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Multizeta values for function fields: A survey

par Dinesh S. THAKUR

In memory of David M. Goss

Résumé. Nous donnons une vue d'ensemble des développements récents concernant la compréhension des valeurs multi-zêta pour les corps de fonctions.

Abstract. We give a survey of the recent developments in the understanding of the multizeta values for function fields.

1. Introduction

The birth of calculus led to a development of a new bag of tools for the treasure hunt that followed for discovering nice evaluations for many interesting infinite sums. While many mathematicians tried and "failed" the problem of evaluating (the zeta values) $\zeta_E(s) := \sum_{n=1}^{\infty} 1/n^s$ for integers $s > 1$, Euler "half succeeded" by his breakthrough result evaluating it, for even such s, in terms of known quantities such as π, Bernoulli numbers and factorials. He also discovered the correct evaluations for its values (defined rigorously only much later by analytic continuation proved by Riemann) at non-positive integers s, and even at some fractions, while checking the functional equation that he conjectured, which was proved much later by Riemann. Partly motivated by attempts to understand the remaining values at odd integers $s > 1$, he introduced (for $r = 2$) multizeta values

$$\zeta_E(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} \in \mathbb{R},$$

for positive integers s_i, with $s_1 > 1$ and discovered interesting relations such as $\zeta_E(3) = \zeta_E(2, 1)$ and sum-shuffle relations.

The last three decades witnessed a huge resurgence of interest in developing understanding of these multizeta values again, as their relations with various fundamental objects and structures in mathematics and mathematical physics were discovered. They occur in the Grothendieck–Ihara program to study the absolute Galois group of the rational number field

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through the fundamental group of the projective line minus three points and other moduli spaces, and in the related studies of iterated extensions of Tate motives. They also occur in various Feynman path integral calculations that show up in physics. We refer the reader to papers on this subject by Broadhurst, Brown, Cartier, Deligne, Drinfeld, Ecalle, Furusho, Gangl, Goncharov, Hoffman, Ihara, Kaneko, Kreimer, Racinet, Terasoma, Unver, Waldschmidt, Zagier, Zhao, Zudilin to mention just a few names. (See [55], its extensive bibliography, [24] and the reference site `www.usna.edu/Users/math/meh/biblio.html` maintained by Hoffman).

Number theorists often study number fields and function fields (over finite fields) together [23, 47] for various reasons. The zeta story in the function field context was developed in two independent directions. The first direction was taken up by Artin, who studied complex valued zeta associated to function fields in his thesis, by essentially interpreting "n" in the zeta definition as the number of residue classes modulo n and thus replacing it by q^d which is the number of residue classes mod "a" (i.e., the norm) for a polynomial a of degree d over a finite field of q elements. This has been developed into standard algebraic geometry definitions in much more generality. The functions you get are rational functions of q^s. In [47, 5.10], multizeta with this line of thought were introduced and evaluated fully as explicit rational functions (of several variables). We ignore this direction in this survey and refer the reader to [33, 47].

The second direction was taken by Carlitz, who studied around 1935 (now called) the Carlitz zeta values in the completion of the function field (at the "usual" infinite place), and produced an analog of Euler's result at "even" (see §2.2 for the definition) positive integers. David Goss, around 1975, generalized, developed analytic continuation, found evaluation at non-positive integers and its connection to the class group components of cyclotomic fields etc. This produces rich transcendental functions in contrast to the rational functions of the first case.

The author was fortunate to learn about the rich interconnections of the multizeta with various fundamental structures in the lecture course by Deligne at the Institute for Advanced Study in 2000–2001 and at the Arizona Winter School in 2002. This led him to develop the multizeta in the function field context, by generalizing both the directions mentioned above, but we will only concentrate on the second one below.

We give a survey of various developments since then due to the efforts of several mathematicians, focusing not only on the end results and conjectures, but also explaining how they arose. This survey has been written for the readers who are somewhat familiar with the story for the Riemann–Euler case and curious about the function field case. Thus we describe some technical results only informally, by suggestive terminology, analogies and references. For the general background on that, we refer to [23, 47, 50].

2. Some basic analogies and the Zeta story

We start by setting up some basic notations and analogies.

2.1. Base rings, fields, places. Let $\mathbb{F}_q$ be a finite field of characteristic p consisting of q elements.

As analogs of the rational number field $\mathbb{Q}$, its archimedean place ∞, the ring of integers $\mathbb{Z}$ (whose primes correspond to the finite places), the completion $\mathbb{R}$ of $\mathbb{Q}$ at ∞ and its algebraic closure $\mathbb{C}$, which is also complete; we have the rational function field $\mathbb{F}_q(t)$, its usual place ∞ corresponding to the degree valuation with $1/t$ as an uniformizer, the polynomial ring $\mathbb{F}_q[t]$ (whose irreducible polynomials correspond to the finite places), the (finite tailed) laurent series field $\mathbb{F}_q((1/t))$, and the completion C_∞ of its algebraic closure respectively.

More generally, we can take any function field K (of one variable over finite field $\mathbb{F}_q$), ∞ any chosen place of K (taken rational, or equivalently of degree one usually, we will drop this restriction only in §5.4), the ring of integers A (consisting of elements of K with only pole at ∞, the completion K_∞ (with its residue field $\mathbb{F}_\infty$) and the completion C_∞ of its algebraic closure.

We will refer to this set-up as the general A case and to the special case above, as the $A = \mathbb{F}_q[t]$ case, where we will also use the same general notation, when clear from the context. The Euler–Riemann case of the introduction will be referred to as the real case.

Though most of the definitions and the basic properties below work in the general case, more substantial results have been established only in $A = \mathbb{F}_q[t]$ case.

2.2. The signs. The set of units or signs $\mathbb{Z}^*$ then corresponds to $A^* = \mathbb{F}_q^* = \mathbb{F}_\infty^*$ (the last equality, when ∞ is of degree one). The respective cardinalities 2 and $q-1$ lead to analog "even" (or q-even) to mean multiples of $q-1$ in this context, in place of 2 for the usual notion of even. An integer which is not "even" is called "odd" (or q-odd). (Note that when $q = 2$, all integers are "even").

An analog of positive (sign 1) integers is monic (sign 1) polynomials. (Note that whereas the set $\mathbb{Z}+$ of positive integers is closed under addition and multiplication, the set $A+$ of monic polynomials is closed for multiplication, and only under addition in different degrees).

For the rest of this section, except for the last subsection, we will restrict to $A = \mathbb{F}_q[t]$ case. We write $[n] := t^{q^n} - t$.

2.3. Multiplicative and additive actions. Consider the multiplicative and additive groups denoted as usual by $\mathbb{G}_m, \mathbb{G}_a$ respectively. As analog of (n-th power actions) $n \mapsto (x \mapsto x^n) : \mathbb{Z} \to \mathrm{End}\ \mathbb{G}_m$, we have (Carlitz module actions) $a \mapsto (x \mapsto C_a(x)) : \mathbb{F}_q[t] \to \mathrm{End}\ \mathbb{G}_a$, defined by $C_t(x) = tx + x^q$, and $C_{\sum f_i t^i}(x) = \sum f_i C_t^{(i)}(x)$ (where $C_t^{(i)}$ denotes i-fold composition).

2.4. Corresponding exponential, and the Carlitz period. The related notion is the usual exponential $e^z = \sum z^n/n!$ satisfying the functional equations $e^{nz} = (e^z)^n$ with respect to this action and $e^z = 1 \leftrightarrow z \in 2\pi i\mathbb{Z}$. The Carlitz analog is $e(z) = \sum z^{q^n}/d_n$, with $d_n = [n][n-1]^q \ldots [1]^{q^{n-1}}$, satisfying $e(az) = C_a(e(z))$ (thus implying the formula for d_n for $n > 0$, from its empty product case $d_0 = 1$ and recursion) and $e(z) = 0 \leftrightarrow z \in \Lambda := \tilde{\pi}A$, for some $\tilde{\pi} \in C_\infty$ (known [52] to be transcendental over K) considered as analog of $2\pi i$.

