathbb{G}_m^d)$, then E^φ, the *twist* of E by φ, is defined by setting $\rho_{E^\varphi} = \rho_E \circ \varphi$. It is straightforward to verify that $[E^\varphi] = \widehat{\varphi}([E])$.

Proposition 1.4. *The height pairing enjoys the following properties:*

1. $\langle [E \otimes F], g\rangle_h = \langle [E], [g]\rangle_h + \langle [F], g\rangle_h$.
2. $\langle [E], g^m\rangle_h = m\langle [E], g\rangle_h$.
3. $\langle [E], gg'\rangle_h \leq \langle [E], g\rangle_h + \langle [E], g'\rangle_h$.
4. $\langle [E], \varphi(g)\rangle_h = \langle \widehat{\varphi}([E]), g\rangle_h$.
5. *The kernel on the left is the submonoid of isomorphism classes of isotypical* $\mathbb{G}_m^d$*-modules.*
6. *The kernel on the right is the* $\operatorname{Tors}(\mathbb{G}_m^d)$ *the torsion subgroup of* $\mathbb{G}_m^d(\overline{k})$.

Proof. (1), (2), and (3) follow directly from (h6), (h3), and (h4) respectively. (4) is a straightforward computation. By (1.2) and the product formula, the height associated to an isotypical $\mathbb{G}_m^d$-module is trivial. On the other hand if E is not isotypical, then using formula (1.2) it is immediate to verify that h_E is not trivial, proving (5). To prove (6) we first note that (h3) implies that the torsion subgroups of $\mathbb{G}_m^d(\overline{k})$ lies in the right kernel. On the other hand, if g is not torsion, then there exists a character χ such that $\chi(g)$ is not a root of unity. Let $[E]$ be the isomorphism class of $\mathbb{G}_m^d$-modules corresponding to $\mathbf{f} = \chi + \varepsilon_0 \in \mathbb{N}[\mathbf{\Gamma}_d]$, then $h_E(g) \neq 0$. □

Remark 1.5. If in our construction of the height pairing we had used

$$h_s^+ = \sum_{v \in \mathcal{M}_k} d_v \log^+(\rho_v(T))$$

where $\log^+(x) = \max\{\log(x), 0\}$ instead of h_s, then the resulting height pairing would have been a non-degenerate pairing between $\mathrm{Rep}(\mathbb{G}_m^d)$ and $\mathbb{G}_m^d(\overline{k}) / \mathrm{Tors}(\mathbb{G}_m^d)$ (cf. [2, Section 2.2]).

2. Symmetries for $\mathbb{G}_m^d$ as maximal torus of $\mathbf{GL}_d$

We now undertake the study of the group of symmetries for heights associated to $\mathbb{G}_m^d$-modules. After a few general remarks we concentrate, in this and the following section, on specific examples.

We start by fixing our notations and conventions about endomorphism of $\mathbb{G}_m^d$. Given $A = (a_{ij}) \in \mathrm{Mat}_d(\mathbb{Z})$ let φ_A be the endomorphism of $\mathbb{G}_m^d$ defined by:

$$\varphi_A(g) = (g_1^{a_{11}} \dots g_d^{a_{1d}}, \dots, g_1^{a_{d1}} \dots g_d^{a_{dd}}),$$

where $g = (g_1, \dots, g_d) \in \mathbb{G}_m^d$. The assignment $A \mapsto \varphi_A$ gives rise to an isomorphism of $\mathrm{Mat}_d(\mathbb{Z})$ into $\mathrm{End}(\mathbb{G}_m^d)$ allowing us to identify $\mathrm{GL}_d(\mathbb{Z})$ with $\mathrm{Aut}(\mathbb{G}_m^d)$ the full group of (algebraic) automorphisms of $\mathbb{G}_m^d$.

Regarding characters we use the following conventions: $\varepsilon_1, \dots, \varepsilon_d$ always denote the standard basis for $\mathbf{\Gamma}_d$. For $\chi = \prod_{i=1}^d \epsilon_i^{m_i}$, we set $\ell(\chi) := \sum_{i=1}^n m_i$. Note that if we identify $\widehat{\mathbb{G}_m^d}$ with $\mathbb{Z}^d$ by means of the standard basis of characters, then the matrix associated to $\widehat{\varphi}_A := \widehat{(\varphi_A)}$ is ${}^tA$, the transpose of A.

Let E be a $\mathbb{G}_m^d$-module, recall that the group of symmetries of h_E is

$$\mathcal{H}_E = \left\{ \varphi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, h_E(\varphi(g)) = h_E(g),\ \forall\, g \in \mathbb{G}_m^d(k) \right\} .$$

Clearly $\mathcal{H}_E$ depends only on the isomorphism class $[E]$ of E.

