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Representations of quasi-projective groups, flat connections and transversely projective foliations

REPRESENTATIONS OF QUASI-PROJECTIVE GROUPS, FLAT CONNECTIONS AND TRANSVERSELY PROJECTIVE FOLIATIONS

by Frank Loray, Jorge Vitório Pereira & Frédéric Touzet

Abstract. — The main purpose of this paper is to provide a structure theorem for codimension-one singular transversely projective foliations on projective manifolds. To reach our goal, we firstly extend Corlette-Simpson's classification of rank-two representations of fundamental groups of quasi-projective manifolds by dropping the hypothesis of quasi-unipotency at infinity. Secondly we establish a similar classification for rank-two flat meromorphic connections. In particular, we prove that a rank-two flat meromorphic connection with irregular singularities having non trivial Stokes matrices projectively factors through a connection over a curve.

Résumé (Représentations de groupes quasi-projectifs, connexions plates et feuilletages transversalement projectifs)

L'objet de cet article est d'établir un théorème de structure pour les feuilletages singuliers transversalement projectifs de codimension 1 sur une variété projective lisse. Pour ce faire, nous étendons d'abord la classification de Corlette et Simpson de représentations de rang 2 des groupes fondamentaux des variétés quasi-projectives lisses en omettant l'hypothèse de quasi-unipotence à l'infini. Ensuite, nous établissons une classification analogue pour les connexions méromorphes plates de rang 2. En particulier, nous montrons qu'une connexion méromorphe plate de rang 2 avec des singularités irrégulières et des matrices de Stokes non triviales se factorise par une connexion sur une courbe.

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1. Introduction

Let X be a smooth projective manifold over $\mathbb{C}$. A (holomorphic singular) codimension one foliation $\mathscr{F}$ on X is defined by a non zero rational 1-form ω satisfying Frobenius integrability condition $\omega \wedge d\omega = 0$. The foliation is said transversely projective if there are rational 1-forms α, β on X such that the $\mathfrak{sl}_2$-connection defined on the trivial vector bundle $X \times \mathbb{C}^2$ by

$$(1.1) \qquad Z \longmapsto \nabla Z = dZ + AZ \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha \end{pmatrix}$$

is *flat*: $dA + A \cdot A = 0$. Note that this flatness property is intrinsically attached to the foliation and does not depend on the choice of the defining rational one form ω. Indeed, if $\mathscr{F}$ is defined by $\omega' = a\omega$ where a is any rational function not identically zero, the corresponding connection matrix is

$$A' = \begin{pmatrix} \alpha + \frac{1}{2}\frac{da}{a} & \frac{1}{a}\beta \\ \omega' & -(\alpha + \frac{1}{2}\frac{da}{a}) \end{pmatrix}.$$

This definition, equivalent to that in [39], extends to the singular case the classical definition in [18] for smooth foliations. Outside the polar divisor of the connection matrix A, the foliation $\mathscr{F}$ admits distinguished germs of first integrals taking values in $\mathbb{P}^1$, well defined up to left composition with elements of $\operatorname{Aut}(\mathbb{P}^1) = \operatorname{PSL}_2(\mathbb{C})$. Precisely, given a local basis of ∇-horizontal sections $B = (b_{ij}) \in \operatorname{SL}_2(\mathscr{O}(U))$ on some open set U, i.e., satisfying $dB + A \cdot B = 0$, the ratio $\varphi := b_{21}/b_{22}$ provides such a local first integral for $\mathscr{F}$; changing to another basis $B \cdot B_0$ will have the effect of composing φ with a Moebius transformation.

Transversely projective foliations play a singular role in the study of codimension one foliations. They are precisely those foliations whose Galois groupoid in the sense of Malgrange is small (see [8, 28]). They often occur as exceptions or counter examples [4, 22, 42] and played an important role in our study of foliations with numerically trivial canonical bundle [25]. For these foliations, one can define a monodromy representation by considering analytic continuation of distinguished germs of first integrals, making the transverse pseudo-group into a group (see [23]); it coincides with the (projectivization of the) monodromy of the flat connection ∇.

The goal of this paper is to provide a structure theorem for transversely projective foliations in the spirit of [11, 6] and what has been done recently in [16] for transversely affine foliations.

In fact, we mainly work with the connection ∇ defined by (1.1) up to birational bundle transformation. When it has at worst regular singularities, ∇ is characterized by its monodromy representation up to birational bundle transformations [17]. One of the main ingredients that goes into the proof of our structure theorem is an extension of Corlette-Simpson's classification of rank two representations of quasi-projective fundamental groups [13] which we now proceed to explain.

1.1. Rank-two representations of quasi-projective fundamental groups

Let X° be a quasi-projective manifold and consider X a projective compactification of X° with boundary equal to a simple normal crossing divisor D. If D_i is an irreducible component of D, then by a small loop around D_i we mean a loop $\gamma : S^1 \to X^\circ$ that extends to a smooth map $\overline{\gamma} : \overline{\mathbb{D}} \to X$ which intersects D transversely on a unique smooth point of D_i. A representation $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is quasi-unipotent at infinity if for every irreducible component D_i of D and every small loop γ around D_i, the conjugacy class of $\rho(\gamma)$ is quasi-unipotent (eigenvalues are roots of the unity).

A representation $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ projectively factors through an orbifold Y if there exists a morphism $f : X^\circ \to Y$ and a representation $\widetilde{\rho} : \pi_1^{\mathrm{orb}}(Y, f(x)) \to \mathrm{PSL}_2(\mathbb{C})$ such that the diagram

$$
\begin{array}{ccc}
\pi_1(X^\circ, x) & \xrightarrow{\ \rho\ } & \mathrm{SL}_2(\mathbb{C}) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}} \\
\pi_1^{\mathrm{orb}}(Y, f(x)) & \xrightarrow{\ \widetilde{\rho}\ } & \mathrm{PSL}_2(\mathbb{C})
\end{array}
$$

is commutative (we refer to [32, §2.4] for the definition and general properties of orbifolds).

A polydisk Shimura modular orbifold is a quotient $\mathfrak{H}$ of a polydisk $\mathbb{D}^n$ by a group of the form $U(P, \Phi)$ where P is a projective module of rank two over the ring of integers $\mathscr{O}_L$ of a totally imaginary quadratic extension L of a totally real number field F; Φ is a skew hermitian form on $P_L = P \otimes_{\mathscr{O}_L} L$; and $U(P, \Phi)$ is the subgroup of the Φ-unitary group $U(P_L, \Phi)$ consisting of elements which preserve P. This group acts naturally on $\mathbb{D}^n$ where n is half the number of embeddings $\sigma : L \to \mathbb{C}$ such that the quadratic form $\sqrt{-1}\,\Phi(v, v)$ is indefinite. The aforementioned action is explained in details in [13, §9]. Note that there is one tautological representation

$$
\pi_1^{\mathrm{orb}}(\mathbb{D}^n/U(P, \Phi)) \simeq \mathrm{SU}(P, \Phi)/\{\pm \mathrm{Id}\} \hookrightarrow \mathrm{PSL}_2(\mathbb{C}),
$$

which induces for each embedding $\sigma : L \to \mathbb{C}$ one tautological representation $\pi_1^{\mathrm{orb}}(\mathbb{D}^n/U(P, \Phi)) \to \mathrm{PSL}_2(\mathbb{C})$. The quotients $\mathbb{D}^n/U(P, \Phi)$ are always quasi-projective orbifolds, and when $[L : \mathbb{Q}] > 2n$ they are projective (i.e., proper/compact) orbifolds. The archetypical examples satisfying $[L : \mathbb{Q}] = 2n$ are the Hilbert modular orbifolds, which are quasi-projective but not projective. We refer again to [13] for a thorough discussion and point out that our definition of tautological representations differs slightly from loc. cit. as they consider polydisk Shimura modular stacks instead of orbifolds and consequently their representations take values in $\mathrm{SL}_2(\mathbb{C})$. Here we are lead to consider representations with values in $\mathrm{PSL}_2(\mathbb{C})$ because $\pm \mathrm{Id} \in \mathrm{SU}(P, \Phi)$ acts trivially on $\mathbb{D}^n$.

THEOREM 1.1 (Corlette-Simpson). — *Suppose that X° is a quasi-projective manifold and $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is a Zariski dense representation which is quasi-unipotent at infinity. Then ρ projectively factors through*

(1) *a morphism $f : X^\circ \to Y$ to an orbicurve Y (orbifold of dimension one); or*

(2) *a morphism $f : X^\circ \to \mathfrak{H}$ to a polydisk Shimura modular orbifold $\mathfrak{H}$.*

In the latter case, the representation actually projectively factors through one of the tautological representations of $\mathfrak{H}$.

Although their hypothesis is natural, as representations coming from geometry (Gauss-Manin connections) are automatically quasi-unipotent at infinity, Corlette and Simpson asked in [13, §12.1] what happens if this assumption is dropped. Our first main result answers this question.

THEOREM A. — *Suppose that X° is a quasi-projective manifold and $\rho : \pi_1(X^\circ, x) \to$ $\mathrm{SL}_2(\mathbb{C})$ is a Zariski dense representation which is not quasi-unipotent at infinity. Then ρ projectively factors through a morphism $f : X^\circ \to Y$ to an orbicurve Y.*

Our method to deal with representations which are not quasi-unipotent at infinity is considerably more elementary than the sophisticated arguments needed to deal with the quasi-unipotent case. The non quasi-unipotency allows us to prove the existence of effective divisors with topologically trivial normal bundle at the boundary of X°. We then use Malcev's Theorem combined with a result of Totaro about the existence of fibrations to produce the factorization.

COROLLARY B. — *Suppose that X° is a quasi-projective manifold and $\rho : \pi_1(X^\circ, x) \to$ $\mathrm{SL}_2(\mathbb{C})$ is a representation which is not virtually abelian. Then ρ projectively factors through*

(1) *a morphism $f : X^\circ \to Y$ to an orbicurve Y; or*

(2) *a morphism $f : X^\circ \to \mathfrak{H}$ to a polydisk Shimura modular orbifold $\mathfrak{H}$ equipped with one of its tautological representations.*

In case (2)*, the representation is Zariski dense and quasi-unipotent at infinity.*

Proof. — The case of Zariski dense representations is covered by Theorem 1.1 and Theorem A. Non virtually abelian representations in $\mathrm{Aff}(\mathbb{C})$ factors through an orbicurve according to [2], [16, Th. 4.1] and references therein. □

1.2. RICCATI FOLIATIONS. — Since we are interested in PSL_2 rather than SL_2, we will essentially work with the projective connection associated to (1.1), or more geometrically the foliation $\mathscr{H}$ induced by ∇-horizontal sections on the total space of the projective bundle $X \times \mathbb{P}^1$. In an affine chart, $\mathscr{H}$ is defined by the "Riccati" pfaffian equation

$$dz + \omega - 2\alpha z - \beta z^2 = 0.$$

More generally, a *Riccati foliation* over a projective manifold X consists of a pair $(\pi : P \to X, \mathscr{H}) = (P, \mathscr{H})$ where $\pi : P \to X$ is a locally trivial $\mathbb{P}^1$-fiber bundle in the Zariski topology (i.e., P is the total space of the projectivization $\mathbb{P}(E)$ of a rank two vector bundle E) and $\mathscr{H}$ is a codimension one foliation on P which is transverse to a general fiber of π. If the context is clear, we will omit the $\mathbb{P}^1$-bundle P from the notation and call $\mathscr{H}$ a Riccati foliation.

The foliation $\mathscr{H}$ is defined by the projectivization of horizontal sections of a (non unique) flat meromorphic connection ∇ on E: it is the phase portrait of the projective connection $\mathbb{P}(E, \nabla)$. The connection ∇ is uniquely determined by $\mathscr{H}$ and by its trace on $\det(E)$. We say that the Riccati foliation $\mathscr{H}$ has *regular singularities* if it can be lifted to a meromorphic connection ∇ with regular singularities (see [17, Chap. II, Déf. 4.2]), and *irregular* if not.

We will say that a Riccati foliation $(P, \mathscr{H})$ over X factors through a projective manifold X' if there exists a Riccati foliation $(\pi' : P' \to X', \mathscr{H}')$ over X', and rational maps $\phi : X \dashrightarrow X'$, $\Phi : P \dashrightarrow P'$ such that $\pi' \circ \Phi = \phi \circ \pi$, Φ has degree one when restricted to a general fiber of P, and $\mathscr{H} = \Phi^* \mathscr{H}'$.

Our second main result describes the structure of Riccati foliations having irregular singularities.

Theorem C. — *Suppose that X is a projective manifold, and $(P, \mathscr{H})$ is a Riccati foliation over X. If $\mathscr{H}$ is irregular then at least one of the following assertions holds true.*

(1) *Maybe after passing to a (possibly ramified) two-fold covering, the Riccati foliation $\mathscr{H}$ is defined by a closed rational 1-form.*

(2) *The Riccati foliation $(P, \mathscr{H})$ factors through a curve.*

The proof of Theorem C relies on similar ideas as those used in the proof of Theorem A, on an infinitesimal criterion for the existence of fibrations due to Neeman (Theorem 2.3) and on the semi-local study of Riccati foliations at a neighborhood of irregular components of the polar divisor (Section 6).

1.3. Transversely projective foliations. — Our main goal, the description of the structure of transversely projective foliations, is achieved by combining Corollary B and Theorem C. To this end, recall that for a polydisk Shimura modular orbifold $\mathfrak{H}$ and any of its tautological representations ρ, Deligne extension of the associated local system provides a logarithmic flat connection; denote by $(\mathfrak{H} \times_\rho \mathbb{P}^1, \mathscr{H}_\rho)$ the induced Riccati foliation.