2.5. Carlitz–Tate–Anderson *t*-motives. In Greg Anderson's world of t-motives, which are higher dimensional generalizations of Drinfeld modules (such as the Carlitz module above), the Carlitz module C plays the role of the Tate motive $\mathbb{Z}(1)$, and its n-th tensor power $C^{\otimes n}$ plays the role of $\mathbb{Z}(n)$. See [1, 23, 47].

2.6. The Carlitz zeta values. The Carlitz zeta values $\zeta(k)$ are defined for positive integers k as

$$\zeta(k) = \sum_{a \in A+} \frac{1}{a^k} \in K_\infty .$$

2.7. The Carlitz analog of Euler result. Since the Carlitz exponential is an entire function with simple zeros exactly at $\Lambda := \tilde{\pi}A$, the standard non-archimedean analysis shows that $e(z) = z \prod (1 - z/\lambda)$, where the product is over non-zero $\lambda \in \Lambda$. Making parallel with the analogous product formula of the sine function that Euler discovered in this process, and taking the logarithmic derivative to turn the product into sum, we see by the comparison of the coefficients of both the sides of

$$\frac{z}{e(z)} = 1 - \sum_{\lambda \in \Lambda - \{0\}} \frac{z/\lambda}{1 - z/\lambda} = 1 - \sum_{n=1}^{\infty} \sum_{\lambda} \left(\frac{z}{\lambda}\right)^n = 1 + \sum_{n \text{ ``even''}} \frac{\zeta(n)}{\tilde{\pi}^n} z^n$$

the Carlitz analog [11] of the Euler result that $\zeta(n)/\tilde{\pi}^n \in K$ for "even" n.

Interestingly, in the function field case, we can handle all n's, as we will see below.

2.8. Logarithm, polylogarithms, and tensor power logarithms. Consider the Carlitz "logarithm" function $\ell(z) = \sum z^{q^k}/\ell_k$ which is the inverse power series of the Carlitz exponential (Concretely, $\ell_k = \prod_{i=1}^{k}(-[i])$, with the empty product case interpreted as $\ell_0 = 1$ as usual). Similarly, there is a (canonical co-ordinate) "logarithm" [7, 47] $\log_n(z_1, \ldots, z_n)$ associated to $C^{\otimes n}$ which is a deformation of naive n-th polylogarithm $\mathcal{L}_n(z) = \sum z^{q^k}/\ell_k^n$.

2.9. Algebraic incarnation of $\boldsymbol{\zeta(n)}$. In [7], we expressed $\zeta(n)$ as the logarithm of an explicit algebraic point (torsion exactly when n is "even") on $C^{\otimes n}$, or equivalently as a period of an explicit extension of $C^{\otimes n}$ by the trivial module. We also gave an expression in terms of K-linear combinations of polylogarithms evaluated at integral arguments.

2.10. Transcendence implications. Jing Yu's fundamental work proving various transcendence results for exponential, logarithms in the context of Drinfeld modules and t-motives combined with the above expression implied the following theorem.

Theorem 2.1 ([53, 54]). *For all positive integers n, $\zeta(n)$'s are transcendental over K. For all "odd" positive integers n, $\zeta(n)/\tilde{\pi}^n$ and v-adic zeta values $\zeta_v(n)$ are transcendental, where v is any finite prime of A.*

After 15 years, there was a fundamental advance in transcendence theory due to the works of Anderson, Brownawell, Papanikolas, Yu and Chang [5, 16, 37] proving roughly (analog of the Grothendieck period conjecture in this setting) that the algebraic relations between the periods of t-motives (e.g., gamma, zeta or multizeta values) come from the structural relations between the motives involved, and using it to get several interesting special values results.

Theorem 2.2 ([16]). *All the algebraic relations between $\tilde{\pi}$ and $\zeta(n)$'s (positive integers n) come from the Euler–Carlitz relation at "even" n mentioned above and $\zeta(kp) = \zeta(k)^p$, special in characteristic p. Thus, for example, all the $\zeta(n)$'s and $\tilde{\pi}$ are algebraic independent, if we choose n's "odd" not divisible by p.*

For an exposition of the concepts involved, see [50, §6]. We will just say here that in right circumstances, the number of algebraic independent periods is the dimension of the corresponding motivic Galois group accessible through Frobenius-difference Galois theory applied to the matrices defining the t-motives.

2.11. General $\boldsymbol{A}$, $\boldsymbol{L}$-functions etc. In the general case, the Carlitz module is replaced by the sign normalized rank one Drinfeld A-modules, or even more general objects [23, 47].

In [2, 3, 46], some aspects were generalized to the general A case. There are many recent generalizations and interesting important results in different directions due to Taelman, Papanikolas, Pellarin, Perkins, Angles, Ngo Dac, Tavares Riberio, Fang, Derby, see [9, 10, 20, 22, 36, 38, 41, 43] and other papers on arXiv.

Now we turn to the multizeta story.

3. Definitions related to the multizeta values

Consider the general A case, with an infinite place of degree one. Let A_d+ denote the set of monic polynomials in A of degree d. For $k, k_i, d \in \mathbb{Z}$, consider the power sums (sometimes denoted by $S_d(-k)$ in the references)

$$S_d(k) = \sum_{a \in A_d+} \frac{1}{a^k} \in K\,,$$

and extend inductively to the iterated power sums

$$\begin{aligned} S_d(k_1, \ldots, k_r) &= S_d(k_1)\, S_{<d}(k_2, \ldots, k_r) \\ &= S_d(k_1) \sum_{d > d_2 > \cdots > d_r} S_{d_2}(k_2) \ldots S_{d_r}(k_r)\,, \end{aligned}$$

where $S_{<d} = \sum_{i=0}^{d-1} S_i$ as the notation suggests.

For positive integers s_i, we consider the multizeta values

$$\zeta(s_1, \ldots, s_r) := \sum_{d=0}^{\infty} S_d(s_1, \ldots, s_r) = \sum \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in K_\infty\,,$$

of weight $\sum s_i$ and depth r (associated a priori to the tuple s_i rather than the value). (Here the second sum is over monic $a_i \in A$ of strictly decreasing degrees).

In comparison to the real case, the convergence condition $s_1 > 1$ is not necessary, and while in that case, the weight, but not depth (as can be seen from the Euler's result mentioned in the introduction), is conjecturally well-defined from the value, in our case the weight is known to be well-defined (by Chang's transcendence result mentioned below in Section 9) and depth is conjecturally (see §6.3 below) well-defined from its value.

Hence, there are 2^{w-1} convergent multizetas of weight w, in contrast to 2^{w-2} in the real case.

3.1. Non-vanishing. In the real case, being a sum of positive terms, multizeta value is clearly non-zero, while this is a subtle issue in a finite characteristic, especially since the degrees of the power sums $S_d(k)$'s are complicated combinatorial functions of d, k because of cancellations. Non-vanishing was deduced in [48] using some results on these degrees.

3.2. Motivation and other variants. In contrast to $\mathbb{Z}$, there is no "natural" order on A to define multizeta by copying Euler, but degree provides natural partial order, also used by Goss in his analytic continuation and in [7] for interpolations used for motivic interpretation. This is the main motivation for the definition, which seems to be justified by the results below, even though at first sight, not having a total order immediately destroys (a priori) sum shuffle relations.

If one considers analogy between positive and monic, and interprets $a_1 > a_2$ as saying $a_1 - a_2$ is monic, one does not get total order. Variants of multizeta with such notions, or with the computer science natural "lexicographic" order notions were also explored in [47, 5.10]. While one gets some interesting evaluations and relations for these variants also, they do not seem to be as robust.

4. Mixed t-motives and period interpretation

Theorem 4.1 ([8]). *Given multizeta value $\zeta(s_1, \ldots, s_r)$, we can construct explicitly iterated extension of the t-motives $C^{\otimes s_i}$ over $\mathbb{F}_q[t]$ which has a period matrix containing this multizeta value (suitably normalized) as an entry.*

4.1. Special interpolation technique. For the precise definitions of the concepts involved, we refer to [8, 23, 47]. We just mention here, for the use below, that the techniques are generalizations of those used in the zeta case [7], and in particular, involve constructing rational functions on products of curves which specialize to quantities related to S_d's on the graph of the d-th power of Frobenius. The relations of such objects with Anderson's t-motives (related to Dtinfeld shtukas) allows the constructions.