Lemma 2.1. *Let E be $\mathbb{G}_m^d$-module and $\psi \in \mathrm{Aut}(\mathbb{G}_m^d)$. Then*

$$\mathcal{H}_{E^\psi} = \psi^{-1} \mathcal{H}_E \psi .$$

Proof. Let φ belong to $\mathcal{H}_E$, then

$$h_{E^\psi}(\psi^{-1} \circ \varphi \circ \psi)(g) = h_E(\varphi(\psi(g))) = h_E(\psi(g)) = h_{E^\psi}(g)$$

and so $\mathcal{H}_{E^\psi} \supset \psi^{-1} \mathcal{H}_E \psi$. Conversely if $\eta \in \mathcal{H}_{E^\psi}$, set $\varphi = \psi \circ \eta \circ \psi^{-1}$, then

$$\begin{aligned} h_E(\varphi(g)) &= h_E(\psi(\eta(\psi^{-1}(g)))) = h_{E^\psi}(\eta(\psi^{-1}(g))) \\ &= h_{E^\psi}(\psi^{-1}(g)) = h_E(\psi(\psi^{-1}(g))) = h_E(g) . \end{aligned}$$

Therefore φ belongs to $\mathcal{H}_E$, and the lemma follows. $\square$

Let $\mathbf{e}=\sum_{i=1}^n \varepsilon_i \in \mathbb{Z}[\mathbf{\Gamma}_d]$. For ease of notation set $h_{\mathbf{e}} := h_{E_{\mathbf{e}}}$, $\mathcal{H}_{\mathbf{e}} := \mathcal{H}_{E_{\mathbf{e}}}$, and $\rho_{\mathbf{e}} := \rho_{E_{\mathbf{e}}}$. As mentioned before, $\rho_{\mathbf{e}}$ gives the standard realization of $\mathbb{G}_m^d$ as diagonal matrices of GL_d. As in the introduction we let $\mathcal{W}(\rho_e(\mathbb{G}_m^d), \mathrm{GL}(E_{\mathbf{e}}))$ denote the Weyl group of GL_d relative to $\rho_{\mathbf{e}}(\mathbb{G}_m^d)$ and we set

$$\mathcal{W}_{\mathbf{e}} = \left\{\varphi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, \varphi = \rho_{\mathbf{e}}^{-1} \circ \omega \circ \rho_{\mathbf{e}}, \text{ for some } \omega \in \mathcal{W}\big(\rho_{\mathbf{e}}(\mathbb{G}_m^d), \mathrm{GL}(E)\big)\right\}$$

The main result of this section is the explicit determination of $\mathcal{H}_{\mathbf{e}}$. Since the spectral height is invariant under scalar multiplication we have that $h_{\mathbf{e}}$ is invariant under the action of $\mathbb{G}_m(\overline{k})$ on $\mathbb{G}_m^d(\overline{k})$ by diagonal multiplication[2]. Using this action we can define an action of $\widehat{\mathbb{G}_m^d}$ on $\mathrm{End}(\mathbb{G}_m^d)$ as follows: given a character χ and endomorphism φ we let φ^χ be the endomorphism of $\mathbb{G}_m^d$ given by $\varphi^\chi(g) = \chi(g)\varphi(g)$; φ^χ is called the *twist* of φ by χ. Suppose that $\chi = \prod_{i=1}^d \varepsilon_i^{m_i}$. Let J_χ be the matrix whose rows are all equal to $(m_1, \dots, m_d)$, it is straightforward to verify that $(\varphi_A)^\chi = \varphi_{(J_\chi + A)}$. Unfortunately this implies that the above action does not restrict to an action on $\mathrm{Aut}(\mathbb{G}_m^d)$. The next lemma gives a criterion for detecting when $(\varphi_A)^\chi$ is an automorphism. Before stating it we need the following definition. Let $\langle \cdot, \cdot \rangle$ denote the standard pairing that $\widehat{\mathbb{G}_m^d}$ inherits under its identification with $\mathbb{Z}^d$, (i.e. $\langle \prod_{i=1}^d \varepsilon_i^{m_i}, \prod_{i=1}^d \varepsilon_i^{n_i} \rangle = \sum_{i=1}^d m_i n_i$).

Lemma 2.2. *Let χ be a character of $\mathbb{G}_m^d$ and $\varphi_A \in \mathrm{Aut}(\mathbb{G}_m^d)$. Then $(\varphi_A)^\chi$ belongs to $\mathrm{Aut}(\mathbb{G}_m^d)$ if and only if $\langle \chi, \widehat{\varphi}_A^{-1}(\prod_{i=1}^d \varepsilon_i) \rangle \in \{0, -2\}$.*