Theorem D. — *Let $\mathscr{F}$ be a codimension one transversely projective foliation on a projective manifold X. Then at least one of the following assertions holds true.*

(1) *There exists a generically finite Galois morphism $f : Y \to X$ such that $f^* \mathscr{F}$ is defined by a closed rational 1-form.*

(2) *There exists a rational dominant map $f : X \dashrightarrow S$ to a ruled surface $\pi : S \to C$, and a Riccati foliation $\mathscr{H}$ defined on S (i.e., over the curve C) such that $\mathscr{F} = f^* \mathscr{H}$.*

(3) *There exists a polydisk Shimura modular orbifold $\mathfrak{H}$ and a rational map $f : X \dashrightarrow \mathfrak{H} \times_\rho \mathbb{P}^1$ towards one of its tautological Riccati foliations such that $\mathscr{F} = f^* \mathscr{H}_\rho$.*

Remark 1.2. — In the third item, we note (see [37]) that, after blowing-up the ambient, $\mathscr{H}_\rho$ (resp. $\mathscr{F}$) is locally defined at singular points by a 1-form of the type

$$\sum_{i=1}^{k} \lambda_i \frac{dx_i}{x_i}, \quad 1 \leqslant k \leqslant n, \ \lambda_i \in \mathbb{R}_{>0},$$

for local coordinates $(x_1, \ldots, x_n)$ (where n is the dimension of the ambient space). In particular, for instance for X a surface, if $\mathscr{F}$ has (maybe after blowing-up) a hyperbolic singular point, a saddle-node or a non linearizable saddle, then we are in the first two items.

There are previous results on the subject [39] and on the neighboring subject of transversely affine foliations ([6], [11], [16]). With the exception of [16], all the other works impose strong restrictions on the nature of the singularities of the foliation. Our only hypothesis is the projectivity of the ambient manifold.

Theorem D also answers a question left open in [16]. There, a similar classification for transversely affine foliations is established for foliations on projective manifolds with zero first Betti number. Theorem D gives the analogue classification for arbitrary projective manifolds, showing that the hypothesis on the first Betti number is not necessary.

1.4. Flat meromorphic $\mathfrak{sl}_2$-connections. — A meromorphic rank-2 connection (E, ∇) on a projective manifold X is the datum of a rank 2 vector bundle E equipped with a $\mathbb{C}$-linear morphism $\nabla : E \to E \otimes \Omega^1_X(D)$ satisfying Leibniz rule

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \quad \text{for any section } s \text{ and function } f.$$

Here D is the polar divisor of the connection ∇. The connection ∇ is *flat* when the curvature vanishes, that is $\nabla \cdot \nabla = 0$, meaning that it has no local monodromy outside the support of D; we can therefore define its monodromy representation. *Throughout the text, all connections are assumed to be flat.*

When $\det(E) = \mathscr{O}_X$ and the trace connection $\operatorname{tr}(\nabla)$ is the trivial connection on $\mathscr{O}_X$, we say that (E, ∇) is a $\mathfrak{sl}_2$-*connection*. In particular, its monodromy representation takes values into $\operatorname{SL}_2(\mathbb{C})$.

We will say that any two connections (E, ∇) and (E', ∇') are *birationally equivalent* when there is a birational bundle transformation $\phi : E \dashrightarrow E'$ that conjugates the two operators ∇ and ∇'. In other words, $\mathscr{O}_X(\star D) \otimes_{\mathscr{O}_X} E \simeq \mathscr{O}_X(\star D) \otimes_{\mathscr{O}_X} E'$ for some reduced divisor D in X and the respective differential operators coincide. Keep in mind that the polar divisor might be not the same for ∇ and ∇'. The connection (E, ∇) is called regular if local ∇-horizontal sections have moderate growth near the polar divisor $(\nabla)_\infty$; equivalently, the restriction of (E, ∇) to a general (complete) curve $C \subset X$ is birationally equivalent to a connection having only simple poles (see [17, Chap. II, Déf. 4.2] for details). When the polar divisor $(\nabla)_\infty$ has only normal crossings, then regularity is equivalent to be birationally equivalent to a logarithmic connection, in particular having only simple poles. A connection (E, ∇) is said irregular if it is not regular.

We will say that (E, ∇) and (E', ∇') are *projectively equivalent* if the induced $\mathbb{P}^1$-bundles coincide $\mathbb{P}(E) = \mathbb{P}(E')$, and if moreover ∇ and ∇' induce the same projective connection $\mathbb{P}(\nabla) = \mathbb{P}(\nabla')$; equivalently, they induce the same Riccati foliation on the total space of $\mathbb{P}(E)$. This is equivalent to say that $(E', \nabla') = (E, \nabla) \otimes (L, \delta)$ for some flat meromorphic rank 1 connection (L, δ) on X.

We provide a structure theorem for connections up to the combination of projective and birational equivalence. In order to settle our result, we note that any meromorphic rank 2 connection is (projectively birationally) equivalent to a $\mathfrak{sl}_2$-connection (see Remark 4.2). A combination of Corollary B and Theorem C yields:

Theorem E. — *Let (E, ∇) be a flat meromorphic $\mathfrak{sl}_2$-connection on a projective manifold X. Then at least one of the following assertions holds true.*

(1) *There exists a generically finite Galois morphism $f : Y \to X$ such that $f^*(E, \nabla)$ is projectively birationally equivalent to one of the following connections defined on the trivial bundle:*

$$\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad or \quad d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

with ω a rational closed 1-form on X.

(2) *There exists a rational map $f : X \dashrightarrow C$ to a curve and a meromorphic connection (E_0, ∇_0) on C such that (E, ∇) is projectively birationally equivalent to $f^*(E_0, \nabla_0)$.*

(3) *The $\mathfrak{sl}_2$-connection (E, ∇) has at worst regular singularities and there exists a rational map $f : X \dashrightarrow \mathfrak{H}$ which projectively factors the monodromy through one of the tautological representations of a polydisk Shimura modular orbifold $\mathfrak{H}$. In particular, the monodromy representation of (E, ∇) is quasi-unipotent at infinity, rigid, and Zariski dense.*

In particular, when (E, ∇) is irregular, only the former two cases occur. As mentioned in the abstract, the connection projectively factors through a curve whenever it has non trivial Stokes matrices.

Remark 1.3. — As we are working in the meromorphic setting, it is no more relevant to consider orbicurves instead of curves in item (2).

1.5. Structure of the paper. — The paper is divided in two parts, with the first independent of the second. In the first part, Sections 2 and 3, we recall some results on the existence of fibrations which will be used throughout the paper, and present the proof of Theorem A. In this first part we avoided using foliation theory aiming at a wider audience. The second part deals with the irregular case and is organized as follows. Section 4 presents foundational results about Riccati foliations, most of them borrowed from [24] and [16]. In particular, we show how to reduce the general problem to the surface case. Section 5 describes the local structure of a Riccati foliation along its polar divisor (on a surface); we state there, in Theorem 5.6, a reduction of singularities in the spirit of Sabbah's "good formal model" for meromorphic connections by

using blowing-ups and ramified covers; the proof of this technical result is postponed to Section 8, and it uses tools from the theory of transversely projective foliations, like classification of their reduced singular points following [3, 42]. Section 6 analyzes the structure of reduced Riccati foliations over surfaces at a neighborhood of a connected component of its irregular divisor, showing in particular the existence of flat transverse coordinates. This will be essential to produce the fibration in the absence of rich monodromy. Section 7 contains the proofs of Theorems C and E. Section 8 deals with transversely projective foliations, including the proof of Theorem D, and the fact that it is actually equivalent to our structure result on $\mathfrak{sl}_2$-connections. Section 9 presents a series of examples underlining the sharpness of our results. Finally in an appendix we prove a result reminiscent of Sabbah's good formal models in the context of Riccati foliations.

2. Existence of fibrations

In this section we collect three results about the existence of fibrations on projective manifolds which will be used in the sequel. For the proof of Theorem A all we will need is Theorem 2.1 below, which is due to Totaro [40]. Toward the end of the paper (proof of Theorem C), we will make use of the two other results below.

Theorem 2.1. — *Let X be a projective manifold and D_1, D_2, D_3 3 connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line inside of $H^2(X, \mathbb{R})$. Then there exists a non constant morphism $f : X \to C$ to a smooth curve C with connected fibers which contracts every divisor numerically proportional to D_1 to a point. In particular, the D_i's are rational multiple of fibers of f.*

The original proof studies the restriction map $H^1(X, \mathbb{Q}) \to H^1(\widehat{D}_1, \mathbb{Q})$, where $\widehat{D}_1$ is the disjoint union of desingularizations of the irreducible components of D_1. When it is injective, this map leads to a divisor linearly equivalent to zero in the span of $D_2, \ldots, D_r$ which defines the fibration. Otherwise, the fibration is constructed as a quotient of the Albanese map of X. An alternative proof, based on properties of some auxiliary foliations is given in [35]. It goes as follows: given two divisors with proportional Chern classes, one constructs a logarithmic 1-form with poles on these divisors and purely imaginary periods. The induced foliation, although not by algebraic leaves in general, admits a non-constant real first integral. Comparison of the leaves of two of these foliations coming from three pairwise disjoint divisors with proportional Chern classes reveals that they are indeed the same foliation. The proportionality factor of the corresponding two logarithmic 1-forms gives, after Stein factorization, the sought fibration. For details, see respectively [40, 35].

In general, two disjoint divisors with same Chern classes are not fibers of a fibration. Indeed, if L is a non-torsion flat line-bundle over a projective curve C then the surface

$S = \mathbb{P}(L \oplus \mathscr{O}_C)$ admits two homologous disjoint curves, corresponding to the inclusions of L and $\mathscr{O}_C$ in $L \oplus \mathscr{O}_C$, which are not fibers of a fibration. The point is that the normal bundle of these sections are L^* and L. Nevertheless, if the normal bundle of one of the effective divisors is torsion then we do have a fibration containing them as fibers. This example is extracted from [40, p. 613].

Theorem 2.2. — *Let X be a projective manifold and D_1, D_2 be connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line of $H^2(X, \mathbb{R})$. If $\mathscr{O}_X(D_1)|_{D_1}$ is torsion then there exists a non constant map $f : X \to C$ to a smooth curve C with connected fibers which maps the divisors D_i to points.*

A proof of this result is presented in Section 2.1 below. There is also an infinitesimal variant of Theorem 2.1 due to Neeman [34, Art. 2, Th. 5.1] where instead of three divisors we have only one divisor with torsion normal bundle and constraints on a infinitesimal neighborhood of order bigger than the order of the normal bundle. The formulation below is due to Totaro, see [41, paragraph before the proof of Lem. 4.1], and the proof we present is an adaptation of Neeman's proof. For an alternative proof see [12, §3].

Theorem 2.3. — *Let X be a projective manifold and D be a connected effective divisor. Suppose that $\mathscr{O}_X(D)|_D$ is torsion of order m. If*

$$\mathscr{O}_X(mD)|_{(m+1)D} \simeq \mathscr{O}_{(m+1)D},$$

then there exists a non-constant morphism $f : X \to C$ to a smooth curve C with connected fibers which maps the divisor D to a point.

Proof. — Let $I = \mathscr{O}_X(-D)$ be the defining ideal of D. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{O}_X(mD) & \longrightarrow & \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^m & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_D & \longrightarrow & \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^{m+1} & \longrightarrow & \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^m & \longrightarrow & 0
\end{array}
$$

deduced from the standard exact sequence

$$0 \longrightarrow \mathscr{O}_X(-mD) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X/I^m \longrightarrow 0.$$

In cohomology we get the diagram

$$
\begin{array}{ccc}
& & H^0(X, \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^m) \to H^1(X, \mathscr{O}_X) \\
& & \wr \downarrow \qquad\qquad\qquad\qquad \downarrow \\
H^0((m+1)D, \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^{m+1}) \to H^0(mD, \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^m) \to H^1(D, \mathscr{O}_D)
\end{array}
$$

If $\mathscr{O}_X(mD)|_{(m+1)D} \simeq \mathscr{O}_{(m+1)D}$ then $\mathscr{O}_X(mD)|_{mD} \simeq \mathscr{O}_{mD}$ and

$$1 \in H^0(mD, \mathscr{O}_{mD}) = H^0(mD, \mathscr{O}_X(mD) \otimes \mathscr{O}_X/I^m)$$

belongs to the image of the map in the lower left corner. Exactness of the bottom row implies that $1 \in H^0(mD, \mathscr{O}_{mD})$ is mapped to zero in $H^1(D, \mathscr{O}_D)$.

According to the diagram, the morphism $H^0(mD, \mathscr{O}_{mD}) \to H^1(D, \mathscr{O}_D)$ factors through $H^1(X, \mathscr{O}_X) \to H^1(D, \mathscr{O}_D)$.

If $1 \in H^0(mD, \mathscr{O}_{mD})$ is mapped to zero in $H^1(X, \mathscr{O}_X)$ then we deduce from the first row of the first diagram above that $h^0(X, \mathscr{O}_X(mD)) \geqslant 2$. Moreover, $H^0(X, \mathscr{O}_X(mD))$ contains a section nowhere vanishing on D. Therefore mD moves in a pencil without base locus and we have the sought fibration.