4.2. Mechanical proofs of individual relations. As mentioned in §2.10, the multizeta relations are then motivic, and thus should be reducible to relations between the interpolating functions. Hence a given relation should be mechanically provable, if correct. See [49, §5]. But we do not yet understand fully the mechanisms of motivic relations at this level, except in depth one and two (with s_2 "even"). See Section 6 for some more details.

5. Shuffle relations

Understanding the relations between multizeta values thus has many structural implications.

5.1. Relations between multizeta in the real case. In the real case, multizeta is given by iterated sum as in the definition and also by iterated integral of holomorphic forms dz/z's and $dz/(1-z)$'s on projective line minus $0, 1, \infty$. Thus, simple manipulations on sums and integrals give sum-shuffle and integral-shuffle relations, expressing product of multizeta values as sums of (several other) multizeta values. For example, the simplest case is Euler's sum-shuffle

$$\zeta_E(a)\zeta_E(b) = \left(\sum \frac{1}{n_1^a}\right)\left(\sum \frac{1}{n_2^b}\right) = \zeta_E(a,b) + \zeta_E(b,a) + \zeta_E(a+b),$$

which follows by order trichotomy $n_1 > n_2$, or $n_1 < n_2$ or $n_1 = n_2$. (Note that the sum-shuffle is also called quasi-shuffle or stuffle to emphasize that it is not a pure shuffle on s_i's (unlike the integral shuffle) and sometimes s_i's get stuck together as in the last term above).

Either set of these shuffle relations are quadratic relations which reduce the study of polynomial relations between the multizeta relations to linear relations, which are conjectured to be generated from the double shuffle relations obtained by equating what we get from the two kind of shuffles. (Here we are oversimplifying by ignoring the inclusion of renormalized divergent multizeta). There are many other conjectural descriptions (conjecturally equivalent), for which we refer to the papers mentioned above.

5.2. Relations between multizeta in the function field case. Since we are not dealing with a total order, the sum-shuffle argument mentioned above fails. In fact, it is easy to see that the classical sum-shuffle relations and integral-shuffle relations do not usually work, and we have new relations such as $\zeta(ps_1, \ldots, ps_r) = \zeta(s_1, \ldots, s_r)^p$. The author proved families of examples, where in the place of the three multizetas on right side of shuffle above, we could have fewer or many of them. For example, when $q = 2$, we have [49] $\zeta(2)\zeta(2b-1) = \zeta(2b+1) + \sum \zeta(2i+1, 2(b-i))$, where the sum is over $1 \leq i < b$. The patterns get much more complicated in the general case.

At first, it was not clear that we can always reduce to the linear relations by having linear span to be algebra by some substitute for shuffle relations. (In [8], many more "degenerate" multizeta were introduced to achieve this by shuffle over degrees. But we will not use them.)

Then the author gave a full conjectural shuffle recipe [49] (for product of two zetas) in the case $q = 2$. This was a combinatorial recursive recipe. To give a flavour of part of it, given the formula for $\zeta(19)\zeta(b)$ as sum of multizetas, it will tell how to derive exact formula for $\zeta(19)\zeta(b+32)$, which has 8 more multizeta terms.

Alejandro Lara Rodriguez, then the author's masters student, generalized [27, 28] parts of this conjectural recipe to any q, with better control for q prime. It exhibited [44, 3.2] "base q digit permutation symmetries" stressed by David Goss.

In [45], bypassing these conjectural recipes completely, the author directly proved the following theorem.

Consider

$$(*) \qquad S_d(a)S_d(b) - S_d(a+b) = \sum f_i S_d(a_i, a+b-a_i)\,,$$

with $f_i \in \mathbb{F}_p$.

Theorem 5.1.

(1) *Let $A = \mathbb{F}_q[t]$. Given $a, b \in \mathbb{Z}+$, there are $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$, so that $(*)$ holds for $d = 1$.*

(2) *Fix q. If $(*)$ holds for some $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$ for $d = 1$ and $A = \mathbb{F}_q[t]$, then $(*)$ holds for all $d \geq 0$ and for all A (with ∞ of degree one, and corresponding to the given q). In this case, we have the shuffle relation*

$$(**) \qquad \zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum f_i \zeta(a_i, a+b-a_i)\,.$$

(3) *$S_d(a_1, \ldots, a_r)S_d(b_1, \ldots, b_k)$ can be expressed as $\sum g_i S_d(c_{i1}, \ldots, c_{im_i})$, with $g_i \in \mathbb{F}_p$, c_{ij}'s and m_i's being independent of d, and with $\sum a_i + \sum b_j = \sum_j c_{ij}$ and $m_i \leq r + k$.*

(4) *For any A with rational infinite place, the product of multizeta values can be expressed as a sum of some multizeta values, such an expression preserving total weight and keeping depth filtration. In particular, the $\mathbb{F}_p$-span of all the multizeta values is an algebra.*

Note that the mechanism here is of shuffle (or quasi-shuffle, stuffle), not on the polynomials themselves, but on the degrees, and then taking care of the $d_1 = d_2$ case by $(*)$. We take this opportunity to record that while the proof of (3) and (4) follows from $(*)$, splitting up of front entry from S_d's and induction on length, as mentioned in [45, p. 1979], the author makes a mistake there in setting up the vector notation induction, in particular the fourth equality, which is said to follow from shuffle is messed up and should be replaced by what follows from shuffle. The full correct argument is given on the author's update page.

Next, Alejandro Lara Rodriguez, then the author's PhD student, turned this effective procedure into a nice formula for the shuffle relation for the product of two zeta values as a sum of products of two binomial coefficients, or in terms of generating functions.

Interestingly, the day of his PhD thesis defense, the author got an email from his past student (he is proud to say!) Huei Jeng Chen, who using (similar in principle, but much better) a partial fraction decomposition formula, which, as we later learned from M. Kaneko, was used even by Euler [21, p. 145-146] in the real multizeta case) gave a much simpler formula as follows.

Theorem 5.2 ([19])**.** *We have*

$$\zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum \left((-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1} \right) \zeta(a+b-j, j),$$

where the sum is over j which are multiples of $q-1$ and $0 < j < a+b$.

The subtle dependence on the base q expansions of a, b of the values and vanishing of the binomial coefficients may explain why it was not so easy to guess these formulas (different family for each q) from the numerical data. The combinatorics is a little complicated so that most of the original conjectures of [27, 28, 49] on the shuffle descriptions are still open. Some of them have been proved using Chen's formulas, by Chen [19] and Ryotaro Harada (in his thesis from Nagoya).

These relations have coefficients in $\mathbb{F}_p$, which is much smaller than the function field K, whereas in the real case, the coefficients are in $\mathbb{Q}$, which is simultaneously analog of a function field and a prime field.

Combining this depth 1 case, with induction and shuffle on degrees as in the previous theorem, one can describe the product in general. The following clean explicit form for the product was conjectured to the author by Shuji Yamamoto (thanks to him!) after the Lyon lectures. The details of verifications together with extra Hopf algebra structure (with commutative and associative product, and not co-commutative but co-associative explicit co-product) can be found in Shuhui Shi's upcoming PhD thesis, at University of Rochester.