Proof. We start by noting that $\det(A + J_\chi) = \det({}^tA + {}^tJ_\chi)$. Therefore it suffices to prove that $\det({}^tA + {}^tJ_\chi) = \pm 1$ if and only if $\langle \chi, \widehat{\varphi}_A^{-1}(\prod_{i=1}^d \varepsilon_i) \rangle$ equals 0 or -2. Next note that ${}^tJ_\chi$ can be viewed as the product of the $m \times 1$ matrix ${}^t(1, \dots, 1)$ with the $1 \times m$ matrix $(m_1, \dots, m_d)$. Using Sylvester determinant identity $\big(\det(I + BC) = \det(I + CB)\big)$ we find that

$$\det({}^tA + {}^t(1, \dots, 1)(m_1, \dots, m_d)) = 1 + (m_1, \dots, m_d)({}^tA)^{-1}\, {}^t(1, \dots, 1)).$$

But

$$(m_1, \dots, m_d)({}^tA)^{-1}\, {}^t(1, \dots, 1)) = \left\langle \chi, \widehat{\varphi}_A^{-1}\left(\prod_{i=1}^d \varepsilon_i\right)\right\rangle,$$

proving the lemma. $\square$

Let $\mathcal{S}_d$ denote the group of permutations on d elements. Let A_σ be the permutation matrix associated to $\sigma \in \mathcal{S}_d$, we set $\varphi_\sigma := \varphi_{A_\sigma} \in \mathrm{Aut}(\mathbb{G}_m^d)$, so $\varphi_\sigma(g_1, \dots, g_d) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(d)})$. We set

$$\mathcal{P}_d := \left\{\varphi_\sigma \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, \sigma \in \mathcal{S}_d\right\}$$

[2]Which we recall is defined by setting $\lambda g = (\lambda g_1, \dots, \lambda g_d)$ for $\lambda \in \mathbb{G}_m(\overline{k})$ and $g = (g_1, \dots, g_d) \in \mathbb{G}_m^d(\overline{k})$

and

$$\widetilde{\mathcal{P}_d} := \left\{\varphi_\sigma^\chi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, \varphi_\sigma \in \mathcal{P}_d \text{ and } \chi \in \mathbf{\Gamma}_d\right\}.$$

Since $W\big(\rho_e(\mathbb{G}_m^d), \mathrm{GL}(E_{\mathbf{e}})\big)$ acts on $\rho_e(\mathbb{G}_m^d)$ by permutations, it follows that $\mathcal{W}_{\mathbf{e}} = \mathcal{P}_d$. Note that given $\varphi_\sigma \in \mathcal{P}_d$ and $\chi = \prod_{i=1}^d \varepsilon_i^{m_i}$, we have:

$$\left\langle \chi, \widehat{\varphi_\sigma}^{-1}\left(\prod_{i=1}^d \varepsilon_i\right)\right\rangle = \ell(\chi)\,.$$

Therefore by Lemma 2.2, φ_σ^χ belongs to $\mathrm{GL}_d(\mathbb{Z})$, and hence to $\widetilde{\mathcal{P}_d}$, if and only if $\ell(\chi) = 0$ or -2.

Lemma 2.3. *$\widetilde{\mathcal{P}_d}$ is a subgroup of* $\mathrm{Aut}(\mathbb{G}_m^d)$.

Proof. Let $\psi_1 = \varphi_{\sigma_1}^{\chi_1}$ and $\psi_2 = \varphi_{\sigma_2}^{\chi_2}$. Since φ_{σ_1} and φ_{σ_2} are just permutations of the coordinates we have

$$(\psi_1 \circ \psi_2)(g) = \chi_2(g)^{\ell(\chi_1)+1}\widehat{\varphi_2}(\chi_1)(g)(\varphi_1(\varphi_2(g))\,,$$

and

$$\ell(\widehat{\varphi_2}(\chi_1)) = \ell(\chi_1)\,.$$

Now we have two cases:

(a) $\ell(\chi_1) = -2$,
(b) $\ell(\chi_1) = 0$.

(a). We have $(\psi_1 \circ \psi_2)(g) = \eta(g)(\varphi_1(\varphi_2(g))$, where $\eta = \chi_2^{-1}\widehat{\varphi_2}(\chi_1)$ and

$$\ell(\eta) = -\ell(\chi_2) + \ell(\widehat{\varphi_2}(\chi_1)) = -\ell(\chi_2) - 2 \in \{0, -2\}\,.$$

(b). We have $(\psi_1 \circ \psi_2)(g) = \eta(g)(\varphi_1(\varphi_2(g))$, where $\eta = \chi_2\widehat{\varphi_2}(\chi_1)$ and $\ell(\eta) = \ell(\chi_2)$. Thus $(\psi_1 \circ \psi_2) \in \widetilde{\mathcal{P}_d}$. Furthermore the inverse of $\psi = \varphi_\sigma^\chi \in \widetilde{\mathcal{P}_d}$ is $\varphi_{\sigma^{-1}}^\eta$, where

$$\eta = \begin{cases} \widehat{\varphi_{\sigma^{-1}}}(\chi) & \text{if } \ell(\chi) = -2 \\ \big(\widehat{\varphi_{\sigma^{-1}}}(\chi)\big)^{-1} & \text{if } \ell(\chi) = 0 \end{cases}$$

and so $\psi^{-1} \in \widetilde{\mathcal{P}_d}$, completing the proof of the lemma. □

The following notation will be used throughout the rest of the paper. Given a character χ we let $\mathbb{G}_m^d[\chi]$ denote the kernel of χ. Furthermore we set

$$(\mathbb{G}_m^d)_j := \bigcap_{\substack{i=1 \\ i\neq j}}^d \mathbb{G}_m^d[\varepsilon_i]\,.$$

Proposition 2.4. $\mathcal{H}_{\mathbf{e}} = \widetilde{\mathcal{P}_d}$.