If instead $1 \in H^0(mD, \mathscr{O}_{mD})$ is mapped to a nonzero element in $H^1(X, \mathscr{O}_X)$ then $H^1(X, \mathscr{O}_X) \to H^1(D, \mathscr{O}_D)$ is not injective. Thus the same holds true for the map $H^1(X, \mathscr{O}_X) \to \oplus H^1(D_i, \mathscr{O}_{D_i})$ where D_i are the irreducible components of D. It follows that $\oplus \mathrm{Alb}(D_i)$ (direct sum of the Albanese varieties) do not dominate $\mathrm{Alb}(X)$. By the second part of the proof of [40, Th. 2.1], the morphism

$$X \longrightarrow \frac{\mathrm{Alb}(X)}{\oplus \mathrm{Alb}(D_i)}$$

contracts D, and is non constant. It follows (cf. [34] or [40]) that the image is a curve and we get the sought fibration as the Stein factorization of this morphism. $\square$

2.1. Proof of Theorem 2.2. — By assumption the effective divisors D_1 and D_2 have proportional Chern classes. Therefore, there exists non-zero positive integers a_1, a_2 such that the line bundle $\mathscr{L} = \mathscr{O}_X(a_1 D_1 - a_2 D_2)$ lies in $\mathrm{Pic}^0(X)$. If $\mathscr{L}$ is a torsion element of $\mathrm{Pic}^0(X)$ then there exists a rational function $g : X \to \mathbb{P}^1$ with zero set supported on D_1 and polar set supported on D_2. We can take f as the Stein factorization of g.

Suppose now that $\mathscr{L}$ is not a torsion line-bundle. The restriction $\mathscr{L}|_{D_1}$ of $\mathscr{L}$ to D_1 is isomorphic to $\mathscr{O}_{D_1}(a_1 D_1)$ and according to our assumptions is a torsion line-bundle. Hence for some integer $m \neq 0$, $\mathscr{L}^{\otimes m}$ is in the kernel of the restriction morphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D_1)$. Since we are assuming that $\mathscr{L}$ is not a torsion line-bundle, it follows that the restriction morphism $H^1(X, \mathscr{O}_X) \to H^1(D_1, \mathscr{O}_{D_1})$ is not injective. We conclude as in the proof of Theorem 2.3. $\square$

3. Factorization of representations

3.1. Criterion for factorization. — We apply Theorem 2.1 to establish a criterion for the factorization of representations of quasi-projective fundamental groups. In the statement below, we have implicitly fixed an ample divisor A in X and we consider the bilinear pairing defined in $\mathrm{NS}(X)$, the Néron-Severi group of X, defined by $(E, D) = E \cdot D \cdot A^{n-2}$ where $n = \dim X$. According to Hodge index Theorem this bilinear form has signature $(1, \mathrm{rank}\,\mathrm{NS}(X) - 1)$.

Theorem 3.1. — *Let X be a projective manifold, D be a reduced simple normal crossing divisor in X, and $\rho : \pi_1(X \smallsetminus D) \to G$ be a representation to a simple linear*

algebraic group[1] *G with Zariski dense image. Suppose there exists E, a connected component of the support D with irreducible components $E_1, \ldots, E_k$, and an (analytic) open subset $U \subset X$ containing E such that the restriction of ρ to $\pi_1(U - E)$ has solvable image. Then either the intersection matrix (E_i, E_j) is negative definite or the representation factors through an orbicurve.*

Proof. — If the intersection matrix (E_i, E_j) is negative definite then there is nothing to prove. Throughout we will assume that (E_i, E_j) is not negative definite.

Let S be the Zariski closure of $\rho(\pi_1(U \smallsetminus E))$ in G. Since G is simple while $\rho(\pi_1(U \smallsetminus E))$ is solvable, we have that $S \neq G$. Let μ be the derived length of S and choose an element γ of the $(\mu + 1)$-th derived group of $\pi_1(X \smallsetminus D)$ such that $\rho(\gamma) \neq \mathrm{id}$.

We apply Malcev's Theorem (any finitely generated subgroup of G is residually finite) [26] to obtain a morphism $\varrho : \pi_1(X \smallsetminus D) \to \Gamma$ to a finite group Γ such that $\varrho(\gamma) \neq \mathrm{id}$. This choice of ϱ implies that the derived length of $\varrho(\pi_1(X \smallsetminus D))$ is at least $\mu + 2$. Since the derived length of S is μ, it follows that

$$c = [\varrho(\pi_1(X \smallsetminus D)) : \varrho(\pi_1(U \smallsetminus E))]$$

is at least 3.

Let us now consider the covering $p : X_\varrho \to X$, ramified along D, determined by ϱ. Over $X \smallsetminus D$, $p|_{p^{-1}(X \smallsetminus D)}$ is the étale covering of $X \smallsetminus D$ determined by $\ker \varrho$. In particular $p^{-1}(X \smallsetminus D)$ is a smooth quasi-projective manifold. In general, X_ϱ is projective but not necessarily smooth. Notice that by construction $p^{-1}(E)$ has at least $c = [\varrho(\pi_1(X \smallsetminus D)) : \varrho(\pi_1(U \smallsetminus E))] \geqslant 3$ distinct connected components in X_ϱ.

To be able to apply Hodge index Theorem, we now proceed to *desingularize* X_ϱ. In order to keep track of the intersection matrix (E_i, E_j), instead of applying Hironaka's Theorem we will construct a smooth variety Y together with a ramified covering $\phi : Y \to X$ which factors through p.

To each irreducible component D_i of D let m_i be the order of $\varrho(\gamma_i)$ on a short loop γ_i around D_i. Let $\kappa : \widetilde{X} \to X$ be a finite ramified covering of X, with $\widetilde{X}$ smooth, such that $\kappa^*(D_i) = m_i \widetilde{D_i}$ where $\widetilde{D_i}$ is a smooth and irreducible hypersurface and $\widetilde{D} = \sum \widetilde{D_i}$ is a simple normal crossing divisor. For the existence of κ with the above properties, we can use Kawamata coverings (see [20, Prop. 4.1.12]). If we denote the ramification divisor of κ by R and its image under κ by Δ then we can also assume that $\Delta + D$ is a simple normal crossing divisor on X.

The composition $\varrho \circ \kappa_* : \pi_1(\widetilde{X} \smallsetminus \widetilde{D}) \to \Gamma$ sends short loops around the irreducible components of $\widetilde{D}$ to the identity of Γ and therefore induces a representation $\widetilde{\varrho} : \pi_1(\widetilde{X}) \to \Gamma$. Let $\pi : Y \to \widetilde{X}$ be the étale covering of $\widetilde{X}$ determined by the kernel of $\widetilde{\varrho}$. It is a projective manifold endowed with a ramified covering $\phi = \kappa \circ \pi : Y \to X$

---

[1]As the terminology may vary, we precise that a linear algebraic group G is *simple* if it is not commutative and has no normal algebraic subgroups (other than 1 and G), and it is quasi-simple if its centre Z is finite and G/Z is simple.

which factors through $p : X_\varrho \to X$ as wanted. Consequently $\phi^{-1}(E)$ has (at least) as many connected components as $p^{-1}(E)$.

Let B be one of the connected components of $\phi^{-1}(E)$ with irreducible components $B_1, \ldots, B_\ell$. Consider the ample divisor $A' := \phi^* A$ on Y and define $(\cdot, \cdot)_Y$ using it. Notice that

$$(\phi^* C_1, \phi^* C_2)_Y = \phi^* C_1 \cdot \phi^* C_2 \cdot (\phi^* A)^{n-2} = \deg(\phi) \cdot (C_1, C_2),$$

for any divisors C_1, C_2 in X. Since we are assuming that (E_i, E_j) is not negative definite, there exists an effective divisor F supported by B with $(F, F)_Y \geqslant 0$. In fact, there exists such divisor for each connected component of $\phi^{-1}(E)$, therefore at least three. Hence, we can produce F_1, F_2, F_3 pairwise disjoint effective divisors with connected supports on Y satisfying $(F_i, F_i)_Y \geqslant 0$ and $(F_i, F_j)_Y = 0$. Since the signature of the quadratic form $(\cdot, \cdot)_Y$ is $(1, \operatorname{rank} \operatorname{NS}(Y) - 1)$, we deduce that all the three divisors have proportional Chern classes; moreover, $(F_i, F_i)_Y = 0$. Notice also that either $F_i \cap B = \varnothing$ or the support of F_i coincides with B. Indeed, if F is an effective divisor with support contained in B, but not equal to B, then we can choose an irreducible component of B, say C, not contained in the support of F but intersecting it. Therefore $(C, F)_Y > 0$ and, provided that $(F, F)_Y \geqslant 0$, for k large enough $C + kF$ is an effective divisor with

$$(C + kF, C + kF)_Y = (C, C)_Y + 2k(C, F)_Y + k^2(F, F)_Y > 0.$$

But, doing so, we would produce new disjoint F_i's with $(F_i, F_i)_Y > 0$, contradicting Hodge index Theorem.

We can apply Theorem 2.1 to ensure the existence of a curve Σ and a non-constant morphism with irreducible general fiber $g : Y \to \Sigma$ such that the divisors F_1, F_2, and F_3 are multiples of fibers of g. The morphism g is proper and open, thus all the other connected components of $\phi^{-1}(D)$ are mapped by g to points. Let us denote by $\widetilde{\rho} := \rho \circ \phi_* : \pi_1(Y \smallsetminus \phi^* D) \to G$ the lifted representation. We want first to prove that $\widetilde{\rho}$ factors through g. Note that $\widetilde{\rho}$ also has Zariski dense image in G.

Let $U \subset Y \smallsetminus \phi^* D$ be a Zariski open subset such that the restriction of g to U is a smooth and proper fibration, thus locally trivial in the C^∞ category over $\Sigma^o = g(U)$. Let also F be a fiber of $g_{|U}$ and H be the Zariski closure in G of $\widetilde{\rho}(\pi_1(F))$, and consider the following diagram

$$
\begin{array}{ccc}
1 \longrightarrow H \longrightarrow & G \\
\quad \widetilde{\rho} \Big\uparrow \qquad\qquad \widetilde{\rho} \Big\uparrow & \\
\pi_1(F) \longrightarrow \pi_1(U) \xrightarrow{\ g_*\ } \pi_1(\Sigma^o) \longrightarrow 1 &
\end{array}
$$

From this, it is clear that the image of $\pi_1(F)$ is normal in $\pi_1(U)$, and since normality is a (Zariski) closed condition, we deduce that H is a normal subgroup of G. Since G is simple, we conclude that H must be trivial, i.e., the restriction of $\widetilde{\rho}$ to U factors through the curve Σ^o. Following the proof of [13, Lem. 3.5] we see that this suffices

to obtain the factorization through an orbicurve. Still denote by $g : Y \to \Sigma$ the factorization morphism.

We will now prove that the morphism $g : Y \to \Sigma$ descends to a morphism $f : X \to C$ that factors the initial representation ρ. Here, we follow the argument of [13, Lem. 3.6]. By Stein factorization Theorem, we can assume that $g : Y \to \Sigma$ has connected fibers. Assume also $\phi : Y \to X$ is Galois, with Galois group Λ; if not, replace Y by some finite Galois covering $\widetilde{\phi} : \widetilde{Y} \overset{\varphi}{\longrightarrow} Y \overset{\phi}{\longrightarrow} X$ (choose a finite index normal subgroup of $\pi_1(X)$ that contains the subgroup defining ϕ). We note that Y might become singular, but this does not matter in what follows. For any $\gamma \in \Lambda$, we want to prove that γ permutes g-fibers, i.e. for a generic fiber F of g, then $g \circ \gamma(F)$ is a point.

Aiming at a contradiction assume $g \circ \gamma(F)$ is not a point. Then $g_{|\gamma(F)} : \gamma(F) \to \Sigma$ is surjective and, since the representation $\widetilde{\rho}$ factors through the curve Σ, it follows that the representation $\widetilde{\rho}$ has Zariski dense image in restriction to $\gamma(F)$. Strictly speaking, in order to define the restriction $\widetilde{\rho}$ to $\gamma(F)$, we have to move the base point of the fundamental group (that we have omitted so far) to put it into $\gamma(F)$ and the restriction depends on the way we do this, but different choices lead to conjugated subgroups and the property of being Zariski dense is invariant by conjugation. Once this has been done, the image H of $\widetilde{\rho}_{|\gamma(F)}$ must be of finite index in the image of the factorizing representation $\pi_1(\Sigma) \to G$. By hypothesis G is simple and therefore H is Zariski dense. On the other hand, since $\widetilde{\rho}$ comes from a representation on $X = Y/\Lambda$, it follows that $\widetilde{\rho}$ must be trivial in restriction to $\gamma(F)$, since it is in restriction to F. This contradiction shows that Λ, the Galois group of ϕ, permutes the fibers of $g : Y \to \Sigma$, and thus permutes the points of Σ (connectedness of fibers). We get a morphism from X to the orbicurve $C = \Sigma/\Lambda$ through which ρ factors. $\qquad\square$

REMARK 3.2. — We can avoid the factorization through an orbicurve by instead restricting the factorization to a Zariski open subset of $X \smallsetminus D$. In the opposite direction, if we allow one dimensional Deligne-Mumford stacks with general point having non trivial stabilizer as targets of the factorization, then we can replace simple linear algebraic groups by quasi-simple linear algebraic groups in the statement, since our proof shows that for G quasi-simple, there exists a fibration such that the Zariski closure of the images of fundamental groups of fibers of f under ρ are finite.

3.2. Rank-two representations at neighborhoods of divisors

PROPOSITION 3.3. — *Let X be a complex manifold, D a reduced and simple normal crossing divisor in X, and $\rho : \pi_1(X \smallsetminus D) \to \mathrm{SL}_2(\mathbb{C})$ a representation. Let E be a connected divisor with support contained in D such that for each irreducible component E_i of E and any short loop γ_i turning around E_i, the element $\rho(\gamma_i)$ does not lie in the center of $\mathrm{SL}_2(\mathbb{C})$, i.e., $\rho(\gamma_i)$ is distinct from $\pm \mathrm{Id}$. Then there exists an open subset $U \subset X$ containing E such that the restriction of ρ to $\pi_1(U \smallsetminus D)$ has solvable image.*

Proof. — Let $E_1, \ldots, E_k$ be the irreducible components of E and $\gamma_1, \ldots, \gamma_k$ be short loops turning around them. We will denote the set of smooth points of D in E_i by E_i°, i.e., $E_i^\circ = E_i \smallsetminus \bigcup_{j \neq i}(E_j \cap E_i)$.