Let S be the free monoid on set $\{x_n\}_{n\in\mathbb{Z}_+}$ and consider the $\mathbb{F}_p$-vector space R with basis S. We define a product $*$ on R: Let $\phi \in S$ be the empty word, for any $A \in S$, $A * \phi = \phi * A = A$. For $A = x_{a_1} \dots x_{a_n}$, $B = x_{b_1} \dots x_{b_m} \in S$, denote $A_- = x_{a_2} \dots x_{a_n}$, $B_- = x_{b_2} \dots x_{b_m}$, we define product, inductively on the length of A and B, as

$$A * B = x_{a_1}(A_- * B) + x_{b_1}(A * B_-) + x_{a_1+b_1}(A_- * B_-) + \sum_{\substack{0<j<a_1+b_1 \\ (q-1)|j}} \Delta^j_{a_1,b_1} x_{a_1+b_1-j} \cdot ((A_- * B_-) * x_j),$$

where

$$\Delta^j_{a_1,b_1} = (-1)^{a_1-1}\binom{j-1}{a_1-1} + (-1)^{b_1-1}\binom{j-1}{b_1-1},$$

and $w \sum_i f_i w_i = \sum_i f_i(ww_i)$ for $f_i \in \mathbb{F}_p, w, w_i \in S$. Then $(R, +, *)$ is a commutative algebra. Define a linear map $\theta : R \to K_\infty$, $\phi \mapsto 1$, $x_{a_1} \dots x_{a_n} \mapsto \zeta(a_1, \dots, a_n)$. Then θ is an $\mathbb{F}_p$-algebra homomorphism, i.e. $\phi(x_{a_1} \dots x_{a_n} * x_{b_1} \dots x_{b_m}) = \zeta(a_1, \dots, a_n)\zeta(b_1, \dots, b_m)$.

5.3. A fixed classical sum-shuffle relation survives for large enough q. If $a+b \leq q$, Chen's formula specializes to Euler's sum-shuffle relation. In fact, it was already proved in [47, 5.10.6] that given a fixed sum-shuffle relation, it holds for function fields for large enough q. The author wonders whether there is another logic-model theory type proof for this. While the sum-shuffle relations are technically simpler, some mathematicians coming from motives and cohomology sides think (in the real case), the integral-shuffles more natural and sum-shuffles as accidents which need to be explained. So it is interesting to note that in our case, the sum-shuffles, but not the integral-shuffles survive, at least for large enough q.

5.4. Which shuffle relations survive for non-rational ∞. We saw in Theorem 5.1 that the shuffle relations are universal for all A's (for a given q) with ∞ of degree one. When the place ∞ has degree more than one, the residue field $\mathbb{F}_\infty$ at ∞, which determines the signs, is bigger than $\mathbb{F}_q^*$, and there are two natural approaches to extend the notion of monic in this case to define multizeta. Either, one fixes a sign in $\mathbb{F}_\infty^*$, or restricts signs to be in fixed a set of representatives for $\mathbb{F}_\infty^*/\mathbb{F}_q^*$ to define monic. It turns out [32] that if the classical shuffle $\zeta(a)\zeta(b) = \zeta(a,b)+\zeta(b,a)+\zeta(a+b)$ (the product relation $\zeta(a)\zeta(b) = \zeta(a+b)$ respectively) holds for $\mathbb{F}_q[t]$, then it holds for all such A in the first approach (in the second approach respectively), and numerical data seems to indicate that other shuffle relations in depth 2 do not survive!

6. Linear relations: Examples and general conjectural restrictions

6.1. Euler type relations. In [47, 5.10.13], it was proved that for $q = 2$, $\zeta(1,1) = \zeta(2)/(t^2+t)$ and generalized to [49, Thm. 5, 3.4.6]

$$\zeta(1, q-1) = \zeta(q)/(t-t^q),$$

closest in sprit to the Euler relation mentioned above for $q = 3$ (when even agrees with "even"). In fact, we show that $S_d(q)/\ell_1 = S_{d+1}(1, q-1)$, which summing over the degrees gives this multizeta relation.

We also have [49, 3.4.6] $\zeta(1, q^2-1) = \zeta(q^2)(1/(t-t^q)+1/((t-t^q)(t-t^{q^2}))$ and some other such relations.

Now we give some speculations on the restrictions on all possible relations, based on heuristics using motivic interpretation and some other reasons.

6.2. Weight and depth filtration. Just as in the classical case, we had conjectured [49, §5] that the relations between multizeta preserves weight gradation and depth filtration (though not the depth itself) and had given some heuristic structural reasons. By Chang's work recalled below in Section 9, we now know that the weights are indeed preserved.

6.3. Conjecture: Only one shuffle. In contrast to the real case recalled above, we conjectured [49, 5.4] that there is only one shuffle relation (over $\mathbb{F}_p$, as the sums will only provide prime field coefficients) in the function field case and gave some heuristic reasons based on motivic interpretation. In particular, there should be no (non-trivial) linear relations with $\mathbb{F}_p$ coefficients. So, for example, in contrast to the duality relations in the real case, no distinct tuples could give same multizeta, and thus the depth is well-defined by the multizeta value, again in contrast to the real case (e.g., Euler relation in the real case).

We also note here that there should not even be a general different shuffle relation with $\mathbb{F}_q(t)$ coefficients of integral-shuffle type, in the sense of preserving the depth. This is because it would imply, in the simplest case of product of two zetas, by equating its result with the shuffle relation above, that even in weights lower than q we will have a non-trivial linear relation, as the sum shuffle relation in this case does have depth one term appearing with a non-zero coefficient. This would contradict the expectations of Section 8.

6.4. Conjectural parity restriction. In [49, 5.3] we conjectured that multizeta relations, being motivic by the results of Section 4, come through the relations of special interpolating function mentioned there. We constructed [49, §3] such functions for $S_{<d}(k)$, suitably normalized, when k is "even" and gave heuristic reasons why such functions can exist only for k "even". Thus, it was speculated that, apart from the shuffle relations effects, the iterated indices $s_i, i > 1$ in the multizeta linear relations should be "even". This is true e.g., at S_d level in the sum shuffle relations above and for the Euler type relations and the eulerian and zeta-like relations below, because the depth reduction mechanism (see e.g., [49, 3.4.6] or proofs in [30]) seems to come through the special cancellations in the products of such functions. See also [13, Thm. 3.1.1], which proves that in weight-preserving linear relation between values of depth at most 2, the second co-ordinates s_2 are always "even".

As we will see below in Section 8, many linear relations can be generated by complicated sum shuffle processes, and even the first simplest families of examples of §8.4 shows that you can use any indices $m = s_i$ iteratively without any parity restrictions, and so in general, even at S_d level iterated indices need not be "even" in general. It would be interesting to understand the exact scope of the parity restriction, by separating out the shuffle type effects from the relations involving manipulations of the rational functions of the Section 4.

7. Special linear relations: Eulerian and Zeta-like multizeta values

We now focus on the simplest kind of linear relations, those between two multizeta, one of which is the zeta value itself. We discuss the results, conjectures and algorithms to decide when the ratio of a multizeta value with a zeta value is algebraic [12, 15, 18, 25, 30]. As we now know, by Chang's results (only expected when the definition was made) recalled below in Section 9, that no such relation exists, in different weights, and that the ratio is algebraic if and only if it is rational, we make the following definitions.

7.1. Definitions. [30, 47]

We call a multizeta value $\zeta(s_1, \dots, s_r)$ (or the tuple $(s_1, \dots, s_r)$) *zeta-like* if the ratio $\zeta(s_1, \dots, s_r)/\zeta(s_1 + \cdots + s_r)$ is rational. (We use depth $r > 1$ below, sometimes without mention, because in the $r = 1$ case everything is zeta-like by definition). A multizeta value of weight w is called *eulerian*, if it is a rational multiple of $\tilde{\pi}^w$. So zeta-like multizeta values of "even" weight are eulerian by Carlitz' result of §2.7. While the zeta-like multizeta values of "odd" weight w cannot be eulerian, for the simple reason that $\tilde{\pi}^w$ (and thus the ratio) is not even in K_∞ then, by Yu's result (Theorem 2.1), the ratio in this case is not even algebraic. Since $\zeta(ps_1, \dots, ps_r) = \zeta(s_1, \dots, s_r)^p$, we can restrict to the tuples where not all s_i's are divisible by p. We call such tuples *primitive*.

7.2. Some conjectures and results. In [30], we proved several families of zeta-like values and made several conjectures based on the numerical evidence obtained by studying continued fractions of ratio of the multizeta divided by the zeta of the same weight.