Proof. We assume that $d > 2$, for the case $d = 1$ is trivial and the case $d = 2$ can be dealt by computations similar, but simpler, to the one we carry out below. It is straightforward to check that $\mathcal{H}_{\mathbf{e}} \supset \widetilde{\mathcal{P}_d}$. Next suppose that φ_A belongs to $\mathcal{H}_{\mathbf{e}}$, where $A = (a_{ij})$. Set

$$T_j = \min_{1\le i\le n} a_{ij} \qquad \text{and} \qquad R_j = \max_{1\le i\le n} a_{ij}\,.$$

Suppose $g \in (\mathbb{G}_m^d)_j$, Then $h_{\mathbf{e}}(g) = h_{NW}(1, \varepsilon_j(g))$. Thus

$$h_{NW}(1, \varepsilon_j(g)) = h_{\mathbf{e}}(g) = h_{\mathbf{e}}(\varphi_A(g)) = h_{NW}\Big(1, \varepsilon_j(g)^{R_j - T_j}\Big),$$

yielding $R_j - T_j = 1$. Suppose that $a_{hj} = R_j$ and $R_k = a_{hk}$ with $k \neq j$. Let $g_1 \in (\mathbb{G}_m^d)_j$ and $g_2 \in (\mathbb{G}_m^d)_k$, and set $g = g_1 g_2$. Then $h_{\mathbf{e}}(\varphi_A(g)) \geq h_{NW}(1, \varepsilon_j(g)\varepsilon_h(g))$, which for appropriate choice of g_1 and g_2 is strictly bigger than $h_{\mathbf{e}}(g) = h_{NW}(1, \varepsilon_j(g), \varepsilon_k(g))$. Thus it cannot happen $a_{hj} = R_j$ and $R_k = a_{hk}$ with $k \neq j$, hence for each j there exists a unique $h_j \in \{1, \ldots, n\}$ such that $a_{h_j j} = R_j$. Let σ denote the permutation $j \mapsto h_j$. Then φ_A is the twist of φ_σ by $\chi = \prod_{i=1}^d \beta_j^{T_j}$. $\square$

As a consequence of Proposition 2.4 we have that $\mathcal{H}_{\mathbf{e}}$ contains properly $\mathcal{W}_{\mathbf{e}}$.

Corollary 2.5. *Let E be $\mathbb{G}_m^d$-module such that $\rho_E(\mathbb{G}_m^d)$ is a maximal torus of* $\mathrm{GL}(E)$. *Then $\mathcal{H}_E$ and $\widetilde{\mathcal{P}_d}$ are conjugate subgroups.*

Proof. Since $\rho_E(\mathbb{G}_m^d)$ is a maximal torus of $\mathrm{GL}(E)$ then $[E] = \chi_1 + \cdots + \chi_d$ and $\chi_1, \ldots, \chi_d$ generate $\mathbf{\Gamma}_d$ by [1, Theorem 3.2.19]. Therefore there exists $\varphi \in \mathrm{Aut}(\mathbb{G}_m^d)$ such that $\chi_i = \varepsilon_i \circ \varphi$, for all $i = 1, \ldots, d$. It follows that $[E] = [\mathbf{e}^\varphi]$ and the corollary follows from Lemma 2.1 and the fact that the group of symmetries of h_E depends only on $[E]$. $\square$

3. Symmetries for $\mathbb{G}_m^d$-module of classical type

In this section we consider the realizations of $\mathbb{G}_m^d$ as the maximal torus of some classical groups and determine their group of symmetries. We briefly recall that the definiton of the groups under consideration, the reader is advised to consult [5] for details. The *special linear group* $\mathrm{SL}_{d+1}(k)$ consists of the matrices of determinant 1 in $\mathrm{GL}_{d+1}(k)$. The *symplectic group* $\mathrm{Sp}_{2d}(k)$, consists of all $A \in \mathrm{GL}_{2d}(k)$ such that

$$ {}^tA \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \qquad \text{where } J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Next we want to define the *orthogonal groups* $\mathrm{SO}_{2d+1}(k)$ and $\mathrm{SO}_{2d}(k)$, to do this we need two additional matrices:

$$S_{2d+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix} \quad \text{and} \quad S_{2d} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