Suppose first that $\pm\rho(\gamma_1)$ is unipotent. Since, by hypothesis, it is different from the identity, its action on $\mathbb{C}^2$ leaves invariant a one-dimensional subspace L. If U_1 is a small tubular neighborhood of E_1 and $U_1^\circ = U_1 \smallsetminus D$ then U_1° has the homotopy type of a S^1-bundle over E_1° and therefore the subgroup generated by γ_1 in $\pi_1(U_1^\circ)$ is normal. It follows that every $\gamma \in \pi_1(U_1^\circ)$ also leaves L invariant. It follows that the rank two local system induced by ρ admits a unique rank one local subsystem determined by L on U_1°.

To analyze what happens at a non-empty intersection $E_1 \cap E_j$, we can assume that both γ_1 and γ_j have base points near $E_1 \cap E_j$. Thus γ_1 commutes with γ_j, both $\rho(\gamma_1)$ and $\rho(\gamma_j)$ are unipotent, and they both leave L invariant. Thus the rank one local subsystem determined by L on U_1° extends to a rank one local subsystem on $U_1^\circ \cup U_j^\circ$. Repeating the argument above for the other irreducible components $E_2, \ldots, E_k$, we deduce the existence of a neighborhood U of E such that $\rho(\pi_1(U \smallsetminus D))$ is contained in a Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Similarly if $\rho(\gamma_1)$ is semi-simple, then the same holds true for every γ_i. Moreover, the representation now leaves invariant the union of two linear subspaces L_1 and L_2 (but does not necessarily leave invariant any of the two). We deduce that the image of ρ restricted to a neighborhood of E minus D is contained in an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{C}^*$. □

Remark 3.4. — The proof above is very similar to the proof of [13, Lem. 4.5].

Proposition 3.5. — *Let X, D, E, U and ρ be as in Proposition 3.3. Assume also that every short loop γ turning around an irreducible component of $D - E$ which intersects E has monodromy in the center of $\mathrm{SL}_2(\mathbb{C})$. If the intersection matrix of E is invertible, then the restriction of ρ to $\pi_1(U \smallsetminus D)$ is quasi-unipotent at the irreducible components of E.*

Proof. — Let (F, ∇) be a rank-two vector bundle over U with a flat logarithmic connection whose monodromy is given by ρ (see [17, Prop. 5.4, p. 94]). Since the monodromy is solvable, around each point of $U \smallsetminus D$ we have one or two sub-bundles of F which are left invariant by ∇. Modulo passing to a double covering of $U \smallsetminus D$ if necessary, we can assume that (F, ∇) is reducible, i.e., we have a sub-bundle $F_1 \subset F$ and a logarithmic connection ∇_1 on F_1 such that $\nabla_1 = \nabla|_{F_1}$. The monodromy of ∇_1 on a loop γ around irreducible components of E equal to one of the eigenvalues of $\rho(\gamma)$, say λ_γ. If γ_i is a short loop around an irreducible component E_i of E then the residue of ∇_1 along E_i satisfies

$$\exp(2\pi i \operatorname{Res}_{E_1}(\nabla_1)) = \lambda_{\gamma_1}.$$

By the residue formula we can write

$$c_1(F_1) = \sum \operatorname{Res}_{E_1}(\nabla_1)E_i + \sum \operatorname{Res}_{D_j}(\nabla_1)D_j,$$

where $D_1, \ldots, D_s$ are the irreducible components of D intersecting E but not contained in it. Since the eigenvalues around D_j are ± 1, we have that Res_{D_j} is a half-integer. Since $c_1(F_1)$ lies in $H^2(U, \mathbb{Z})$, we have that for every k

$$(c_1(F_1), E_k) = \sum \mathrm{Res}_{E_i}(\nabla_1)(E_i, E_k) + \sum \mathrm{Res}_{D_j}(\nabla_1)(D_j, E_k)$$

is an integer. Therefore the vector $v = (\mathrm{Res}_{E_1}(\nabla_1), \ldots, \mathrm{Res}_{E_k}(\nabla_1))^t$ satisfies a linear equation of the form $A \cdot v = b$ with $A = (E_i, E_j)$ and $2b \in \mathbb{Z}^k$. Since A has integral coefficients and is negative definite, it follows that v is a rational vector. Therefore the restriction of ρ to $U \smallsetminus D$ is quasi-unipotent at the irreducible components of E. $\quad\square$

REMARK 3.6. — The proof above is reminiscent of Mumford's computation [33] of the homology of the plumbing of a contractible divisor on a smooth surface.

3.3. PROOF OF THEOREM A. — Let X be a projective manifold and $D \subset X$ a simple normal crossing hypersurface such that $X^\circ = X \smallsetminus D$. Let $\rho : \pi(X \smallsetminus D) \to \mathrm{SL}_2(\mathbb{C})$ be a Zariski dense representation which is not quasi-unipotent at infinity. Let E be a connected divisor with support contained in[2] $|D|$ such that $\rho(\gamma) \neq \pm\,\mathrm{Id}$ for every small loop around an irreducible component of E, and $\rho(\gamma)$ is not quasi-unipotent for at least one small loop. If E is maximal with respect to these properties, Proposition 3.3 implies that the restriction of the projectivization of ρ to a neighborhood of E is solvable, and Proposition 3.5 implies that the intersection matrix of E is indefinite. Let $D' \subset D$ the union of the components of D around which the monodromy is equal to $\pm\,\mathrm{Id}$. Clearly $|E|$ is a connected component of $|D - D'|$. Since $\mathrm{PSL}_2(\mathbb{C})$ is a simple group we can apply Theorem 3.1 (replacing D by $D - D'$) to factorize the projectivization of ρ through an orbicurve. Theorem A follows. $\quad\square$

4. RICCATI FOLIATIONS

Here we recall basic definitions and properties of Riccati foliations, and provide some reduction lemmata.

4.1. PROJECTIVE CONNECTIONS AND RICCATI FOLIATIONS. — Let E be a rank 2 vector bundle on a complex manifold X and $\nabla : E \to E \otimes \Omega^1_X(D)$ be a meromorphic connection on E with (effective) polar divisor D: the operator ∇ is $\mathbb{C}$-linear and satisfying Leibniz rule

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \quad \text{for any local section } s \text{ and function } f.$$

In any local trivialization $Z : E \to \mathbb{C}^2$ of the bundle, the connection is defined by

$$Z \longmapsto \nabla(Z) = dZ + AZ \quad \text{with} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ and } A = \begin{pmatrix} \alpha & \beta \\ \omega & \delta \end{pmatrix},$$

where A is a matrix of meromorphic 1-forms (sections of $\Omega^1_X(D)$).

The operator ∇, being linear, commutes with the fiberwise action of $\mathbb{G}_m = \mathbb{C}^*$ on E and induces a projective connection on the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$. In local trivialization

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[2]If H is a divisor, $|H|$ stands for the *support* of H

above, after setting $(1 : z) = (z_1 : z_2)$, the differential equation $\nabla Z = 0$ takes the form $\Omega = 0$ for the Riccati 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2,$$

with $(\omega_0, \omega_1, \omega_2) = (\omega, \delta - \alpha, -\beta)$. Any two connections (E, ∇) and (E', ∇') induce the same projective connection if, and only if there is a rank 1 meromorphic connection (L, ζ) such that $(E', \nabla') = (L, \zeta) \otimes (E, \nabla)$. The connection (E, ∇) is actually determined by the (independent) data of the projective connection and the trace connection $(\det(E), \mathrm{tr}(\nabla))$ (defined in local trivialization by $d + \alpha + \delta$).

The distribution $\Omega = 0$ defines a (singular codimension one) foliation $\mathscr{H}$ on the total space P of the $\mathbb{P}^1$-bundle if, and only if it satisfies Frobenius integrability condition $\Omega \wedge d\Omega = 0$, which is equivalent to

$$(4.1) \qquad \begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \omega_1 \wedge \omega_2. \end{cases}$$

The flatness condition $\nabla \cdot \nabla = 0$ for the linear connection, which writes $dA + A \cdot A = 0$ in local trivialization, is equivalent to (4.1) (flatness of the projective connection) and the flatness $d(\alpha + \delta) = 0$ for the trace connection. In this case, ∇-horizontal sections project onto the leaves of $\mathscr{H}$.

We say that $(P \to X, \mathscr{H})$ is a Riccati foliation since over a general point of X, the $\mathbb{P}^1$-fiber is transverse to $\mathscr{H}$.

Remark 4.1. — A codimension one foliation $\mathscr{H}$ on (the total space of a) $\mathbb{P}^1$-bundle $P \to X$ is a Riccati foliation if and only if it is transverse to a general $\mathbb{P}^1$-fiber. Indeed, in local trivialization, $\mathscr{H}$ must be defined by a 1-form $P(x, z)dz + \widetilde{\Omega}(x, z)$ polynomial in z. One easily checks that transversality to $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$ implies that P does not depend on the variable z. On the other hand, transversality near $z = \infty$ implies that $\widetilde{\Omega}$ has degree at most 2 in z.

Remark 4.2. — One could define define Riccati foliations on arbitrary $\mathbb{P}^1$-bundles, not necessarily of the form $\mathbb{P}(E)$. However, in this paper, *all Riccati foliations are defined on projectivized vector bundles* $\mathbb{P}(E)$. In particular, when X is projective, they are birationally equivalent to the trivial bundle $X \times \mathbb{P}^1$ where all above formula makes sense globally; moreover, they come from the $\mathfrak{sl}_2$-connection on $X \times \mathbb{C}^2$ obtained by setting $\alpha + \delta = 0$.

Remark 4.3. — Any two $\mathfrak{sl}_2$-connections (E, ∇) and (E', ∇') are equivalent if, and only if, there exists a flat rank 1 logarithmic connection (L, δ) on X having monodromy into the binary group $\{\pm 1\} \subset \mathbb{G}_m$ such that (E, ∇) is birationally equivalent to $(L, \delta) \otimes (E', \nabla')$. In particular, (E, ∇) and (E', ∇') are birationally equivalent after pulling them back to the ramified two-fold cover $Y \to X$ determined by the monodromy representation of (L, δ).

Remark 4.4. — If X is a projective manifold and $H \subset X$ is an hypersurface, there may exist representations

$$\rho : \pi_1(X \smallsetminus H) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$$

which cannot be realized as the monodromy of a Riccati foliation (see [24, Ex. 5.2]). However, when X is a projective curve, then $H = \{p_1, \ldots, p_l\}$ is a finite union of points, this obstruction does not exist because every local monodromy (around each p_i) can be realized as the projectivization of the monodromy of a meromorphic connection. This enables us to construct over X a $\mathbb{P}^1$ bundle (hence a projectivization of a rank 2 vector bundle) equipped with a Ricatti foliation with the right monodromy (see [21] for the details).

The (effective) polar divisor $(\mathscr{H})_\infty$ of the Riccati foliation is defined as the direct image under $\pi : P \to X$ of the tangency divisor between $\mathscr{H}$ and the vertical foliation defined by the fibers of π. It corresponds to the vertical part of the polar divisor of Ω in local trivialization. We have $(\nabla)_\infty \geqslant (\mathscr{H})_\infty$ since the trace $\mathrm{tr}(\nabla)$ may have more, or higher order poles; however, in the $\mathfrak{sl}_2$-case, the two divisors coincide.

The *monodromy representation* of the Riccati foliation $(P \to X, \mathscr{H})$ is the representation

$$\rho_{\mathscr{H}} : \pi_1(X \smallsetminus |(\mathscr{H})_\infty|) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$$

defined by lifting paths on $X \smallsetminus |(\mathscr{H})_\infty|$ to the leaves of $\mathscr{H}$. We note that $\rho_{\mathscr{H}}$ is just the projectivization of the linear monodromy of ∇.

4.2. Riccati foliations defined by a closed 1-form. — Let us start with two criteria.

Lemma 4.5. — *Let $(P, \mathscr{H})$ be a Riccati foliation over a projective manifold X. If there exists a birational map $\Phi : P \dashrightarrow P$ distinct from the identity such that $\Phi^* \mathscr{H} = \mathscr{H}$ and $\pi \circ \Phi = \pi$ then there exists a generically finite morphism of degree at most two $f : Y \to X$ such that $f^*(P, \mathscr{H})$ is defined by a closed rational 1-form.*

Proof. — Since Φ commutes with projection $\pi : P \to X$ it follows that over a general fiber of π, Φ is an automorphism. Let $F = \{z \in P \smallsetminus \mathrm{indet}(\Phi) | \Phi(z) = z\}$ be the set of fixed points of Φ. Since we are dealing with a family of automorphism of $\mathbb{P}^1$, the projection of F to X is generically finite of degree one or two. Assume first that the degree is one. Then we can birationally trivialize P in such a way that F becomes the section at infinity and Φ is of the form $z \mapsto z + \tau$ for some $\tau \in \mathbb{C}(X)$. Let $\Omega = dz + \omega_0 + \omega_1 z + \omega_2 z^2$ be a rational form defining $\mathscr{H}$. The invariance of $\mathscr{H}$ under Φ reads as

$$\Omega \wedge \Phi^* \Omega = 0 \iff \omega_2 = 0 \text{ and } \omega_1 = -d \log \tau \iff d(\tau \Omega) = 0.$$

If the degree of $\pi|_Z$ is two then after replacing X by (a resolution of) a ramified double covering we can assume that P is trivial and that Φ is given by $z \mapsto \lambda(x)z$. The invariance of $\mathscr{H}$ under Φ reads as

$$\Omega \wedge \Phi^* \Omega = 0 \iff \omega_0 = \omega_2 = d\lambda = 0 \iff d(z^{-1} \Omega) = 0.$$

This concludes the proof of the lemma. $\qquad \square$

Lemma 4.6. — *Let $(P, \mathscr{H})$ be a Riccati foliation over a complex manifold X. Let $H \subset P$ be a $\mathscr{H}$-invariant (maybe singular) hypersurface which intersects the generic fiber of $P \to X$ at $2 \leqslant n < \infty$ distinct points.*

– If $n \geqslant 3$, then $\mathscr{H}$ has a (non constant) meromorphic first integral.