Conjecture 7.1 (Size restrictions and Slicing)**.** *If $(s_1, \dots, s_r)$ is zeta-like, then*

(1) $s_i \leq s_{i+1}$*, and* $(q-1)s_i \leq s_{i+1} \leq (q^2-1)s_i$*.*
(2) $(s_2, \dots, s_r)$ *is eulerian and* $(s_1, \dots, s_{r-1})$ *is zeta-like.*

Conjecture 7.2 (Splicing constructions)**.**

(1) *Let $q = 2$. If $(s_1, \dots, s_k)$ and $(s_k, \dots, s_r)$ are zeta like and the total weight $\sum_{i=1}^r s_i$ is a power of 2 or a power of 2 minus one, then $(s_1, \dots, s_r)$ is zeta-like, except when the two tuples to be spliced are (1,1) and (1,1).*
(2) *Let q be any prime power. If $(s_1, \dots, s_k)$ and $(s_k, \dots, s_r)$ are eulerian and the total weight $\sum_{i=1}^r s_i$ is q^n-1 or $q(q-1)$, then $(s_1, \dots, s_r)$ is eulerian.*

Conjecture 7.3 (Weight restrictions)**.** *Primitive eulerian values occur only in weights $q(q-1)$ or q^n-1, if $q>2$ and also possibly in weight q^n, if $q=2$.*

We take this opportunity to mention that in Conjecture 4.4 of [30] which gives full conjectural list of the zeta-like values of depth at most q^2, the word “primitive” should have been dropped and that the remaining part of 4.4 also follows from Conjecture 7.1 (1) (i.e. Conjecture 4.1 (i) of [30]) and the parity conjecture mentioned in §6.4, which as mentioned there, follows in this case from the result of Chang.

There are also combinatorially involved conjectures [30, 4.3] on weight restrictions on zeta-like tuples, depending on depths etc. which we do not give here. We characterized eulerian tuples completely by these conjectures and depth 2 list below.

When only the first part of the conjecture was made, I heard from Papanikolas of the following nice result they had just proved (at that time only in the eulerian (“even” weight) case, which they soon generalized after learning about the rest of the conjectures) using the transcendence analysis of the t-motives involved. We had only checked and expected (by the parity conjecture mentioned above) the second statement on our data.

Theorem 7.4 ([15])**.** *If $\zeta(s_1,\ldots,s_r)$ is zeta-like, then $\zeta(s_2,\ldots,s_r)$ is eulerian. In particular, $s_2,\ldots,s_r$ are “even”, and if the value is eulerian, then s_1 is also “even”.*

In [15] ([25] respectively), efficient algorithms to effectively check whether a given tuple is eulerian (zeta-like respectively), for a given q , were developed using motivic interpretation ideas and were used to produce much more (than [30]) data, which was then used to verify and make conjectures.

7.3. Explicit conjectural description for eulerian values.

7.3.1. ***Let $q>2$.*** In [30], we proved (for $q\geq 2$) that in the depth 2 case,

$$(q-1,(q-1)^2),(q^n(q-1),q^{n+2}-1-q^n(q-1)),(q^n-1,(q-1)q^n)$$

are eulerian. We conjectured that these are the only primitive eulerian tuples in depth 2. We also conjectured that

$$T_r:=(q^n-1,(q-1)q^n,\ldots,(q-1)q^{n+r-2})$$

of depths $r>2$ ($q\geq 2$) are eulerian, and this was proved by Chen [18]. Conjecturally these are all the primitive eulerian tuples in depth more than 2.

The last conjecture was made in this simple explicit form in [15] by checking on the extensive data generated by the efficient algorithm constructed there, but is immediately seen to be equivalent (as mentioned in [30, Notes added in the proof]) to the depth 2, slicing/splicing, weight conjectures of [30] as follows.

If $S := (s_1, \ldots, s_r)$ is primitive eulerian with $r > 2$, then (s_1, s_2) and (s_2, s_3) are eulerian by the tuple restriction conjecture 7.1 (2), but the second co-ordinate of the first two in depth 2 list cannot be the first co-ordinate of eulerian (even up to p-powers), so that $(s_1, s_2) = (q^n - 1, (q-1)q^n)$ for some n, without loss of generality. By induction on $r > 2$, S is T_{r-1} followed by s_r. By weight part of the conjecture, it has weight $q^k - 1$, T_{r-1} has weight $q^{n+r-2} - 1$, so $s_r = q^k - q^{n+r-2}$, so comparison with the depth 2 list above with (s_{r-1}, s_r) proves the claim.

7.3.2. ***Let $q = 2$.*** In depth 2, the primitive eulerian (same as zeta-like now) list above specializes to $(1, 1), (2^n, 2^{n+2} - 2^n - 1), (2^n - 1, 2^n)$, there are two more: $(1, 3)$ and $(3, 5)$ The conjecture [30] is that these are exactly the primitive eulerian values in depth 2. In depth $r > 2$, the family above specializes to $(2^n - 1, 2^n, \ldots, 2^{n+r-2})$. There are two more, $(1, 1, 2, \ldots, 2^{r-2})$ [30, Thm. 3.2] and $(1, 3, 2^2, \ldots, 2^{r-1})$ ([30, Conj. 4.6 (2)] proved in [18]). The conjecture in [15, 6.2] (again it can similarly be deduced from slicing, splicing conjectural characterization above) is that, apart from the known $(1, 2, 5)$, there are no other primitive eulerian tuples.

We refer to [15, 6.2] for careful weight, depth cataloging details.

7.4. Results and conjectures on Zeta-like values. We do not yet have a conjectural description of all zeta-like values of "odd" weight.

In [30], it was proved that $(q^n - \sum_{i=1}^{s} q^{k_i}, (q-1)q^n)$ for $1 \leq s < q$, and $0 \leq k_i < n$, and a few other depth 2 tuples are zeta-like. There was also an explicit conjecture on all depth 2 zeta-like values of weight at most q^2.

It was also proved that any depth family $(1, (q-1), (q-1)q, \ldots, (q-1)q^n)$ is zeta-like. It was also conjectured (proved in [18]) that

$$\begin{aligned}&(1, q^2 - 1, (q-1)q^2, \ldots, (q-1)q^{n+1}),\\&((q-1)q^n - 1, (q-1)q^{n+1}, \ldots, (q-1)q^{n+r-1})\end{aligned}$$

are also zeta-like.

In [25], an algorithm for determining whether the given multizeta value is zeta-like, examples, data, conjectural zeta-like families, and more conjectural weight restrictions on the zeta-like values are given. We only give one conjecture from there to give a flavour: For any $n > 0$, $r \geq 2$ and $m \geq 1$ such that $p^m \leq q$,

$$\left(q^n - \sum_{i=1}^{s} q^{k_i}, p^m q^{n-1}(q-1), \ldots, p^m q^{n+r-3}(q-1)\right)$$

is zeta-like, where $s < q$ and $k_i < n$ such that $(q-1)s_1 \leq s_2$, where s_1, s_2 are the first two co-ordinates of the tuple.

7.5. Comparison with the real multizeta case. As mentioned in [30], it seems that only zeta-like multizeta (of depth more than one) known in the real case are $\zeta(2, 1_k) = \zeta(k+2)$ which is not Eulerian if k is odd, and others are Eulerian $\zeta((3,1)_k) = \zeta(2_{2k})/(2k+1)$, $\zeta((2n)_k) = \zeta((2, 1_{2n-2})_k)$ and $\zeta((2_m, 1, 2_m, 3)_n, 2_m) = \zeta(2_m, (2, 1, 2_m, 3, 2_{m-1})_n)$. (Here the subscript denotes the number of times the value or the tuple is repeated. The first and the last two equalities are the result of duality, whereas $(3,1)_k$ is self-dual).

The last family was mentioned as 1997 conjecture in [30], but it has since been proved by Charlton [17] (and also independently by Francis Brown (in a letter to the author) soon after the author mentioned it to him).

The author would like to thank M. Kaneko, who checked that there are no more zeta-like values, (and pointed out missing dual tuples which give the same values) in weights at most 16, and Broadhurst, who told the author that he does not know any more zeta-like examples, and that he had calculated data for weights up to 22 but had not checked this question systematically.

If these are the only such tuples, this description (namely $(2, 1_{2n+1})$) of "odd weight" zeta-like multizeta seems much simpler in the real case than the function field case, which is not yet understood even conjecturally.

Our results also show that in contrast to the real case, the multizeta of weight "odd" and depth 2 in our case are not generated (over rationals or algebraics) by the zeta values. For $q = 2$, when everything is "even", exactly the non-eulerian ones are not generated in this way.