Then:

$$\mathrm{SO}_{2d+1}(k) = \left\{ A \in \mathrm{SL}_{2d+1}(k) \,\middle|\, {}^tAS_{2d+1}A = S_{2d+1} \right\},$$

and

$$\mathrm{SO}_{2d}(k) = \left\{ A \in \mathrm{SL}_{2d}(k) \,\middle|\, {}^tAS_{2d}A = S_{2d} \right\}.$$

In Tables 3.1 and 3.2 we list the $\mathbb{G}_m^d$-modules that we are going to investigate. In the first column of each table we list the $\mathbb{G}_m^d$-module in question, then in Table 3.1 we list the vector space underlying the $\mathbb{G}_m^d$-module, the element of $\mathbb{Z}[\mathbf{\Gamma}_d]$ corresponding to the $\mathbb{G}_m^d$-module and the classical group of which $\rho_E(\mathbb{G}_m^d)$ is a maximal torus. In the second table we list the action and the Weyl group $\mathcal{W}(\rho_E(\mathbb{G}_m^d), \mathrm{GL}(E)) = \mathrm{N}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d))/\,\mathrm{C}_{\mathrm{GL}(E)}(\rho_E(\mathbb{G}_m^d))$.

TABLE 3.1.

$\mathbb{G}_m^d$-module	u.v.	$\mathbb{Z}[\mathbf{\Gamma}_d]$	Classical group
$E_\mathbf{a}$	k^{d+1}	$\mathbf{a}=\varepsilon_1+\cdots+\varepsilon_d+(\varepsilon_1\ldots\varepsilon_d)^{-1}$	$\mathrm{SL}_{d+1}(k)$
$E_\mathbf{b}$	k^{2d+1}	$\mathbf{b}=\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_d+\varepsilon_1^{-1}+\cdots+\varepsilon_d^{-1}$	$\mathrm{SO}_{2d+1}(k)$
$E_\mathbf{c}$	k^{2d}	$\mathbf{c}=\varepsilon_1+\ldots\varepsilon_d+\varepsilon_1^{-1}+\ldots\varepsilon_d^{-1}$	$\mathrm{Sp}_{2d}(k)$
$E_\mathbf{d}$	k^{2d}	$\mathbf{d}=\varepsilon_1+\ldots\varepsilon_d+\varepsilon_1^{-1}+\cdots+\varepsilon_d^{-1}$	$\mathrm{SO}_{2d}(k)$

TABLE 3.2.

$\mathbb{G}_m^d$-module	Action	Weyl Group
$E_\mathbf{a}$	$g.\mathbf{x}=(g_1x_1,\ldots,g_dx_d,(g_1\ldots g_d)^{-1}x_{d+1})$	$\mathcal{S}_{d+1}$
$E_\mathbf{b}$	$g.\mathbf{x}=(x_1,g_1x_2,\ldots,g_dx_{d+1},g_1^{-1}x_{d+2}\ldots g_d^{-1}x_{2d+1})$	$(\mathbb{Z}/2\mathbb{Z})^d \rtimes \mathcal{S}_d$
$E_\mathbf{c}$	$g.\mathbf{x}=(g_1x_1,\ldots,g_dx_d,g_1^{-1}x_{d+1},\ldots g_d^{-1}x_{2d})$	$(\mathbb{Z}/2\mathbb{Z})^d \rtimes \mathcal{S}_d$
$E_\mathbf{d}$	$g.\mathbf{x}=(g_1x_1,\ldots,g_dx_d,g_1^{-1}x_{d+1},\ldots g_d^{-1}x_{2d})$	$(\mathbb{Z}/2\mathbb{Z})^d \rtimes \mathcal{S}_d$

Definition 3.1. If E is a $\mathbb{G}_m^d$-module such that $[E]$ is one of the isomorphism class of Table 1, we then say that E is of *cmt-type*. If E is of cmt-type, we then set

$$\mathcal{W}_E = \left\{ \varphi \in \mathrm{Aut}(\mathbb{G}_m^d) \,\middle|\, \varphi = \rho_E^{-1} \circ \omega \circ \rho_E, \text{ for some } \omega \in \mathcal{W}(\rho_E(\mathbb{G}_m^d), \mathrm{GL}(E)) \right\}.$$

Theorem 3.2. *Let E be of cmt-type. Then $\mathcal{H}_E = \mathcal{W}_E$.*

The rest of this section is devoted to the proof of Theorem 3.2. Since both $\mathcal{H}_E$ and $\mathcal{W}_E$ actually depend only on the isomorphism class E, it suffices to prove Theorem 3.2 for the $\mathbb{G}_m^d$-modules listed in Table 3.1.

