ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

Journal de l'École polytechnique
Mathématiques

Ramla Abdellatif & Auguste Hébert

Completed Iwahori-Hecke algebras and parahoric Hecke algebras for Kac-Moody groups over local fields

Tome 6 (2019), p. 79-118.

<http://jep.centre-mersenne.org/item/JEP_2019__6__79_0>

Publié avec le soutien
du Centre National de la Recherche Scientifique

CENTRE
MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Journal de l'École polytechnique
Mathématiques
Tome 6, 2019, p. 79–118 DOI: 10.5802/jep.88

COMPLETED IWAHORI-HECKE ALGEBRAS AND PARAHORIC HECKE ALGEBRAS FOR KAC-MOODY GROUPS OVER LOCAL FIELDS

by Ramla Abdellatif & Auguste Hébert

Abstract. — Let G be a split Kac-Moody group over a non-Archimedean local field. We define a completion of the Iwahori-Hecke algebra of G, then we compute its center and prove that it is isomorphic (via the Satake isomorphism) to the spherical Hecke algebra of G. This is thus similar to the situation for reductive groups. Our main tool is the masure $\mathscr{I}$ associated to this setting, which plays here the same role as Bruhat-Tits buildings do for reductive groups. In a second part, we associate a Hecke algebra to each spherical face F of type 0, extending a construction that was only known, in the Kac-Moody setting, for the spherical subgroup and for the Iwahori subgroup.

Résumé (Algèbres d'Iwahori-Hecke complétées et algèbres de Hecke parahoriques pour les groupes de Kac-Moody sur les corps locaux)

Soit G un groupe de Kac-Moody déployé sur un corps local non archimédien. Nous définissons une complétion de l'algèbre d'Iwahori-Hecke de G, puis nous prouvons que son centre est isomorphe (via l'isomorphisme de Satake) à l'algèbre de Hecke sphérique de G, ce qui est analogue au cas des groupes réductifs. Notre outil principal est la masure associée à G, qui joue ici un rôle similaire à celui de l'immeuble de Bruhat-Tits dans le cas réductif. Dans une seconde partie, nous associons une algèbre de Hecke à chaque face sphérique F de type 0. Jusqu'à présent cette construction n'était connue que pour le sous-groupe sphérique et le sous-groupe d'Iwahori.

Contents

1. Introduction 80
2. Masures: general framework 82
3. A topological restriction on parahoric subgroups 88
4. The completed Iwahori-Hecke algebra 91
5. Hecke algebra associated with a parahoric subgroup 106
References 117

2010 Mathematics Subject Classification. — 20G44, 20C08, 20E42.
Keywords. — Kac-Moody groups, Hecke algebras, masure, local fields, Iwahori-Hecke algebras.

The first author was supported by the ANR grants ANR-14-CE25-0002-01 and ANR-16-CE40-0010-01, by the GDR TLAG and by a CNRS grant PEPS-JCJC. The second author was supported by the ANR grant ANR-15-CE40-0012.

1. Introduction

Let $\mathbf{G}_0$ be a split reductive group over a non-Archimedean local field $\mathscr{K}$ and set $G_0 = \mathbf{G}_0(\mathscr{K})$. An important tool to study complex representations of G_0 are Hecke algebras attached to each open compact subgroup of G_0: if K is such a subgroup, the Hecke algebra $\mathscr{H}_K$ associated to K is the convolution algebra of complex-valued K-bi-invariant functions on G_0 with compact support. Two choices of K are of particular interest: the first one is when $K = K_s = \mathbf{G}(\mathscr{O})$ with $\mathscr{O}$ being the ring of integers of $\mathscr{K}$. In this case, K_s is a maximal open compact subgroup of G and $\mathscr{H}_s = \mathscr{H}_{K_s}$ is a commutative algebra called the *spherical Hecke algebra* of G_0. This algebra can be explicitly described through the Satake isomorphism: indeed, if W^v denotes the Weyl group of G_0 and $Q^\vee$ is the coweight lattice of G_0, then $\mathscr{H}_s$ is isomorphic to the subalgebra $\mathbb{C}[Q^\vee]^{W^v}$ of W^v-invariant elements in the group algebra of $(Q^\vee, +)$. A second interesting choice is when $K = K_I$ is an Iwahori subgroup of G_0: then $\mathscr{H} := \mathscr{H}_{K_I}$ is called the *Iwahori-Hecke algebra* of G_0. This algebra comes with a basis (called the Bernstein-Lusztig basis) indexed by the affine Weyl group of G_0 and such that the product of two elements of this basis can be expressed via the Bernstein-Lusztig presentation [Lus89]. This presentation enables us to compute the center of $\mathscr{H}$ and to check that it is isomorphic to the spherical Hecke algebra of G_0. These results can be summarized as follows:

$$\mathscr{H}_s \xrightarrow[\sim]{S} \mathbb{C}[Q^\vee]^{W^v} \xhookrightarrow{g} \mathscr{H}, \quad \text{and} \quad \mathrm{Im}(g) = \mathscr{Z}(\mathscr{H}),$$

where S denotes the Satake isomorphism and g comes from the Bernstein-Lusztig basis.

This article aims to study how far this theory can be extended to split Kac-Moody groups. Among the different definitions of Kac-Moody groups that are available in the literature, we choose to use the definition given by Tits in [Tit87] as it is more algebraic. Given $\mathbf{G}$ a split Kac-Moody group over $\mathscr{K}$, set $G = \mathbf{G}(\mathscr{K})$. To study G, Gaussent and Rousseau built in [GR08] an object $\mathscr{I} = \mathscr{I}(G)$ that they called a *masure* (also known as *affine ordered hovel*), and later extended by Rousseau [Rou16, Rou17]. This masure is a generalization of Bruhat-Tits buildings introduced in [BT72, BT84] as it gives back the Bruhat-Tits building of G when $\mathbf{G}$ is reductive. As a set, $\mathscr{I}$ is a union of apartments that are all isomorphic to a standard one (denoted by $\mathbb{A}$ in the sequel) and G acts on $\mathscr{I}$. We still have an arrangement of hyperplanes, called *walls*, but in general this arrangement is not locally finite anymore. This explains why faces in $\mathbb{A}$ are not sets anymore but filters. Another main difference with Bruhat-Tits buildings is that in general, two points of $\mathscr{I}$ are not necessarily contained in a common apartment.

Analogues of K_s and K_I (and more generally of parahoric subgroups) can be defined as fixers of some specific faces in $\mathscr{I}$. When $\mathbf{G}$ is an affine Kac-Moody group, Braverman, Kazhdan and Patnaik attached to G a spherical Hecke algebra and an Iwahori-Hecke algebra [BK11, BKP16], and they obtained a Satake isomorphism as well as Bernstein-Lusztig relations. All these results were generalized to arbitrary Kac-Moody groups by Bardy-Panse, Gaussent and Rousseau [GR14, BPGR16]. In this framework, the Satake isomorphism appears as an isomorphism between $\mathscr{H}_s := \mathscr{H}_{K_s}$ and $\mathbb{C}[\![Y]\!]^{W^v}$, where Y denotes a lattice that can be, in first approximation, thought of as the coroot lattice (though it can be really different, even in the affine case) and $\mathbb{C}[\![Y]\!]$ is its Looijenga's algebra (which is a completion of the group algebra $\mathbb{C}[Y]$, see Definition 4.6). So far, the analogy with the reductive case stops here. Indeed, let $\mathscr{H}$ be the Iwahori-Hecke algebra of G: following what happens in the reductive case [Par06], one would expect the center of $\mathscr{H}$ to be isomorphic to the spherical Hecke algebra $\mathscr{H}_s$. Unfortunately, this is not the case, as we prove that the center of $\mathscr{H}$ is actually more or less trivial (see Lemma 4.31). Moreover, in the non-reductive case, $\mathbb{C}[\![Y]\!]^{W^v}$ is a set of infinite formal series that cannot embed into $\mathscr{H}$, where all elements have finite support. All these reasons explain why we define a completion $\widehat{\mathscr{H}}$ of $\mathscr{H}$ as follows: letting $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ be the Bernstein-Lusztig basis of $\mathscr{H}$, we define $\widehat{\mathscr{H}}$ as the set of formal series $\widehat{\mathscr{H}}$ whose support satisfies some conditions similar to what appears in the definition of $\mathbb{C}[\![Y]\!]$. One of our main results states that $\widehat{\mathscr{H}}$ can be turned into an algebra when endowed with a well-defined convolution product compatible with the canonical inclusion of $\mathscr{H}$ into $\widehat{\mathscr{H}}$ (see Theorem 4.21 and Corollary 4.23). We then determine the center of $\widehat{\mathscr{H}}$ and show that it is isomorphic to $\mathbb{C}[\![Y]\!]^{W^v}$, as wanted (see Theorem 4.30). As before, these results can be summarized as follows:

$$\mathscr{H}_s \underset{\sim}{\xrightarrow{S}} \mathbb{C}[\![Y]\!]^{W^v} \overset{g}{\hookrightarrow} \widehat{\mathscr{H}}, \quad \text{and} \quad \mathrm{Im}(g) = \mathscr{Z}(\widehat{\mathscr{H}}),$$

where S denotes the Satake isomorphism and g comes again from the Bernstein-Luzstig basis.

Another part of this paper is devoted to the construction of Hecke algebras attached to more general subgroups than K_s and K_I. Recall that K_s is the fixer of $\{0\}$ while K_I is the fixer of the type 0 chamber C_0^+. When G is reductive, any face F between $\{0\}$ and C_0^+ corresponds to an open compact subgroup of G (namely the parahoric subgroup associated with F) contained in its fixer K_F, hence one can use it to attach a Hecke algebra to F. This explains why it seems natural, in the non-reductive case, to wonder whether one can attach a Hecke algebra to K_F for all faces F between $\{0\}$ and C_0^+. We succeed in defining such an algebra when F is spherical, which means that its fixer under the action of the Weyl group is finite. Our construction is very close to what is done for the Iwahori-Hecke algebra in [BPGR16]. We also prove that when F is not spherical and different from $\{0\}$ (that cannot happen if G is affine), this construction fails because the structural constants are infinite.

Finally, recall that up to now, there was no known topology on G that generalizes the usual topology on $\mathbf{G}_0(\mathscr{K})$, for $\mathbf{G}_0$ a split reductive group over $\mathscr{K}$, in which K_s and K_I are open compact subgroups of $\mathbf{G}_0(\mathscr{K})$. We prove (see Theorem 3.1) that when $\mathbf{G}$ is not reductive, there is no way to turn G into a topological group such that K_s or K_I (or, more generally, any given parahoric subgroup of G)

is open compact. This result implies in particular that one cannot define smooth representations of G in the same way as in the reductive case.

The paper is organized as follows: we first recall the definition of masures in Section 2. The reader only interested in Iwahori-Hecke algebras can read the two first sections and the last one, and skip the rest of this section. Section 3 is devoted to prove that G cannot be turned into a topological group in which K_s or K_I is open compact. In Section 4, we define the completed Iwahori-Hecke algebra $\widehat{\mathscr{H}}$ of G and compute its center, as well as the center of $\mathscr{H}$. Finally, we use Section 5 to attach a Hecke algebra to any spherical face between $\{0\}$ and C_0^+ and to prove that this construction fails if F is not spherical and different from $\{0\}$.

Remark. — As explained in Section 2, this paper is actually written in a more general framework, as we only need $\mathscr{I}$ to be an abstract masure and G to be a strongly transitive group of (positive, type-preserving) automorphisms of $\mathscr{I}$. In particular, this applies to almost split (and not only split) Kac-Moody groups over local fields.

Acknowledgements. — We warmly thank Stéphane Gaussent for suggesting our collaboration, for multiple discussions and for his useful comments on previous versions of this manuscript. We also thank Nicole Bardy-Panse and Guy Rousseau for discussions on this topic and for their comments on a previous version of this paper. Finally, we thank the referee for their valuable comments and suggestions, and for his/her interesting questions.

2. Masures: general framework

We recollect here some well-known facts. Further details are available in the first two sections of [Rou11].

2.1. Root generating system and Weyl groups. — A *Kac-Moody matrix* (or *generalized Cartan matrix*) is a square matrix $A = (a_{i,j})_{i,j\in I}$ indexed by a finite set I, with integral coefficients, and such that:

(i) $\forall\, i \in I,\ a_{i,i} = 2$;
(ii) $\forall\,(i,j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leqslant 0)$;
(iii) $\forall\,(i,j) \in I^2,\ (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$.

A *root generating system* is a 5-tuple $\mathscr{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix A indexed by the finite set I, of two dual free $\mathbb{Z}$-modules X and Y of finite rank, and of a free family $(\alpha_i)_{i\in I}$ (respectively $(\alpha_i^\vee)_{i\in I}$) of elements in X (resp. Y) called *simple roots* (resp. *simple coroots*) that satisfy $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all i, j in I. Elements of X (respectively of Y) are called *characters* (resp. *cocharacters*).

Fix such a root generating system $\mathscr{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and set $\mathbb{A} := Y \otimes \mathbb{R}$. Each element of X induces a linear form on $\mathbb{A}$, hence X can be seen as a subset of the dual $\mathbb{A}^*$. In particular, the α_i's (with $i \in I$) will be seen as linear forms on $\mathbb{A}$. This allows us to define, for any $i \in I$, an involution r_i of $\mathbb{A}$ by setting $r_i(v) := v - \alpha_i(v)\alpha_i^\vee$ for any $v \in \mathbb{A}$. Note that the points fixed by r_i are exactly the elements of $\ker\alpha_i$. We define the *Weyl group of $\mathscr{S}$* as the subgroup W^v of $\mathrm{GL}(\mathbb{A})$ generated by the finite set $\{r_i,\ i\in I\}$. The pair $(W^v,\{r_i,\ i\in I\})$ is a Coxeter system, hence we can consider the length $\ell(w)$ with respect to $\{r_i,\ i\in I\}$ of any element w of W^v.

For any $x\in\mathbb{A}$, we set $\underline{\alpha}(x)=(\alpha_i(x))_{i\in I}\in\mathbb{R}^I$. Let $P^\vee:=\{v\in\mathbb{A}\mid\underline{\alpha}(v)\in\mathbb{Z}^I\}$ be the *coweight-lattice*, which is not a lattice when $\mathbb{A}_{\mathrm{in}}:=\bigcap_{i\in I}\ker\alpha_i$ is non-zero, $Q^\vee:=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$ be the *coroot-lattice* and $Q^\vee_{\mathbb{R}}=\bigoplus_{i\in I}\mathbb{R}\alpha_i^\vee$. Furthermore setting $Q^\vee_+:=\bigoplus_{i\in I}\mathbb{N}\alpha_i^\vee$, we can define a pre-order $\leqslant_{Q^\vee}$ on $\mathbb{A}$ as follows: for any $x,y\in\mathbb{A}$, we say that $x\leqslant_{Q^\vee}y$ iff $y-x\in Q^\vee_+$. We also set $Q^\vee_-:=-Q^\vee_+$ for future reference.

There is an action of the Weyl group W^v on $\mathbb{A}^*$ given by the following formula:

$$\forall\, x\in\mathbb{A},\ w\in W^v,\ \alpha\in\mathbb{A}^*,\quad (w\cdot\alpha)(x):=\alpha(w^{-1}\cdot x).$$

Let $\Phi:=\{w\cdot\alpha_i\mid(w,i)\in W^v\times I\}$ be the set or *real roots*: then Φ is a subset of $Q:=\bigoplus_{i\in I}\mathbb{Z}\alpha_i$. We will also use the set $\Delta:=\Phi\cup\Delta^+_{\mathrm{im}}\cup\Delta^-_{\mathrm{im}}\subset Q$ of all roots, as defined in [Kac94]. Note that Δ is stable under the action of W^v. For any root $\alpha\in\Delta$, we set $\Lambda_\alpha:=\mathbb{Z}$ if $\alpha\in\Phi$, and $\Lambda_\alpha:=\mathbb{R}$ otherwise (i.e., if $\alpha\in\Delta_{\mathrm{im}}:=\Delta^+_{\mathrm{im}}\cup\Delta^-_{\mathrm{im}}$). For any pair $(\alpha,k)\in\Delta\times\mathbb{Z}$, we set

$$D(\alpha,k):=\{t\in\mathbb{A}\mid\alpha(t)+k\geqslant 0\}\quad\text{and}\quad D^\circ(\alpha,k):=\{t\in\mathbb{A}\mid\alpha(t)+k>0\}.\tag{2.1}$$

For any root $\alpha\in\Delta$, we also set $D(\alpha,+\infty):=\mathbb{A}$.

Finally, we let $W:=Q^\vee\rtimes W^v\subset\mathrm{GA}(\mathbb{A})$ be the *affine Weyl group of $\mathscr{S}$*, where $\mathrm{GA}(\mathbb{A})$ denotes the group of affine isomorphisms of $\mathbb{A}$. Note that $W\subset P^\vee\rtimes W^v$ and that $\alpha(P^\vee)$ is contained in $\mathbb{Z}$ for any $\alpha\in Q$. Consequently, if τ is a translation of $\mathbb{A}$ of vector $p\in P^\vee$, then for any $\alpha\in Q$, τ acts by permutations on the set $\{D(\alpha,k),\ k\in\mathbb{Z}\}$. On the other hand, as W^v stabilizes Δ, any element of W^v permutes the sets of the form $D(\alpha,k)$, where α runs over Δ. Hence we have an action of W on $\{D(\alpha,k),\ (\alpha,k)\in\Delta\times\mathbb{Z}\}$.

2.2. Vectorial faces and Tits cone. — As in the reductive case, define the *fundamental chamber* as $C^v_f:=\{v\in\mathbb{A}\mid\forall\, i\in I,\,\alpha_i(v)>0\}$. For any subset J in I, set

$$F^v(J)=\{v\in\mathbb{A}\mid\forall\, j\in J,\,\alpha_j(v)=0\ \text{ and }\forall\, i\in I\smallsetminus J,\,\alpha_i(v)>0\};$$

then the closure $\overline{C^v_f}$ of C^v_f is exactly the union of all $F^v(J)$'s for $J\subset I$. The *positive vectorial faces* (resp. *negative vectorial faces*) are defined as the sets $w\cdot F^v(J)$ (resp. $-w\cdot F^v(J)$) for $w\in W^v$ and $J\subset I$, and a *vectorial face* is either a positive vectorial face or a negative one. A *positive chamber* (resp. *negative chamber*) is a cone of the form $w\cdot C^v_f$ (resp. $-w\cdot C^v_f$) for some $w\in W^v$. Note that for any $x\in C^v_f$ and any $w\in W^v$, we have $(w\cdot x=1)\Rightarrow(w=1)$, which ensures that the action of $w\in W^v$ on the set of positive chambers is simply transitive.

Let $\mathscr{T}:=\bigcup_{w\in W^v}w\cdot\overline{C^v_f}$ be the *Tits cone* and $-\mathscr{T}$ be the negative cone. We can use it to define a W^v-invariant pre-order $\leqslant$ on $\mathbb{A}$ as follows:

$$\forall\,(x,y)\in\mathbb{A}^2,\quad x\leqslant y\iff y-x\in\mathscr{T}.$$

We also set $Y^+ := \mathscr{T} \cap Y$ and $Y^{++} := Y \cap \overline{C^v_f}$. We can now recall the following simple but very useful result [GR14, Lem. 2.4 a)].

Lemma 2.1. — *For any $\lambda \in Y^{++}$ and any $w \in W^v$, we have $w \cdot \lambda \leqslant_{Q^\vee} \lambda$.*

2.3. Filters and masures. — This section aims to recall what masures are. As stated in the introduction, the reader only interested in the completion of Iwahori-Hecke algebras can skip this section and go directly to Section 4.

Masures were first introduced for symmetrizable split Kac-Moody groups over a valued field whose residue field contains $\mathbb{C}$ by Gaussent and Rousseau [GR08]. Later, Rousseau axiomatized this construction in [Rou11], then generalized it with further developments to almost-split Kac-Moody groups over non-Archimedean local fields in [Rou16, Rou17]. For the reader familiar with this work, let us mention that we consider here semi-discrete masures which are thick of finite thickness.

2.3.1. *Filters, sectors and rays*

Definition 2.2. — A *filter on a set E* is a non-empty set F of non-empty subsets of E that satisfies the following conditions:

– for any subsets S_1, S_2 of E that both belong to F, then $S_1 \cap S_2$ belongs to F;
– for any subsets S_1, S_2 of E with S_1 in F and $S_1 \subset S_2$, then S_2 belongs to F.

Given a filter F on a set E and a subset E' of E, we say that F *contains* E' if every element of F contains E'. If E' is non-empty, then the set $F_{E'}$ of all subsets of E containing E' is a filter on E called the *filter associated to* E'. By language abuse and to ease notations, we will sometimes write that E' is a filter, by identification of E' with $F_{E'}$.

Now, let F be a filter on a finite-dimensional real affine space E. We define its *closure* $\overline{F}$ as the filter of all subsets of E that contain the closure of some arbitrary element of F, and its *convex hull* $\mathrm{conv}(F)$ as the filter of all subsets of E that contain the convex hull of some arbitrary element of F. Said differently, we have:

$$\overline{F} := \{S \subset E \mid \exists\, S' \in F,\ \overline{S'} \subset S\} \text{ and } \mathrm{conv}(F) := \{S \subset E \mid \exists\, S' \in F,\ \mathrm{conv}(S') \subset S\}.$$

Given two filters F_1 and F_2 on the same set E, we say that F_1 *is contained in* F_2 iff any subset of E contained F_2 is in F_1. Similarly, we say that the filter F_1 is *contained in a subset* Ω of E iff any subset of E contained in Ω is in F_1.