7.6. Some techniques. We have focused on the results and refer to original papers for the proofs and techniques. But we will just mention that some of the techniques are the special interpolations [7, 8] of power sums S_d, the results of Sections 4 and 9, the generating functions [49, 3.2] for $S_d, S_{<d}$, and the new multiplicative identities [31] for the coefficients of logarithmic derivatives of $\mathbb{F}_q$-linear functions, as well as the shuffle relations.

8. General linear relations: Dimensions, basis and generation

Many years ago, Zagier conjectured $d_w = d_{w-2} + d_{w-3}$ for the dimension d_w of the $\mathbb{Q}$-linear span of the multizeta (in the real case) of a given weight w, by using the well-known LLL (Lenstra, Lenstra, Lovasz) method of lattice reduction, implemented then on pari, by calculating the multizeta approximately numerically and using the method to guess the linear relations between them.

8.1. Todd's dimension conjecture. George Todd, another PhD student of the author, undertook [51] similar calculations, writing up codes for function field implementations of LLL-method and running extensive calculations for each small q, trying to enumerate all linear relations between multizeta values with small degree coefficients. (One can do this with approximate multizeta computations or even with "degree-wise" computations, because of the motivic nature of the relations. But see §8.3 for how we may need to mix a few degrees.)

The author had conjectured that there are no linear relations in weights less than q, and the only weight q relation is the Euler type relation above. Todd's numerical calculations were consistent with this expectation, and based on them he further conjectured in his thesis the following formula for the relevant dimensions for any weight.

Conjecture 8.1 ([51])**.** *If $d(w)$ denotes the dimension of the span of multizeta values of weight w, then $d(w) = 2^{w-1}$ for $1 \leq w < q$, $d(w) = 2^{w-1} - 1$ for $w = q$ and for $w > q$, we have $d(w) = \sum_{i=1}^{q} d(w-i)$.*

8.1.1. ***Evidence.*** We have already mentioned that the conjectural bound is upper bound for $w \leq q$. By finding the required number of relations, Todd extends this to $w \leq q+3$. Also, he proves it for $w \leq 11$, for any q, by calculating sufficient relations for remaining low q's.

8.2. Conjectural basis for the weight spaces. The author's earlier speculation that $\zeta(s_1, \ldots, s_r)$'s with all $s_i < q$ should all be linearly independent seemed consistent with Todd's calculation results. Inspired by the Euler type relation, Todd's dimension conjecture and Hoffman–Brown's work in the real case, the author has the following refined speculation on the possible basis for the K-linear span L_w of weight w multizetas.

Conjecture 8.2. *A basis for L_w can be given explicitly inductively, by collection of exactly those $\zeta(s_1, \ldots, s_r)$ of weight w with all $s_i < q$, for $w \leq q$, and for $w > q$ by adding $s_1 = i$ in front of the tuples in the basis for weights $w - i$, for $i \leq q$.*

8.3. Complicated combinatorics of the relations. We have already seen that since the relations are motivic, they should come from relations between S_d's for some d's. For example, as we have seen, the sum shuffle comes from a single d in each term (such relations will be called "*fixed*"), while there are examples already in [49], of relations needing terms for both d and $d+1$ (such relations will be called "*binary*"). Todd's computations showed that the situation is even more complicated. We give one example, for $q = 2$. The known relation $(\ell_1 + \ell_2)\zeta(4) + \ell_1\ell_2\zeta(1,3) = 0$, follows, for example, by summing either of the following three (rational function) family of relations (one for each $d \in \mathbb{Z}$) for all $d \in \mathbb{Z}$:

$$\ell_1 S_d(4) + \ell_2 S_{d+1}(4) + \ell_1\ell_2 S_{d+2}(1,3) = 0\,.$$

$$\ell_1 S_d(4) + \ell_2 S_d(4) + \ell_1^2 S_d(2,2) + \ell_1^2 S_{d+1}(2,2) + \ell_1\ell_2 S_{d+1}(1,3) = 0\,.$$

$$\begin{aligned}(\ell_1+\ell_2)S_d(4)&+(\ell_1+\ell_2)\ell_1S_{d+1}(1,3)+\ell_1^2S_d(1,3)\\&+\ell_1^2S_d(1,1,2)+\ell_1^2S_{d+1}(1,1,2)\\&+\ell_1\ell_2S_d(1,2,1)+\ell_1\ell_2S_{d+1}(1,2,1)\\&+\ell_1\ell_2S_d(1,1,1,1)+\ell_1\ell_2S_{d+1}(1,1,1,1)=0\,.\end{aligned}$$

Thus, if you insist to involve only tuples present in the original relations at the zeta level, you may need to deal with several d's at once, while the best you can do in general, is to allow binary relations with extra tuples which cancel out eventually, but still there is no uniqueness.

8.4. Generating relations. There are various simple ways to generate more relations from the known relations, such as the shuffle relations and the relations mentioned above, and in [29, 30, 49]. For example, suppose we know a relation $\sum k_iS_d(X_i)=0$, (for all d, thus implying a linear multizeta relation by summing) for some tuples X_i, then summing over $d<D$ gives the same relation for $S_{<D}(X_i)$'s. Multiplying "on the left" by $S_D(m)$ and summing gives then the same relation for $S_D(m,X_i)$'s.

8.5. Todd's generation conjecture. Todd's numerical investigations suggest that all relations can be reduced to binary relations. Start from any binary relation $\sum a_iS_d(X_i)+\sum b_iS_{d+1}(X_i)=0$ (of given weight), and let U be a tuple of s_i's of weight w. Todd identifies two special processes called B_U, C_U respectively, which by multiplying "on the left" by $S_D(U)$ the corresponding relation for $S_{<D}$'s or by multiplying it "on the right" by $S_{<d+1}(U)$ respectively, gives new fixed or binary relation (of weight w more) respectively, after manipulations using decompositions and shuffle relations.

Let R be the relation [49] $S_d(q)-\ell_1S_{d+1}(1,q-1)=0$ of weight q behind the "Euler type" relation of 6.1. Todd has the following conjecture. (We refer to [51] for some fine details).

Conjecture 8.3 ([51]). *All the relations of weight $q+w$ are in the linear span of the collection of $B_U(R)$, $C_U(R)$ and $B_X(C_Y(R))$'s, as U run over all the tuples of weight w and X, Y over all the tuples of weight adding to w.*

This was a surprise for the author, whose optimistic hope, when Todd's calculation started was just that probably some relations for $\zeta(1,q^k-1)$, generalizing those in 6.1 and [49], could be identified by the calculations which would generate all. Todd's calculations suggested that just the first one is enough!

8.5.1. *Evidence.* As evidence, Todd gives another proofs by this processes of some results proved by the author and Lara Rodriguez, and of many relations found by his computations.

It still remains to have a good combinatorial description and structural understanding of all the relations.

9. Transcendence and algebraic independence results

Some isolated transcendence results on multizeta were obtained by the author [47, 10.5], [49, 2.7.8] by relations to the zeta case (as in the zeta-like case or through more involved relations) (and applying Jing Yu's results), or relations to the logarithmic values (and applying Papanikolas' results) or to non-torsion on $C^{\otimes n}$ (and applying Jing Yu's results) etc.

But let us now state very nice and strong results, compared to the real case, that are obtained by Chieh-Yu Chang and Yoshinori Mishiba, by using the motivic interpretation, the work on multi-polylogarithm values by Chang and the strong transcendence machine development mentioned in 2.10.

Theorem 9.1 ([12])**.** *Linear combination over the algebraic closure of K of monomials in multizeta of several weights vanishes, then a sub-linear combination over K of fixed weight monomials vanishes. In particular, any ratio of different weight monomials is transcendental, and non-trivial monomials in multizeta are transcendental. "Odd" weight monomial is algebraically independent to $\tilde{\pi}$.*

In fact, Chang has even more general results on multi-polylogarithm values that he defines. He also shows that the multizeta values are K-linear combination of their values on A^n. In [13], he also gives an efficient algorithm for the dimension calculation in depth 2, in any weight.