– If $n = 2$, maybe passing to a two-fold cover $X' \to X$, then $\mathscr{H}$ is defined after a convenient bundle trivialization by $\Omega = dz/z + \omega$ with ω a closed 1-form.

In particular, the monodromy of $\mathscr{H}$ is virtually abelian.

Since H is $\mathscr{H}$-invariant, the intersection set between H and a fiber must be (globally) invariant by the monodromy group computed on the same fiber. For $n = 2$, this implies that the monodromy is dihedral, and for $n > 2$, that it is finite. The conclusion of the Lemma is much stronger in case $\mathscr{H}$ is irregular, since $\mathscr{H}$ could have no monodromy but transcendental leaves in that case.

Proof. — Maybe passing to a finite cover $X' \to X$, we can assume that H splits into n meromorphic sections. For $n \geqslant 3$ one can send three of them to $z = 0, 1, \infty$ and observe that their $\mathscr{H}$-invariance implies $\omega_0 = \omega_1 = \omega_2 = 0$. Therefore, z is a first integral and all leaves have algebraic closure. For $n = 2$ one can send the two sections to $z = 0$ and $z = \infty$ and we get that the Riccati 1-form defining $\mathscr{H}$ takes the form $dz + \omega z$; Frobenius integrability implies $d\omega = 0$.   □

Let us now describe Riccati foliations defined by closed 1-forms.

Proposition 4.7. — *Let $(P, \mathscr{H})$ be a Riccati foliation over a projective complex manifold X. If $\mathscr{H}$ is defined by a closed 1-form Ω on P, then:*

– either $\mathscr{H}$ has a (non constant) meromorphic first integral,

– or after a convenient bimeromorphic bundle trivialization, we have

$$(4.2) \qquad \Omega = c\left(\frac{dz}{z} + \omega\right) \quad or \quad \Omega = dz + \omega,$$

with ω a meromorphic closed 1-form on X and $c \in \mathbb{C}^*$.

Proof. — Denote by $(\Omega)_0$ and $(\Omega)_\infty$ the zero and polar divisor of Ω. Since Ω is closed, the support H of $(\Omega)_0 - (\Omega)_\infty$ is $\mathscr{H}$-invariant. The hypersurface H intersects the generic fiber of $P \to X$ in $n \geqslant 1$ distinct points (1-forms have non trivial divisor on $\mathbb{P}^1$). If $n \geqslant 3$, then $\mathscr{H}$ has a first integral.

If $n = 2$, then H splits into the union of two meromorphic sections, H_0 and H_∞ say; indeed, either they are zero and polar locus of Ω, or they are both simple pole, but with opposite residue. After trivialization of P sending them to $z = 0$ and $z = \infty$ respectively, we get that $\mathscr{H}$ is defined by a closed 1-form $\Omega' := dz/z + \omega$; but $\Omega = f \cdot \Omega'$ is closed as well, which implies that f is a first integral. We are thus in one of the two items depending on f is constant or not.

Finally, when $n = 1$, we can assume $H = \{z = \infty\}$ and we get that

$$\Omega = f(dz + \omega_0 + z\omega_1 + z^2\omega_2)$$

restricts to a general fiber as $c \cdot dz$, with $c \in \mathbb{C}^*$. This means that f does not depend on z and, by gauge transformation $z \mapsto cz$, we can assume $f = 1$. The 1-form Ω is closed if, and only if, $d\omega_0 = \omega_1 = \omega_2 = 0$. $\qquad\square$

4.3. GENERICALLY FINITE MORPHISMS AND FACTORIZATIONS. — In order to prove Theorems C and D, it will be useful to blow-up the manifold X and pass to a finite cover in order to simplify the foliation. At the end, we will need the following descent lemma to come back to a conclusion on X.

PROPOSITION 4.8. — *Let $(P, \mathscr{H})$ be a Riccati foliation over X and assume that $f^*(P, \mathscr{H})$ is not defined by a closed rational 1-form for any dominant morphism $f : Y \to X$. If the pull-back $\varphi^*(P, \mathscr{H})$ via generically finite morphism $\varphi : \widetilde{X} \to X$ factors through a curve, then the same holds true for $(P, \mathscr{H})$.*

Proof. — It is very similar to the proof of [13, Lem. 3.6]. Maybe composing by a generically finite morphism, we can assume that

(1) φ is Galois in the sense that there is a finite group G of birational transformations acting on $\widetilde{X}$ and acting transitively on a general fiber;

(2) there is a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ with connected fibers and a Riccati foliation $(\widetilde{P}_0, \widetilde{\mathscr{H}_0})$ over $\widetilde{C}$ such that its pull-back on $\widetilde{X}$ is birationally equivalent to $(\widetilde{P}, \widetilde{\mathscr{H}}) := \varphi^*(P, \mathscr{H})$.

In consequence, for any $g \in G$, $g^*(\widetilde{P}, \widetilde{\mathscr{H}})$ is birationally equivalent to $(\widetilde{P}, \widetilde{\mathscr{H}})$ and the Riccati foliation $(\widetilde{P}, \widetilde{\mathscr{H}})$ factors through $\widetilde{f} \circ g : \widetilde{X} \dashrightarrow \widetilde{C}$. Consider a general fiber Z of $\widetilde{f}$. The Riccati foliation $(\widetilde{P}, \widetilde{\mathscr{H}})|_Z$ restricted to Z is birationally equivalent to the trivial Riccati foliation on Z, and thus admits a rational first integral. If the map $\widetilde{f} \circ g$ were dominant in restriction to Z, this would imply that $(\widetilde{P}_0, \widetilde{\mathscr{H}_0})$ also has a rational first integral, and the same for $(\widetilde{P}, \widetilde{\mathscr{H}})$ and $(P, \mathscr{H})$, contradiction. Thus g (and G) must permute general fibers of $\widetilde{f}$ and acts on $\widetilde{C}$. Moreover, $g^*(\widetilde{P}_0, \widetilde{\mathscr{H}_0})$ is birational to $(\widetilde{P}_0, \widetilde{\mathscr{H}_0})$ for all $g \in G$. Lemma 4.5 implies that over each $g : \widetilde{X} \dashrightarrow \widetilde{X}$ there exists a unique birational map $\widehat{g} : \widetilde{P} \dashrightarrow \widetilde{P}$ such that $\widehat{g}^* \widetilde{\mathscr{H}} = \widetilde{\mathscr{H}}$, and a similar statement holds true for the action of G on $\widetilde{C}$. Therefore, we get an action of G on the diagram

$$
\begin{array}{ccc}
\widetilde{P} & \longrightarrow & \widetilde{P}_0 \\
\downarrow & & \downarrow \\
\widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{C}
\end{array}
$$

which preserves the Riccati foliations. Passing to the quotient, we get a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & P_0 \\
\downarrow & & \vdots \\
X & \xrightarrow{f} & C
\end{array}
$$

where $P_0 \dashrightarrow C$ has $\mathbb{P}^1$ as a general fiber. Moreover, the quotient foliation $\mathscr{H}_0$ on P_0 is transverse to the general fiber and thus of Riccati type. $\qquad\square$

4.4. Monodromy and factorization. — We say that a Riccati foliation $(P, \mathscr{H})$ has regular singularities when it can be locally induced by a flat meromorphic linear connection having regular singularities in the sense of [17, Chap. II]. Like in the linear case, any two Riccati foliations $(P, \mathscr{H})$ and $(P', \mathscr{H}')$ having regular singularities have the same monodromy if and only if there exists a birational bundle map $\phi :$ $P \dashrightarrow P'$ such that $\phi^* \mathscr{H}' = \mathscr{H}$, see [16, Lem. 2.13]. In particular, if the monodromy factors through a curve, then so does the Riccati foliation by Remark 4.4. The next proposition, borrowed from [16, Prop. 2.14], tells us what remains true in the irregular case.

Proposition 4.9. — *Let $(P, \mathscr{H})$ be a Riccati foliation over X. Suppose there exists a morphism $f : X \to C$ with connected fibers such that the polar divisor $(\mathscr{H})_\infty$ of $\mathscr{H}$ intersects the general fiber of f at most on regular singularities; assume moreover that the monodromy representation ρ of $(P, \mathscr{H})$ factors through f, i.e., there exists a divisor F supported on finitely many fibers of f and a representation ρ_0 from the fundamental group of $C_0 = f(X \smallsetminus |(\mathscr{H})_\infty + F|)$ to $\mathrm{PSL}_2(\mathbb{C})$ fitting in the diagram below.*

$$
\begin{array}{ccc}
\pi_1(X \smallsetminus |(\mathscr{H})_\infty + F|) & \xrightarrow{\;\;\rho\;\;} & \mathrm{PSL}_2(\mathbb{C}) \\
{\scriptstyle f_*} \downarrow & \nearrow{\scriptstyle \rho_0} & \\
\pi_1(C_0) & &
\end{array}
$$

Then $(P, \mathscr{H})$ factors through $f : X \to C$.

Remark 4.10. — For a Riccati or a linear connection, to have regular singularities is a local property which, if satisfied at some point of the polar divisor, remains true all along the irreducible component. The support of the polar divisor splits into regular and irregular components. The assumption in Proposition 4.9 that the general fiber of f intersect only regular singularities just means that the fiber intersects only regular components of the polar locus. Equivalently, the connection (or Riccati foliation) restricts to the fiber as a regular singular connection (resp. Riccati foliation).

4.5. Reduction to the two-dimensional case. — The proposition below allows us to reduce our study of Riccati foliations over arbitrary projective manifolds to study of Riccati foliation over projective surfaces.

Proposition 4.11. — *Let $(P, \mathscr{H})$ be a Riccati foliation over a projective manifold X and assume it has no rational first integral. If the restriction of $(P, \mathscr{H})$ to a sufficiently general surface $S \subset X$ factors through a curve, then the same holds true for $(P, \mathscr{H})$ over X.*

Proof. — Denote by $\pi : P \to X$ the natural projection. Let $(P_S, \mathscr{H}_S)$ be the restriction of $(P, \mathscr{H})$ to $S \subset X$, i.e. $P_S = P|_{\pi^{-1}(S)}$ and $\mathscr{H}_S = \mathscr{H}|_{\pi^{-1}(S)}$. By assumption, we get a rational bundle map $\phi : P_S \dashrightarrow P_0$ such that $\phi^* \mathscr{H}_0 = \mathscr{H}_S$. This shows that $\mathscr{H}_S$ contains a foliation by algebraic curves.

Applying the same argument for different choices of S, we obtain that through a general point $p \in P$ the leaf of $\mathscr{H}$ through p contains an algebraic subvariety A_p. It is known that the germ of leaf at a general point contains an unique germ of algebraic subvariety maximal with respect to inclusion which turns out to be irreducible, see the proof of [25, Lem. 2.4]. In our setup, we can thus assume that A_p has codimension at most two, since otherwise we would be able to enlarge A_p by choosing an appropriate S and applying the above argument.

It follows from [25, Lem. 2.4] that $\mathscr{H}$ contains a foliation $\mathscr{G}$ with all leaves algebraic and having codimension at most two, and that $\mathscr{H}$ is the pull-back of a foliation on a curve or a surface. In the former case, we get a rational first integral for $\mathscr{H}$, contradiction. Hence the codimension of $\mathscr{G}$ must be equal to two. Let now L be a general leaf of $\mathscr{G}$ and consider its projection to X. Since $\mathscr{H}$ is transverse to the general of fiber of π, this projection is generically étale and $\pi(L)$ is (a Zariski open subset) of a divisor D_L on X. The construction method of $\mathscr{G}$ makes clear that $\pi^{-1}(D_L)$ is invariant by $\mathscr{G}$, and also that the restriction of $\mathscr{H}$ to $\pi^{-1}(D_L)$ is birationally equivalent to the foliation on a trivial $\mathbb{P}^1$-bundle over D_L given by the natural projection to $\mathbb{P}^1$. This is sufficient to show that $\mathscr{H}$ is the pull-back of a Riccati foliation on a $\mathbb{P}^1$-bundle over a curve. □

Proposition 4.12. — *Let $(P, \mathscr{H})$ be a Riccati foliation over a projective manifold X, $\dim(X) \geqslant 2$. If the restriction of $(P, \mathscr{H})$ to a general hyperplane section $Y \subset X$ is defined by a closed rational 1-form, then the same holds true for $(P, \mathscr{H})$ over X.*

Proof. — Since being defined by a closed rational 1-form is invariant under birational transformation, we can assume without loss of generality that $P = X \times \mathbb{P}^1$. Set $Z = Y \times \mathbb{P}^1 \subset P$. If Y is sufficiently general, then Z is transversal to $\mathscr{H}$ outside a proper algebraic subset of $\mathscr{H}$ and we can consider ω, the closed (non zero) rational 1-form defining the restriction $\mathscr{H}|_Z$ on Z. Then, following [10, p. 47–50] (see also [7, §5.6]), ω extends (uniquely) as closed a meromorphic 1-form Ω defining $\mathscr{H}$ on the neighborhood of Z. Let U some connected neighborhood of Y over which Ω is defined and then can be written as

$$\Omega = a_0 dz + a_1 \omega_1 + \cdots + a_n \omega_n,$$

where $\omega_1, \ldots, \omega_n$ are $n = \dim_{\mathbb{C}} X$ meromorphically independent 1-forms on X and the a_i's are rational functions of z (a projective coordinate on $\mathbb{P}^1$) with coefficients lying in the field of meromorphic functions on U. The extension of those coefficients on the whole X is then automatic and implies the extension of Ω through $X \times \mathbb{P}^1$. □

5. Polar divisor and reduction of Riccati foliations

In the next two sections, we will restrict our attention to Riccati foliations over projective surfaces. Proposition 4.11 allows to transfer the conclusions to Riccati foliations over arbitrary projective manifolds. The general case will come back into play only at the proofs of Theorems C and D.