Let Ω be a subset of $\mathbb{A}$ containing an element x in its closure $\overline{\Omega}$ of Ω. The *germ of* Ω *in* x is defined as the filter $\mathrm{germ}_x(\Omega)$ of all subsets of $\mathbb{A}$ containing some neighborhood of x in Ω. A *sector* in $\mathbb{A}$ is a translate $\mathfrak{s} := x + C^v$ of a vectorial chamber $C^v = \pm w \cdot C^v_f$ (with $w \in W^v$) by an element $x \in \mathbb{A}$. The point x is called the *base point* of the sector $\mathfrak{s}$ and the chamber C^v is called its *direction*. One can easily check that the intersection of two sectors having the same direction is a sector with the same direction.

Given a sector $\mathfrak{s} := x + C^v$ as above, the *sector-germ of* $\mathfrak{s}$ is the filter $\mathfrak{S}$ of all subsets of $\mathbb{A}$ containing an $\mathbb{A}$-translate of $\mathfrak{s}$. Note that it only depends on the direction C^v of $\mathfrak{s}$. In particular, we denote by $\pm\infty$ the sector-germ of $\pm C^v_f$.

Finally, let δ be a ray with base point x and let $y \neq x$ be another point on δ (which amounts to say that δ contains the interval $]x, y] = [x, y] \smallsetminus \{x\}$, as well as the interval $[x, y]$). We say that δ is *pre-ordered* (resp. *generic*) if either $y \leqslant x$ or $x \leqslant y$ (resp. if $y - x \in \pm\mathring{\mathscr{T}}$, where $\pm\mathring{\mathscr{T}}$ denotes the interior of the cone $\pm\mathscr{T}$).

2.3.2. *Enclosures and faces*. — We keep the notations introduced at the end of Section 2.1. Given a filter F on $\mathbb{A}$, we define its *enclosure* $\mathrm{cl}_{\mathbb{A}}(F)$ as the filter of all subsets of $\mathbb{A}$ containing some element of F of the form $\bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha)$, with $k_\alpha \in \mathbb{Z} \cup \{+\infty\}$ for any $\alpha \in \Delta$. Sets of the form $D(\alpha, k)$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ are called *half-apartments* in $\mathbb{A}$. Sets of the form $M(\alpha, k) := \{t \in \mathbb{A} \mid \alpha(t) + k = 0\}$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ are called *walls* in $\mathbb{A}$.

A *local face in* $\mathbb{A}$ is a filter F^ℓ of the form $\mathrm{germ}_x(x + F^v)$, where $x \in \mathbb{A}$ is called the *vertex* of F^ℓ and a vectorial face $F^v \subset \mathbb{A}$ is a vectorial face called the *direction* of F^ℓ. To keep track of the elements x and F^v, such a local face may be denoted $F^\ell(x, F^v)$. A local face is said to be *spherical* when its direction is spherical: in this case, its pointwise stabilizer under the action of W^v is a finite group.

A *face in* $\mathbb{A}$ is a filter $F = F(x, F^v)$ associated with a point $x \in A$ and a vectorial face $F^v \subset A$ as follows: a subset S of $\mathbb{A}$ belongs to F iff it contains an intersection of half-spaces $D(\alpha, k_\alpha)$ or open half-spaces $\mathring{D}(\alpha, k_\alpha)$, with $\alpha \in \Delta$ and $k_\alpha \in \mathbb{Z}$, that also contains the local face $F^\ell(x, F^v)$. Note that (local) faces can be ordered as follows: given two such faces F, F' in $\mathbb{A}$, we say that F *is a face of* F' (or F' *contains* F, or F' *dominates* F) when $F \subset \overline{F'}$.

As explained at the end of Section 2.1, the action of W on $\mathbb{A}$ permutes the sets of the form $D(\alpha, k)$, where (α, k) runs over $\Delta \times \mathbb{Z}$. In particular, this implies that W permutes enclosures (resp. walls, faces) of $\mathbb{A}$.

The *dimension* of a face F is the smallest dimension of an affine space generated by some element S of F. Such an affine space is unique and is called the *support* of F. A *local chamber* (or local alcove) is a maximal local face, i.e., a local face of the form $F^\ell(x, \pm w \cdot C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local chamber* is $C_0^+ := F^\ell(0, C^v_f)$. A *local panel* is a spherical local face which is maximal among faces that are not chambers. Equivalently, a local panel is a spherical local face of dimension $\dim \mathbb{A} - 1$. Analogue definitions of chambers and panels exist, see for instance [GR08, §1.4]. Finally, a local face is *of type* 0 (or: is a *type* 0 *local face*) if its vertex lies in Y. We denote by F_0 the local face $F^\ell(0, \mathbb{A}_{\mathrm{in}})$, where $\mathbb{A}_{\mathrm{in}} = F^v(I) = \bigcap_{i\in I} \ker(\alpha_i)$. From now on, we will write *type* 0 *face* instead of *type* 0 *local face* to make it shorter.

Remark 2.3. — In [Rou11], Rousseau defines a notion of *chimney* that he uses in his axiomatization of masures. We do not define here what chimneys are: we only recall that each sector-germ is a splayed, solid chimney-germ, that each spherical face is contained in a solid chimney and that the action of W on $\mathbb{A}$ permutes the chimneys and preserve their properties (being splayed or solid for instance). For more details about this, see [Rou11, §1.10].

2.3.3. *Apartments and masure of type* $\mathbb{A}$. — An *apartment of type* $\mathbb{A}$ is a set A together with a non-empty set $\mathrm{Isom}(\mathbb{A}, A)$ of bijections (called *Weyl-isomorphisms*) such that, given $f_0 \in \mathrm{Isom}(\mathbb{A}, A)$, the elements of $\mathrm{Isom}(\mathbb{A}, A)$ are exactly the bijections of the form $f_0 \circ w$ with $w \in W$. All the isomorphisms considered in the sequel will be Weyl-isomorphisms, hence we will only write isomorphism instead of Weyl-isomorphism.

An isomorphism $\phi : A \to A'$ between two apartments of type $\mathbb{A}$ is a bijection such that: $f \in \mathrm{Isom}(\mathbb{A}, A) \iff \phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$. By construction, all the notions that are preserved under the action of W can be extended to any apartment of type A. For instance, we can define sectors, enclosures, faces or chimneys in any apartment A of type $\mathbb{A}$, as well as a pre-order $\leqslant_A$ on A.

We can now define the most important object of this section: the masures of type $\mathbb{A}$.

Definition 2.4. — A *masure of type* $\mathbb{A}$ is a set $\mathscr{I}$ endowed with a covering $\mathscr{A}$ of subsets (called *apartments*) such that the five following axioms hold.

(MA1) Any $A \in \mathscr{A}$ can actually be endowed with a structure of apartment of type $\mathbb{A}$.

(MA2) If F is a point (resp. a germ of a preordered interval, a generic ray, a solid chimney) contained in an apartment A and if A' is an other apartment containing F, then $A \cap A'$ contains $\mathrm{cl}_A(F)$ and there exists an isomorphism from A to A' that fixes $\mathrm{cl}_A(F)$.

(MA3) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains both $\mathfrak{R}$ and F.

(MA4) If two apartments A, A' contain both $\mathfrak{R}$ and F as in (MA3), then there exists an isomorphism from A to A' that fixes $\mathrm{cl}_A(\mathfrak{R} \cup F)$.

(MAO) If x and y are two points that are both contained in two apartments A and A' and such that $x \leqslant_A y$, then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

Recall that saying that an apartment contains a germ of a filter means that it contains at least one element of this germ. Similarly, a map fixes a germ when it fixes at least one element of this germ.

From now on, $\mathscr{I}$ will denote a masure of type $\mathbb{A}$. We assume that $\mathscr{I}$ is thick of *finite thickness*, which means that the number of chambers (= alcoves) containing a given panel is finite and greater or equal to three. We also assume that there exists a group G of automorphisms of $\mathscr{I}$ that acts strongly transitively on $\mathscr{I}$, which implies that all the isomorphisms involved in the axioms above are induced by the action of elements of G. We fix an apartment in $\mathscr{I}$ that we identify with $\mathbb{A}$ and call the *fundamental apartment of* $\mathscr{I}$. As the action of G on $\mathscr{I}$ is strongly transitive, the apartments of $\mathscr{I}$ are exactly the sets $g \cdot \mathbb{A}$ with $g \in G$. Let N be the stabilizer of $\mathbb{A}$ in G: it defines a group of affine automorphisms of $\mathbb{A}$, and we assume that this group is $W^v \ltimes Y$. As we will see in Section 2.4, these assumptions are not very restrictive for our purpose as they are all satisfied by the masure attached to a split Kac-Moody group G over a non-Archimedean local field (see also [GR08] and [Rou16]).

Remark 2.5. — In a recent work [Héb17a, §5], the second author gives a much simpler axiomatic for masures. To simplify the arguments, the reader can assume that G is an affine Kac-Moody group, in which case the three axioms (MA2), (MA4) and (MAO) can be replaced by the following statement [Héb17a, Th. 5.38]: *for any two apartments A and A' in $\mathscr{I}$, we have $A\cap A'=\mathrm{cl}(A\cap A')$ and there exists an isomorphism from A to A' that fixes $A\cap A'$.* This partially explains why the affine case is less technical, as it does not require to question the existence of isomorphisms that fix subsets of an intersection of apartments.

Remark 2.6. — Let F be a local face of an apartment A and A' be another apartment that contains F. Then F is also a local face of A' and there exists an isomorphism from A to A' that fixes F. Indeed, if x is the vertex of F and if J is a germ of a pre-ordered segment based at x and contained in F, then the enclosure of J contains F and the application of (MA2) to J now proves the claim.

Remark 2.7. — Pick $w\in W$ and $\mathscr{F}$ a filter on $\mathbb{A}$ fixed by w: then w fixes $\mathrm{cl}(\mathscr{F})$. Combined with the argument used in Remark 2.6, this proves here that for any vectorial face F^v and any base point x, the fixer in G of the face $F(x,F^v)$ is exactly the fixer in G of the corresponding local face $F^\ell(x,F^v)$.

Remark 2.8. — As we noticed earlier, each apartment A of $\mathscr{I}$ can be endowed with a pre-order $\leqslant_A$ induced by $\leqslant_{\mathbb{A}}$. Let A be an apartment of $\mathscr{I}$ and x,y be two points in A such that $x\leqslant_A y$. By [Rou11, Prop. 5.4], we know that for any apartment A' of $\mathscr{I}$ that contains both x and y, we also have $x\leqslant_{A'} y$. We hence get a relation $\leqslant$ on $\mathscr{I}$ and [Rou11, Th. 5.9] ensures that this relation is a G-invariant pre-order on $\mathscr{I}$.

2.4. Masure attached to a split Kac-Moody group. — As in [Tit87] or in [Rém02, Chap. 8], we consider the group functor $\mathbf{G}$ associated with the root generating system $\mathscr{S}$ fixed in Section 2.1. This functor goes from the category of rings to the category of groups and satisfies axioms (KMG1)–(KMG9) of [Tit87]. For any field R, the group $\mathbf{G}(R)$ is uniquely determined by these axioms [Tit87, Th. 1']. Furthermore, this functor $\mathbf{G}$ contains a toric functor $\mathbf{T}$ (denoted by $\mathscr{T}$ in [Rém02]) that goes from the category of rings to the category of abelian groups, and two functors $\mathbf{U}^{\pm}$ going from the category of rings to the category of groups.

In particular, let $\mathscr{K}$ be a non-Archimedean local field. Denote by $\mathscr{O}$ its ring of integers, by ϖ a fixed uniformizer of $\mathscr{O}$, by q the cardinality of the residue class field $\mathscr{O}/\varpi\mathscr{O}$ and set $G:=\mathbf{G}(\mathscr{K})$ (as well as $U^{\pm}:=\mathbf{U}^{\pm}(\mathscr{K})$, $T:=\mathbf{T}(\mathscr{K})$, etc.). For any sign $\varepsilon\in\{+,-\}$ and any root $\alpha\in\Phi^{\varepsilon}$, there is an isomorphism x_α from $\mathscr{K}$ to a root group U_α. For any integer $k\in\mathbb{Z}$, we get a subgroup $U_{\alpha,k}:=x_\alpha(\varpi^k\mathscr{O})$ of U_α (see [GR08, §3.1] for precise definitions). Let $\mathscr{I}$ denote the masure attached to G in [Rou17]: then the following properties hold.

– The fixer of $\mathbb{A}$ in G is $H:=\mathbf{T}(\mathscr{O})$ [GR08, Rem. 3.2].

– The fixer of $\{0\}$ in G is $K_s:=\mathbf{G}(\mathscr{O})$ [GR08, Exam. 3.14]. Applying (MA2) to $\{0\}$ and using Remark 2.7, we get that K_s is also the fixer in G of the face F_0.

– For any pair $(\alpha, k) \in \Phi \times \mathbb{Z}$, the fixer of $D(\alpha, k)$ in G is $H.U_{\alpha,k}$ [GR08, Exer. 4.2.7)].

– For any sign $\varepsilon \in \{+, -\}$, U^ε is the fixer in G of $\varepsilon\infty$ [GR08, Exam. 4.2.4)].

Moreover, each panel is contained in $q+1$ chambers, hence $\mathscr{I}$ is thick of finite thickness.

Remark 2.9. — The group G is reductive iff W^v is finite. In this case, $\mathscr{I}$ is the usual Bruhat-Tits building of G and we have $\mathscr{T} = \mathbb{A}$ and $Y^+ = Y$.

3. A topological restriction on parahoric subgroups

3.1. Statement of the result and idea of the proof. — In this section, we will prove that beside the reductive case, it is impossible to endow G with a structure of topological group for which K_s or K_I are open compact subgroups, where K_I denotes the (standard) Iwahori subgroup.(1) In fact, we will prove the following result, which is slightly more general.

Theorem 3.1. — *Let F be a type 0 face of $\mathbb{A}$ and let K_F be its fixer in G. If W^v is infinite, then there is no topology of topological group on G for which K_F is open and compact.*

Let F be a type 0 face of $\mathbb{A}$, i.e., a local face whose vertex lies in Y. First note that, up to replacing F by $h \cdot F$ for some well-chosen $h \in G$, which leads to consider the conjugate of K_F under h instead of K_F, we can assume that F is contained in $C_0^{\pm}$. As the treatment of both cases is similar, we assume that F is contained in C_0^+. To prove Theorem 3.1, it is enough to prove the existence of $g \in G$ such that $K_F/(K_F \cap g \cdot K_F.g^{-1})$ is infinite.

To explain the strategy of proof, we need to introduce some more notations. Let $\alpha \in \Phi^+$ and $(w, i) \in W^v \times I$ be such that $\alpha = w \cdot \alpha_i$. For any $k \in \mathbb{Z}$, set

$$M_k^\alpha := \{t \in \mathbb{A} \mid \alpha(t) = k\}, \quad D_k^\alpha := \{t \in \mathbb{A} \mid \alpha(t) \leqslant k\},$$

and let $K_{\alpha,k}$ be the fixer of D_k^α in G. Furthermore, pick a panel P_k^α in M_k^α and a chamber C_k^α contained in $\operatorname{conv}(M_k^\alpha, M_{k+1}^\alpha)$ that dominates P_k^α.

For any $i \in I$, we let q_i+1 (resp. $q_i'+1$) be the number of chambers containing $P_0^{\alpha_i}$ (resp. $P_1^{\alpha_i}$). By [Rou11, Prop. 2.9] and [Héb16, Lem. 3.2], q_i and q_i' do not depend on the choices of the panels $P_0^{\alpha_i}$ and $P_1^{\alpha_i}$. (This fact will be explained in the proof of Lemma 3.4.) As $\alpha_i(\alpha_i^\vee) = 2$, and as there exists an element of G that induces on $\mathbb{A}$ a translation of vector $\alpha_i^\vee$ (because we assumed that the stabilizer N of $\mathbb{A}$ in G induces $W^v \ltimes Y$ for group of affine automorphisms), the value of $1 + q_i$ (resp. $1 + q_i'$) is also the number of chambers that contain $P_{2k}^{\alpha_i}$ (resp. $P_{2k+1}^{\alpha_i}$) for any integer k.

Let us now explain the basic idea of the proof. Pick $g \in G$ such that $g \cdot 0 \in C_f^v$ and set $F' := g \cdot F$: then $K_F/(K_F \cap K_{F'})$ is in bijection with $K_F \cdot F'$. For $\alpha = w \cdot \alpha_i$, set $\widetilde{K_\alpha} := \bigcup_{k \in \mathbb{Z}} K_{\alpha,k}$: then $\mathbb{T}_\alpha := \widetilde{K_\alpha} \cdot \mathbb{A}$ is a semi-homogeneous extended tree with

(1)We recall that K_I is the fixer in G of the fundamental local chamber C_0^+.

parameters q_i and q_i'. Using the thickness of $\mathscr{I}$, we can prove that the number n_α of walls between 0 and $g \cdot 0$ that are parallel to $\alpha^{-1}(\{0\})$ satisfies $|K_{\alpha,1} g \cdot 0| \geqslant 2^{n_\alpha}$, which implies that $|K_F g \cdot 0| \geqslant 2^{n_\alpha}$. As n_α can be made arbitrarily large (for a suitable choice of α) when W^v is infinite, this will end the proof.

3.2. Detailed proof of Theorem 3.1. — Fix for now $\alpha = w \cdot \alpha_i$. Set $K_\alpha := K_{\alpha,1}$ and pick a sector-germ $\mathfrak{q}$ contained in D_0^α. By (MA3), we know that for any $x \in \mathscr{I}$, there exists an apartment A_x that contains both x and $\mathfrak{q}$. Axiom (MA4) implies the existence of an isomorphism $\phi_x : A_x \to \mathbb{A}$ that fixes $\mathfrak{q}$, and [Rou11, §2.6] ensures that $\phi_x(x)$ does not depend on the choice of the apartment A_x nor on the isomorphism ϕ_x, hence we can denote this element by $\rho_{\mathfrak{q}}(x)$. The map $\rho_q : \mathscr{I} \to \mathbb{A}$ is the retraction of $\mathscr{I}$ onto $\mathbb{A}$ centered at $\mathfrak{q}$, and its restriction to $\mathbb{T}_\alpha$ does not depend on the choice of $\mathfrak{q} \in D_0^\alpha$.

Remark 3.2. — Let A be an apartment containing $\mathfrak{q}$ and $\phi = \rho_q|_A$ be the restriction to A of the retraction map ρ_q. Then ϕ is the unique isomorphism of apartments that fixes $A \cap \mathbb{A}$. Indeed, (MA4) implies the existence of an isomorphism of apartments $\psi : A \to \mathbb{A}$ that fixes $\mathfrak{q}$. By definition, $\rho_{\mathfrak{q}}$ coincides with ψ on A, hence $\phi = \psi$ is an isomorphism of apartments, and fixes $\mathbb{A}$, hence ϕ fixes $\mathbb{A} \cap A$. If $f : A \to \mathbb{A}$ is an isomorphism of apartments that fixes $A \cap \mathbb{A}$, then $f \circ \phi^{-1} : \mathbb{A} \to \mathbb{A}$ is an isomorphism of affine spaces that fixes $\mathfrak{q}$, hence it must be trivial, which proves that $f = \phi$ is unique.

Lemma 3.3. — *Let $v \in \widetilde{K}_\alpha$ and $x \in \mathbb{A}$ be such that $v \cdot x \in \mathbb{A}$. Then v fixes $D^\alpha_{\lceil \alpha(x) \rceil}$.*

Proof. — Set $A := v \cdot \mathbb{A}$ and let $k \in \mathbb{Z}$ be such that $A \cap \mathbb{A}$ contains D_k^α. By [Héb16, Lem. 3.2], $A \cap \mathbb{A}$ is a half-apartment,(2) hence there exists an integer $k_1 \in \mathbb{Z}$ such that $A \cap A = D^\alpha_{k_1}$. Let $\phi : \mathbb{A} \to A$ be the isomorphism of apartments induced by v: Remark 3.2 ensures that ϕ fixes $A \cap \mathbb{A}$, which means that v fixes $A \cap \mathbb{A}$. As $v \cdot x$ belongs to $A \cap \mathbb{A}$, we obtain that $v \cdot (v \cdot x) = v \cdot x$, hence $v \cdot x = x$ belongs to $D^\alpha_{k_1}$. This implies that $D^\alpha_{\lceil \alpha(x) \rceil}$ is contained in $D^\alpha_{k_1} = A \cap \mathbb{A}$, hence is fixed by v. □

Lemma 3.4. — *Let $x \in \mathbb{A}$ and $M_x := M^\alpha_{\lceil \alpha(x) \rceil}$. Then the map $f : K_\alpha \cdot x \to K_\alpha . M_x$ defined by $f(v \cdot x) := v \cdot M_x$ is well-defined and bijective.*

Proof. — Let $v, v' \in K_\alpha$ be such that $v \cdot x = v' \cdot x$: by Lemma 3.3, we get that $v^{-1} \cdot v'$ fixes M_x, hence f is well-defined. Now assume that $v, v' \in K_\alpha$ satisfy $v \cdot M_x = v'.M_x$. Set $u := v^{-1} v'$ and $A := u \cdot \mathbb{A}$: then $u \cdot M_x = M_x$ is contained in $A \cap \mathbb{A}$ hence u fixes $D^\alpha_{\lceil \alpha(x) \rceil}$ by Lemma 3.3. In particular, we have $u \cdot x = x$, i.e., $v \cdot x = v' \cdot x$ and f is injective. As f is surjective by definition, the lemma is proved. □

Set $\alpha_{\mathscr{I}} := \alpha \circ \rho_{\mathfrak{q}}$ and, for any integer $k \geqslant 0$, let $\mathscr{C}_k^\alpha$ be the set of all chambers C that dominate some element of $K_\alpha . P_k$ and satisfy $\alpha_{\mathscr{I}}(C) > k$ (which means that there

(2)Our definition of half-apartments is a bit different from the definition of [Héb16]: what we call half-apartments correspond to the *true* half-apartments of [Héb16].

exists $X \in C$ such that $\alpha_{\mathscr{I}}(x) > k$ for all $x \in X$). Assume also that the chamber C_k^α chosen in Section 3.1 is not contained in D_k^α.