Theorem 9.2 ([34])**.** *Let $n_1, \ldots, n_d$ be distinct positive "odd" integers with n_i/n_j not an integral power of p for each $i \neq j$, then the $1 + d(d+1)$ elements*

$$\tilde{\pi}, \zeta(n_i), \zeta(n_j, n_{j+1}), \zeta(n_k, n_{k+1}, n_{k+2}), \ldots, \zeta(n_1, \ldots, n_d)$$

are algebraically independent over K.

Hence for $q \neq 2$, and $d \geq 2$, there are infinitely many multizeta of depth d algebraically independent over the field K adjoined with the zeta values, and this also implies [34] a good lower bound on the transcendence degree over K of the field generated by the multizeta values of bounded weight. For some more general results, see [35].

10. Ihara power series analog, Hidden structures

In the real case [24], the multizeta, as the periods coming from iterated extensions of Tate motives, is the de Rham–Betti side of the mixed motive structure provided by Deligne on the (Malcev completion) arithmetic fundamental group of projective line minus $0, 1, \infty$. The etale-galois side has developed understanding of Ihara power series giving a Galois co-cycle, at least at the "meta-abelian" level, if not at the full nilpotent level of the completion.

10.1. Ihara power series analog. We first describe the result in suggestive terms using analogies and without defining precisely all the terms to give the flavour.

Theorem 10.1 ([6]). *Let $A = \mathbb{F}_q[t]$, v be a prime of A and let G_K denote the absolute Galois group of K, and A_v the completion of A at v.*

We can construct 1-cocycle $I_\sigma \in A_v[\![Z]\!]^*$ ("Ihara power series"), $\sigma \in G_K$ satisfying $I_{\sigma\tau}(Z) = I_\tau(C_{\chi(\sigma)}(Z))I_\sigma(Z)$, where χ is the v-adic cyclotomic character of the Carlitz–Drinfeld cyclotomic theory and C is the extended Carlitz action. (Thus we have a $A_v[\![Z]\!]^*$-valued Galois representation of the Galois group of the separable closure of K over v^∞ cyclotomic extension of K).

The logarithmic derivative with respect to Z of I_σ, evaluated at the normalized exponential $e(Z)$, has a power series expansion with divided power coefficients being the v-adic limits of analogs of Deligne–Soule co-cycles (at finite levels), which connect to the extensions giving the zeta values as periods, and to the action on cyclotomic unit module analog of Anderson.

These power series interpolate Gauss sum analogs in tower in the sense that for a prime $\wp$ of $A[v^n]$, not dividing v, $I_{Frob_\wp}(\zeta_{v^n})$ is a normalized Gauss sum (analog) above $\wp$ for $K(\zeta_{v^n})$, where ζ_v^n is the v^n-torsion of the Carlitz module, and $Frob_\wp$ is the Frobenius element at $\wp$.

10.2. Techniques. While the original Ihara power series had two variables as in the Beta functions, and interpolated Jacobi sums, this is really an analog of Anderson's refinement in real case to a one variable series using his hyper-adelic Gamma functions. The "hyper-adelic" extension is not necessary in function fields. This construction, due to Anderson, which ties up a lot of our work on Gamma values, Gauss sums, Zeta values, cyclotomic theory, poly-logs etc. in v^∞ tower, uses the technology of Coleman power series, generalizing it from the finite to transcendental residue fields, for putting norm compatible families of "solitons" (interpolating partial products of gamma products by algebraic functions of certain kinds on products of curves) into division compatible families.

10.3. Hidden structures. In the function field case, analog of the fundamental group connection understanding, if any, is missing, but we still see multizeta (at the nilpotent completion level of de Rham–Betti side) and Ihara power series (at meta-abelian level of Galois side), so the motivic structures have nice and useful analogs, even if a little ad-hoc and quite different in the structural details. Similarly, in the real case, the motivic extensions are (conjecturally) related to K-theory (Deligne–Soule cocycles also have description in those terms), whose analog which would work in this context is also missing so far. The Hopf algebra structure mentioned in §5.2 suggests an underlying algebraic group still to be understood properly.

We should soon have better understanding of the motivic extensions here due to the recent works of Vincent Lafforgue [26] and Lenny Taelman [42, 43]. Another contrast is that the analog of $\mathbb{Z}_p$ in the Theorem above is $A_\wp$, which being huge, there is no suitable well-developed Iwasawa structure theory for it in the function field case. (See [4]).

11. Finite variants

Hoffman, Kaneko, Zagier and many others have considered finite variants of multizeta in the real case: Given a prime p, we define, for $s_i \in \mathbb{Z}$,

$$Z_p(s_1, \ldots, s_r) := \sum_{p>n_1>\cdots>n_r>0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

considered modulo p. Thus, s_i's matter only modulo $p-1$, but for a fixed s_i's there is an "universality" phenomena for large enough p.

Given a prime $\wp$ of $A = \mathbb{F}_q[t]$ of degree D, we can similarly define (modulo $\wp$)

$$Z_\wp(s_1, \ldots, s_r) = \sum_{\substack{a_i \in A+, \\ D>deg(a_1)>\cdots>\deg(a_r)}} \frac{1}{a_1^{s_1} \ldots a_r^{s_r}}.$$

11.1. Some differences. In depth one, while $p > n > 0$ covers the full range of (non-zero) residues modulo p, the monics of degree less that D do not cover such a range modulo $\wp$ of degree D. Hence, we get the analog of the fact that $Z_p(k)$ vanishes modulo p, for $p > k+1$, if we drop "monic" conditions on a_i, but as such values vanish or are related to $Z_\wp$ by an easy sign, we keep the monic condition.

11.2. Depth one Bernoulli–Goss connection. Let $q^D - 1 > s > 0$, then we have (modulo $\wp$)

$$\begin{aligned} Z_\wp(s) = Z_\wp(-(q^D-1-s)) &= \sum_{d=0}^{D-1} S_d(-(q^D-1-s)) \\ &= \sum_{i=0}^{\infty} S_d(-(q^D-1-s)) = \zeta(-(q^D-1-s)), \end{aligned}$$

where the last but one equality makes sense since the terms of the infinite sum are zero for $d > D-1$, by applying the result [47, 5.6.2] of Carlitz.

Thus there is a nice "Bernoulli-type" connection with the zeta values of Goss, related to Bernoulli–Goss polynomials.

Note that because of lack of a simple functional equation there are two analogs [47, 5.3] of Bernoulli numbers, the Bernoulli–Carlitz numbers coming from the zeta values at positive integers and the Bernoulli–Goss numbers coming from those at negative integers. The finite variant, this connection and the remark is also independently noted in [41] which also deals with irrationality results for them when considered in a suitable ring (following analog of Zagier work in the real case) which puts all $\wp$ together.

11.3. Nice congruences modulo $\wp$. We can push this further, and consider ∞-adic $(1-1/\wp^s)\zeta(s)$ naturally as $\sum_{d=0}^{\infty} S_{d,\wp}(s)$, where $S_{d,\wp}(s) = \sum \frac{1}{a^s}$, with the sum over $a \in A+$ prime to $\wp$. But the same sum considered $\wp$-adically converges to the $\wp$-adic zeta value $\zeta_\wp(s)$ of Goss.

Next, for a positive integer s and modulo $\wp$, we have

$$S_{d,\wp}(s) = S_d(s), \quad (d < D), \quad S_{D+i,\wp}(s) = 0, \quad (i > 0),$$
$$S_{D,\wp}(s) = -\sum_{i=0}^{D-1} S_i(s), \quad \text{if } s \text{ is "even" and 0 otherwise,}$$

where the first follows immediately as an equality from the definitions, the second can be seen by considering the terms modulo $\wp$, noting that they occur with a multiplicity which is a multiple of q which is zero in characteristic p, and the last follows by considering modulo $\wp$ and using that $\sum f^{-s}$, where f runs over elements of $\mathbb{F}_q^*$, is -1 or 0 according to whether s is "even" or not.