3.1. Special linear group. Let $E_{\mathbf{a}}$ denote the $\mathbb{G}_m^d$-module associated to $\mathbf{a}$ and set $h_{\mathbf{a}} := h_{E_{\mathbf{a}}}$, $\mathcal{W}_{\mathbf{a}} := \mathcal{W}_{E_{\mathbf{a}}}$, $\mathcal{H}_{\mathbf{a}} := \mathcal{H}_{E_{\mathbf{a}}}$ and $\rho_{\mathbf{a}} := \rho_{E_{\mathbf{a}}}$. The height $h_{\mathbf{a}}$ is given by:

$$\begin{aligned} h_{\mathbf{a}}(g) &= h_{NW}\left(\varepsilon_1(g),\dots,\varepsilon_d(g),(\varepsilon_1(g)\dots\varepsilon_d(g))^{-1}\right) \\ &= h_{NW}(\varepsilon_1(g)\prod_{j=1}^{d}\varepsilon_j(g),\dots,\varepsilon_d(g)\prod_{j=1}^{d}\varepsilon_j(g),1)\,. \end{aligned} \tag{3.1}$$

We start by giving an explicit description of $\mathcal{W}_{\mathbf{a}}$. Firstly we ecall that $\mathcal{W}\big(\rho_{\mathbf{a}}(\mathbb{G}_m^d),\mathrm{GL}_{d+1}\big) \simeq \mathcal{S}_{d+1}$. Next the elements of $\mathcal{W}\big(\rho_{\mathbf{a}}(\mathbb{G}_m^d),\mathrm{GL}_{d+1}\big)$ come in two guises: those who leave the last coordinate fixed and those who do not.

Those who leave the last coordinate fixed correspond to permutations of the coordinates on $\mathbb{G}_m^d$. Of those who do not fix the last coordinate it is enough to consider, in order to generate $\mathcal{S}_{d+1}$, the transpositions of the form $(\ell, d+1)$. We denote by $\gamma_\ell \in \mathcal{W}_{\mathbf{a}}$ the automorphism corresponding to the transposition $(\ell, d+1)$, its action on $\mathbb{G}_m^d$ is given by:

$$(g_1,\dots,g_d) \xmapsto{\gamma_\ell} (g_1,\dots,\, g_{\ell-1},(g_1\cdots g_d)^{-1}, g_{\ell+1},\dots,g_d)\,.$$

Thus $\mathcal{W}_{\mathbf{a}}$ is the subgroup of $\mathrm{Aut}(\mathbb{G}_m^d)$ generated by $\mathcal{P}_d$ and the γ_ℓ's. In order to be able to prove that actually $\mathcal{H}_{\mathbf{a}} = \mathcal{W}_{\mathbf{a}}$ we need the following explicit characterization for the elements of $\mathcal{W}_{\mathbf{a}} \subset \mathrm{Aut}(\mathbb{G}_m^d)$.

Lemma 3.3. *Let $A = (a_{ij}) \in \mathrm{GL}_d(\mathbb{Z})$. Then $\varphi_A \in \mathcal{W}_{\mathbf{a}}$ if and only if A satisfies one of the following conditions:*

1. *A is a permutation matrix*
2. *There exist $h, k \in \{1,\dots,d\}$ such that the following conditions hold*
   - (a) *$a_{hj} = -1$ for all $j = 1,\dots,d$.*
   - (b) *$a_{ik} = 0$ for all $i = 1,\dots,d, i \neq h$.*
   - (c) *If we delete the h-th row and the k-th column the $(d-1)\times(d-1)$ the resulting matrix is a permutation matrix.*

Proof. A straightforward computation shows that the set of matrices satisfying either (1) or (2) forms a group. Moreover let $C = (c_{ij})$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq d \text{ and } j = i \\ -1 & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases}$$

then $\gamma_\ell = \varphi_C$ and so every γ_ℓ satisfies condition (2). Thus every element of $\mathcal{W}_{\mathbf{a}}$ satisfies either (1) or (2). It remains to show that any φ_A, with A

satisfying either (1) or (2) belongs to $\mathcal{W}_{\mathbf{a}}$. We only have to worry about the matrices that satisfy (2). Suppose we have an automorphism φ_A with $A = (a_{ij})$ satisfying condition (2) for $h, k \in \{1, \ldots, d\}$. Let $B = (b_{ij})$ be the matrix obtained by replacing the k-th column of A with the column $(0, \ldots, 0, 1, 0, \ldots, 0)$ the 1 occurring at the h-th spot. Then φ_B belongs to $\mathcal{P}_d$, and $\gamma_h \varphi_B = \varphi_A$, yielding $\varphi_A \in \mathcal{W}_{\mathbf{a}}$. □

Lemma 3.4. $\mathcal{H}_{\mathbf{a}} = \mathcal{W}_{\mathbf{a}}$.