Lemma 3.5. — *For any integer $k \geqslant 0$, the map $g_k : K_\alpha M_{k+1}^\alpha \to \mathscr{C}_k^\alpha$ sending $v \cdot M_{k+1}^\alpha$ onto $v \cdot C_k^\alpha$ is well-defined and bijective.*

Proof. — The proof of the first part of the assertion is as in Lemma 3.4: if $v, v' \in K_\alpha$ are such that $v \cdot M_{k+1}^\alpha = v'.M_{k+1}^\alpha$, then $u := v^{-1} \cdot v'$ satisfies $u \cdot M_{k+1}^\alpha \subset \mathbb{A}$ and Lemma 3.3 implies that $u \cdot C_k^\alpha = C_k^\alpha$, i.e., $v \cdot C_k^\alpha = v' \cdot C_k^\alpha$. As we moreover have $\alpha_{\mathscr{I}}(v \cdot C_k^\alpha) = \alpha(C_k^\alpha) > k$, $v \cdot C_k^\alpha = v' \cdot C_k^\alpha$ belongs to $\mathscr{C}_k^\alpha$ and the map g_k is well-defined.

Assume now that $v, v' \in K_\alpha$ are such that $v \cdot C_k^\alpha = v' \cdot C_k^\alpha$. Set $u := v^{-1}v'$ and let $X \in C_k^\alpha$ be an element fixed by u. Let $x \in X$ be such that $\alpha(x) > k$: by Lemma 3.3, u fixes $M_{k+1}^\alpha \subset D_{\lceil \alpha(x) \rceil}^\alpha$, hence g_k is injective.

It remains to check that g_k is surjective, i.e., that $\mathscr{C}_k^\alpha = K_\alpha \cdot C_k^\alpha$. If $C \in \mathscr{C}_k^\alpha$, then there exists $u \in K_\alpha$ such that C dominates $u \cdot P_k^\alpha$. By [Rou11, Prop. 2.9.1)], there is an apartment A that contains both $u \cdot D_k^\alpha$ and C, and Remark 3.2 now gives an explicit isomorphism $\phi : A \to \mathbb{A}$ that fixes $A \cap \mathbb{A}$. If $v \in K_\alpha$ induces ϕ, then $\alpha_{\mathscr{I}}(C) = \alpha(v \cdot C)$, hence $\alpha(v \cdot C) > k$, and $v \cdot C \subset \mathbb{A}$ dominates P_k^α, hence $v \cdot C = C_k^\alpha$, i.e., $C = v^{-1} \cdot C_k^\alpha$, which ends the proof. □

Combining Lemmas 3.4 and 3.5, we get the following corollary.

Corollary 3.6. — *For any $x \in \mathbb{A}$, if $k := \max(1, \lceil \alpha(x) \rceil)$, then $|K_\alpha \cdot x| = q_i' q_i q_i' \cdots$ ($k-1$ factors).*

Until the end of this section, we assume that W^v is infinite.

Lemma 3.7. — *Let F be a type 0 face of $\mathbb{A}$. If W^v is infinite, then there exists $g \in G$ such that $K_F / K_F \cap K_{g \cdot F}$ is infinite.*

Proof. — Let $g \in G$ be such that $a := g \cdot 0$ belongs to C_f^v and set $F' := g \cdot F$. Let $(\alpha_n)_{n \geqslant 0}$ be an injective sequence of positive real roots (i.e., $\alpha_n \in \Phi^+$ for any non-negative integer n). As we have $K_{\alpha_n} \subset K_F$, hence $|K_F \cdot F'| \geqslant |K_{\alpha_n}.a|$, for all $n \geqslant 0$, it is enough to check (by Corollary 3.6 and thickness of $\mathscr{I}$) that $\alpha_n(a) \to +\infty$ as $n \to +\infty$.

By definition, any α_n can be written as $\sum_{i \in I} \lambda_{n,i} \alpha_i$ with $\lambda_{n,i} \in \mathbb{Z}_+$ for all $(n, i) \in \mathbb{Z}_+ \times I$. The injectivity of the sequence $(\alpha_n)_{n \geqslant 0}$ implies that $\sum_{i \in I} \lambda_{n,i} \to +\infty$ as n goes to $+\infty$, hence $\lim_{n \to +\infty} \alpha_n(a) = +\infty$ as required. □

Corollary 3.8. — *Let F be a type 0 face of $\mathscr{I}$. If W^v is infinite, then there is no topology of topological group on G for which K_F is open and compact.*

Proof. — If there was such a topology, then for any $g \in G$, K_F and $K_{g \cdot F} = gK_F g^{-1}$ would be open and compact in G, hence $K_F \cap K_{g \cdot F}$ would have the same properties. This would imply the finiteness of the quotient $K_F / K_F \cap K_{g \cdot F}$ for any $g \in G$, which contradicts Lemma 3.7. □

Considering $F = F_0$ (resp. $F = C_0^+$), we obtain that K_s (resp. K_I) cannot be open and compact in G when W^v is infinite, i.e., when G is not reductive. This shows how different reductive groups and (non-reductive) Kac-Moody groups are from this point of view.

4. The completed Iwahori-Hecke algebra

4.1. Definition of the usual Iwahori-Hecke algebra. — Let us first recall briefly the construction of the Iwahori-Hecke algebra via its Bernstein-Lusztig presentation, as done in [BPGR16, §6.6]. Note that this definition requires some restrictions on the possible choices for the ring of scalars; nevertheless, choosing $\mathbb{C}$ or $\mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}]$ is allowed when G is a split Kac-Moody group over $\mathscr{K}$. Another definition of the Iwahori-Hecke algebra (as an algebra of functions on pairs of type 0 chambers in a masure) is given in [BPGR16, Def. 2.5] and allows more flexibility in the choice of scalars. This will be recalled in Section 5.

Let $\mathscr{R}_1 := \mathbb{Z}[(\sigma_i, \sigma_i')_{i\in I}]$, where $(\sigma_i)_{i\in I}$ and $(\sigma_i')_{i\in I}$ denote indeterminates that satisfy the following relations:

– if $\alpha_i(Y) = \mathbb{Z}$, then $\sigma_i = \sigma_i'$;

– if $(i,j) \in I^2$ are such that r_i and r_j are conjugate (i.e., such that $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$), then $\sigma_i = \sigma_j = \sigma_i' = \sigma_j'$.

To define the Iwahori-Hecke algebra $\mathscr{H}$ associated with $\mathbb{A}$ and $(\sigma_i, \sigma_i')_{i\in I}$, we first introduce the Bernstein-Lusztig-Hecke algebra. Let ${}^{\mathrm{BL}}\mathscr{H}$ be the free $\mathscr{R}_1$-module with basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$. For short, set $H_i := H_{r_i}$ for $i \in I$, as well as $H_w = Z^0 H_w$ for $w \in W^v$ and $Z^\lambda = Z^\lambda H_1$ for $\lambda \in Y$. The *Bernstein-Lusztig-Hecke algebra* ${}^{\mathrm{BL}}\mathscr{H}$ is the module ${}^{\mathrm{BL}}\mathscr{H}$ equipped with the unique product $*$ that turns it into an associative algebra and satisfies the following relations (known as *Bernstein-Lusztig relations*):

(BL1) $\forall\, (\lambda, w) \in Y \times W^v, \quad Z^\lambda * H_w = Z^\lambda H_w;$

(BL2) $\forall\, i \in I, \forall\, w \in W^v,\ H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) = \ell(w) + 1, \\ (\sigma_i - \sigma_i^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) = \ell(w) - 1; \end{cases}$

(BL3) $\forall\, (\lambda, \mu) \in Y^2, \quad Z^\lambda * Z^\mu = Z^{\lambda+\mu};$

(BL4) $\forall\, \lambda \in Y, \forall\, i \in I, \quad H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)}),$

$$\text{with} \quad b(t, u; z) = \frac{(t - t^{-1}) + (u - u^{-1})z}{1 - z^2}.$$

The existence and unicity of such a product $*$ comes from [BPGR16, Th. 6.2]. Following [BPGR16, §6.6], the *Iwahori-Hecke algebra* $\mathscr{H}_{\mathscr{R}_1}$ associated with $\mathbb{A}$ and $(\sigma_i, \sigma_i')_{i\in I}$ is now defined as the $\mathscr{R}_1$-submodule of ${}^{\mathrm{BL}}\mathscr{H}$ spanned by $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ (recall that $Y^+ = Y \cap \mathscr{T}$ with $\mathscr{T}$ being the Tits cone). Note that for G reductive, we recover the usual Iwahori-Hecke algebra of G.

Remark 4.1. — This construction is compatible with extension of scalars. Let indeed $\mathscr{R}$ be a ring that contains $\mathbb{Z}$ and $\phi : \mathscr{R}_1 \to \mathscr{R}$ be a ring homomorphism such that $\phi(\sigma_i)$ and $\phi(\sigma'_i)$ are invertible in $\mathscr{R}$ for all $i \in I$: then the Iwahori-Hecke algebra associated with $\mathbb{A}$ and $(\phi(\sigma_i), \phi(\sigma'_i))_{i\in I}$ over $\mathscr{R}$ is $\mathscr{H}_{\mathscr{R}} = \mathscr{R} \otimes_{\mathscr{R}_1} \mathscr{H}_{\mathscr{R}_1}$.

Remark 4.2. — When G is a split Kac-Moody group over $\mathscr{K}$, we can (and will) set $\sigma_i = \sigma'_i = \sqrt{q}$ for all $i \in I$ and $\mathscr{R} = \mathbb{Z}[\sqrt{q}^{\pm 1}]$. The corresponding Iwahori-Hecke algebra $\mathscr{H}_{\mathscr{R}}$ will simply be denoted by $\mathscr{H}$.

4.2. Almost finite sets in Y and Y^+. — We fix a pair $(\mathscr{R}, \phi)$ as in Remarks 4.1 and 4.2. In this section, we introduce a notion of almost finite sets in Y and Y^+, that will be used to define the Looijenga algebra $\mathscr{R}[\![Y]\!]$ in the next section and the completed Iwahori-Hecke algebra $\widehat{\mathscr{H}} = \widehat{\mathscr{H}_{\mathscr{R}}}$ in Section 4.4.

4.2.1. *Definition of almost finite sets*

Definition 4.3. — A subset E of Y is *almost finite (in Y)* if there is a finite set $J \subset Y$ such that: $\forall\, \lambda \in E,\ \exists\, \nu \in j \mid \lambda \leqslant_{Q^\vee} \nu$.

Replacing Y by Y^+ in the previous definition, we have the definition of almost finite sets in Y^+. Nevertheless, the following lemma (applied to $F = Y^+$) justifies why we do not set this other definition apart: it shows indeed that almost finiteness for Y^+ can already be seen in Y, which explains why we will just write *almost finite sets* with no more specification.

Lemma 4.4. — *Let $E \subset Y$ be an almost finite set. For any subset F of Y, there exists a finite set $J \subset F$ such that $F \cap E \subset \bigcup_{j\in J}(j - Q^\vee_+)$.*

Proof. — As E is almost finite, we can assume that E is contained in $y - Q^\vee_+$ for some well-chosen $y \in Y$. Let J be the set of all elements in $F \cap E$ that are maximal in $F \cap E$ for the pre-order $\leqslant_{Q^\vee}$. As E is almost finite, we already have: $\forall\, x \in E,\ \exists\, \nu \in K \mid x \leqslant_{Q^\vee} \nu$. Let us prove that J is finite, which will conclude the proof. To do this, we identify $Q^\vee$ with $\mathbb{Z}^I$ and set $J' := \{u \in Q^\vee \mid y - u \in J\}$. We define a comparison relation $\prec$ on $Q^\vee$ as follows: for all $x = (x_i)_{i\in I}$ and $x' = (x'_i)_{i\in I}$, we write $x \prec x'$ when $x_i \leqslant x'_i$ (in $\mathbb{Z}$) for all $i \in I$ and $x \neq x'$. By definition of J, elements of J' are pairwise non comparable, hence [Héb17b, Lem. 2.2] implies that J' is finite, which requires that J itself is finite and completes the proof. □

4.2.2. *Examples of almost finite sets in Y^+*

In the affine case. — Suppose that $\mathbb{A}$ is associated with an affine Kac-Moody matrix A. By [Rou11, Rem. 5.10], we know that $\mathscr{T} = \delta^{-1}(\mathbb{R}^*_+) \sqcup \mathbb{A}_{\mathrm{in}}$, where δ denotes the smallest positive imaginary root of A, and that δ is W^v-invariant, thus $\delta(\alpha_i^\vee) > 0$ for all $i \in I$. Therefore, an almost finite set of Y^+ is a set E contained in $\bigcup_{i=1}^k (y_i - Q^\vee_+)$ for some integer $k \geqslant 1$ and some $y_1, \ldots, y_k \in Y^+$.

In the indefinite case. — Unlike the finite or the affine case, when $\mathbb{A}$ is associated with an indefinite Kac-Moody matrix A, we have: $\forall\, y \in Y,\ y - Q^\vee_+ \not\subseteq Y^+$. Indeed, due to the proof and the statement of [GR14, Lem. 2.9], there exists a linear form $\delta : \mathbb{A} \to \mathbb{R}$ such that $\delta(\mathscr{T}) \geqslant 0$ and $\delta(\alpha_i^\vee) < 0$ for all $i \in I$. Consequently, if $y \in Y$ and $i \in I$, then $\delta(y-n\alpha_i^\vee)<0$ for n large enough, hence $y-Q_+^\vee$ is not contained in Y^+. However, Y^+ may be contained in $Q_-^\vee$, as stated by the following lemma.

Lemma 4.5. — *We have $Y^+\subset Q_-^\vee$ iff we have $Y^{++}\subset Q_-^\vee$.*

Proof. — As Y^{++} is contained in Y^+, the direct implication is obvious. Assume conversely that $Y^{++}\subset Q_-^\vee$. By Lemma 2.1, we know that $\lambda\leqslant_{Q^\vee}\lambda^{++}$ for any $\lambda\in Y^+$, with λ^{++} in Y^{++}, hence we must have $Y^+\subset Q_-^\vee$ and the proof is complete. □

We say that $\mathbb{A}$ is the *essential realization of the Kac-Moody matrix A* when $\mathbb{A}_{\mathrm{in}}=\{0\}$, or equivalently when $\dim_{\mathbb{R}}\mathbb{A}$ equals the size of the matrix A. If $\mathbb{A}$ is the essential realization of an indefinite matrix $A=\left(\begin{smallmatrix}2 & a_{1,2}\\ a_{2,1} & 2\end{smallmatrix}\right)$ of size 2, with $a_{1,2}$ and $a_{2,1}$ negative integers, then $\mathbb{A}=\mathbb{R}\alpha_1^\vee\oplus\mathbb{R}\alpha_2^\vee$. For any integers λ and μ of opposite sign (i.e., such that $\lambda\mu<0$), we have $(2\lambda+a_{1,2}\mu)(a_{2,1}\lambda+2\mu)<0$, hence Y^{++} is contained in $Q_+^\vee\cup Q_-^\vee$. By [Kac94, Th. 4.3], we get that $Q_+^\vee\cap Y^{++}=\{0\}$, hence Y^{++} is contained in $Q_-^\vee$, and Lemma 4.5 implies that Y^+ is also contained in $Q_-^\vee$. Consequently, every subset of Y^+ is almost finite.

Note that this conclusion does not always hold when A is of size $n\geqslant 3$. Indeed, assume for instance that $\mathbb{A}$ is the essential realization of the matrix

$$A=\begin{pmatrix}2 & 0 & -2\\ 0 & 2 & 0\\ -5 & 0 & 2\end{pmatrix}.$$

Then $-2\alpha_1^\vee+\alpha_2^\vee-\alpha_3^\vee$ is in Y^{++} but not in $Q_-^\vee$.

4.3. The Looijenga algebra $\mathscr{R}[\![Y]\!]$. — We keep the previous notations. The next definition follows the definition of the algebra A given in [Loo80, §4].

Definition 4.6. — Let $(e_\lambda)_{\lambda\in Y}$ be a family of symbols that satisfy $e^\lambda e^\mu=e^{\lambda+\mu}$ for all $\lambda,\mu\in Y$. The *Looijenga algebra* $\mathscr{R}[\![Y]\!]$ of Y over $\mathscr{R}$ is defined as the set of formal series $\sum_{\lambda\in Y}a_\lambda e^\lambda$ with $(a_\lambda)_{\lambda\in Y}\in\mathscr{R}^Y$ having almost finite support.

For any element $\lambda\in Y$, let $\pi_\lambda:\mathscr{R}[\![Y]\!]\to\mathscr{R}$ be the "λ-the coordinate map" defined by $\pi_\lambda\bigl(\sum_{\mu\in Y}a_\mu e^\mu\bigr):=a_\lambda$. Define $\mathscr{R}[\![Y^+]\!]$ and $\mathscr{R}[\![Y]\!]^{W^v}$ as follows:

$$\begin{cases}\mathscr{R}[\![Y^+]\!]:=\{a\in\mathscr{R}[\![Y]\!]\mid\forall\,y\in Y\smallsetminus Y^+,\pi_\lambda(a)=0\}\\ \mathscr{R}[\![Y]\!]^{W^v}:=\{a\in\mathscr{R}[\![Y]\!]\mid\forall\,(\lambda,w)\in Y\times W^v,\ \pi_\lambda(w\cdot a)=\pi_\lambda(a)\}.\end{cases}$$

One can check that $\mathscr{R}[\![Y^+]\!]$ and $\mathscr{R}[\![Y]\!]^{W^v}$ are $\mathscr{R}$-subalgebras of $\mathscr{R}[\![Y]\!]$.

Definition 4.7. — A family $(a_j)_{j\in J}\in(\mathscr{R}[\![Y]\!])^J$ is *summable* if:
- for all $\lambda\in Y$, $\{j\in J\mid\pi_\lambda(a_j)\neq 0\}$ is finite;
- the set $\{\lambda\in Y\mid\exists\,j\in J,\,\pi_\lambda(a_j)\neq 0\}$ is almost finite.

Given a summable family $(a_j)_{j\in J}\in(\mathscr{R}[\![Y]\!])^J$, we set $\sum_{j\in J}a_j:=\sum_{\lambda\in Y}b_\lambda e^\lambda$, with $b_\lambda:=\sum_{j\in J}\pi_\lambda(a_j)$ for any $\lambda\in Y$. For $\lambda\in Y^{++}$, set $E(\lambda):=\sum_{\mu\in W^v\cdot\lambda}e^\mu\in\mathscr{R}[\![Y]\!]$. (Note that this is well-defined by Lemma 2.1.) Finally, for any $\lambda\in\mathscr{T}$, let λ^{++} be the unique element in $\overline{C_f^v}$ that has the same W^v-orbit as λ (i.e., such that $W^v \cdot \lambda = W^v \cdot \lambda^{++}$).

Lemma 4.8. — *Let $y \in Y$. Then $W^v \cdot y$ is upper-bounded (for $\leqslant_{Q^\vee}$) iff y belongs to Y^+.*

Proof. — If y belongs to Y^+, then Lemma 2.1 implies that $W^v \cdot y$ is upper-bounded by y^{++}. Assume conversely that $y \in Y$ is such that $W^v \cdot y$ is upper-bounded for $\leqslant_{Q^\vee}$ and let $x \in W^v \cdot y$ be a maximal element. For any $i \in I$, we have $r_i(x) \leqslant_{Q^\vee} x$, hence $\alpha_i(x) \geqslant 0$, which proves that x belongs to $\overline{C_f^v}$ and implies that y is in Y^+. □

Denote by $\mathrm{AF}_{\mathscr{R}}(Y^{++})$ the set of elements of $\mathscr{R}^{Y^{++}}$ having almost finite support.