Let $(P, \mathscr{H})$ be a Riccati foliation over a projective surface S. In this section, we review the local structure of $(P, \mathscr{H})$ over a neighborhood of a general point of $(\mathscr{H})_\infty$. Maybe blowing-up S and passing to a ramified cover, we arrive to a list of nice local models that either are defined by a first integral, or factor through a curve (a local version of Theorem C). Let us first recall the one dimensional classification following the description of [5, Chap. 4, §1].

5.1. A review of the one dimensional case. — Here, we follow the description of [5, Chap. 4, §1] which corresponds to Levelt-Turrittin normalization.

Proposition 5.1. — *Let $(P \to \Delta, \mathscr{H})$ be a Riccati foliation over a disc $0 \in \Delta \subset \mathbb{C}$. Maybe after passing to the two-fold cover*

$$\Delta \longrightarrow \Delta \; ; \quad x \longmapsto x^2,$$

and after bimeromorphic bundle trivialization and change of x coordinate, the Riccati 1-form Ω defining the foliation fits into one of the following types:

  – $\Omega = dz$ *(trivial case)*
  – $\mathrm{Log}^{\mathbb{G}_m}$: $\Omega/z = dz/z + \lambda dx/x$, *with $\lambda \in \mathbb{C} \smallsetminus \mathbb{Z}$;*
  – $\mathrm{Log}^{\mathbb{G}_a}$: $\Omega = dz + dx/x$;
  – Irreg: $\Omega = dz + \left(dx/x^{k+1} + \lambda dx/x \right) z + (a(x)z^2 + b(x)z + c(x))dx$ *with a, b, c holomorphic, $k \in \mathbb{Z}_{\geqslant 1}$ and $\lambda \in \mathbb{C}$.*

Since the statement does not appear in this form in the literature, we provide a sketch of proof as well as definitions and useful remarks.

Let us start with a Riccati foliation $\mathscr{H}$ defined near $x = 0$ in $\mathbb{C}_x \times \mathbb{P}^1_z$ by

$$dz + \left(a(x)z^2 + b(x)z + c(x) \right) dx = 0.$$

Here, a, b, c are meromorphic at $x = 0$. If a, b, c are holomorphic, then the Riccati foliation is equivalent to $dz = 0$ by a biholomorphic bundle transformation (choose three distinct solutions and send them to $z = 0, 1, \infty$). If a, b, c have at most simple poles, then by a bimeromorphic bundle transformation we can reduce $\mathscr{H}$ to one of the following models (see [5, loc. cit.])

$$dz + \lambda \frac{dx}{x} z = 0, \quad dz + \frac{dx}{x} = 0 \quad \text{or} \quad dz = 0,$$

with $\lambda \in \mathbb{C} \smallsetminus \mathbb{Z}$ (for $\lambda \in \mathbb{Z}$, we get $d\widetilde{z} = 0$ for $\widetilde{z} = x^\lambda z$). Note that the trivial case can really occur even if the original Riccati admits a pole at $x = 0$ (see [5, loc. cit.]). So far, the coordinate x remains unchanged. The monodromy of $\mathscr{H}$ around $x = 0$ is respectively given by $z \mapsto e^{2i\pi\lambda}z$, $z \mapsto z + 1$ or trivial. This normal form is unique except the choice of λ: bundle transformations $z \mapsto x^n z^{\pm 1}$ change λ into $\pm\lambda + n$; the birational invariant is $\cos(2\pi\lambda)$.

From now on, let us assume that the Riccati equation has a multiple pole at $x = 0$, even up to bimeromorphic bundle transformations. We are in the so-called *irregular case.*

We may assume $z = \infty$ is not $\mathscr{H}$-invariant, i.e., $a \not\equiv 0$. Then, after a (unique) bimeromorphic bundle transformation of the form $z \mapsto f(x)z + g(x)$, with f, g meromorphic at $x = 0$, $f \not\equiv 0$, we may assume the Riccati equation in the form

$$(5.1) \qquad dz + \left(z^2 + \frac{\phi(x)}{x^{n+2}}\right)dx = 0.$$

(the corresponding $\mathfrak{sl}_2$-matrix connection is in companion form). Here, we assume ϕ holomorphic non vanishing and $n \in \mathbb{Z}_{\geqslant 0}$ is the order of pole. Define the *irregularity index* (or Poincaré-Katz rank) to be

$$\kappa := \frac{n}{2} \in \tfrac{1}{2}\mathbb{Z}_{\geqslant 0}$$

(see [1]). By an additional bundle transformation $z \mapsto x^{k+1}z$ with $k = \kappa$ or $\kappa + \frac{1}{2}$ (depending on the parity of 2κ) we get the form

$$dz + \frac{z^2 + (k+1)x^k z + \phi(x)}{x^{k+1}}\, dx = 0.$$

Then we cannot decrease the order of poles by further bimeromorphic bundle transformations (here we actually assume $\kappa > 0$ otherwise we are back to the logarithmic case).

We say that we are in the *unramified case* if $\kappa \in \mathbb{Z}_{>0}$, which is equivalent to say that $\phi(0) \neq 0$. In this case, the Riccati foliation has two singular points, and through each of them passes a (unique) formal $\mathscr{H}$-invariant section, i.e., a solution $z = f(x)$ of the Riccati equation with $f(x)$ a possibly divergent power series. After sending these two formal leaves to $z = 0$ and $z = \infty$ by a formal bundle transformation, the Riccati equation becomes

$$\frac{dz}{z} + \omega = 0 \quad \text{with} \quad \omega = \frac{\widetilde{\phi}(x)}{x^{k+1}}\, dx,$$

with $\widetilde{\phi}(x)$ formal power series. Usually, we kill the holomorphic part of ω by a last (formal) bundle modification of the form $z \mapsto h(x)z$, and the principal part of ω is an invariant by birational bundle transformations. Instead of this, we can use a (formal) change of x-coordinate to put the 1-form into the normal form $\omega = dx/x^{k+1} + \lambda dx/x$ (see [23, proof of Prop. 1.1.3]). In fact, a combination of both operations shows that, after analytic change of x-coordinate and holomorphic bundle transformation, we can put the Riccati equation into the form *Irreg* of Proposition 5.1: we use analytic approximation of formal solutions to get a, c holomorphic and then change x-coordinate up to the order k to normalize the principal part of ω. The formal normal form (see also [29, §IV.1.2])

$$(5.2) \qquad \frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}.$$

can be obtained by another formal bundle transformation. As in the logarithmic case, the residue λ is not unique: bundle transformations $z \mapsto x^n z^{\pm 1}$ change λ into $\pm\lambda + n$.

In the *ramified case* (called nilpotent in [5]) $\kappa = k - \frac{1}{2}$ we get, after bimeromorphic bundle transformation, the following normal form

– *Irreg^{ram}*: $\Omega = dz + \dfrac{z^2 - 2x^k z - x}{4x^{k+1}}dx + (a(x)z^2 + b(x)z + c(x))dx$ with a, b, c holomorphic and $k \in \mathbb{Z}_{\geqslant 1}$.

After ramification $x = \widetilde{x}^2$, we get a singularity of irregular (unramified) type *Irreg* with irregularity index $\widetilde{k} = 2\kappa$ (and $\lambda = 0$). More generally, all three types $Log^{\mathbb{G}m}$, $Log^{\mathbb{G}a}$ and *Irreg* are stable under ramifications $x = \widetilde{x}^n$; moreover λ and the irregularity index κ are both multiplied by n.

REMARK 5.2. — As explained in [5, Chap. 4, §1], there is a more geometric algorithmic way to minimize the order of pole of a Riccati foliation $(\pi : P \to \Delta, \mathscr{H})$ at $0 \in \Delta$. Recall that an elementary transformation of P at a point $p \in P_0 := \pi^{-1}(0)$ consists in blowing-up the point $p \in P$ and then contracting the strict transform of the fiber P_0. Assume $\mathscr{H}$ has a multiple pole at $0 \in \Delta$.

– If $\mathscr{H}$ has 2 singular points on the fiber P_0, then the order of poles is minimal, i.e., cannot be decreased by bimeromorphic bundle transformation: we are in the irregular unramified case.

– If $\mathscr{H}$ has only 1 singular point p on the fiber P_0, then apply an elementary transformation at p; if moreover the order of pole has not decreased, then again it is minimal: we are in the irregular ramified case.

Finally, when the order of pole is not minimal, and we want to minimize it, we just have to apply an elementary transformation to the unique singular point of $\mathscr{H}$, and possibly repeat this operation, until we arrive at a simple pole or at one of the aforementioned two cases. This algorithmic procedure can easily be generalized for Riccati foliations $(P \to S, \mathscr{H})$ over a surface S (see [24]).

5.2. Irregular singular points and Stokes matrices. — For details on what follows, see [29, Chap. VI] or [23, §5]. In the normal form *Irreg*, coefficients a, b, c can be killed by formal (generally divergent) gauge transformation and we arrive at the formal normal form (5.2):
$$\frac{\Omega}{z} = \frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}.$$
In the setting of $\mathfrak{sl}_2$-connections, this normal form writes
$$d + \begin{pmatrix} -\alpha/2 & 0 \\ 0 & \alpha/2 \end{pmatrix},$$
where $\alpha = dx/x^{k+1} + \lambda\, dx/x$.

In general this last normalization is not convergent. Nevertheless, there are $2k$ closed sectors covering a neighborhood of 0 in the x-variable such that the differential equation is holomorphically conjugate to the normal form over the interior of the sector, and the conjugation extends continuously to the boundary. Each of the sectors contains exactly one of the arcs $\{x \in \mathbb{D}_\varepsilon \mid x^k \in i\mathbb{R}\} \smallsetminus \{0\}$ where $\mathbb{D}_\varepsilon = \{|x| < \varepsilon\}$. Over each of these sectors there are only two solutions with well defined limit when x approaches zero which correspond to the solutions $\{z = 0\}$ and $\{z = \infty\}$ for the normal form. When we change from a sector intersecting $x^k \in i\mathbb{R}_{>0}$ to a sector intersecting $x^k \in i\mathbb{R}_{<0}$ in the counter clockwise direction then we can continuously extend the solution corresponding to $\{z = +\infty\}$ but the same does not hold true for the solution corresponding to $\{z = 0\}$. Similarly, when changing from sectors

Sectors containing $x^3 \in i\mathbb{R}_{>0}$ — Sectors containing $x^3 \in i\mathbb{R}_{<0}$
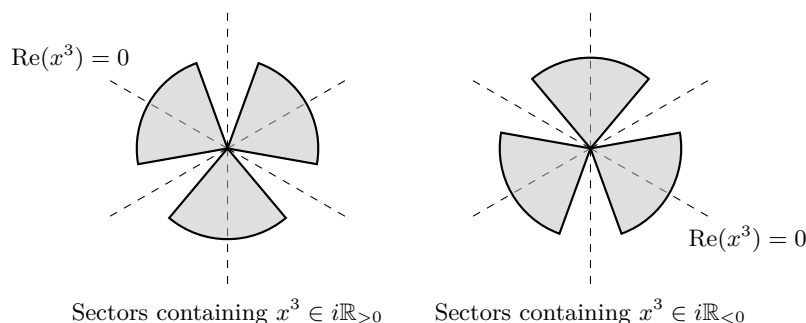
Figure 5.1.

intersecting $x^k \in i\mathbb{R}_{<0}$ to sectors with points in $x^k \in i\mathbb{R}_{>0}$ in the counter clockwise direction we can extend continuously the solution corresponding to $\{z = 0\}$ but not the one corresponding to $\{z = \infty\}$.

The obstructions to glue continuously the two distinguished solutions over adjacent sectors are the only obstructions to analytically conjugate the differential equation to its normal form. These obstructions are codified by the Stokes matrices (the matrix point of view is more convenient here):

$$\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_k & 1 \end{pmatrix}$$

well-defined up to simultaneous conjugacy by a diagonal matrix. Precisely, the b_i's (resp. the c_i's) are responsible for the divergence of the central manifold at $z = \infty$ (resp. $z = 0$). In other terms, the b_i's (resp. the c_i's) are the obstructions to kill the coefficient $a(x)$ (resp. $c(x)$). The monodromy around $x = 0$ is given by multiplying this sequence of Stokes matrices (in this cyclic order) with the formal monodromy

$$\begin{pmatrix} e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{pmatrix}$$

(on the left or the right, as this does not matter up to diagonal conjugacy).

If $x = \varphi(\widetilde{x})$ is a change of coordinate compatible with normal form *Irreg*, then the linear part $\varphi'(0)$ is a kth root of unity; it permutes the sectors, and consequently induces a cyclic permutation of (indices of) Stokes matrices. All these classical results can be found in [29, §VI.2]. Furthermore, one also finds there explicit examples of Riccati foliations with non trivial Stokes matrices. The most famous of them is undoubtedly Euler's equation that can be interpreted as a Riccati foliation over $\mathbb{P}^1$ defined by the rational 1-form

$$dz - \frac{(z-x)}{x^2}dx.$$

The pole over $\{x = 0\}$ is irregular unramified with non-trivial Stokes matrices, since the weak separatrix through $(0,0)$ is divergent.