Proof. Since $\mathcal{H}_{\mathbf{a}} \supseteq \mathcal{W}_{\mathbf{a}}$ we only have to show that if $\varphi_A \in \mathcal{H}_{\mathbf{a}}$, with $A = (a_{ij})$, the A satisfies either (1) or (2) of the previous lemma. Set $s_j = \sum_{i=1}^d a_{ij}$ and $m_j = -\min\{a_{1j}, \ldots, a_{dj}, -s_j\} > 0$. If $g \in (\mathbb{G}_m^d)_j$, then

$$h_{\mathbf{a}}\left(\varphi_A(g)\right) = h_{NW}\left(\varepsilon_j(g)^{a_{1j}}, \ldots, \varepsilon_j(g)^{a_{dj}}, \varepsilon_k(g)^{-s_k}\right).$$

Therefore

$$\begin{aligned} 2h_{NW}(1, \varepsilon_j(g)) &= h_{\mathbf{a}}(g) = h_{\mathbf{a}}\left(\varphi_A(g)\right) \\ &= h_{NW}\left(\varepsilon_j(g)^{a_{1k}+m_j}, \ldots, \varepsilon_j(g)^{a_{dj}+m_j}, \varepsilon_j(g)^{-s_j+m_j}\right). \end{aligned} \tag{3.2}$$

On the other hand either $a_{ij} + m_j = 0$ for some i or $-s_j + m_j = 0$, which, when put together with (3.2), gives $0 \leq a_{ij} + m_j \leq 2$ and $0 \leq -s_j + m_j \leq 2$. Combining these inequalities with the definition of s_j yields:

$$\begin{aligned} -1 \leq s_j \leq 1 &\quad \forall\, j = 1, \ldots, d\,, \\ m_j = 1 &\quad \forall\, j = 1, \ldots, d\,, \\ -1 \leq a_{ij} \leq 1 &\quad \forall\, i, j = 1, \ldots, d\,. \end{aligned}$$

In particular if $a_{ik} \geq 0$ for all $i = 1, \ldots, d$ then there exists i_k such that $a_{ik} = \delta_i^{i_k}$ where δ is the Kronecker's delta. Now choose $g \in \mathbb{G}_m^d$ so that $\varepsilon_i(g)$ is a positive integer for $i = 1, \ldots, d$. Then

$$h_{\mathbf{a}}(g) = \log\left(\max_{1 \leq i \leq n}\left\{\varepsilon_i(g)\prod_{j=1}^d \varepsilon_j(g)\right\}\right) \tag{3.3}$$

and

$$h_{\mathbf{a}}\left(\varphi_A(g)\right) = h_{NW}\left(\prod_{j=1}^d \varepsilon_j(g)^{a_{1j}+1}, \ldots, \prod_{j=1}^d \varepsilon_j(g)^{a_{dj}+1}, \prod_{j=1}^d \varepsilon_j(g)^{-s_j+1}\right). \tag{3.4}$$

For each k we can choose a suitable $g_k \in \mathbb{G}_m^d$ in such a way that

$$\max_{1 \leq i \leq d}\left\{\varepsilon_i(g_k)\prod_{j=1}^d \varepsilon_j(g_k)\right\} = \varepsilon_1(g_k) \ldots \varepsilon_k(g_k)^2 \ldots \varepsilon_d(g_k)\,. \tag{3.5}$$

Set $a_{d+1j} = -s_j$. It follows from (3.3), (3.4) and (3.5) that for each $k \in \{1, \dots, d\}$ it must exist $i_k \in \{1, \dots, d+1\}$ such that

$$\prod_{j=1}^{d} \varepsilon_j(g_k)^{a_{i_k j}+1} = \varepsilon_1(g_k) \dots \varepsilon_k(g_k)^2 \dots \varepsilon_d(g_k). \tag{3.6}$$

But then (3.6) implies that $a_{i_k j} = 1$ if $j = k$ and is otherwise zero. A being invertible implies that $i_k \neq i_\ell$ for $h \neq k$. Therefore there exists only one $h \in \{1, \dots, n+1\}$ such that $h \neq i_k$ for all $k = 1, \dots, d$. It follows that $a_{hj} = -1$ for all $j = 1, \dots, d$. If $h = d+1$ then A consists only of non-negative elements and as we already remarked this implies that A is a permutation matrix. If $h \leq n$, then there exists $k \in \{1, \dots, d\}$ such that $i_k = d+1$. Then $a_{ik} = 0$ for $i < d+1$. Deleting the h-th row and k-th column from A we get a permutation matrix proving that $\varphi_A \in \mathcal{W}_{\mathbf{a}}$. □

3.2. Symplectic and Orthogonal Groups. Let $E_{\mathbf{c}}$ denote the $\mathbb{G}_m^d$-module associated to $\mathbf{c}$ and set $h_{\mathbf{c}} := h_{E_{\mathbf{c}}}$, $\mathcal{W}_{\mathbf{c}} := \mathcal{W}_{E_{\mathbf{c}}}$, $\mathcal{H}_{\mathbf{c}} := \mathcal{H}_{E_{\mathbf{c}}}$ and $\rho_{\mathbf{c}} := \rho_{E_{\mathbf{c}}}$