Proposition 4.9. — *The map $E : \mathrm{AF}_{\mathscr{R}}(Y^{++}) \to \mathscr{R}[\![Y]\!]^{W^v}$ that sends $a \in \mathrm{AF}_{\mathscr{R}}(Y^{++})$ to $\sum_{\lambda \in Y^{++}} \pi_\lambda(a) E(\lambda)$ is well-defined and bijective. In particular, we have $\mathscr{R}[\![Y]\!]^{W^v} \subset \mathscr{R}[\![Y^+]\!]$.*

Proof. — Let $a = (a_\lambda)_{\lambda \in Y^{++}}$ be an element of $\mathrm{AF}_{\mathscr{R}}(Y^{++})$. As a has almost finite support, there exists a finite set $J \subset Y$ such that: $\forall\, \lambda \in \mathrm{supp}(a),\ \exists\, \nu \in J \mid \lambda \leqslant_{Q^\vee} \nu$. We start by proving that $(a_\lambda E(\lambda))_{\lambda \in Y^{++}}$ is summable. Let $\nu \in Y$ and set

$$F_\nu := \{\lambda \in Y^{++} \mid \pi_\nu(a_\lambda E(\lambda)) \neq 0\}.$$

For any $\lambda \in F_\nu$, ν belongs to $W^v \cdot \lambda$, hence Lemma 2.1 implies that $\nu \leqslant_{Q^\vee} \lambda$. As there exists moreover some $j \in J$ such that $\lambda \leqslant_{Q^\vee} j$, we get the finiteness of F_ν. Now let $F := \bigcup_{\nu \in Y} F_\nu$. We just saw that any element of F is dominated (for $\leqslant_{Q^\vee}$) by some element of J, hence F is by definition almost finite, and $(a_\lambda E(\lambda))_{\lambda \in Y^{++}}$ is summable.

By construction, $E(\lambda)$ is in $\mathscr{R}[\![Y]\!]^{W^v}$ for any $\lambda \in Y^{++}$, hence $\sum_{\lambda \in Y^{++}} a_\lambda E(\lambda)$ is in $\mathscr{R}[\![Y]\!]^{W^v}$ too and E is well-defined. Now assume that $a \in \mathrm{AF}_{\mathscr{R}}(Y^{++})$ is non-zero and let ν be maximal (for $\leqslant_{Q^\vee}$) among the elements λ of Y^{++} such that $\pi_\lambda(a) \neq 0$. Then $\pi_\nu(E(a)) = \pi_\nu(a) \neq 0$, hence $E(a)$ is non-zero and E is injective. To prove E is surjective, let $u = \sum_{\lambda \in Y} u_\lambda e^\lambda$ be any element of $\mathscr{R}[\![Y]\!]^{W^v}$ and let $\lambda \in \mathrm{supp}(u)$. As $\mathrm{supp}(u)$ is almost finite and W^v-invariant, $W^v \cdot \lambda$ is upper-bounded, hence Lemma 4.8 implies that λ belongs to Y^+. This proves that $\mathrm{supp}(u)$ is contained in Y^+, and that $u = E((\pi_\lambda(u))_{\lambda \in \mathrm{supp}(u)^{++}})$ is in the image of E, which completes the proof. □

Remark 4.10. — Lemma 4.8 and Proposition 4.9 are not explicitly stated in [GR14], but their proof is basically contained in the proof of [GR14, Th. 5.4].

4.4. The completed Iwahori-Hecke algebra $\widehat{\mathscr{H}}$.

— In this subsection, we define an $\mathscr{R}$-algebra $\widehat{\mathscr{H}}$ as a "completion" of the usual Iwahori-Hecke algebra, what justifies the name of *completed Iwahori-Hecke algebra* given to $\widehat{\mathscr{H}}$. In the next section, we will compute the centers of $\mathscr{H}$ and of $\widehat{\mathscr{H}}$, and recover the reasons that motivated the introduction of $\widehat{\mathscr{H}}$ in this context.

Endow W^v with its Bruhat order $\leqslant$ and, for any $w \in W^v$, set

$$[1, w] := \{u \in W^v \mid u \leqslant w\}.$$

This notation makes sense as $1 \leqslant w$ for all $w \in W^v$. Let $\mathscr{B} := \prod_{\lambda\in Y^+,\ w\in W^v} \mathscr{R}$.

Definition 4.11. — For any $a = (a_{\lambda,w})_{(\lambda,w)\in Y^+\times W^v}$ in $\mathscr{B}$, the *support of* a is the set

$$\operatorname{supp}(a) := \big\{(\lambda, w) \in Y^+ \times W^v \mid a_{\lambda,w} \neq 0\big\}.$$

The *support of a along Y* is the set

$$\operatorname{supp}_Y(a) := \big\{\lambda \in Y^+ \mid \exists\, w \in W^v, a_{\lambda,w} \neq 0\big\}$$

and the *support of a along W^v* is the set

$$\operatorname{supp}_{W^v}(a) := \big\{w \in W^v \mid \exists\, \lambda \in Y^+, a_{\lambda,w} \neq 0\big\}.$$

Definition 4.12. — A subset Z of $Y^+ \times W^v$ is *almost finite* if

$$\{w \in W^v \mid \exists\, \lambda \in Y^+, (\lambda, w) \in Z\}$$

is finite and if, for all $w \in W^v$, the set $\{\lambda \in Y^+ \mid (\lambda, w) \in Z\}$ is almost finite (in the sense of Definition 4.3).

Let $\widehat{\mathscr{H}}$ be the set of all elements in $\mathscr{B}$ with almost finite support. An element $(a_{\lambda,w})_{(\lambda,w)\in Y^+\times W^v}$ of $\widehat{\mathscr{H}}$ will also be written as $\sum_{(\lambda,w)\in Y^+\times W^v} a_{\lambda,w} Z^\lambda H_w$. Any pair $(\lambda, w) \in Y^+ \times W^v$ defines a projection map $\pi_{\lambda,w} : \widehat{\mathscr{H}} \to \mathscr{R}$ defined by $\pi_{\lambda,w}\big(\sum_{(\nu,u)\in Y^+\times W^v} a_{\nu,u} Z^\nu H_u\big) := a_{\lambda,w}$.

To extend the product $*$ to $\widehat{\mathscr{H}}$, we start by proving that for any elements

$$\sum_{(\lambda,w)\in Y^+\times W^v} a_{\lambda,w} Z^\lambda H_w \quad \text{and} \quad \sum_{(\lambda,w)\in Y^+\times W^v} b_{\lambda,w} Z^\lambda H_w$$

of $\widehat{\mathscr{H}}$, and any pair $(\mu, v) \in Y^+ \times W^v$, the sum

$$\sum_{(\lambda,w),(\lambda',w')\in Y^+\times W^v} \pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$$

is a finite sum, i.e., that only finitely many terms $\pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$ are non-zero. The key fact to prove this is that for any pair $(\lambda, w) \in Y \times W^v$, the support of $H_w * Z_\lambda$ along Y^+ is in the convex hull of $\{u \cdot \lambda, u \in [1, w]\}$. This fact comes from Lemma 4.15 below.

For any subset E of Y and any $i \in I$, set $R_i(E) := \operatorname{conv}(\{E, r_i(E)\}) \subset E + Q^\vee$. When $E = \{\lambda\}$ is reduced to a single element, we write $R_i(\lambda)$ instead of $R_i(\{\lambda\})$. For any $w \in W^v$ and any $\lambda \in Y$, set $R_w(\lambda) := \bigcup R_{i_1}(R_{i_2}(\ldots(R_{i_k}(\lambda))\ldots))$, where the union is taken over all the reduced writings $r_{i_1}.r_{i_2}\ldots r_{i_k}$ of w.

Remark 4.13. — For any pair $(\lambda, w) \in Y \times W^v$, the set $R_w(\lambda)$ is actually finite. Indeed, given any finite set E and any $i \in I$, the set $R_i(E)$ is bounded and contained in $E + Q^\vee$, hence must be finite. By induction, we get that for any integer $k \geqslant 0$ and any list $(i_1, \ldots, i_k)$ of elements of I, the set $R_{i_1}(R_{i_2}(\ldots(R_{i_k}(E))\ldots))$ is also finite. As w has only finitely many reduced writings,[3] we obtain that $R_w(\lambda)$ is finite.

[3] The number of reduced writings of w is upper-bounded by $|I|^{\ell(w)}$.

Lemma 4.14. — *For all $i\in I$ and all $\lambda\in Y$, the product H_i*Z^λ is in*
$$\bigoplus_{(\nu,t)\in R_i(\lambda)\times\{1,r_i\}}\mathscr{R}\cdot Z^\nu H_t.$$

Proof. — Let $i\in I$ and $\lambda\in Y$. If $\sigma_i=\sigma_i'$, then (BL4) implies that
$$H_i*Z^\lambda=Z^{r_i(\lambda)}*H_i+(\sigma_i-\sigma_i^{-1})Z^\lambda\,\frac{1-Z^{-\alpha_i(\lambda)\alpha_i^\vee}}{1-Z^{-\alpha_i^\vee}},$$
hence we have the following alternative.

– Either $\alpha_i(\lambda)=0$, and $H_i*Z^\lambda=Z^\lambda*H_i$, i.e., H_i and Z^λ commute to each other.
– Or $\alpha_i(\lambda)>0$, in which case we have
$$H_i*Z^\lambda=Z^{r_i(\lambda)}*H_i+(\sigma_i-\sigma_i^{-1})\sum_{k=0}^{\alpha_i(\lambda)-1}Z^{\lambda-k\alpha_i^\vee}$$
with $r_i(\lambda)$ and $\lambda-k\alpha_i^\vee$ in $R_i(\lambda)$ for all $k\in[\![0,\alpha_i(\lambda)-1]\!]$.
– Or $\alpha_i(\lambda)<0$, in which case we have
$$H_i*Z^\lambda=Z^{r_i(\lambda)}*H_i+(\sigma_i-\sigma_i^{-1})\sum_{k=1}^{-\alpha_i(\lambda)}Z^{\lambda+k\alpha_i^\vee}$$
with $r_i(\lambda)$ and $\lambda+k\alpha_i^\vee$ in $R_i(\lambda)$ for all $k\in[\![1,-\alpha_i(\lambda)]\!]$.

In any case, we have proven that H_i*Z^λ is in $\bigoplus_{(\nu,t)\in R_i(\lambda)\times\{1,r_i\}}\mathscr{R}Z^\nu H_t$ when $\sigma_i=\sigma_i'$.

If $\sigma_i\neq\sigma_i'$, then $\alpha_i(Y)=2\mathbb{Z}$, so we have now
$$H_i*Z^\lambda=Z^{r_i(\lambda)}*H_i+Z^\lambda\left((\sigma_i-\sigma_i^{-1})+(\sigma_i'-\sigma_i'^{-1})\right)Z^{-\alpha_i^\vee},$$
and similar computations as those done in the $\sigma_i=\sigma_i'$ case complete the proof. □

Lemma 4.15. — *For all $w,w'\in W^v$ and all $\lambda\in Y$, $H_{w'}*Z^\lambda H_w$ is in*
$$\bigoplus_{(\nu,t)\in R_w(\lambda)\times[1,w]\cdot w'}\mathscr{R}\cdot Z^\nu H_t.$$

Proof. — The proof goes by induction on $\ell(w')\geqslant 0$. There is nothing to prove if $\ell(w')=0$, and if $\ell(w')=1$, this is exactly Lemma 4.14. Now, fix an integer $k\geqslant 1$ and a pair $(w,\lambda)\in W^v\times Y^+$, and assume that for any $u\in W^v$ satisfying $\ell(u)\leqslant k-1$, the product $H_u*Z^\lambda H_w$ belongs to $\bigoplus_{(\nu,t)\in R_w(\lambda)\times[1,w]\cdot u}\mathscr{R}\cdot Z^\nu H_t$. Let $w'\in W^v$ be an element of length k and write $w'=r_iu$ for $i\in I$ and $u\in W^v$ of length $k-1$. Then $H_{w'}*Z^\lambda H_w=H_i*H_u*Z^\lambda H_w$ belongs to $\bigoplus_{(\nu,t)\in R_u(\lambda)\times[1,u]\cdot w}\mathscr{R}H_i*Z^\lambda H_t$, with
$$\bigoplus_{(\nu,t)\in R_u(\lambda)\times[1,u]\cdot w}\mathscr{R}H_i*Z^\lambda H_t\subset\bigoplus_{(\nu,t)\in R_u(\lambda)\times[1,u]\cdot w}\ \bigoplus_{(\nu',t')\in R_i(\nu)\times\{1,r_i\}}\mathscr{R}\cdot Z^{\nu'}*H_{t'}*H_t$$
by Lemma 4.14. Using (BL2) for $t'\in\{1,r_i\}$ and $t\in[1,u]\cdot w$, we get that $H_{t'}*H_t$ belongs to $\mathscr{R}H_{r_it}\oplus\mathscr{R}H_t$, hence to $\bigoplus_{v\in[1,w']\cdot w}\mathscr{R}H_v$, and the lemma follows. □

Lemma 4.16. — *Let $(\lambda,w)\in Y\times W^v$. For any $\nu\in R_w(\lambda)$, there is a family $(a_u)_{u\leqslant w}\in[0,1]^{[1,w]}$ such that $\sum_{u\in[1,w]}a_u=1$ and $\nu=\sum_{u\in[1,w]}a_u\cdot u\cdot\lambda$.*

Proof. — This proof goes again by induction on $\ell(w)\geqslant 0$. There is nothing to do when $\ell(w)=0$. Let $k\in\mathbb{Z}$ be non-negative and assume that the statement of the lemma is true whenever $w\in W^v$ is of length k. Let $w'\in W^v$ be of length $k+1$, fix $\lambda\in Y$ and pick $\nu\in R_{w'}(\lambda)$. Then there exists some triple $(i,w,\nu')\in I\times W^v\times W^v$, with w of length k and $\nu'\in R_w(\lambda)$, such that $\nu\in R_i(\nu')$. As we can write $\nu'=\sum_{u\in[1,w]}a_u u\cdot\lambda$ for some family $(a_u)_{u\in[1,w]}\in[0,1]^{[1,w]}$ and $\nu=s\nu'+(1-s)r_i\nu'$ for some $s\in[0,1]$, we finally get that $\nu=\sum_{u\in[1,w]}(sa_u u\cdot\lambda+(1-s)a_u r_i u\cdot\lambda)$ with $r_i u\in[1,w']$ for all $u\in[1,w]$, hence the lemma follows. $\square$

Lemma 4.17

(1) *For all $\lambda,\mu\in Y^+$, we have $(\lambda+\mu)^{++}\leqslant_{Q^\vee}\lambda^{++}+\mu^{++}$.*
(2) *Let $(\lambda,w)\in Y^+\times W^v$. For all $\nu\in R_w(\lambda)$, we have $\nu^{++}\leqslant_{Q^\vee}\lambda^{++}$.*

Proof. — Let $\lambda,\mu\in Y^+$ and $w\in W^v$ be such that $(\lambda+\mu)^{++}=w\cdot(\lambda+\mu)$. By Lemma 2.1, we have $w\cdot\lambda\leqslant_{Q^\vee}\lambda^{++}$ and $w\cdot\mu\leqslant_{Q^\vee}\mu^{++}$, hence we get the first statement. Together with Lemma 2.1, Remark 4.13 and Lemma 4.16, the first statement implies the second one. $\square$

Define the *height* $h(x)$ of any $x=\sum_{i\in I}x_i\alpha_i^\vee\in Q^\vee$ by $h(x):=\sum_{i\in I}x_i$. For any $\lambda\in Y^+$, let $w_\lambda\in W^v$ be the element of minimal length such that $w^{-1}\cdot\lambda\in\overline{C^v_f}$: then we have $\lambda=w_\lambda\cdot\lambda^{++}$.

Lemma 4.18. — *Let $\lambda\in Y^{++}$ and $(\mu_n)_{n\geqslant 0}\in(W^v\cdot\lambda)^{\mathbb{Z}_+}$ be such that*

$$\lim_{n\to+\infty}\ell(w_{\mu_n})=+\infty.$$

Then $\lim_{n\to+\infty}h(\mu_n-\lambda)=-\infty$.

Proof. — By Lemma 2.1, we know that $h(\mu_n-\lambda)$ is well-defined. For all $\alpha\in\Phi_+$, we have $\alpha(\lambda)\geqslant 0$. Assume that $\alpha(\lambda)=0$: then $\alpha_i(\lambda)=0$ for all $i\in I$, hence $r_i\lambda=\lambda$ for all $i\in I$ and $W^v\cdot\lambda$ is then reduced to $\{\lambda\}$, which contradicts the fact that $\lim_{n\to+\infty}\ell(w_{\mu_n})=+\infty$. This proves that $\alpha(\lambda)>0$ for all $\alpha\in\Phi_+$.

Now let $\rho:\mathbb{A}\to\mathbb{R}$ be a linear form satisfying $\rho(\alpha_i^\vee)=1$ for all $i\in I$. Pick $n\in\mathbb{Z}_+$ and set $w:=w_{\mu_n}$. Then we have

$$h(\mu_n-\lambda)=h(w\cdot\lambda-\lambda)=\rho(w\cdot\lambda-\lambda)=(w^{-1}\rho-\rho)(\lambda).$$

Thanks to [Kum02, Cor. (3)], we know that $w^{-1}\cdot\rho-\rho=-\sum_{\alpha\in\Phi_{w^{-1}}}\alpha$ with $\Phi_{w^{-1}}:=\Delta^+\cap w^{-1}\cdot\Delta^-$. By [Kum02, Lem. 3.14], we also know that $|\Phi_{w^{-1}}|=\ell(w^{-1})=\ell(w)$. Letting n go to $+\infty$, we obtain that

$$h(\mu_n-\lambda)=-\sum_{\alpha\in\Phi_{w_{\mu_n}^{-1}}}\alpha(\lambda)$$

goes to $-\infty$, as required. $\square$

Lemma 4.19. — *For all $(\lambda,\mu,w)\in Y^+\times Y^+\times W^v$, the set $\{\nu\in W^v\cdot\lambda\mid\mu\in R_w(\nu)\}$ is finite.*

Proof. — Let $N \geqslant 0$ be an integer that satisfies the following property:

$$\forall \nu' \in W^v \cdot \lambda \mid \ell(w_{\nu'}) \geqslant N,\ h(\nu' - \lambda) < h(\mu - \lambda). \tag{4.1}$$

(Such an integer exists by Lemma 4.18.) Let $\nu \in W^v \cdot \lambda$ be such that μ belongs to $R_w(\nu)$ and set $u := w_\nu$. Following Lemma 4.16, write $\mu = \sum_{x\in[1,w]} a_x x \cdot \nu$ with $(a_x)_{x\in[1,w]} \in [0,1]^{[1,w]}$ such that $\sum_{x\in[1,w]} a_x = 1$. For all $x \in [1,w]$, set $v(x) := w_{x\cdot\nu}$. Then there exists $x \in [1,w]$ such that $\ell(v(x)) < N$. Indeed, suppose by contradiction that $\ell(v(x)) \geqslant N$ for all $x \in [1,w]$. As

$$\mu - \lambda = \sum_{x\in[1,w]} a_x(x\cdot\nu - \lambda) = \sum_{x\in[1,w]} a_x(v(x)\nu - \lambda),$$

we obtain from (4.1) that

$$h(\mu - \lambda) = \sum_{x\in[1,w]} a_x h(v(x) - \lambda) < \sum_{x\in[1,w]} a_x h(\mu - \lambda) = h(\mu - \lambda),$$

which is absurd. We can hence pick $\overline{u} \in [1,w]$ such that $\ell(v(\overline{u})) < N$. As $\overline{u}\cdot\nu = v(\overline{u})\cdot\nu^{++}$, we have $\ell(\overline{u}^{-1}v(\overline{u})) \geqslant \ell(u)$ by definition of u. It implies that $\ell(v(\overline{u})) + \ell(u) \geqslant \ell(v(\overline{u})) + \ell(\overline{u}) \geqslant \ell(u)$, hence we have $\ell(u) \leqslant N + \ell(\overline{u})$, so $\ell(u)$ is upper-bounded and the lemma follows. $\square$

To allow infinite sums in $\widehat{\mathscr{H}}$, we need a suitable notion of summable families, as we have by Definition 4.7 for the Looijenga algebra $\mathbb{R}[\![Y]\!]$. This is the purpose of the next definition.

Definition 4.20. — A family $(a_j)_{j\in J} \in \widehat{\mathscr{H}}^J$ is *summable* when the following properties hold.

- For all $\lambda \in Y^+$, the set $\{j \in J \mid \exists\, w \in W^v,\ \pi_{\lambda,w}(a_j) \neq 0\}$ is finite.
- $\bigcup_{j\in J} \operatorname{supp}(a_j)$ is almost finite.

If $(a_j)_{j\in J} \in \widehat{\mathscr{H}}^J$ is a summable family, we define $\sum_{j\in J} a_j \in \widehat{\mathscr{H}}$ by

$$\sum_{j\in J} a_j := \sum_{(\lambda,w)\in Y^+\times W^v} a_{\lambda,w} Z^\lambda H_w, \qquad \text{with} \quad a_{\lambda,w} := \sum_{j\in J} \pi_{\lambda,w}(a_j) \text{ for all } (\lambda,w) \in Y^+ \times W^v.$$

The next result claims that the product of two summable families is well-defined. This is the crucial step in the process that turns $\widehat{\mathscr{H}}$ into a convolution algebra for $*$. Recall that elements of $\mathscr{H}$ corresponds to elements of $\widehat{\mathscr{H}}$ with finite support.