5.3. Local factorization at a generic point of the polar divisor. — After reviewing the situation in dimension one we now move back to the two dimensional case. Let $(\pi : P \to S, \mathscr{H})$ be a Riccati foliation over a surface S and let $H \subset S$ be an irreducible component of the polar divisor $(\mathscr{H})_\infty$. The hypersurface $\pi^{-1}(H)$ can be $\mathscr{H}$-invariant or not. In fact, the following assertions are equivalent (see [24, Lem. 4.1])

  – H is not $\mathscr{H}$-invariant,
  – the polar order of $d\Omega$ along $\pi^{-1}(H)$ is strictly greater than the polar order of Ω.

Moreover, in this case, if H is smooth (e.g. if $(\mathscr{H})_\infty$ is simple normal crossing), then we can make a bundle modification over H such that H is no more a polar component of $\mathscr{H}$.

The singular set of $\mathscr{H}$ is located over the polar divisor, and in restriction to $\pi^{-1}(H)$, it consists in a curve Γ intersecting the generic fiber $\pi^{-1}(p)$, $p \in H$ in 1 or 2 points. Vertical irreducible components $\pi^{-1}(p)$ of Γ occur over singular points of $(\mathscr{H})_\infty$ and so-called *turning points* for the connection. Let us call them *special points*. The remaining (non vertical) part of Γ can consist in one or two sections, or also of an irreducible double section. When H is $\mathscr{H}$-invariant, then we have (see [24, Lem. 4.1])

Lemma 5.3. — *Let H be an irreducible component of the polar divisor of $(P, \mathscr{H})$. If $\pi^{-1}(H)$ is $\mathscr{H}$-invariant then the foliation $\mathscr{H}$ locally factors through a curve along H minus its set of special points.*

For the sake of completeness, we will give a short proof in Section 8.1. By local factorization we mean the following. At any non special point $p \in H$, and for a sufficiently small disk $\Delta \subset S$ transverse to H at p, there exists a neighborhood $U \subset S$ of p and a submersion $f : U \to \Delta$ with $f^{-1}(p) = H \cap U$ such that the Riccati $(P, \mathscr{H})|_U$ over the neighborhood U is the pull-back of its restriction over Δ through a fibre bundle isomorphism. In other words, the Riccati foliation is locally a product of a Riccati foliation over a disk by a disk (or a polydisk in higher dimension). In this situation, the isomorphism class of the Riccati foliation $(P, \mathscr{H})|_\Delta$ is called *the transverse type* of $(P, \mathscr{H})$ along the component H.

Remark 5.4. — For a linear $\mathfrak{sl}_2$-connection (E, ∇), the corresponding Riccati foliation obtained by projectivization satisfies the assumption of Lemma 5.3 at a non special point $p \in H$ of the polar divisor if and only if the polar divisors of the matrix connection A of ∇ and its differential dA satisfies in any local trivialization of E, $(dA)_\infty \leqslant (A)_\infty$. In the case of simple poles, this just means that the pole is actually logarithmic.

The following result follows from the existence of a canonical lattice (see [27]) and is also proved in [24] for the Riccati setting.

Proposition 5.5. — *Let $(\pi : P \to S, \mathscr{H})$ be a Riccati foliation over a projective surface S. Up to birational bundle modification, we can assume that all irreducible components H of $(\mathscr{H})_\infty$ satisfy assumption of Lemma 5.3 and the Riccati foliation locally factors through a curve at the neighborhood of any non special point $p \in S$.*

We now explain how special points can be simplified by blowing-up and finite flat morphism.

5.4. Reduction of singularities for Riccati foliations over surfaces. — The following result will be proved in Appendix A.

Theorem 5.6. — *Let $(\pi : P \to S, \mathscr{H})$ be a Riccati foliation over a projective surface S. There exists a generically finite morphism $\phi : \widetilde{S} \to S$ (with $\widetilde{S}$ smooth) and a birational bundle modification of the pull-back bundle $\phi^* P$ such that the pull-back Riccati foliation has normal crossing polar divisor D, and is locally defined over any point $p \in |D|$ by a Riccati 1-form having one of the following types:*

(1) $\mathrm{Log}^{\mathbb{G}_m}$: $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \lambda_x \dfrac{dx}{x}$, *or* $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \lambda_x \dfrac{dx}{x} + \lambda_y \dfrac{dy}{y}$;

(2) $\mathrm{Log}^{\mathbb{G}_a}$: $\Omega = dz + \dfrac{dx}{x}$, *or* $\Omega = dz + \dfrac{dx}{x} + \lambda \dfrac{dy}{y}$;

(3) Irreg^0: $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \dfrac{df}{f^{k+1}} + \lambda_x \dfrac{dx}{x} + \lambda_y \dfrac{dy}{y}$, *where $f = x$ or xy;*

(4) Irreg: $\Omega = dz + \Big(\dfrac{df}{f^{k+1}} + \lambda \dfrac{df}{f}\Big) z + \big(a(f)z^2 + b(f)z + c(f)\big) \, df$, *where $f = x$ or xy.*

In all cases, we have $\lambda \in \mathbb{C}^*$, $\lambda_x, \lambda_y \in \mathbb{C} \setminus \mathbb{Z}$, a, b, c holomorphic and $k \in \mathbb{Z}_{\geqslant 1}$.

The first part of the proof has to be compared with Sabbah's result in [38]. He defines, for linear meromorphic connections, a notion of good formal model at a point of a normal crossing divisor; this allows him to define Stokes matrices. Bad formal models occur in codimension 2; they correspond to special points of Section 5.3. When the base manifold X is a surface (and up to rank 5 connection), Sabbah proved that, maybe blowing-up X, we can assume that all points of the polar divisor have good formal model; this was generalized by Kedlaya and Mochizuki for any rank [19, 31].

In Appendix A, for the surface and rank 2 case, we provide a proof using an auxiliary (transversely projective) foliation, Seidenberg's resolution of singular points and the classification of reduced singular points of transversely projective foliations by Berthier and the third author in [3, 42]. We use ramified coverings to get rid of ramified irregular singular points.

In the sequel, a Riccati foliation $(\pi : P \to S, \mathscr{H})$ will be said *reduced* if it satisfies the conclusion of Theorem 5.6.

5.5. Closed 1-form and Stokes matrices. — Let $(P, \mathscr{H})$ be a reduced Riccati foliation on a complex surface S. Let $p \in S$ be a point of the support of the divisor $(\mathscr{H})_\infty$. If the local model for $\mathscr{H}$ at p is of type (1), (2) or (3) in Theorem 5.6, we see that $\mathscr{H}$ is locally defined by a meromorphic closed 1-form. In the last item (4) of Theorem 5.6, the coefficients $a(f), b(f), c(f)$ can be killed by a formal bundle transformation $\widehat{z} = \widehat{\phi}(z)$, i.e., with $\widehat{\phi} \in \mathrm{PGL}_2(\mathbb{C}[\![f]\!])$. By this way, we arrive at the normal form (see §5.1 or [29, §IV.1.2])

$$\widehat{\Omega} = \frac{d\widehat{z}}{\widehat{z}} + \Big(\frac{\lambda}{f} + \frac{1}{f^{k+1}}\Big) df$$

which is closed. From this formal model, one easily checks that $\widehat{z} = 0$ and $\widehat{z} = \infty$ are the only formal sections on the bundle that are $\mathcal{H}$-invariant (i.e., in restriction to which $\widehat{\Omega}$ identically vanishes). Therefore, Ω has exactly two $\mathcal{H}$-invariant formal sections of P at p. The obstruction to the convergence of these two formal sections is given by non diagonal coefficients of Stokes matrices (see [29, Chap. III, Th. 4.4]).

Lemma 5.7. — *At a sufficiently small neighborhood V_p of p, the following assertions are equivalent:*

- *$\mathcal{H}|_{V_p}$ is defined by a meromorphic closed 1-form on the restriction $P|_{V_p}$,*
- *there are two analytic sections $V_p \to P$ of the bundle that are $\mathcal{H}$-invariant,*
- *Stokes matrices are all trivial at p if the type is (4).*

In this case, the two analytic $\mathcal{H}$-invariant sections induce the formal sections at p.

Proof. — If $\mathcal{H}|_{V_p}$ is defined by a meromorphic closed 1-form Ω, then it follows from Proposition 4.7 that $\Omega = c \cdot (dz/z + \omega)$ after convenient bundle trivialization, with ω a meromorphic closed 1-form on X and $c \in \mathbb{C}^*$. Indeed, $\mathcal{H}$ cannot have a meromorphic first integral over p since it is irregular, and it cannot be defined by $dz + \omega$ since it has two $\mathcal{H}$-invariant formal sections. Therefore, $z = 0$ and $z = \infty$ are two $\mathcal{H}$-invariant sections which must coincide with the two formal ones at p. Conversely, if $\mathcal{H}$ has two analytic sections, then it is defined by a closed 1-form (see Lemma 4.6 and its proof). Finally, that the two last assertions are equivalent is well-known (see [29, Chap. III, Th. 4.4]). $\qquad\square$

Remark 5.8. — At the neighborhood of an irregular singular point p, a Riccati foliation $\mathcal{H}$ has no other local (formal) multi-section than the two formal sections discussed above. It has no meromorphic (formal) first integral and the unique formal 1-form defining $\mathcal{H}$ is, up to a scalar constant, the 1-form $\widehat{\Omega}$ above.

6. Irregular divisor

Throughout this section, $(P, \mathcal{H})$ is a *reduced* Riccati foliation over a projective surface S, i.e., $\mathcal{H}$ is as in the conclusion of Theorem 5.6. We study the irregular part of the polar divisor of $\mathcal{H}$. Precisely, since $\mathcal{H}$ is reduced, we have $(\mathcal{H})_\infty = \sum_i (1 + k_i) D_i$ where k_i is the irregularity index of $\mathcal{H}$ along D_i; in particular, $k_i = 0$ precisely if $\mathcal{H}$ is logarithmic at the generic point of D_i. Then we can decompose

$$(\mathcal{H})_\infty = (\mathcal{H})_{\infty,\mathrm{red}} + I,$$

where $(\mathcal{H})_{\infty,\mathrm{red}}$ is a reduced divisor with support $|(\mathcal{H})_\infty|$, and $I := \sum_i k_i D_i$ denotes the *irregular divisor*.

We note that the irregularity index k_i must be constant along each connected component of I, due to the local model of $\mathcal{H}$ at intersection points $p \in D_i \cap D_j$ (see (3) and (4) of Theorem 5.6).

We also note that the singular set $\mathrm{sing}(\mathcal{H})$ over the irregular divisor I consists of finitely many $\mathbb{P}^1$-fibers, located over intersection points of the polar divisor, and a smooth curve $Z \subset \pi^{-1}(I)$ on which π induces a 2-fold étale covering $\pi|_Z : Z \to I$. If

we restrict this picture to a connected component D of I, the curve $Z|_D$ may split as a union of two sections, say Z_0 and Z_∞, or be irreducible (i.e., unsplit). Throughout the section, we will refer to this dichotomy as the *split* or *unsplit case*. Note that, if we are working on a small analytic neighborhood V of D, with V and D having the same homotopy type, then maybe passing to a 2-fold étale covering $\widetilde{V} \to V$, we can always assume that we are in the split case. Indeed, the monodromy of $\pi|_Z$ induces a representation $\pi_1(V) \simeq \pi_1(|D|) \to \mathbb{Z}/2\mathbb{Z}$ which defines such a covering.

6.1. Flat coordinates. — Here, we show that any connected component D of the irregular divisor I has torsion normal bundle, in particular $D \cdot D = 0$. Indeed something even stronger holds true as proved in the proposition below.

Proposition 6.1. — *Let $(P, \mathscr{H})$ be a reduced Riccati foliation on a projective surface S. Let $D = k \cdot D_{\mathrm{red}}$ be a connected component of the irregular divisor I. Then the normal bundle of D_{red} is torsion of order r dividing $2k$ (dividing k in the split case) and moreover*
$$\mathscr{O}_S(rD_{\mathrm{red}})|_{kD_{\mathrm{red}}} \simeq \mathscr{O}_{kD_{\mathrm{red}}}.$$
In particular, in any case, we have $D \cdot D = 0$.

Proof. — Take an open covering of a neighborhood of $|D|$ by sufficiently small open sets V_i on which $\mathscr{H}$ is defined by models of Theorem 5.6, for convenient analytic coordinates x_i, y_i; the restriction $|D| \cap V_i$ is defined by $\{f_i = 0\}$ with $f_i = x_i$ or $f_i = x_i y_i$. We can assume that intersections $V_i \cap V_j$ are simply connected and do not contain singular points of the polar divisor $(\mathscr{H})_\infty$. We want to prove that, on $V_i \cap V_j$, we have
$$f_i = a_{ij} f_j + b_{ij} f_j^{k+1},$$
with constant $a_{ij}^{2k} = 1$ and b_{ij} holomorphic; moreover, in the split case, we want to show that we can choose f_i's such that $a_{ij}^k = 1$. This will prove the Proposition.