Since $\rho_{\mathbf{c}}$ embeds $\mathbb{G}_m^d$ as the subgroup of diagonal matrices of Sp_{2d}, we have that

$$h_{\mathbf{c}}(g) = h_{NW}(\varepsilon_1(g), \dots, \varepsilon_d(g), \varepsilon_1^{-1}(g), \dots, \varepsilon_d^{-1}(g)).$$

Also in this case the relative Weyl group $\mathcal{W}(\rho_{\mathbf{c}}(\mathbb{G}_m^d(k)), \mathrm{GL}_{2d}(k))$ is equal to the Weyl group of Sp_{2d} and thus is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d \rtimes \mathcal{S}_d$ (with $\mathcal{S}_d$ acting on $(\mathbb{Z}/2\mathbb{Z})^d$ by permutations). The action of $(\mathbb{Z}/2\mathbb{Z})^d \rtimes \mathcal{S}_d$ on $\rho_{\mathbf{c}}(\mathbb{G}_m^d(k))$ is as follows:

$$\begin{aligned}((b_1, \dots, b_d), \sigma).\Big(t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}\Big) \\ = \Big(t_{\sigma^{-1}(1)}^{b_1}, \dots, t_{\sigma^{-1}(d)}^{b_d}, t_{\sigma^{-1}(1)}^{-b_1}, \dots, t_{\sigma^{-1}(d)}^{-b_d}\Big).\end{aligned}$$

Therefore, if we denote by $\mathfrak{S}_d^{\pm}$ the subgroup of $\mathrm{GL}(d, \mathbb{Z})$ formed by the monomial matrices with entries ± 1, it follows that

$$\mathcal{W}_{\mathbf{c}} = \Big\{\varphi_A \in \mathrm{Aut}(\mathbb{G}_m^d) \,\Big|\, A \in \{\mathfrak{S}_d^{\pm}\Big\},$$

and $\mathcal{W}_{\mathbf{c}} \subset \mathcal{H}_{\mathbf{c}}$.

Lemma 3.5. *$\mathcal{H}_{\mathbf{c}} = \mathcal{W}_{\mathbf{c}}$,*

Proof. Suppose that $\varphi_A \in \mathcal{H}_{\mathbf{c}}$, with $A = (a_{ij}) \in \mathrm{GL}_d(\mathbb{Z})$. Let $g \in (\mathbb{G}_m^d)_k$. As before we have $h_{\mathbf{c}}(g) = 2h_{NW}(1, \varepsilon_k(g))$, and so

$$\begin{aligned}2h_{NW}(1, \varepsilon_k(g)) &= h_{\mathbf{c}}(\varphi_A(g)) \\ &= h_{NW}(\varepsilon_k(g)^{a_{1k}}, \dots, \varepsilon_k(g)^{a_{dk}}, \varepsilon_k(g)^{-a_{1k}}, \dots, \varepsilon_k(g)^{-a_{dk}}).\end{aligned}$$

which immediately yields $a_{ij} \in \{1,0,-1\}$ for $i,j = 1,\dots,d$. It remains to prove that $\alpha = (a_{ij})$ is a monomial matrix. Suppose that $a_{ik} \neq 0$ and $a_{ih} \neq 0$ for $k \neq h$. By symmetry we reduce to examine the following two cases:

(a) $a_{ik} = 1$ and $a_{ih} = 1$,
(b) $a_{ik} = 1$ and $a_{ih} = -1$.

(a). Chose $g \in \mathbb{G}_m^d$ such that $\varepsilon_i(g) = 1$ if $i \neq h,k$ and $\varepsilon_h(g) = \varepsilon_k(g) = \lambda$ a positive integer. Then $h_{\mathbf{c}}(g) = 2h_{NW}(1, \varepsilon_k(g))$. On the other hand the usual computation shows that $h_{\mathbf{c}}(\varphi_A(g))$ is equal to

$$h_{NW}(\lambda^{a_{1k}+a_{1h}}, \dots, \lambda^2, \dots \lambda^{a_{dk}+a_{dk}}, \lambda^{-a_{1k}-a_{1h}}, \dots, \lambda^{-2}, \dots \lambda^{-a_{dk}-a_{dk}}).$$

Since $-2 \leq a_{jk} + a_{jh} \leq 2$ we find $h_{\mathbf{c}}(\varphi_A(g)) = \lambda^4$, which is a contradiction.

(b). Select $g \in \mathbb{G}_m^d$ such that $\varepsilon_i(g) = 1$ if $i \neq h,k$ and $\varepsilon_h(g) = \varepsilon_k(g)^{-1} = \lambda$ is a positive integer. The same computation done in (a) yields a contradiction, completing the proof of the theorem. □

The proof of the similar result for the orthogonal groups is identical to that of Lemma 3.5, thus the proof of Theorem 3.2 is completed.

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