Theorem 4.21. — *Let $(a_j)_{j\in J} \in \mathscr{H}^J$ and $(b_k)_{k\in J} \in \mathscr{H}^K$ be two summable families. Then $(a_j * b_k)_{(j,k)\in J\times K}$ is summable and $\sum_{(j,k)\in J\times K} a_j * b_k$ only depends on the two elements $\sum_{j\in J} a_j$ and $\sum_{k\in K} b_k$ of $\widehat{\mathscr{H}}$.*

Proof. — For any $j \in J$ and $k \in K$, we can decompose a_j and b_k as follows:

$$a_j = \sum_{(\lambda,u)\in Y^+\times W^v} a_{j,\lambda,u} Z^\lambda H_u \quad \text{and} \quad b_k = \sum_{(\mu,v)\in Y^+\times W^v} b_{k,\mu,v} Z^\mu H_v.$$

For any $\lambda\in Y^+$, we set

$$J(\lambda):=\{j\in J\mid \exists\, u\in W^v,\ a_{j,\lambda,u}\neq 0\}\text{ and }K(\lambda):=\{k\in K\mid \exists\, v\in W^v,\ b_{k,\mu,v}\neq 0\}.$$

For any triple $(u,v,\mu)\in W^v\times W^v\times Y^+$, the application of Lemma 4.15 to $H_u*Z^\mu H_v$ gives a family $(z^{u,v,\mu}_{\nu,t})_{(\nu,t)\in R_u(\mu)\times[1,u]\cdot v}$ of scalars that satisfy

$$H_u*Z^\mu H_v=\sum_{(\nu,t)\in R_u(\mu)\times[1,u]\cdot v} z^{u,v,\mu}_{\nu,t}Z^\nu H_t.$$

Given $j\in J$ and $k\in K$, we then have

$$a_j*b_k=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{(\nu,t)\in R_u(\mu)\times[1,u]\cdot v} a_{j,\lambda,u}b_{k,\mu,v}z^{u,v,\mu}_{\nu,t}Z^{\lambda+\nu}H_t. \tag{4.2}$$

This equality implies that $\mathrm{supp}_{W^v}(a_j*b_k)$ is contained in $S^a_j.S^b_k$, where $S^c_n:=\bigcup_{w\in\mathrm{supp}_{W^v}(c_n)}[1,w]$ for $c\in\{a,b\}$ and $n\in\{j,k\}$. This already gives the finiteness of

$$S_{W^v}(a,b):=\bigcup_{(n,m)\in J\times K}\mathrm{supp}_{W^v}(a_n*b_m)\subset(\bigcup_{n\in J}S^a_n).(\bigcup_{m\in K}S^b_m).$$

If we set $S:=\bigcup_{j\in J}\mathrm{supp}(a_j)\cup\bigcup_{k\in K}\mathrm{supp}(b_k)$ and $S_Y:=\pi_Y(S)$, where $\pi_Y\colon Y\times W^v\to Y$ is the projection on the first coordinate, then S and S_Y are by construction both almost finite. We can hence choose an integer $N\geqslant 0$ and elements $\kappa_1,\dots,\kappa_N\in Y^{++}$ such that: for all $x\in S_Y$, there exists $i\in [\![1,N]\!]$ such that $x^{++}\leqslant_{Q^\vee}\kappa_i$.

Now pick a pair $(\rho,s)\in Y^+\times W^v$. The image of a_j*b_k by the projection $\pi_{\rho,s}$ is given by

$$\pi_{\rho,s}(a_j*b_k)=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{\nu\in R_u(\mu)\mid\lambda+\nu=\rho} a_{j,\lambda,u}b_{k,\mu,v}z^{u,v,\mu}_{\nu,s}.$$

Set

$$F(\rho):=\{(\lambda,\nu)\in S_Y\times Y^+\mid \exists\,(\mu,u)\in S_Y\times S_{W^v},\nu\in R_u(\mu)\text{ and }\lambda+\nu=\rho\}.$$

If (λ,ν) is an element of $F(\rho)$, choose some $(\mu,u)\in S_Y\times S_{W^v}$ such that $\nu\in R_u(\mu)$ and $\lambda+\nu=\rho$. Then Lemma 4.17 implies the existence of $n,m\in[\![1,N]\!]$ such that $\lambda\leqslant_{Q^\vee}\lambda^{++}\leqslant_{Q^\vee}\kappa_n$ and $\nu\leqslant_{Q^\vee}\mu^{++}\leqslant_{Q^\vee}\kappa_m$, which proves that $F(\rho)$ is finite.

Set

$$F'(\rho):=\{\mu\in S_Y\mid \exists\,(u,(\lambda,\nu))\in S_{W^v}\times F(\rho),\ \nu\in R_u(\mu)\}.$$

If μ is an element of $F'(\rho)$ and if $(u,(\lambda,\nu))\in S_{W^v}\times F(\rho)$ is such that $\nu\in R_u(\mu)$, applying again Lemma 4.17 gives an integer $i\in[\![1,N]\!]$ such that $\nu^{++}\leqslant_{Q^\vee}\mu^{++}\leqslant_{Q^\vee}\kappa_i$. This implies the finiteness of $F'(\rho)^{++}$, which implies itself the finiteness of $F'(\rho)$ by Lemma 4.19.

Set

$$F_1(\rho):=\{\lambda\in Y^+\mid\exists\,\nu\in Y^+,(\lambda,\nu)\in F(\rho)\text{ and }L(\rho):=\bigcup_{(\lambda,\mu)\in F_1(\rho)\times F'(\rho)}J(\lambda)\times K(\mu).$$

By construction, $L(\rho)$ is finite and for all $(j,k)\in J\times K$, the non-vanishing of $\pi_{\rho,s}(a_j*b_k)$ implies that (j,k) belongs to $L(\rho)$. Also, if (ρ,s) is in

$$\bigcup_{(j,k)\in J\times K}\mathrm{supp}(a_j*b_k),$$

then there exists $(\lambda,\mu)\in S_Y\times S_Y$, $u\in S_{W^v}$ and $\nu\in R_u(\mu)$ such that $\lambda+\nu=\rho$. Applying once more Lemma 4.17, we get integers $n,m\in [\![1,N]\!]$ such that

$$\rho^{++}\leqslant_{Q^\vee}\lambda^{++}+\nu^{++}\leqslant_{Q^\vee}\kappa_n+\kappa_m.$$

Summed up, all this shows that $\bigcup_{(j,k)\in J\times K}\operatorname{supp}(a_j*b_k)$ is almost finite and that $(a_j*b_k)_{(j,k)\in J\times K}$ is a summable family.

Moreover, we have

$$\begin{aligned}\pi_{\rho,s}\Big(\sum_{(j,k)\in J\times K}a_j*b_k\Big)&=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{\nu\in R_u(\mu)\mid\lambda+\nu=\rho}\ \sum_{(j,k)\in J\times K}a_{j,\lambda,u}b_{k,\mu,v}z^{u,v,\mu}_{\nu,s}\\&=\sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\ \sum_{\nu\in R_u(\mu)\mid\lambda+\nu=\rho}a_{\lambda,u}b_{\mu,v}z^{u,v,\mu}_{v,s},\end{aligned}$$

where we set

$$\sum_{j\in J}a_j=\sum_{(\lambda,u)\in Y^+\times W^v}a_{\lambda,u}Z^\lambda H_u\quad\text{and}\quad\sum_{k\in K}b_k=\sum_{(\mu,v)\in Y^+\times W^v}b_{\mu,v}Z^\mu H_v,$$

hence the lemma is proved. □

Definition 4.22. — For any summable families $(a_j)_{j\in J}\in\mathscr{H}^J$ and $(b_k)_{k\in J}\in\mathscr{H}^K$, we set

$$a*b:=\sum_{(j,k)\in J\times K}a_j*b_k\in\widehat{\mathscr{H}},\quad\text{with } a:=\sum_{j\in J}a_j\text{ and } b:=\sum_{k\in K}b_k.$$

Corollary 4.23. — *The convolution product $*$ provides $\widehat{\mathscr{H}}$ with a structure of associative $\mathscr{R}$-algebra.*

Proof. — Theorem 4.21 ensures that $(\widehat{\mathscr{H}},*)$ is an $\mathscr{R}$-algebra. The associativity of $*$ in $\widehat{\mathscr{H}}$ comes from Theorem 4.21 and from the associativity of $*$ in $\mathscr{H}$. □

The algebra $\widehat{\mathscr{H}}$ is called the *completed Iwahori-Hecke algebra of* $(\mathbb{A},(\sigma_i,\sigma_i')_{i\in I})$ *over* $\mathscr{R}$.

Example 4.24. — Let $\mathscr{I}$ be a thick masure of finite thickness on which a group G acts strongly transitively. For any $i\in I$, pick a panel P_i of $\{x\in\mathbb{A}\mid\alpha_i(x)=0\}$ and a panel P_i' of $\{x\in\mathbb{A}\mid\alpha_i(x)=1\}$. Let $1+q_i$ (resp. $1+q_i'$) be the number of chambers in $\mathscr{I}$ that contain P_i (resp. P_i') and set $\sigma_i=\sqrt{q_i}$, $\sigma_i'=\sqrt{q_i'}$. Then $(\sigma_i,\sigma_i')_{i\in I}$ satisfy the relations stated at the beginning of Section 4 and the completed Iwahori-Hecke algebra of $(\mathbb{A},(\sigma_i,\sigma_i')_{i\in I})$ over $\mathscr{R}$ is called the *completed Iwahori-Hecke algebra of $\mathscr{I}$ over $\mathscr{R}$*.

4.5. Center of the Iwahori-Hecke algebras.

— The goal of this section is to compute the center of the Iwahori-Hecke algebra $\mathscr{H}$ and of its completed version $\widehat{\mathscr{H}}$. Our proof is basically an adaptation to this context of the proof of [NR03, Th. 1.4]. In the sequel, we denote by $\mathscr{Z}(A)$ the center of any $\mathscr{R}$-algebra A.

4.5.1. *The completed Bernstein-Lusztig bimodule* $\overline{{}^{\mathrm{BL}}\mathscr{H}}$. — To determine $\mathscr{Z}(\widehat{\mathscr{H}})$, we want to compute elements of the form $Z^\mu * z * Z^{-\mu}$ for $z \in \mathscr{Z}(\widehat{\mathscr{H}})$ and $\mu \in Y^+$. However, left and right multiplication by $\mathbb{Z}^\lambda$ are only defined in $\widehat{\mathscr{H}}$ for $\lambda \in Y^+$. To extend multiplication by Z^λ for arbitrary $\lambda \in Y$, we need to pass to a bigger space: indeed, if $\lambda \in Y$ is not in Y^+, multiplication by Z^λ obviously does not stabilize $\widehat{\mathscr{H}}$, as $Z^\lambda * 1 = Z^\lambda$ is not in $\widehat{\mathscr{H}}$ in this case. The bigger space aforementioned is a "completion" $\overline{{}^{\mathrm{BL}}\mathscr{H}}$ of ${}^{\mathrm{BL}}\mathscr{H}$ that contains $\widehat{\mathscr{H}}$. Note that $\overline{{}^{\mathrm{BL}}\mathscr{H}}$ will not be equipped with a structure of algebra, but with a structure of $\mathscr{R}[Y]$-bimodule compatible with the convolution product $*$ on $\widehat{\mathscr{H}}$.

Any $a = (a_{\lambda,w}) \in \mathscr{R}^{Y\times W^v}$ will also be written as $a = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$. For such an a, we define the *support of a along W^v* as

$$\mathrm{supp}_{W^v}(a) := \{w \in W^v \mid \exists\, \lambda \in Y, a_{\lambda,w} \neq 0\}.$$

We set $\overline{{}^{\mathrm{BL}}\mathscr{H}} := \{a \in \mathscr{R}^{Y\times W^v} \mid \mathrm{supp}_{W^v}(a) \text{ is finite}\}$; note that ${}^{\mathrm{BL}}\mathscr{H}$ and $\widehat{\mathscr{H}}$ can be seen as subspaces of $\overline{{}^{\mathrm{BL}}\mathscr{H}}$. For any pair $(\rho, s) \in Y \times W^v$, we have again a projection map $\pi_{\rho,s} : \overline{{}^{\mathrm{BL}}\mathscr{H}} \to \mathscr{R}$ defined by:

$$\forall \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w \in \overline{{}^{\mathrm{BL}}\mathscr{H}}, \qquad \pi_{\rho,s}\Bigg(\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w\Bigg) = a_{\rho,s}.$$

Definition 4.25. — A family $(a_j)_{j\in J} \in \overline{{}^{\mathrm{BL}}\mathscr{H}}$ is *summable* if the following properties hold.

- For all pair $(\rho, s) \in Y \times W^w$, the set $\{j \in J \mid \pi_{s,\rho}(a_j) \neq 0\}$ is finite.
- $\bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$ is almost finite.

If $(a_j)_{j\in J} \in \overline{{}^{\mathrm{BL}}\mathscr{H}}$ is a summable family, we define $\sum_{j\in J} a_j \in \overline{{}^{\mathrm{BL}}\mathscr{H}}$ as

$$\sum_{j\in J} a_j := \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w, \quad \text{with } a_{\lambda,w} := \sum_{j\in J} \pi_{\lambda,w}(a_j) \text{ for all } (\lambda, w) \in Y\times W^v.$$

Lemma 4.26. — *Let $(a_j)_{j\in J} \in (\overline{{}^{\mathrm{BL}}\mathscr{H}})^J$ be a summable family in $\overline{{}^{\mathrm{BL}}\mathscr{H}}$ and $a := \sum_{j\in J} a_j \in \overline{{}^{\mathrm{BL}}\mathscr{H}}$. For any $\mu \in Y$, $(a_j * Z^\mu)_{j\in J}$ and $(Z^\mu * a_j)_{j\in J}$ are summable families of $\overline{{}^{\mathrm{BL}}\mathscr{H}}$, and the elements $\sum_{j\in J} Z^\mu * a_j$ and $\sum_{j\in J} a_j * Z^\mu$ only depend on a and μ (but not on the choice of the family $(a_j)_{j\in J}$.*

Moreover, setting $a \mathbin{\overline{*}} Z^\mu := \sum_{j\in J} a_j * Z^\mu$ and $Z^\mu \mathbin{\overline{*}} a := \sum_{j\in J} Z^\mu * a_j$, we define a convolution product that provides $\overline{{}^{\mathrm{BL}}\mathscr{H}}$ with a structure of $\mathscr{R}[Y]$-bimodule.

Proof. — Let $(a_j)_{j\in J} \in (\overline{{}^{\mathrm{BL}}\mathscr{H}})^J$ be a summable family and set $S := \bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$. For all pair $(\lambda, w) \in Y\times W^v$, set $J(\lambda, w) := \{j \in J \mid \pi_{\lambda,w}(a_j) \neq 0\}$. For all $(\mu, \rho, s, j) \in Y \times Y \times W^v \times J$, we have $\pi_{\rho,s}(Z^\mu * a_j) = \pi_{\rho-\mu,s}(a_j)$, hence the summability of $(Z^\mu * a_j)_{j\in J}$ directly comes from the summability of $(a_j)_{j\in J}$ and

$$\pi_{\rho,s}\Bigg(\sum_{j\in J} Z^\mu * a_j\Bigg) = \pi_{\rho-\mu,s}(a)$$

only depends on a and μ.

The corresponding statement for $(a_j * Z^\mu)_{j\in J}$ is a little bit trickier to prove. Given $w\in W^v$, Lemma 4.15 gives a family $(z^w_{\nu,t})_{(\nu,t)\in R_w(\mu)\times[1,w]}$ of coefficients in $\mathscr{R}$ such that
$$H_w * Z^\mu = \sum_{(\nu,t)\in R_w(\mu)\times[1,w]} z^w_{\nu,t} Z^\nu H_t.$$
For $j\in J$, write $a_j=\sum_{(\lambda,w)\in Y\times W^v} a_{j,\lambda,w}Z^\lambda H_w$ with $a_{j,\lambda,w}\in\mathscr{R}$ for any pair $(\lambda,w)\in Y\times W^v$. Then we have, for all $(\mu,\rho,s,j)\in Y\times Y\times W^v\times J$:
$$\begin{aligned}\pi_{\rho,s}(a_j*Z^\mu) &= \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S} a_{j,\lambda,w}Z^\lambda H_w * Z^\mu\Big)\\ &= \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S}\Big(\sum_{(\nu,t)\in R_w(\mu)\times[1,w]} a_{j,\lambda,w} z^w_{\nu,t} Z^{\nu+\lambda}H_t\Big)\Big)\\ &= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu)\mid\nu+\lambda=\rho} a_{j,\lambda,w}z^w_{\nu,s}\Big).\end{aligned}$$
Fix $\mu\in Y$ and set $F(\rho,s):=\{j\in J\ |\pi_{\rho,s}(a_j*Z^\mu)\neq 0\}$ for all pair $(\rho,s)\in Y\times W^v$. By the previous computation, we have $F(\rho,s)\subset\bigcup_{(w,\nu)\in S\times R_w(\mu)} J(\rho-\nu,w)$, hence $F(\rho,s)$ is finite. Moreover, for any $j\in J$, $\operatorname{supp}_{W^v}(a_j*Z^\mu)$ is contained in $\bigcup_{w\in S}[1,w]$, hence $\bigcup_{j\in J}\operatorname{supp}_{W^v}(a_j*Z^\mu)$ is almost finite and $(a_j*Z^\mu)_{j\in J}$ is summable. Also note that the calculation of $\pi_{\rho,s}\,(a_j*Z^\mu)$ we did above implies that for all $(\rho,s)\in Y\times W^v$, we have
$$\begin{aligned}\pi_{\rho,s}\Big(\sum_{j\in J}a_j*Z^\mu\Big) &= \sum_{j\in J}\Big(\sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu)\mid\nu+\lambda=\rho} a_{j,\lambda,w}z^v_{\nu,s}\Big)\Big)\\ &= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu)\mid\nu+\lambda=\rho}\Big(\sum_{j\in J} a_{j,\lambda,w}z^w_{\nu,s}\Big)\Big),\end{aligned}$$
hence if $a=\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w}Z^\lambda H_w$, then
$$\pi_{\rho,s}\Big(\sum_{j\in J}a_j*Z^\mu\Big)=\sum_{(\lambda,w)\in Y\times S}\ \sum_{\nu\in R_w(\mu)\mid\nu+\lambda=\rho} a_{\lambda,w}z^w_{\nu,s}$$
only depends on a and μ.

To conclude the proof, we are left to show that for any $(b,\mu,\mu')\in Y^3$, we have
$$Z^\mu\mathbin{\overline{*}}(Z^{\mu'}\mathbin{\overline{*}}b)=(Z^{\mu+\mu'})\mathbin{\overline{*}}b,\ \ (b\mathbin{\overline{*}}Z^\mu)\mathbin{\overline{*}}Z^{\mu'}=b\mathbin{\overline{*}}(Z^{\mu+\mu'})$$
and
$$Z^\mu\mathbin{\overline{*}}(b\mathbin{\overline{*}}Z^{\mu'})=(Z^\mu\mathbin{\overline{*}}b)\mathbin{\overline{*}}Z^{\mu'}.$$
To do this, write $b=\sum_{(\lambda,w)\in Y\times W^v} b_{\lambda,w}Z^\lambda H_w$ with $(b_{\lambda,w})\in\mathscr{R}^{Y\times W^v}$ and apply the first part of this lemma to $J=Y\times W^v$: by associativity of $*$ in ${}^{\mathrm{BL}}\mathscr{H}$, we get the required identities. $\square$

Corollary 4.27. — *For all $a\in\widehat{\mathscr{H}}$ and $\mu\in Y^+$, we have $Z^\mu*a=Z^\mu\mathbin{\overline{*}}a$ and $a*Z^\mu=a\mathbin{\overline{*}}Z^\mu$.*

This statement justifies that we will from now on denote $*$ instead of $\overline{*}$.

4.5.2. *Computation of the centers*

Lemma 4.28. — *For all $a \in \mathscr{Z}(\widehat{\mathscr{H}})$ and $\mu \in Y$, we have $a * Z^\mu = Z^\mu * a$.*

Proof. — Write $\mu = \mu_+ - \mu_-$ with $\mu_+, \mu_- \in Y^+$. The associativity of $*$ proven in Lemma 4.26 implies that $Z^{\mu_-} * (Z^{-\mu_-} * a) = a$, hence $Z^{-\mu_-} * a = a * Z^{-\mu_-}$ and $Z^\mu * a = Z^{\mu_+} * a * Z^{-\mu_-} = a * Z^\mu$. □

For any $w \in W^v$, we introduce the following subsets of $\overline{{}^{\mathrm{BL}}\mathscr{H}}$:

$$\begin{cases} \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w} := \Big\{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in \overline{{}^{\mathrm{BL}}\mathscr{H}} \mid (a_{\lambda,v} \neq 0) \Rightarrow (v \not\geq w)\Big\}; \\ \overline{{}^{\mathrm{BL}}\mathscr{H}}_{=w} := \Big\{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in \overline{{}^{\mathrm{BL}}\mathscr{H}} \mid (v \neq w) \Rightarrow (a_{\lambda,v} = 0)\Big\}. \end{cases}$$

We let $\widehat{\mathscr{H}}_{\not\geq w} := \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w} \cap \widehat{\mathscr{H}}$ and $\widehat{\mathscr{H}}_{=w} := \overline{{}^{\mathrm{BL}}\mathscr{H}}_{=w} \cap \widehat{\mathscr{H}}$ be the corresponding subspaces in $\widehat{\mathscr{H}}$.

Lemma 4.29. — *Let $w \in W^v$ and $\lambda \in Y$.*

(1) *We have*

$$\overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w} * Z^\lambda \subset \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w}, \quad Z^\lambda * \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w} \subset \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w} \quad \text{and} \quad Z^\lambda * \overline{{}^{\mathrm{BL}}\mathscr{H}}_{=w} \subset \overline{{}^{\mathrm{BL}}\mathscr{H}}_{=w}.$$

(2) *There exists $S \in \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w}$ such that $H_w * Z^\lambda = Z^{w(\lambda)} H_w + S$.*

Proof. — These statements are consequence of [BPGR16, Th. 6.2], of Lemma 4.15 and of Lemma 4.26. □

The following theorem is the heart of this section, as it describes the center of the completed Iwahori-Hecke algebra $\widehat{\mathscr{H}}$. This generalizes a well-known theorem of Bernstein (see [Lus83, Th. 8.1], which seems to be the first published version of this result) and gives a recovery of the spherical Hecke algebra $\mathscr{H}_s$ as center of a natural Iwahori-Hecke algebra.