With the same understanding of handling the infinite sums having only finitely many zero terms as above, but now modulo $\wp$, we have a nice equality of four different kinds of zeta values below (defined with values naturally in K_∞, $K_\wp, A, K$ respectively, before reducing modulo $\wp$), for $q^{\deg(\wp)} - 1 > s > 0$:

$$\left(1 - \frac{1}{\wp^s}\right)\zeta(s) = \zeta_\wp(s) = 2\zeta\left(-\left(q^{\deg(\wp)} - 1 - s\right)\right) = 2Z_\wp(s) \mod \wp\,.$$

Note that these are all zero, if $p = 2$ or if s is "even".

11.4. Restrictions due to extra symmetries. In contrast to the real case finite variants where the values belong to the prime field $\mathbb{Z}/p\mathbb{Z}$, in our case, while the values a priori lie in $A/\wp A$, by the translation invariance and Galois $\mathbb{F}_q/\mathbb{F}_p$- invariance, the values belong to $\mathbb{F}_p(t^q - t) \subset A/\wp A$. This is just the prime field $\mathbb{F}_p$, for example, for the Artin–Schreier primes of the form $\wp = t^q - t - \alpha$, with $\alpha \in \mathbb{F}_p^*$ (e.g., any such if $q = p$).

12. Some ongoing projects and some other directions pursued

We briefly mention some projects being pursued.

12.1. Values at negative integers and Interpolations. In [47, 5.10] it was already explained how to consider multizeta at negative integers s_i, by grouping by degrees and how to interpolate by also removing $\wp$-factors to $\wp$-adic $\zeta_\wp$ at finite primes $\wp$ of A, following Goss' ideas for the zeta case, in a straight-forward manner. We only mention that the definition we give in Section 3 for the multizeta values for positive integers s_i, also makes sense for $s_i \in \mathbb{Z}$, because of vanishing of $S_d(s)$, for large enough d, if s is not positive. In fact, these multizeta are finite sums (and thus rational) if $s_1 \leq 0$, and convergent in general. But the second formula there will not work in general.

12.1.1. ***The real case.*** In contrast, in the real case, analytic continuation leads to singularities and points of indeterminacy at negative integers and there are several candidates of renormalized values there, due to several groups of mathematicians: Akiyama, Egami, Tanigawa; Guo, Zhang; Machon, Pachya; or Furusho, Komori, Matsumoto, Tsumura (for twisted entire versions). Because of this, there are no clear p-adic interpolations in general, but there are some candidates in special situations via interpolation and limit mechanisms due to Deligne, Unver etc. On the other hand, we have a few (related) motivic p-adic multizeta candidates independently due to Deligne, Furusho, Jourassay. Because of the motivic origin, these satisfy the same relations as conjectured for the real case (and one more coming from $\zeta_p(2n) = 0$) and conjecturally no more.

12.1.2. ***The function field situation.*** Author's current PhD students Shuhui Shi and Qibin Shen are working on developing understanding of vanishing and relations between the multizeta values at negative integers, of $\wp$-adic interpolated and motivic values, of "colored" variants (two analogs of the real case [55, Ch. 13], of "level" $n \in \mathbb{Z}$ or $a \in A$), of finite variants values, as well as of higher genus situations. So far, they have some results and conjectures on vanishing at negative integers, as well as on universal (working for all $\wp$ of large enough degree) relations for finite variants. They have discovered some families of relations between v-adic interpolated values, and found that the connecting formulas in the real case between p-adic and finite variants do not hold in our case without modifications.

12.2. Dimensions and transcendence degrees. We are investigating and verifying some conjectures on dimensions and transcendence degrees of various natural spaces of multizeta values taking into account depths and weights.

Let d_w denote the dimension of the K-span of $\tilde{\pi}^w$ and all the multizeta values of weight w and depth 2. (In the analogous situation the dimensions of cusp forms come up in the real case). Then [13] gives a formula for this in terms of motivic special points and also gives an algorithm to compute this and proves that $d_w \geq w - \lfloor (w-1)/(q-1) \rfloor$.

Based on the data of [13] based on the MAGMA program written by Yi-Hsuan Lin, the author has made the following observations/speculations:

Case $q = 2$: $d_2 = 1$ and d_w increases monotonically with jumps 0 or 1, so that it takes all values. Given d_w occurs 2^r times, for some $r \in \mathbb{Z}_{\geq 0}$. More precisely, it seems that starting at weight 2, the sequence giving the number of times the dimension 1, 2, 3 ... is repeated is (respectively) exactly 1, 2, 4, followed by two 4's, $2^0 - 1$ times 8's, two 4's, $2^1 - 1$ times 8's, ..., two 4's, $2^n - 1$ times 8's ... So the dimensions are 1, 2,2, 3,3,3,3, 4,4,4,4, 5,5,5,5, 6,6,6,6, 7,7,7,7, 8 occurring 8 times, then 9 and 10 occuring 4 times, followed by the dimensions 11, 12, 13 occurring 8 times and so on. (Checked by Lin, Chen up to weight 160, but we need to get to weights 197 (or rather 261) to get more confidence).

Case $q > 2$: Then $d_w = w$ for $w \leq q(q-1) - 1$, and the next is $w - 1$. d_w increases monotonically with possible (all occur infinitely often) jumps $0, 1$ or 2. There are no consecutive jumps of 0 or 2. Ignoring initial $q(q-1)$ segment, there are at most $2(q-1)$ consecutive values for d_w. (The bound seems best and seems to occur in bounded blocks of $q^2(q-1)$ magnitude size as mentioned below and at least for $q \leq 9$ that I checked, it occurs before weight $2q(q-1)$).

A little imprecise observation is that there seems to be some approximate pattern of length $q^2(q-1)$. For example, when $q = 3$, for every 18 steps starting from $w = 18n$, the pattern (which we do not describe here) follows one of the two possibilities (same for first twelve steps) and weight increases in the first pattern (which works for most of the time) by 11 and by 12, for the second pattern holding rarely by jumps differing by powers of 3. If this were true, the asymptotic dimension is 11/18 times weight. But this seems to work perfectly till weight 226, and then it changes. For example, $d_{234} - d_{216} = 10$, rather than 11 or 12. For 235 to 241 and 243 the difference is 9, but $d_{247} - d_{229} = 11$. So it is still unclear whether the asymptotic bound is the lowest possible $(9/18)w$ or more. Probably it is between $(9/18)w$ and $(11/18)w$. But this is very near our calculating capacity, so it is hard to make numerical progress right now.

Note that for $q = 2$, d_w also denotes (within 1) the dimension of span of multizeta of depth 2, weight w, modulo the span of sub-space generated by any products of zeta values.

12.3. Multi-polylogarithms. In [12], Chang introduced analogs of multi-polylogarithms $\sum_{n_1>\cdots>n_r}(z_1^{q^{n_1}}\ldots z_r^{q^{n_r}})/(\ell_{n_1}^{s_1}\ldots\ell_{n_r}^{s_r})$ and generalizing [7] formulas, he showed that the multizeta are K-linear combination of their values at tuples with integral co-ordinates. We refer to [12] for some general transcendence results on their values, and to [14], where he and Mishiba study a v-adic analog and show correspondence between the v-adic vanishing and the ∞-adic Eulerian nature, connecting to the torsion phenomena.

12.4. Many variable generalizations. The interesting contrast between the function fields and the number fields, where we can simultaneously deal with several independent copies of function fields at the same time, has been used as a powerful tool in many function field studies, including those of Drinfeld modules and Shtukas. In the context of Zeta and L functions [38] and multizeta [39, 40] (and his recent letter to the author), Pellarin has developed interesting multi-variable generalizations and has proved several interesting evaluations, and results about the relations.

12.5. Mixed motivic zeta category. In an ongoing work, building on the ideas of Lafforgue and Taelman mentioned in §10.3, Kirti Joshi is developing a suitable tannakian categorical framework of mixed Carlitz–Tate–Anderson t-motives suited to multizeta studies.

This expository paper is based on the series of talks on this subject at the conference at Imperial college in 2015, in honor of my friend David Goss, and at the conference at Lyon in 2016. David Goss was a constant source of encouragement to all in the subject area of function field arithmetic whose development owes a lot to him. This article was meant for the volume in his honor, but the volume has sadly turned into a memorial volume.

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