Theorem 4.30. — *The center of the completed Iwahori-Hecke algebra $\widehat{\mathscr{H}}$ is $\mathscr{Z}(\widehat{\mathscr{H}}) = R[\![Y]\!]^{W^v}$.*

Proof. — Let $a = \sum_{\lambda\in Y^+} a_\lambda Z^\lambda$ be an element of $R[\![Y]\!]^{W^v}$ and $i \in I$. We can write $a = x + y$ with $x = \sum_{\lambda\in Y^+\cap\ker\alpha_i} a_\lambda Z^\lambda$ and $y = \sum_{\lambda\in Y^+|\alpha_i(\lambda)>0} a_\lambda (Z^\lambda + Z^{r_i(\lambda)})$. As x and y commute with H_i, we obtain that a commutes with H_i for all $i \in I$, hence we have $a \in \mathscr{Z}(\widehat{\mathscr{H}})$ and $R[\![Y]\!]^{W^v} \subset \mathscr{Z}(\widehat{\mathscr{H}})$.

Conversely, let z be an element of $\mathscr{Z}(\widehat{\mathscr{H}}) \subset \overline{{}^{\mathrm{BL}}\mathscr{H}}$ and write

$$z = \sum_{(\lambda,w)\in Y\times W^v} c_{\lambda,w} Z^\lambda H_w.$$

First assume that the set

$$F = \{(\lambda, w) \in Y \times W^v \mid w \neq 1 \text{ and } c_{\lambda,w} \neq 0\}$$

is non empty and choose a pair $(\nu, m) \in F$ with m maximal in W^v (for the Bruhat order). Write $z = x+y$ with $x = \sum_{\lambda\in Y} x_{\lambda,m} Z^\lambda H_m \in \widehat{\mathscr{H}_{=m}}$ and $y \in \widehat{\mathscr{H}_{\not\geq m}}$. Lemmas 4.28 and 4.29 imply that for all $y \in Y$, we have

$$z = Z^\mu * z * Z^{-\mu} = \sum_{\lambda\in Y} c_{\lambda,m} Z^{\lambda+\mu-m(\mu)} H_m + y_1$$

with $y_1 \in \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq m}$. By projection on $\overline{{}^{\mathrm{BL}}\mathscr{H}}_{=m}$, we get that

$$x = \sum_{\lambda\in Y} c_{\lambda,m} Z^{\lambda+\mu-m(\mu)} H_m.$$

Now let $J \subset Y$ be a finite set that satisfies the following property:

$$\forall\,(\lambda, w) \in Y \times W^v,\ (c_{\lambda,w} \neq 0) \Longrightarrow (\exists\,\nu \in J \mid \lambda \leqslant_{Q^\vee} \nu).$$

Pick $\gamma \in Y$ such that $c_{\gamma,m} \neq 0$. Then for all $\mu \in Y$, we have $c_{\gamma+\mu-m(\mu),m} \neq 0$, hence there exists $\nu(\mu) \in J$ such that $\gamma+\mu-m(\mu) \leqslant_{Q^\vee} \nu(\mu)$. In particular, pick $\mu \in Y \cap C^v_f$ and let $\nu \in J$ be such that for all integer $n \geqslant 0$, we have $\gamma + \sigma(n)(\mu - m(\mu)) \leqslant_{Q^\vee} \nu$, where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is such that $\lim_{n\to+\infty} \sigma(n) = +\infty$. Then $\gamma + \sigma(1)(\mu - m(\mu)) - \nu$ belongs to $Q^\vee$, and Lemma 2.1 implies that $\mu - m(\mu)$ is a non-zero element of $Q^\vee_+$. Hence for n large enough, we have

$$\gamma + \sigma(n)(\mu - m(\mu)) = \gamma + \sigma(1)(\mu - m(\mu)) + (\sigma(n) - \sigma(1))(\mu - m(\mu)) >_{Q^\vee} \nu,$$

which contradicts the definition of ν. Consequently, F is empty and z belongs to $R[\![Y]\!]$. We can hence simplify the above decomposition of z and write $z = \sum_{\lambda\in Y} c_\lambda Z^\lambda$ with $c_\lambda = c_{\lambda,1}$. By Lemma 4.29, we know that for any $w \in W^v$, we have

$$H_w * z = \sum_{\lambda\in Y} Z^{w(\lambda)} H_w + y$$

for some $y \in \overline{{}^{\mathrm{BL}}\mathscr{H}}_{\not\geq w}$. But z commutes with H_w, so we also have

$$H_w * z = z * H_w = \sum_{\lambda\in Y} c_\lambda Z^\lambda H_w.$$

By projection on $\widehat{\mathscr{H}_{=w}}$, we get that

$$\sum_{\lambda\in Y} c_\lambda Z^{w(\lambda)} H_w = \sum_{\lambda\in Y} c_\lambda Z^\lambda H_w,$$

hence z is in $R[\![Y]\!]^{W^v}$, which ends the proof. □

As a consequence of Theorem 4.30, we get a description of the center of the usual Iwahori-Hecke algebra $\mathscr{H}$. Note that the proof relies on a characterization of finite W^v-orbits in $\mathbb{A}$ that will be proven independently at the end of Section 5 (see Corollary 5.23).

Before we state the result, let us recall some notations. If $A_1, \dots, A_r$ denote the indecomposable components of the Kac-Moody matrix A, we let J^f be the set of all $j \in [\![1, r]\!]$ such that A_j is of finite type [Kac94, Th. 4.3] and J^∞ be the complement of J^f in $[\![1, r]\!]$. Set $\mathbb{A}^f := \bigoplus_{j\in J^f} \mathbb{A}_j$, let Φ_j be the root system of $\mathbb{A}_j$ and

$\mathbb{A}_{j,in} := \bigcap_{\phi\in\Phi_j} \ker\phi$ for all $j \in J^f$. Finally, set $Y^f := Y \cap \mathbb{A}^f$, $\mathbb{A}_{\mathrm{in}}^\infty := \bigoplus_{j\in J^\infty} \mathbb{A}_{j,in}$ and $Y_{\mathrm{in}}^\infty := Y \cap A_{\mathrm{in}}^\infty$.

Lemma 4.31. — *We have $\mathscr{Z}(\mathscr{H}) = \mathscr{Z}(\widehat{\mathscr{H}}) \cap \mathscr{H} = R[Y^f \oplus Y_{\mathrm{in}}^\infty]$.*

Proof. — Any $a \in \mathscr{Z}(\mathscr{H})$ is in $\mathscr{H}$ and satisfies $a * Z^\lambda H_w = Z^\lambda H_w * a$ for all $(\lambda, w) \in Y \times W^v$, hence belongs to $\mathscr{Z}(\widehat{\mathscr{H}})$ by Theorem 4.21. As the other inclusion is clear, we already have that $\mathscr{Z}(\mathscr{H}) = \mathscr{Z}(\widehat{\mathscr{H}}) \cap \mathscr{H}$. By Theorem 4.30, we get that $\mathscr{Z}(\mathscr{H}) = \mathscr{H} \cap R[\![Y]\!]^{W^v}$, and Corollary 5.23 now implies that this intersection is reduced to $R[Y^f \oplus Y_{\mathrm{in}}^\infty]$, which ends the proof. $\square$

Remark 4.32. — When W^v is finite, it is well-known that $\mathscr{H}$ is a finitely generated $\mathscr{Z}(\mathscr{H})$-module, and it is natural to wonder whether the corresponding statement holds in the infinite case. Unfortunately, when W^v is infinite, $\widehat{\mathscr{H}}$ is not of finite type over $\mathscr{Z}(\widehat{\mathscr{H}})$. Indeed, let J be any finite set and pick any finite family $(h_j)_{j\in J} \in \widehat{\mathscr{H}}^J$. For all $(z_j)_{j\in J} \in \mathscr{Z}(\widehat{\mathscr{H}})^J$, we have

$$\mathrm{supp}_{W^v}\Big(\sum_{j\in J} z_j h_j\Big) \subset \bigcup_{j\in J} \mathrm{supp}_{W^v}(h_j) \subsetneq W^v,$$

hence $(h_j)_{j\in J}$ cannot span $\widehat{\mathscr{H}}$ over $\mathscr{Z}(\widehat{\mathscr{H}})$.

4.6. Some further remarks

4.6.1. *The special case of reductive groups*. — Assume in this paragraph that G is reductive, in which case $\mathscr{T} = \mathscr{A}$ and $Y = Y^+$. Then almost finite sets as defined in [GR14] are finite sets: indeed, the Kac-Moody matrix A is in this case a Cartan matrix, hence it satisfies condition (FIN) in [Kac94, Th. 4.3]. In particular, Y^{++} is contained in $Q_+^\vee \oplus \mathbb{A}_{\mathrm{in}}$, so to be a subset of some $\big(\bigcup_{i=1}^k (y_i - Q_+^\vee)\big) \cap Y^{++}$ amounts to be finite. Though the algebra $\widehat{\mathscr{H}}$ is still different from $\mathscr{H}$, as $\sum_{\mu\in Q_+^\vee} Z^{-\mu}$ is for instance an element of $\widehat{\mathscr{H}}$ that is not in $\mathscr{H}$, they both have the same center. Indeed, we have the following result.

Proposition 4.33. — *Let R be any ring. Then $R[\![Y]\!]^{W^v} = R[Y]^{W^v}$ if and only if W^v is finite.*

Proof. — If W^v is infinite, then for any $y \in Y \cap C_f^v$, the element $\sum_{w\in W^v} e^{w\cdot y}$ belongs to $R[\![Y]\!]^{W^v}$ but not to $R[Y]^{W^v}$, hence $R[\![Y]\!]^{W^v} \neq R[Y]^{W^v}$.

If W^v is finite, let w_0 be the longest element of W^v. By [Hum92, §1.8], we know that $w_0.Q_+^\vee = Q_-^\vee$. If $E \subset Y$ is almost finite, there is some finite set J and a family $(y_j)_{j\in J} \in Y^J$ such that E is contained in $\bigcup_{j\in J}(y_j - Q_+^\vee)$. If E is furthermore W^v-invariant, then $E = w_0 \cdot E$ is also contained in $\bigcup_{j\in J}(w_0 \cdot y_j + Q_+^\vee)$, hence any element $x \in E$ satisfies $w_0 \cdot y_j \leqslant_{Q^\vee} x \leqslant_{Q^\vee} y_{j'}$ for some $j, j' \in J$. This implies that E is finite and completes the proof. $\square$

Using [Lus83, Th. 8.1], Theorem 4.30 and Lemma 4.31, we get from Proposition 4.33 that when W^v is finite, we have:

$$\mathscr{Z}(\widehat{\mathscr{H}}) = R[Y]^{W^v} = \mathscr{Z}(\mathscr{H}).$$

4.6.2. *Iwahori-Hecke algebras and K_I double cosets*. — In the completion process we used to define $\widehat{\mathscr{H}}$, we used the Bernstein-Lusztig relations of $\mathscr{H}$. However, the Iwahori-Hecke algebra $\mathscr{H}$ is initially defined in a different way, namely as a convolution algebra of K_I-bi-invariant functions. In particular, a natural basis of $\mathscr{H}$ is given by characteristic functions of K_I double cosets, and the Bernstein-Lusztig presentation comes afterwards. This leads naturally to address the following question: can we see the completed algebra at the level of K_I double cosets, as it is the case for the spherical Hecke algebra?

Fix a ring $\mathscr{R}$ as before and let $\mathscr{C}_0$ be the set of positive type 0 chambers. Set $W^+ := W^v \ltimes Y^+$ and let d^W be the distance defined in [BPGR16] (see also Section 5.2 below). Recall that $\mathscr{H}$ is isomorphic to $\bigoplus_{\boldsymbol{w}\in W^+} \mathscr{R}T_{\boldsymbol{w}}$ for the product defined by $T_{\boldsymbol{w}} * T_{\boldsymbol{v}} = \sum_{\boldsymbol{u}\in W^+} a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ for all elements $\boldsymbol{w}, \boldsymbol{v} \in W^+$, provided that we set, for all $\boldsymbol{u} \in W^+$,

$$a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} := \big|\{C \in \mathscr{C}_0 \mid C_0^+ \leqslant C \leqslant \boldsymbol{u}\cdot C_0^+,\ d^W(C_0^+, C) = \boldsymbol{w} \text{ and } d^W(C, \boldsymbol{u}\cdot C_0^+) = \boldsymbol{v}\}\big|.$$

For $x = (x_{\boldsymbol{w}})_{\boldsymbol{w}\in W^+} \in \mathscr{R}^{W^+}$, also write $x = \sum_{\boldsymbol{w}\in W^+} x_{\boldsymbol{w}} T_{\boldsymbol{w}}$. For now, we do not know whether it is possible to endow $\mathscr{R}^{W^+}$, or some subspace $\overline{\mathscr{H}} \subset \mathscr{R}^{W^+}$ containing $\mathscr{H}$, with a product that extends the convolution product of $\mathscr{H}$. At least in general, it seems difficult to embed $\widehat{\mathscr{H}}$ into $\mathscr{R}^{W^+}$. Indeed, assume for instance that $\mathscr{R} = \mathbb{C}$ and let $\pi : \mathbb{C}^{W^+} \to \mathbb{C}$ be the map defined by

$$\pi\bigg(\sum_{\boldsymbol{w}\in W^+} x_{\boldsymbol{w}} T_{\boldsymbol{w}}\bigg) := x_1 \quad \text{for all } x = (x_{\boldsymbol{w}})_{\boldsymbol{w}\in W^+} \in \mathbb{C}^{W^+}.$$

(Here, 1 denotes the identity element in W^+.) When G is reductive, we know for instance by [Opd03, Cor. 1.9] that for any $\lambda \in -Q^\vee$, $\pi(Z^\lambda)$ is a positive real number, which makes it apparently hard to consider $\sum_{\lambda\in -Q^\vee} (1/\pi(Z^\lambda))Z^\lambda \in \widehat{\mathscr{H}}$ as an element of $\mathbb{C}^{W^+}$. In the non-reductive case, we do not know so far whether an analogue of [Opd03, Cor. 1.9] is true.

5. Hecke algebra associated with a parahoric subgroup

The goal of this section is to attach a Hecke algebra to other subgroups than K_s or K_I, by generalizing previous constructions of [BKP16] and [BPGR16] for the Iwahori subgroup K_I. Our motivation comes from the reductive case, where Hecke algebras can be associated with any open compact subgroup (see Section 5.1 below). When G is not reductive, we know from Theorem 3.1 that there is no reasonable topology on G, hence we cannot define “open compact” in our context. Nevertheless, there is still a notion of *special parahoric subgroup*, defined as the fixer of a type 0 face of the masure $\mathscr{I}$.

Given a special parahoric subgroup $K = K_F$ that fixes a spherical type 0 face F satisfying $F_0 \subset \overline{F} \subset \overline{C_0^+}$, we will generalize the construction done for K_I by Bardy-Panse, Gaussent and Rousseau [GR14, BPGR16] to build a Hecke algebra associated with K_F. This requires some finiteness results that fails anytime $F \neq F_0$ is not spherical (see Section 5.4).

5.1. Motivation from the reductive case. — To motivate our definition in the Kac-Moody case (see Definition 5.14), we start by recalling the classical setting for reductive groups. This section follows [Vig96, I.3.3], though the idea of considering Hecke algebras as spaces of bi-invariant functions goes back at least to Weil and Shimura [Shi59], and to Iwahori [Iwa64] and Iwahori-Matsumoto [IM65].

Assume that G is reductive, in which case it is naturally endowed with a structure of topological group induced by the topology of $\mathscr{K}$. For any open compact subgroup K of G, let $\mathbb{Z}_c(G/K)$ be the space of compactly supported functions $G \to \mathbb{Z}$ that are K-invariant under right multiplication. Define an action of G on this space by setting $g \cdot f := [x \mapsto f(g \cdot x)]$ for all pairs $(g, f) \in G \times \mathbb{Z}_c(G/K)$. The algebra $H(G,K) := \mathrm{End}_G(\mathbb{Z}_c(G/K))$ of G-equivariant endomorphisms of $\mathbb{Z}_c(G/K)$ is called the *Hecke algebra of G relative to K*. If $\mathbb{Z}_c(G//K)$ is the ring of compactly supported functions $G \to \mathbb{Z}$ that are K-bi-invariant (for left and right multiplications), then we have a natural isomorphism of algebras $\Upsilon : H(G,K) \to \mathbb{Z}_c(G//K)$ given by $\Upsilon(\phi) := \phi(\mathbb{1}_K)$ for all $\phi \in H(G,K)$. This shows that $H(G,K)$ is a free $\mathbb{Z}$-algebra with canonical basis $\{e_g = \mathbb{1}_{KgK},\ g \in K\backslash G/K\}$. The product of two elements $e_g, e_{g'}$ of this basis is given by

$$(5.1)\quad e_g.e_{g'} = \sum_{g'' \in K\backslash G/K} m(g,g';g'')e_{g''} \quad \text{with } m(g,g';g'') := |(KgK \cap g''Kg'^{-1}K)/K|.$$

(Note that the non-vanishing of $m(g,g';g'')$ implies that $Kg''K$ is contained in $KgKg'K$.)

Extension of scalars works as follows: for any commutative ring R, the algebra $H_R(G,K) := H(G,K) \otimes_{\mathbb{Z}} R$ is called the *Hecke algebra of G over R relative to K*.

When G is not reductive, we will replace open compact subgroups (that are not defined) by special parahoric subgroups. More precisely, let $K = K_F$ be the fixer in G of a type 0 face F that satisfies $F_0 \subset \overline{F} \subset \overline{C_0^+}$. Following what is done in [BPGR16] in the Iwahori case, we will see $(KgK \cap g''Kg'^{-1}K)/K$ as intersection of "spheres" in $\mathscr{I}$ and prove that this intersection is finite when F is spherical (see Lemma 5.11) but infinite when $F \neq F_0$ is not spherical (see Proposition 5.21). Hence for F spherical, we will be able to define the Hecke algebra ${}^F\mathscr{H}$ associated with K_F as the free $\mathbb{Z}$-module with basis $\{e_g = \mathbb{1}_{KgK},\ g \in K\backslash G^+/K\}$, where $G^+ := \{g \in G \mid g \cdot 0 \geqslant 0\}$, equipped with the convolution product given by the analogue of formula (5.1) with $g'' \in G^+$. To prove this, we use the fact that these results are already known when F is a type 0 chamber (by [BPGR16]), and the finiteness of the number of type 0 chambers dominating F as above. Note that the use of G^+ instead of G in the definition of ${}^F\mathscr{H}$ is related to the fact that two points of $\mathscr{I}$ do not always lie in a same apartment.

This change of group already shows up in the spherical and the Iwahori cases (see [BPGR16, BK11, BKP16, GR14]).

From now on, we fix a type 0 face F that satisfy $F_0 \subset \overline{F} \subset \overline{C_0^+}$. We denote by $K = K_F$ its fixer in G and by W_F its pointwise fixer in W^v. Then F is *spherical* when W_F is finite. We also let $\Delta_F = G \cdot F$ be its orbit under the action of G on $\mathscr{I}$. Note that we have a bijection $\Upsilon_F : G/K \to \Delta_F$ that maps $g \cdot K_F$ to $g \cdot F$.

5.2. Distance and spheres associated with a type 0 face. — In this section, we define an "F-distance" (or "W_F-distance") that generalizes the W^v-distance introduced in [GR14] and the W-distance defined in [BPGR16].

If A (resp. A') is an apartment of $\mathscr{I}$ and if $E_1, \dots, E_k$ (resp. $E'_1, \dots, E'_k$) are subsets or filters of A (resp. A'), we denote by $\phi : (A, E_1, \dots, E_k) \to (A', E'_1, \dots, E'_k)$ any isomorphism of apartment $\phi : A \to A'$ induced by some element of G and such that: $\forall\, i \in [\![1, k]\!]$, $\phi(E_i) = E'_i$. When we do not want to precise which apartments A and A' are chosen, we simply write $\phi : (E_1, \dots, E_k) \to (E'_1, \dots, E'_k)$.

We define a relation $\leqslant$ on Δ_F as follows: for F_1, F_2 in Δ_F, we write $F_1 \leqslant F_2$ when $a_1 \leqslant a_2$, where a_i denotes the vertex of F_i for $i \in \{1, 2\}$. We then set

$$\Delta_F \times_{\leqslant} \Delta_F := \{(F_1, F_2) \in \Delta_F^2 \mid F_1 \leqslant F_2\}.$$

For any $F' \in \Delta_F \cap \mathbb{A}$, we set $[F'] := W_F \cdot F'$.

Proposition 5.1. — *For all $(F_1, F_2) \in \Delta_F \times_{\leqslant} \Delta_F$, there exists an apartment A containing F_1 and F_2, and an isomorphism $\phi : (A, F_1) \to (\mathbb{A}, F)$. Moreover, $d^F(F_1, F_2) := [\phi(F_2)]$ only depends on the pair (F_1, F_2).*

Proof. — Given $(F_1, F_2) \in \Delta_F \times_{\leqslant} \Delta_F$, the existence of an apartment A containing F_1 and F_2 comes from [Rou11, Prop. 5.1]. By construction, there is some $g \in G$ such that $F_1 = g \cdot F$. Let $A' := g \cdot \mathbb{A}$: by (MA2), there exists an isomorphism $\psi : (A, F_1) \to (A', F_1)$. Set $\psi' := g_{|\mathbb{A}}^{|A'}$: then $\phi := \psi'^{-1} \circ \psi$ has the required properties.

Let now A_1 be another apartment containing F_1 and F_2 and $\phi_1 : (A_1, F_1) \to (\mathbb{A}, F)$ be another suitable isomorphism. By [Héb17a, Th. 5.18], there exists an isomorphism $f : (A, F_1, F_2) \to (A_1, F_1, F_2)$. We hence have the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{A}, F_1, F_2) \arrow[r, "f"] \arrow[d, "\phi"] & (A_1, F_1, F_2) \arrow[d, "\phi_1"] \\
(\mathbb{A}, F, \phi(F_2)) \arrow[r] & (\mathbb{A}, F, \phi_1(F_2)),
\end{tikzcd}$$

with the lower horizontal arrow that is induced by an element of W_F, hence $[\phi(F_2)] = [\phi_1(F_2)]$ does not depend on any choice and the proof is complete. □

Remark 5.2. — Proposition 5.1 does not require F to be spherical, thought it is the most important case for us. When F is spherical, one can use [BPGR16, Prop. 1.10c)] instead of [Héb17a, Th. 5.18]. Note that in the sequel, we will only use the F-distance attached to a non-spherical face for pairs of type 0 faces based at the same vertex. In this special case, [Héb17a, Th. 5.18] could be replace by [Rou11, Prop. 5.2].

Remark 5.3. — When $F = F_0$, we can identify d^F with the "vectorial distance" d^v of [GR14] through the usual bijections $\Delta_{F_0} \simeq G\cdot 0$ and $Y^{++} \simeq Y^+/W^v$.

When $F = C_0^+$, we have $W_{C_0^+} = \{1\}$, hence $[C] = \{C\}$ for all chamber $C \in \Delta_{C_0^+}$ and $d^{C_0^+}$ can be identified with the distance d^W of [BPGR16], provided that each element w of $W^v \ltimes Y^+$ is identified with the type 0 chamber $w \cdot C_0^+$.

Set

$$\Delta^{\mathbb{A}}_{\geqslant F} := \{E \in \Delta_F \cap \mathbb{A} \mid E \geqslant F\} \quad \text{and} \quad [\Delta_F] := \{[F'], F' \in \Delta^{\mathbb{A}}_{\geqslant F}\}.$$

Moreover, for any pair $(E,[R]) \in \Delta_F \times [\Delta_F]$, set

$$\mathscr{S}^F(E,[R]) := \{E' \in \Delta_F \mid E' \geqslant E \ \text{ and } d^F(E,E') = [R]\},$$
$$\mathscr{S}^F_{\mathrm{op}}(E,[R]) := \{E' \in \Delta_F \mid E' \leqslant E, \ \text{ and } d^F(E,E') = [R]\}.$$

For any $E \in \Delta^{\mathbb{A}}_{\geqslant F}$, we choose some $g_E \in N$ such that $E = g_E \cdot F$. Such an element exists: indeed, let $g \in G$ be such that $E = g\cdot F$ and set $A := g\cdot\mathbb{A}$. By (MA2) and [Rou11, 2.2.1)], we get an isomorphism $\phi : (A, g\cdot F) \to (\mathbb{A}, g\cdot F)$. Letting $\psi = g^{|A}_{|\mathbb{A}}$, we have $\phi\circ\psi \in N$ such that $(\phi\circ\psi)(F) = \phi(E) = E$, hence $\phi\circ\psi$ comes from an element g_E as required.

Lemma 5.4. — *For all $[R] \in [\Delta_F]$, we have*

$$\Upsilon_F^{-1}(\mathscr{S}^F(F,[R])) = K_F g_R K_F/K_F \quad \textit{and} \quad \Upsilon_F^{-1}(\mathscr{S}^F_{\mathrm{op}}(F,[R])) = K_F g_R^{-1} K_F/K_F.$$

Proof. — For any $E \in \mathscr{S}^F(F,[R])$, there exists some $g \in K_F$ such that $g\cdot E = R = g_R\cdot F$, hence $\Upsilon_F^{-1}(E)$ belongs to $K_F g_R K_F/K_F$. Now let $x = k_1 g_R k_2$ be an element of $K_F g_R K_F$: then we have $\Upsilon_F(x) := \Upsilon_F(k_1 g_R) = k_1 g_R\cdot F = k_1\cdot R$. As d^F is G-invariant, we obtain that

$$d^F(k_1\cdot F, k_1\cdot R) = d^F(F,R) = d^F(F, k_1\cdot R),$$

hence x is in $\Upsilon_F^{-1}(S^F(F,[R]))$ and the proof of the first equality is complete. The proof of the second equality is similar and left to the reader. □

5.3. Hecke algebra associated with a spherical type 0 face. — Let C and C' be two positive type 0 chambers base at some common vertex $x \in \mathscr{I}_0 := G\cdot 0$. We can (and will) identify W with the set of type 0 chambers of $\mathbb{A}$ whose vertex lies in Y^+. Thus $d^W(C,C') = d^{C_0^+}(C,C')$ is in W^v and we can set $d(C,C') := \ell(d^W(C,C'))$.

Lemma 5.5. — *Let C be a positive type 0 chamber of $\mathscr{I}$ and let x be its vertex. For any integer $n \geqslant 0$, the set $B_n(C)$ of all positive type 0 chambers C' of $\mathscr{I}$ based at x and such that $d(C,C') \leqslant n$ is a finite set.*

Proof. — The argument goes by induction on $n \geqslant 0$, noticing that $B_n(C)$ contains $B_m(C)$ each time we have $n \geqslant m \geqslant 0$. As $\mathscr{I}$ is of finite thickness, the set $B_1(E)$ is finite for all $E \in G\cdot C_0^+$. Now let $n \geqslant 0$ be such that $B_n(E)$ is finite for all $E \in G\cdot C_0^+$

and take $C' \in B_{n+1}(C)$. By [Rou11, Prop. 5.1], we can choose an apartment A that contains C and C'. Let $\phi : (A, C) \to (\mathbb{A}, C_0^+)$ be an isomorphism of apartments: then we have $\phi(C') = w \cdot C_0^+$ for some $w \in W^v$ of length at most $n+1$. We can assume that $\ell(w) = n+1$, otherwise C' is in $B_n(C)$ and there is nothing more to do. In this case, let $\widetilde{w} \in W^v$ be such that $\ell(\widetilde{w}) = n$ and $d(\widetilde{w} \cdot C_0^+, \phi(C')) = 1$. Then $\widetilde{C} := \phi^{-1}(\widetilde{w} \cdot C_0^+)$ satisfies $d(C', \widetilde{C}) = 1$, hence C' belongs to $\bigcup_{C'' \in B_n(C)} B_1(C'')$, which is a finite set, and the proof is complete. $\square$

5.3.1. *Type of a type* 0 *face*. — Let $\mathscr{F}^v_{\mathbb{A}}$ be the set of all positive vectorial faces of $\mathbb{A}$ and $\mathscr{F}^0_{\mathbb{A}}$ be the set of all positive type 0 faces of $\mathbb{A}$ based at 0.

Lemma 5.6. — *The map $f : \mathscr{F}^v_{\mathbb{A}} \to \mathscr{F}^0_{\mathbb{A}}$ that sends $F^v \in \mathscr{F}^v$ to $F^\ell(0, F^v) \in \mathscr{F}^0_{\mathbb{A}}$ is bijective.*

Proof. — The definition of local faces ensures that f is well-defined and surjective. Now let F^v_1, F^v_2 be two distinct elements of $F^v_{\mathbb{A}}$. As 0 is special, we have $F^v_i \in f(F^v_i)$ for $i \in \{1, 2\}$. But $F^v_1 \cap F^v_2 = \varnothing$ implies that $f(F^v_1) \neq f(F^v_2)$ (for otherwise, we would have $\varnothing \in f(F^v_1)$, which does not make sense) and f is thus injective, which ends the proof. $\square$

For any positive type 0 face F' of $\mathscr{I}$, there is some type 0 face $F_1 \leqslant C_0^+$ and some $g_1 \in G$ such that $F' = g_1 \cdot F_1$. The set $J \subset I$ such that $F_1 = F^\ell(0, F^v(J))$ is called the *type of* F' and denoted by $\tau(F')$. This notion is well-defined: indeed, if we also have $F' = g_2 F^\ell(0, F^v(J'))$ for some $g_2 \in G$ and $J' \subset I$, then $g := g_2^{-1} g_1$ is such that $F^\ell(0, F^v(J)) = g \cdot F^\ell(0, F^v(J')$. By (MA2) and [Rou11, 2.2.1)], we can assume that g lies in N, hence $g_{|\mathbb{A}}$ is in W^v and Lemma 5.6 then implies that $F^v(J) = F^v(J')$. By [Rou11, 1.3], this requires that $J = J'$, as wanted.

Remark 5.7. — The type of a face is invariant under the action of G. Also note that for any type 0 chamber C and any subset J of I, there exists exactly one sub-face of C with type J.

5.3.2. *Finiteness results for spherical type* 0 *faces*. — *From now on, we assume that the face F is furthermore spherical.*

Lemma 5.8. — *For all $F' \in \Delta_F$, the set $\mathscr{C}_{F'}$ of all type* 0 *chambers of $\mathscr{I}$ containing F' is finite.*

Proof. — Fix a chamber $C \in \mathscr{C}_{F'}$, denote by x its vertex and pick another chamber C' in $\mathscr{F}'$. By [Rou11, Prop. 5.1], there exists an apartment A that contains C and C'. We identify A with $\mathbb{A}$ and fix the origin of $\mathbb{A}$ at x: then there exists $w \in W^v$ such that $C' = w \cdot C$. If J denotes the type of F', then $w \cdot F'$ is also a sub-face of type J in C', hence we have $w \cdot F' = F'$ by the unicity property stated in Remark 5.7. This means that w belongs to $W_{F'}$, which is a finite group as $F' \in \Delta_F$ is spherical (because F is). In particular, we must have $d(C, C') \leqslant \max\{\ell(u),\ u \in W_{F'}\}$, which ends the proof by Lemma 5.5. $\square$

Lemma 5.9. — *Let (E_1, E_2) and (E_1', E_2') be in $\Delta_F \times_\leqslant \Delta_F$. Then $d^F(E_1, E_2) = d^F(E_1', E_2')$ iff there exists an isomorphism $\phi : (E_1, E_2) \to (E_1', E_2')$.*

Proof. — Assume that $d^F(E_1, E_2) = d^F(E_1', E_2') =: [R]$. For any choice of

$$\psi : (E_1, E_2) \longrightarrow (F, R) \quad \text{and} \quad \psi' : (E_1, E_2) \longrightarrow (F, R),$$

the map $\phi := \psi'^{-1} \circ \psi$ satisfies the required property.

Conversely, suppose that there exists an isomorphism $\phi : (E_1, E_2) \mapsto (E_1', E_2')$. Pick $R \in d^F(E_1, E_2)$ and choose $\psi : (E_1, E_2) \to (F, R)$: then $\phi^{-1} \circ \psi$ is an isomorphism that maps (E_1', E_2') to (F, R). By Proposition 5.1, we thus have $d^F(E_1', E_2') = [R] = d^F(E_1, E_2)$, which ends the proof of the lemma. □

Lemma 5.10. — *Let $(F_1, F_2) \in \Delta_F \times_\leqslant \Delta_F$ and $r := d^F(F_1, F_2)$. Let $R \in \Delta^{\mathbb{A}}_{\geqslant F}$ be such that $r = [R] = W_F \cdot R$ and let $\mathscr{C}_{\mathbb{A}}(r)$ be the set of all chambers of $\mathbb{A}$ containing an element of r. For any type 0 chambers C_1 and C_2 respectively dominating F_1 and F_2, we have $d^W(C_1, C_2) \in \mathscr{C}_{\mathbb{A}}(r)$. Moreover, the set $\mathscr{C}_{\mathbb{A}}(r)$ is finite.*

Proof. — Pick an apartment A containing C_1 and C_2 and an isomorphism

$$\phi : (A, C_1) \longrightarrow (\mathbb{A}, C_0^+).$$

By Remark 5.7, $\phi(F_1)$ is the unique sub-face of type $\tau(F)$ in C_0^+, hence we have $F = \phi(F_1)$ and $\phi(F_2)$ belongs to $W_F \cdot R = r$, which implies that $d^W(C_1, C_2)$ lies in $\mathscr{C}_{\mathbb{A}}(r)$.

Now note that the map sending a positive type 0 face F' on its type $\tau(F')$ induces a bijection from the set of type 0 chambers of $\mathbb{A}$ containing $w \cdot R$ onto the fixer W_R of R in W. As W_R is a conjugate of W_F, it is finite (as F is spherical), hence so is $\mathscr{C}_{\mathbb{A}}(r)$. □

Lemma 5.11. — *For any pair $(F_1, F_2) \in \Delta_F \times_\leqslant \Delta_F$ and any elements $r_1, r_2 \in [\Delta_F]$, the set $\mathscr{S}^F(F_1, r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2, r_2)$ is finite and its cardinality only depends on r_1, r_2 and $r := d^F(F_1, F_2)$.*

Proof. — Denote by $\mathscr{S}$ the set of type 0 chambers containing an element of $\mathscr{S}^F(F_1, r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2, r_2)$ and let C_1 (resp. C_2) be a type 0 chamber that contains F_1 (resp. F_2). By Lemma 5.10, any chamber $C \in \mathscr{S}$ satisfies $d^W(C_1, C) \in \mathscr{C}_{\mathbb{A}}(r_1)$ and $d^W(C, C_2) \in \mathscr{C}_{\mathbb{A}}(r_2)$. This implies that $\mathscr{S}$ is contained in

$$\bigcup_{(w_1, w_2) \in \mathscr{C}_{\mathbb{A}}(r_1) \times \mathscr{C}_{\mathbb{A}}(r_2)} \{C \in \mathscr{C}_0^+ \mid C_1 \leqslant C \leqslant C_2,\ d^W(C_1, C) = w_1 \text{ and } d^W(C, C_2) = w_2\}.$$

By [BPGR16, Prop. 2.3] and Lemma 5.10, this inclusion implies the finiteness of $\mathscr{S}$, which itself implies that $\mathscr{S}^F(F_1, r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2, r_2)$ is finite.

To prove the independence of the cardinality, assume that $(F_1', F_2') \in \Delta_F \times_\leqslant \Delta_F$ is such that $d^F(F_1', F_2') = r$. By Lemma 5.9, there exists an isomorphism

$$\phi : (F_1, F_2) \longrightarrow (F_1', F_2').$$

Thus we have

$$\mathscr{S}^F(F_1', r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2', r_2) = \phi\left(\mathscr{S}^F(F_1, r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2, r_2)\right),$$

which ends the proof. □

Following Lemma 5.11, and with the same notations, we set

$$a^r_{r_1,r_2} := |\mathscr{S}^F(F_1, r_1) \cap \mathscr{S}^F_{\mathrm{op}}(F_2, r_2)|.$$

Lemma 5.12. — *For any elements r_1, r_2 in $[\Delta_F]$, the set*

$$P_{r_1,r_2} := \big\{d^F(F_1, F_2),\ (F_1, F', F_2) \in \Delta_F \times_{\leqslant} \Delta_F \times_{\leqslant} \Delta_F \mid \\ d^F(F_1, F') = r_1 \text{ and } d^F(F', F_2) = r_2\big\}$$

is finite.

Proof. — Denote by $\mathscr{E}$ the set of all triples (C_1, C', C_2) of type 0 chambers such that there exist sub-faces $F_1 \subset C_1$, $F' \subset C'$ and $F_2 \subset C_2$ of these chambers that satisfy $d^F(F_1, F') = r_1$ and $d^F(F', F_2) = r_2$. If (C_1, C', C_2) is in $\mathscr{E}$, then Lemma 5.10 implies that $d^W(C_1, C') \in \mathscr{C}_{\mathbb{A}}(r_1)$ and $d^W(C', C_2) \in \mathscr{C}_{\mathbb{A}}(r_2)$. This proves that $P := \{d^W(C_1, C_2),\ (C_1, C', C_2) \in \mathscr{E}\}$ is contained in $\bigcup_{(\mathbf{w_1},\mathbf{w_2}) \in \mathscr{C}_{\mathbb{A}}(r_1) \times \mathscr{C}_{\mathbb{A}}(r_2)} P_{\mathbf{w_1},\mathbf{w_2}}$, where the $P_{\mathbf{w_1},\mathbf{w_2}}$'s are the finite sets introduced in the proof of [BPGR16, Prop. 2.2]. By Lemma 5.10, this proves that P is finite.

Let $(F_1, F', F_2) \in \Delta_F \times_{\leqslant} \Delta_F \leqslant \Delta_F$ be such that $d^F(F_1, F') = r_1$ and $d^F(F', F_2) = r_2$. Then there is a triple $(C_1, C', C_2) \in \mathscr{E}$ such that F_i is a face of C_i for $i \in \{1, 2\}$. The distance $d^F(F_1, F_2)$ is of the form $W_F \cdot F''$ for some face F'' of $d^W(C_1, C_2)$, hence the lemma follows. □

5.3.3. *Definition of the Hecke algebra ${}^F\mathscr{H}$*. — Let R be a commutative unitary ring.(4) Denote by ${}^F\mathscr{H} = {}^F\mathscr{H}_R^{\mathscr{I}}$ the set of all functions $\varphi : G\backslash\Delta_F \times_{\leqslant} \Delta_F \to R$. For any $r \in [\Delta_F]$, let $T_r : \Delta_F \times_{\leqslant} \Delta_F \to R$ be defined as follows (where $\delta_{\cdot,\cdot}$ denotes the Kronecker symbol):

$$\forall\, (F_1, F_2) \in \Delta_F \times_{\leqslant} \Delta_F,\ T_r(F_1, F_2) := \delta_{d^F(F_1,F_2),r}.$$

One directly checks that ${}^F\mathscr{H}$ is a free R-module with basis $\{T_r,\ r \in [\Delta_F]\}$.

Theorem 5.13. — *Define a product $* : {}^F\mathscr{H} \times {}^F\mathscr{H} \to {}^F\mathscr{H}$ by the following formula:*

$$\forall\, (\varphi_1, \varphi_2) \in {}^F\mathscr{H} \times {}^F\mathscr{H}, \quad \varphi_1 * \varphi_2 := \Bigg[(F_1, F_2) \longmapsto \sum_{F' \in \Delta_F \mid F_1 \leqslant F' \leqslant F_2} \varphi_1(F_1, F')\varphi_2(F', F_2)\Bigg].$$

Then the product $*$ is well-defined and endow ${}^F\mathscr{H}$ with a structure of associative algebra that has $T_{[F]}$ for identity element. Moreover, the product of any two elements of the basis $\{T_r,\ r \in [\Delta_F]\}$ is given by the following formula:

$$\forall\, (r_1, r_2) \in [\Delta_F] \times [\Delta_F], \quad T_{r_1} * T_{r_2} = \sum_{r \in P_{r_1,r_2}} a^r_{r_1,r_2} T_r.$$

(4)Note that we do not require here any of the additional assumptions made on $\mathscr{R}$ in Section 4.

Proof. — Lemmas 5.11 and 5.12 imply that $*$ is well-defined and give the required formula for $T_{r_1} * T_{r_2}$ for any $(r_1, r_2) \in [\Delta_F] \times [\Delta_F]$. The associativity of $*$ directly comes from the definition, and a direct computation shows that $T_{[F]}$ is the identity element as we have $\mathscr{S}^F(F_1, [F]) = \{F_1\}$ for all $F_1 \in \Delta_F$. $\square$

Definition 5.14. — The algebra ${}^F\mathscr{H} = {}^F\mathscr{H}_R^{\mathscr{I}}$ is called the *Hecke algebra of $\mathscr{I}$ associated to F (or: to K_F) over R.*

Remark 5.15. — Given $g \in G^+$, there exists some element $R_g \in \Delta^{\mathbb{A}}_{\geqslant F}$ such that

$$\{F' \in K_F g K_F \cdot F,\ F' \subset \mathbb{A}\} = [R_g].$$

Let $(F_1, F_2) \in G\backslash\Delta_F \times_{\leqslant} \Delta_F$. We can always assume that $F_1 = F$ and write $F_2 = g \cdot F$ for some $g \in G$. One easily checks that $f(g) := K_F g K_F$ only depends on (F_1, F_2), and that the corresponding map $f : G\backslash\Delta_F \times_{\leqslant} \Delta_F \to K_F\backslash G^+/K_F$ is bijective. Via f, we can identify ${}^F\mathscr{H}$ with the set of all functions $K_F\backslash G^+/K_F \to R$. Under this identification, T_{R_g} corresponds to $e_g := \mathbb{1}_{K_F g K_F}$ for all $g \in G^+$. Moreover, for any $g, g' \in K_F\backslash G^+/K_F$, we have

$$e_g * e_{g'} = \sum_{g'' \in K_F\backslash G^+/K_F} m(g, g'; g'') e_{g''},$$

where $m(g, g'; g'') = a^{[R_{g''}]}_{[R_g],[R_{g'}]}$ for all $g'' \in K_F\backslash G^+/K_F$. Using Lemmas 5.4 and 5.11, we get that $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$, as in the reductive case (compare with (5.1)).

Remark 5.16. — For now, we do not know whether it is possible to define a completed Hecke algebra $\widehat{{}^F\mathscr{H}}$ for any spherical face F as above in the similar manner as what we did for the Iwahori-Hecke algebra. To generalize our completion process to this context, one would in particular need an analogue of Bernstein-Lusztig relations for arbitrary F.

5.4. What about non-spherical type 0 faces? — In [GR14], Gaussent and Rousseau defined the spherical Hecke algebra as a Hecke algebra associated with the non-spherical type 0 face F_0, and we noticed in Remark 5.3 that their distance d^v matches with our d^{F_0}. Consequently, it seems natural to try to associate a Hecke algebra with any type 0 face F between F_0 and C_0^+, i.e., to see whether the extra assumption of being spherical can be suppressed.

In this section, we consider a non-spherical type 0 face F such that $F_0 \subsetneq F \subsetneq C_0^+$. Note that this implies that A is an indefinite Kac-Moody matrix of size $\geqslant 3$: indeed, when A is of finite type, then any type 0 face is spherical, and when A is of affine type, the only non-spherical type 0 face of C_0^+ is F_0. In this last section, we will prove that the coefficients involved in the definition of the convolution product introduced earlier (see Theorem 5.13) are now infinite. The proof of this result requires the injectivity of the restriction map that sends $w \in W^v$ to $w_{|Q^\vee}$. This property is proved in [Kac94] for less general realizations of A than the one we use, hence we will start by extending this property to our framework: this is the point of Lemma 5.18 below.

5.4.1. *Realizations of a Kac-Moody matrix*. — Let $A = (a_{i,j})_{i,j\in[\![1,n]\!]}$ be a Kac-Moody matrix. Following [Kac94, Chap. 1], we say that a *realization of* A is a triple $(\mathscr{A}, \Pi, \Pi^\vee)$ where $\mathscr{A}$ denotes an $\mathbb{R}$-vector space,(5) $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a family of n elements in $\mathscr{A}^*$ (the dual space of $\mathscr{A}$) and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ a family of n elements in $\mathscr{A}$, such that the following three properties hold.

(F) The elements of Π (resp. $\Pi^\vee$) are linearly independent in $\mathscr{A}^*$ (resp. in $\mathscr{A}$).
(C) For all $i, j \in [\![1, n]\!]$, $\alpha_j(\alpha_i^\vee) = a_{i,j}$.
(D) We have $n - \mathrm{rk}(A) = \dim_{\mathbb{R}} \mathscr{A} - n$.

A *generalized free realization of* A is a triple $(\mathscr{A}, \Pi, \Pi^\vee)$ with $\mathscr{A}, \Pi, \Pi^\vee$ defined as above but only satisfying properties (F) and (C). Two realizations $(\mathscr{A}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathscr{A}_2, \Pi_2, \Pi_2^\vee)$ are said *isomorphic* if there exists an isomorphism of vector spaces $\phi : \mathscr{A}_1 \to \mathscr{A}_2$ such that $\phi^*(\Pi_1) = \Pi_2$ and $\phi(\Pi_1^\vee) = \Pi_2^\vee$. We know by [Kac94, Prop. 1.1] that up to (non unique in general) isomorphism, A admits a unique realization $(\mathscr{A}_0, \Pi_0, \Pi_0^\vee)$.

Given a generalized free realization $(\mathscr{A}, \Pi, \Pi^\vee)$ of A, we let the *inessential part of* $\mathscr{A}$ be the subspace $\mathscr{A}_{\mathrm{in}} := \bigcap_{i=1}^n \ker \alpha_i$. We also set

$$Q_{\mathscr{A}}^\vee := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee \quad \text{and} \quad Q_{\mathbb{R},\mathscr{A}}^\vee := \bigoplus_{i=1}^n \mathbb{R}\alpha_i^\vee.$$

The next lemma is easy to prove and thus left to the reader.

Lemma 5.17. — *For any generalized free realization $\mathscr{A}$ of A, there exist subspaces $\mathscr{A}' \subset \mathscr{A}$ and $B \subset \mathscr{A}_{\mathrm{in}}$ such that $\mathscr{A}'$ is isomorphic to $\mathscr{A}_0$ (as realizations of A), $Q_{\mathscr{A}}^\vee \subset \mathscr{A}'$ and $\mathscr{A} = \mathscr{A}' \oplus B$.*

Let $W_{\mathscr{A}}^v$ be the Weyl group of $\mathscr{A}$, i.e., the subgroup of $\mathrm{GL}(\mathscr{A})$ generated by the r_i, $i \in I$ (where $r_i : \mathscr{A} \to \mathscr{A}$ sends any $x \in \mathscr{A}$ to $x - \alpha_i(x)\alpha_i^\vee$).

Lemma 5.18. — *For any generalized free realization $\mathscr{A}$ of A, the map*

$$\big[w \in W_{\mathscr{A}}^v \longmapsto w_{|Q^\vee} \in \mathrm{Aut}_{\mathbb{Z}}(Q_{\mathscr{A}}^\vee)\big]$$

is injective.

Proof. — Write $\mathscr{A} = \mathscr{A}' \oplus B$ with $\mathscr{A}'$ and B as in Lemma 5.17. For any $x \in \mathscr{A}$ and $w \in W_{\mathscr{A}}^v$, we have $w(x) - x \in Q_{\mathbb{R},\mathscr{A}}^\vee$, hence $\mathscr{A}'$ is stable under the action of $W_{\mathbb{A}}^v$. Moreover, $W_{\mathscr{A}}^v$ fixes pointwise $\mathscr{A}_{\mathrm{in}}$, hence the restriction map $W_{\mathscr{A}}^v \to W_{\mathscr{A}'}^v$ is a an isomorphism. As a consequence, we can assume that $\mathscr{A} = \mathscr{A}_0$. Now apply assertion (3.12.1) of the proof of [Kac94, Prop. 3.12] to $\Delta^\vee$ instead of Δ: we get that the only $w \in W_{\mathscr{A}_0}^v$ satisfying $w_{|\Delta^\vee} = 1$ is $w = 1$. As $\Delta^\vee$ is contained in $Q_{\mathscr{A}}^\vee$, this ends the proof. ☐

(5) Note that in [Kac94], complex vector spaces are used instead of real vector spaces.

5.4.2. *Infinite intersections of spheres.* — From now on, we assume that F is a non-spherical type 0 face of $\mathbb{A}$ that satisfies $F_0 \subsetneq F \subsetneq C_0^+$. Recall that this implies that the fixer W_F of W is infinite. Indeed, we can assume that F has 0 for vertex, which identifies W_F with a subgroup of W^v. Let F^v be the vectorial face such that $F = F^\ell(0, F^v)$ and let us prove that W_F is also the fixer W_{F^v} of F^v (which will prove the claim as F^v is non spherical, hence W^{F^v} is infinite by definition). If $w \in W_F$, let $X \in F$ be fixed by w: then w fixes $\mathbb{R}_+^*.X \supset F^v$ hence w is in W_{F^v}. Conversely, we have $W_{F^v} \subset W_F$ because $F^v \in F$ (as 0 is special), hence $W_F = W_{F^v}$ is infinite.

Remark 5.19. — By [Rou11, §1.3], the vectorial faces based at 0 form a partition of the Tits cone. Therefore, for any vectorial face F^v, if there exist some $u \in F^v$ and some $w \in W^v$ such that $w \cdot u \in F^v$, then $w \cdot F^v = F^v$. Consequently, for any $W' \subset W^v$, $W' \cdot F^v$ is infinite if and only if $W' \cdot u$ is infinite for some $u \in F^v$, if and only if $W' \cdot u$ is infinite for all $u \in F^v$.

The proof of the next proposition uses the *graph of the matrix A*, whose vertices are the elements $i \in I$ and whose arrows are the pairs $\{i, j\}$ such that $a_{i,j} \neq 0$.

Lemma 5.20. — *Suppose that the matrix A is indecomposable. For any non-spherical type 0 face F of $\mathbb{A}$ that satisfies $F_0 \subsetneq F \subsetneq C_0^+$, there exists $w \in W^v$ such that $W_F \cdot w \cdot F$ is infinite.*

Proof. — Write $F = F^\ell(0, F^v)$ with

$$F^v = F^v(J) = \{x \in \mathbb{A} \mid \forall\, j \in J, \alpha_j(x) = 0 \text{ and } \forall\, i \notin J, \alpha_i(x) > 0\}$$

for some subset J of I. Note that $J \neq I$ as F_0 is strictly contained in F. Let $k \in I$ be such that $W_F \cdot \alpha_k^\vee$ is infinite (such a k exists by Lemma 5.18). As the graph of A is connected [Kac94, 4.7], any element $j \in I \smallsetminus J$ can be linked to k via a finite sequence $j = j_1, \ldots, j_\ell = k$ of elements of I that satisfy $\prod_{m=1}^{\ell-1} a_{j_m, j_{m+1}} \neq 0$. We fix such a j and such a sequence $j_1, \ldots, j_\ell$.

Now pick $u \in F^v$ and let us show the existence of some $w \in W^v$ such that $\alpha_k(w \cdot u) \neq 0$. Given $x \in \mathbb{A}$ and $m \in [\![1, \ell]\!]$, we say that x satisfies P_m when $\alpha_{j_m}(x) \neq 0$ and $\alpha_{j_{m'}}(x) = 0$ for all $m' \in [\![m+1, l]\!]$. If $x \in \mathbb{A}$ satisfies P_m for some $m \in [\![1, \ell - 1]\!]$, then $x' := r_{j_m}(x)$ satisfies $\alpha_{j_{m+1}}(x') = -\alpha_{j_m}(x) a_{j_m, j_{m+1}} \neq 0$ (recall that $x' = x - \alpha_{j_m}(x)\alpha_{j_m}^\vee$), hence x' satisfies P_s for some $s \in [\![m+1, \ell]\!]$. As u is in F^v and $j_1 = j$ is in $I \smallsetminus J$, we have $\alpha_{j_1}(u) > 0$, hence u satisfies P_m for some $m \in [\![1, \ell]\!]$. Replacing u by x in the previous argument and using successive iterations, we finally get some $w \in W^v$ such that $w \cdot u$ satisfies P_ℓ, i.e., such that $\alpha_k(w \cdot u) \neq 0$.

We conclude as follows: if $W_F \cdot w \cdot u$ is finite, then $W_F.r_k(w \cdot u) = W_F.(u - \alpha_k(w \cdot u)\alpha_k^\vee)$ is infinite, hence at least one of the orbits $W_F \cdot w \cdot u$ or $W_F.r_k(w \cdot u)$ is infinite, which implies the required result by Remark 5.19. $\square$

Let $A_1, \ldots, A_r$ be the indecomposable components of the Kac-Moody matrix A. For any $i \in [\![1, r]\!]$, pick a realization $\mathscr{A}_i$ of A_i: then $\mathbb{A} = \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_r$. Also note that W^v decomposes as $W^v = W_1^v \times \cdots \times W_r^v$, where W_i^v denotes the vectorial Weyl group of $\mathscr{A}_i$, and that we can decompose any face F' of $\mathbb{A}$ as $F' = F'_1 \oplus \cdots \oplus F'_r$ with $F'_i \subset \mathscr{A}_i$.

Proposition 5.21. — *Let $F' = \bigoplus_{i=1}^r F'_i$ be a type 0 face of $\mathbb{A}$. The following are equivalent.*

(i) *There exists $w \in W^v$ such that $W_{F'} \cdot w \cdot F'$ is infinite.*

(ii) *There exists $i \in [\![1, r]\!]$ such that F'_i is non-spherical and different from $F_{i,0} := F^\ell(0, \mathscr{A}_{i,in})$. (Recall that $F_{i,0}$ is the minimal type 0 face of $\mathscr{A}_i$ based at 0.)*

Proof. — The decomposition of F' induces a decomposition of its fixer as $W_{F'} = W_{F'_1} \times \cdots \times W_{F'_r}$. First assume the existence of some $w \in W^v$ such that $W_{F'} \cdot w \cdot F'$ is infinite and decompose w as $w = (w_1, \ldots, w_r)$. Then

$$W_{F'} \cdot w \cdot F' = W_{F'_1} \cdot w_1 \cdot F'_1 \oplus \cdots \oplus W_{F'_r} \cdot w_r \cdot F'_r,$$

hence there is (at least) an integer $i \in [\![1, r]\!]$ such that $W_{F'_i} \cdot w_i \cdot F'_i$ is infinite. For such an i, F'_i must be non-spherical (otherwise $W_{F'_i}$ would be finite) and different from $F_{i,0}$ (otherwise $W_{F'_i} \cdot w_i \cdot F'_i = F_{i,0}$). Hence (i) implies (ii). The reverse implication is a consequence of Lemma 5.20. □

The next proposition gives a counterexample to Lemma 5.11 for non-spherical faces, which explains why we needed this restriction in our construction.

Proposition 5.22. — *Let F' be a face based at 0 for which there exists some $w \in W^v$ such that $W_{F'} w \cdot F'$ is infinite. Then $\mathscr{S}^{F'}(F', [w \cdot F']) \cap \mathscr{S}^{F'}_{\mathrm{op}}(F', [w^{-1} \cdot F'])$ is infinite.*

Proof. — It is enough to check that $W_{F'} w \cdot F'$ is contained in

$$\mathscr{S}^{F'}(F', [w \cdot F']) \cap \mathscr{S}^{F'}_{\mathrm{op}}(F', [w^{-1} \cdot F']).$$

Let $E \in W_{F'} \cdot w \cdot F'$ and let $w_E \in W_{F'}$ be such that $E = w_E \cdot w \cdot F'$. As $w_E \cdot F' = F'$, we have $F' \leqslant E \leqslant F'$, hence $d^{F'}(F', E) = [E] = [w \cdot F']$ by definition of $d^{F'}$. Now the isomorphism $(w_E \cdot w)^{-1} : \mathbb{A} \to \mathbb{A}$ maps E to F' and F' to $w^{-1} \cdot w_E^{-1} \cdot F' = w^{-1} \cdot F'$, thus $d^{F'}(E, F') = [w^{-1} \cdot F']$. This shows that E belongs to

$$\mathscr{S}^{F'}(F', [w \cdot F']) \cap \mathscr{S}^{F'}_{\mathrm{op}}(F', [w^{-1} \cdot F']),$$

hence the proposition. □

Recall that the notations Y^f and Y^∞_{in} used in the next result were introduced in Section 4.5.2.

Corollary 5.23. — *Let $\lambda \in Y^+$. Its W^v-orbit $W^v \cdot \lambda$ is finite iff λ belongs to $Y^f \oplus Y^\infty_{\mathrm{in}}$.*

Proof. — Given $\lambda \in Y^+$, write $\lambda = \sum_{j=1}^r \lambda_j$ with $\lambda_j \in \mathscr{A}_j$ for all $j \in [\![1, r]\!]$. First assume that λ is in $Y^f \oplus Y^\infty_{\mathrm{in}}$: then

$$W^v \cdot \lambda = \bigoplus_{j \in J^f} W^v_j \cdot \lambda_j \oplus \bigoplus_{j \in J^\infty} \lambda_j.$$

As W^v_j is finite for any $j \in J^f$, the finiteness of $W^v \cdot \lambda$ follows from its decomposition above and the converse implication is proved.

Now assume that λ is not in $Y^f \oplus Y^\infty_{\text{in}}$. Let $j \in J^\infty$ be such that $\lambda_j \notin \mathscr{A}_{j,\text{in}}$ and let F^v_j be the vectorial face of $\mathscr{A}_j$ that contains λ_j. By Remark 5.19, the map $W^v_j \cdot F^v_j \lambda_j \to W^v_j \lambda_j$ that sends $w \cdot F^v_j$ onto $w \cdot \lambda_j$ is well-defined and bijective. If F^v_j is spherical, then its stabilizer is finite and $W^v_j \cdot F^v_j$ is thus infinite as W^v_j is. If F^v_j is non-spherical, then Lemma 5.20 produces an element $w_j \in W^v_j$ such that $W_{F_j} \cdot w_j \cdot F^v_j$ is infinite, where W_{F_j} is also the fixer of F^v_j in W^v_j. In any case, $W^v_j \cdot F^v_j$ is infinite, hence so is $W^v_j \cdot \lambda_j$, which ends the proof. $\square$

References

[BPGR16] N. Bardy-Panse, S. Gaussent & G. Rousseau – "Iwahori-Hecke algebras for Kac-Moody groups over local fields", *Pacific J. Math.* **285** (2016), no. 1, p. 1–61.

[BK11] A. Braverman & D. Kazhdan – "The spherical Hecke algebra for affine Kac-Moody groups I", *Ann. of Math. (2)* (2011), p. 1603–1642.

[BKP16] A. Braverman, D. Kazhdan & M. M. Patnaik – "Iwahori–Hecke algebras for p-adic loop groups", *Invent. Math.* **204** (2016), no. 2, p. 347–442.

[BT72] F. Bruhat & J. Tits – "Groupes réductifs sur un corps local", *Publ. Math. Inst. Hautes Études Sci.* **41** (1972), no. 1, p. 5–251.

[BT84] ———, "Groupes réductifs sur un corps local", *Publ. Math. Inst. Hautes Études Sci.* **60** (1984), no. 1, p. 5–184.

[GR08] S. Gaussent & G. Rousseau – "Kac-Moody groups, hovels and Littelmann paths", *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 7, p. 2605–2657.

[GR14] ———, "Spherical Hecke algebras for Kac-Moody groups over local fields", *Ann. of Math. (2)* **180** (2014), no. 3, p. 1051–1087.

[Héb16] A. Hébert – "Distances on a masure (affine ordered hovel)", `arXiv:1611.06105`, 2016.

[Héb17a] ———, "Convexity in a masure", `arXiv:1710.09272`, 2017.

[Héb17b] ———, "Gindikin-Karpelevich finiteness for Kac-Moody groups over local fields", *Internat. Math. Res. Notices* (2017), no. 22, p. 7028–7049.

[Hum92] J. E. Humphreys – *Reflection groups and coxeter groups*, vol. 29, Cambridge Univ. Press, Cambridge, 1992.

[Iwa64] N. Iwahori – "On the structure of a Hecke ring of a Chevalley group over a finite field", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **10** (1964), p. 215–236.

[IM65] N. Iwahori & H. Matsumoto – "On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups", *Publ. Math. Inst. Hautes Études Sci.* **25** (1965), p. 5–48.

[Kac94] V. G. Kac – *Infinite-dimensional Lie algebras*, Cambridge Univ. Press, Cambridge, 1994.

[Kum02] S. Kumar – *Kac-Moody groups, their flag varieties and representation theory*, Progress in Math., vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Loo80] E. Looijenga – "Invariant theory for generalized root systems", *Invent. Math.* **61** (1980), no. 1, p. 1–32.

[Lus83] G. Lusztig – "Singularities, character formulas, and a q-analog of weight multiplicities", in *Analyse et topologie sur les espaces singuliers, II, III (Luminy, 1981)*, Astérisque, vol. 101-102, Société Mathématique de France, Paris, 1983, p. 208–229.

[Lus89] ———, "Affine Hecke algebras and their graded version", *J. Amer. Math. Soc.* **2** (1989), no. 3, p. 599–635.

[NR03] K. Nelsen & A. Ram – "Kostka-Foulkes polynomials and Macdonald spherical functions", in *Surveys in combinatorics, 2003 (Bangor)*, London Math. Soc. Lecture Note Ser., vol. 307, Cambridge Univ. Press, Cambridge, 2003, p. 325–370.

[Opd03] E. M. Opdam – "A generating function for the trace of the Iwahori-Hecke algebra", in *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, Progress in Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, p. 301–323.

[Par06] J. Parkinson – "Buildings and Hecke algebras", *J. Algebra* **297** (2006), no. 1, p. 1–49.

[Rém02] B. Rémy – *Groupes de Kac-Moody déployés et presque déployés*, Astérisque, vol. 277, Société Mathématique de France, Paris, 2002.

[Rou11] G. Rousseau – "Masures affines", *Pure and Applied Mathematics Quarterly* **7** (2011), no. 3, p. 859–921.

[Rou16] ———, "Groupes de Kac-Moody déployés sur un corps local II. Masures ordonnées", *Bull. Soc. math. France* **144** (2016), no. 4, p. 613–692.

[Rou17] ———, "Almost split Kac-Moody groups over ultrametric fields", *Groups Geom. Dyn.* **11** (2017), p. 891–975.

[Shi59] G. Shimura – "Sur les intégrales attachées aux formes automorphes", *J. Math. Soc. Japan* **11** (1959), p. 291–311.

[Tit87] J. Tits – "Uniqueness and presentation of Kac-Moody groups over fields", *J. Algebra* **105** (1987), no. 2, p. 542–573.

[Vig96] M.-F. Vignéras – *Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$*, Progress in Math., vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996.

Manuscript received 4th December 2017
accepted 17th January 2019

Ramla Abdellatif, LAMFA – UPJV, UMR CNRS 7352
80 039 Amiens Cedex 1, France
E-mail : ramla.abdellatif@u-picardie.fr
Url : http://www.lamfa.u-picardie.fr/abdellatif/

Auguste Hébert, Université de Lyon, UJM-Saint-Étienne CNRS, UMR CNRS 5208
F-42023, Saint-Étienne, France
E-mail : auguste.hebert@ens-lyon.fr
Url : http://perso.ens-lyon.fr/auguste.hebert/