We are going to work at the formal completion of V along D. Over each intersection $V_i \cap V_j$, we can choose formal trivializing coordinates for the bundle $z_i, z_j : P|_{V_i \cap V_j} \to \mathbb{P}^1$ such that the foliation $\mathscr{H}$ is defined by the closed formal 1-forms
$$\Omega_i = \frac{dz_i}{z_i} + \left(\lambda \frac{df_i}{f_i} + \frac{df_i}{f_i^{k+1}} \right) \quad \text{and} \quad \Omega_j = \frac{dz_j}{z_j} + \left(\lambda \frac{df_j}{f_j} + \frac{df_j}{f_j^{k+1}} \right).$$
Indeed, if $\mathscr{H}$ has model (4) over V_i, then this follows from §5.1 (see (5.2)); on the other hand, for the model (3), with $f_i = x_i$ or $x_i y_i$, then we have
$$\frac{dz}{z} + \frac{df_i}{f_i^{k+1}} + \lambda_x \frac{dx_i}{x_i} + \lambda_y \frac{dy_i}{y_i} = \left(\frac{dz}{z} + \widetilde{\lambda}_y \frac{dy_i}{y_i} \right) + \frac{df_i}{f_i^{k+1}} + \lambda_x \frac{df_i}{f_i},$$
and we get the normal form Ω_i by setting $z_i = z \cdot \exp(\widetilde{\lambda}_y \log(y_i))$ for a convenient determination of the logarithm (note that $y_i \neq 0$ on $V_i \cap V_j$). We do exactly the same over V_j.

We claim that we can choose $\Omega_i = \pm \Omega_j$ over $V_i \cap V_j$. Indeed, since Ω_i and Ω_j define the same foliation, they must be proportional: we have $\Omega_i = g_{ij} \Omega_j$ for some (formal)

function g_{ij}. From closedness condition for these 1-forms, we deduce the g_{ij} must be a first integral for $\mathscr{H}$; therefore, g_{ij} is constant. But since residues of Ω_i and Ω_j on non vertical poles $z_i = 0, \infty$ are equal to ± 1, we thus get $g_{ij} = \pm 1$.

We note that the non vertical poles of Ω_i define two formal sections over $V_i \cap V_j$ that coincide over $|D| \cap V_i \cap V_j$ with the singular locus of $\mathscr{H}$. In the split case, the non vertical poles therefore globally define two sections so that we can globally fix the residues of all Ω_i's on these. Consequently, in the split case, we can moreover assume $\Omega_i = \Omega_j$ over $V_i \cap V_j$.

If $\lambda \neq 0$, then $\Omega_i = \Omega_j$ because of the sign of the residue over D. Therefore, in the unsplit case, we necessarily have $\lambda = 0$.

Let us first deal with the split case, therefore assuming $\Omega_i = \Omega_j$. We must have $f_i = f_{ij}(x_j, y_j)f_j$ and $z_i = g_{ij}(x_j, y_j)z_j$ for some invertible functions f_{ij} and g_{ij} on $V_i \cap V_j$. Then we get

$$0 = \Omega_i - \Omega_j = \frac{dg_{ij}}{g_{ij}} + \lambda \frac{df_{ij}}{f_{ij}} + \frac{1}{f_j^k}\frac{df_{ij}}{f_{ij}^{k+1}} + \left(\frac{1}{f_{ij}^k} - 1\right)\frac{df_j}{f_j^{k+1}}.$$

We deduce that the 1-form

$$\frac{1}{f_j^k}\frac{df_{ij}}{f_{ij}^{k+1}} + \left(\frac{1}{f_{ij}^k} - 1\right)\frac{df_j}{f_j^{k+1}} = \frac{1}{f_{ij}^{k+1}}\underbrace{\left(\frac{df_{ij}}{f_j^k} + \left(f_{ij} - f_{ij}^{k+1}\right)\frac{df_j}{f_j^{k+1}}\right)}_{\Theta}.$$

cannot have poles.

Notice that the residue of $f_j^k \Theta$ along $\{f_j = 0\}$ is nothing but $f_{ij} - f_{ij}^{k+1} \mod f_j$. Therefore $f_{ij}^k = 1 \mod f_j$. Now, let us write $f_{ij} = a_{ij}(1 + b_{ij}f_j^n)$ for some $n > 0$ and b_{ij} holomorphic (and $a_{ij}^k = 1$). Then we get

$$\Theta = \frac{(n - k)a_{ij}b_{ij}f_j^n df_j + o(f_j^n)}{f_j^{k+1}}.$$

If $n < k$, then b_{ij} must vanish along D and we can set $f_{ij} = a_{ij}(1 + \widetilde{b}_{ij}f_j^{n+1})$. By induction, we arrive at $n = k$. This establishes the proposition in the split case.

Finally, in the unsplit case, whenever we have $\Omega_i = -\Omega_j$, we can replace Ω_i by $\widetilde{\Omega}_i = -\Omega_i$, and (f_i, z_i) by $(\widetilde{f}_i, \widetilde{z}_i) = (af_i, 1/z_i)$ with $a^k = -1$, so that we are back to the previous discussion: we deduce that

$$f_i = \frac{\widetilde{a}_{ij}f_j + \widetilde{b}_{ij}f_j^{k+1}}{a} = a_{ij}f_j + b_{ij}f_j^{k+1},$$

with $a_{ij}^{2k} = (a_{ij}/a)^{2k} = 1$, ending the proof. $\qquad\square$

REMARK 6.2. — In the system of "transverse coordinates" $f_i : V_i \to \mathbb{C}$ constructed in the course of the proof, the Riccati foliation $\mathscr{H}|_{V_i}$ is, except at points of type (3), locally defined by

$$\Omega_i = \frac{dz_i}{z_i} + \left(\alpha_i(f_i)z_i + \lambda \frac{1}{f_i} + \frac{1}{f_i^{k+1}} + \beta_i(f_i) + \frac{\gamma_i(f_i)}{z_i}\right)df_i$$

for a holomorphic bundle coordinate z_i, where $\alpha_i, \beta_i, \gamma_i$ are holomorphic functions of f_i. We can assume the functions $\alpha_i, \beta_i, \gamma_i$ vanishing at arbitrary high order along $|D|$. Recall from Section 5.2 that the obstruction to make them just zero is given by the non triviality of Stokes matrices.

Remark 6.3. — On $V_i \cap V_j$ we get

$$\begin{cases} f_i = a_{ij}(f_j + b_{ij}f_j^{k+1}) \\ z_i = c_{ij}z_j^{\pm 1}, \end{cases}$$

with $a_{ij}^k = \pm 1$ (same sign) and $b_{ij}, c_{ij} \in \mathscr{O}_{V_i \cap V_j}$ satisfying $c_{ij}\exp(b_{ij}a_{ij}^k)|_{|D|} \equiv 1$. Considering transition multiplicators a_{ij}, we have an induced representation

$$\psi : \pi_1(|D|) \longrightarrow \mathbb{C}^*$$

taking values in the group of (respective) roots of the unity, which describe how the differentials df_i change when we follow closed paths along D.

6.2. Smooth fibration. — We have just proved that a connected component D of the irregular divisor I has local equations $f_i^{2k} : V_i \to \mathbb{C}$ that patch together up to order k, where k is the irregularity index. In the setting of real C^∞ functions, we can modify these local equations so that they can patch together to define a fibration on V having D as a singular fiber and which is smooth elsewhere.

Lemma 6.4. — *Let $(P, \mathscr{H})$ be a reduced Riccati foliation on a projective surface S. Let D be a connected component of the irregular divisor I like in Proposition 6.1 and r be the torsion order of D_{red}. Then, there exists a C^∞ map $f : V \to \mathbb{D}$ defined on an analytic neighborhood V of $|D|$ such that:*

  - *f induces a C^∞ locally trivial fibration over $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$,*
  - *f coincides with local equations $f_i^r : V_i \to \mathbb{C}$ of Remark 6.2 up to order k.*

Proof. — The transition functions $\{f_{ij} \in \mathscr{O}_V^*(V_i \cap V_j)\}$ of the proof of Proposition 6.1 define an element in $H^1(V, \mathscr{O}_V^*)$ which corresponds to $\mathscr{O}_V(D)$. Let $\mathscr{A}_V$ and $\mathscr{A}_V^*$ denote, respectively, the sheaves of C^∞ functions and of C^∞ invertible functions. Notice that $H^1(V, \mathscr{A}_V^*)$ is isomorphic to $H^2(V, \mathbb{Z})$, as $\mathscr{A}_V$ has no cohomology in positive degree ($\mathscr{A}_V$ is a fine sheaf). Notice also that the restriction morphism from $H^2(V, \mathbb{Z})$ to $H^2(|D|, \mathbb{Z})$ is an isomorphism since the inclusion of $|D|$ in V is a homotopy equivalence.

Since $f_{ij}^r|_{|D| \cap V_i} = 1$, the element $\{f_{ij}^r\} \in H^1(V, \mathscr{O}_V^*)$ maps to the trivial element of $H^1(V, \mathscr{A}_V^*) \simeq H^2(V, \mathbb{Z}) \simeq H^2(|D|, \mathbb{Z})$. Thus we can find non-vanishing C^∞-functions $\{g_i \in \mathscr{A}_V^*(V_i)\}$ such that $f_{ij}^r = g_i/g_j$. Moreover, we can choose the functions g_i satisfying $g_i|_{|D| \cap V_i} = 1$. Our assumptions imply that we can further assume that the function f_{ij}^r are constant equal to one when restricted to D.

Therefore we can define a C^∞ function $f : V \to \mathbb{C}$ by the formulas $f|_{V_i} = (f_i)^r/g_i$. The function f clearly satisfies $f^{-1}(0) = |D|$ set-theoretically. We claim that, after perhaps shrinking V, the critical set of f is contained in $|D|$. At a neighborhood of a smooth point of $|D|$ the function f is a power of a submersion. At a neighborhood

of a singular point of $|D|$, the function f is of the form $h(x, \overline{x}, y, \overline{y})xy$ and therefore $df = xydh + h(ydx + xdy)$. Since this expression has an isolated singularity at zero (the singular point of $|D|$) it follows that the critical set of f is indeed contained in D. Replacing V by $f^{-1}(\mathbb{D}_\varepsilon)$ for a sufficiently small ε we have just proved the existence of a C^∞ proper map $f : V \to \mathbb{D}_\varepsilon$ from V to the disk of radius ε which maps $|D|$ to the origin in $\mathbb{D}_\varepsilon$ and, when restricted to $V \smallsetminus |D|$, becomes a locally trivial fibration over $\mathbb{D}_\varepsilon^*$.   □

6.3. Closed 1-form and Riccati foliations. — We present now a semi-local version of Lemma 5.7

Proposition 6.5. — *Let $(P, \mathscr{H})$ be a reduced Riccati foliation on a projective surface S. Let D be a connected component of the irregular divisor of $(P, \mathscr{H})$ and V be a sufficiently small neighborhood of $|D|$. Assume we are in the split case. Then the following assertions are equivalent:*

  – *$\mathscr{H}|_V$ is defined by a meromorphic closed 1-form on the restriction $P|_V$,*
  – *there are two analytic sections $V \to P$ of the bundle that are $\mathscr{H}$-invariant,*
  – *at some point $p \in |D|$ (in fact any), the local model for $\mathscr{H}$ is of type Irreg^0, or of type Irreg with trivial Stokes matrices (see Theorem 5.6).*

Proof. — If the local model for $\mathscr{H}$ has trivial Stokes matrices at p (which is automatic in the first item, see Lemma 5.7), then it has two invariant analytic sections. By using local trivializations of $\mathscr{H}$ given in Section 5.3 (see also Remark 6.2), we see that this property propagates all along $|D|$. By the way, we get a two-fold section of the bundle; since they are locally attached to the two irreducible components of Z, we get actually two distinct global $\mathscr{H}$-invariant sections. It follows from Lemma 4.6 that $\mathscr{H}$ is defined by a meromorphic closed 1-form. Finally, it follows from Lemma 5.7 that, if defined by a closed 1-form, then Stokes matrices are trivial at any point $p \in |D|$.   □

Proposition 6.6. — *Let $(P, \mathscr{H})$ be an irregular Riccati foliation on a connected complex surface S. If there exists a generically finite morphism $f : \widetilde{S} \to S$ such that $f^*(P, \mathscr{H})$ is defined by a closed rational 1-form, then there exists another one $f' : \widetilde{S}' \to S$ of degree at most two with the same property.*

Proof. — The pull-back $(\widetilde{P}, \widetilde{\mathscr{H}}) := f^*(P, \mathscr{H})$ must be also irregular; in particular, $\widetilde{\mathscr{H}}$ does not admit non constant rational first integral. By Proposition 4.7, we deduce that, up to a birational bundle transformation, $\widetilde{\mathscr{H}}$ is defined by a Riccati of the form $dz/z + \omega = 0$ for a closed rational 1-form ω on S (note that $dz + \omega = 0$ is regular). In particular, the two sections $z = 0, \infty$ are $\widetilde{\mathscr{H}}$-invariant. Pushing them down via f, we get a n-section of P which is $\mathscr{H}$-invariant, $n \geqslant 2$. We can apply Lemma 4.6: by irregularity, $\mathscr{H}$ cannot have non constant meromorphic first integral and is therefore defined by a closed rational 1-form maybe passing to a two-fold cover $X' \to X$.   □

6.4. Monodromy around the irregular divisor

Proposition 6.7. — *Let $(P, \mathscr{H})$ be a reduced Riccati foliation on a complex surface S. Let D be a connected component of the irregular divisor I of the form $D = kD_{\mathrm{red}}$. Then, there is a neighborhood V of $|D|$ in S in restriction to which the monodromy of $(P, \mathscr{H})$ is virtually abelian. More precisely, at least one of the following assertions holds true.*

(1) *Maybe after passing to an étale covering $\widetilde{V} \to V$ of degree two, the (pulled-back) Riccati foliation $\mathscr{H}$ is defined by a closed meromorphic 1-form over $\widetilde{V}$ (and the monodromy is abelian).*

(2) *The monodromy factors through a C^∞-fibration $f : V \smallsetminus D \to \mathbb{D}^*$ and is therefore cyclic.*

If the supports of $(\mathscr{H})_\infty$ and D do not coincide on V, then we are in case (1).

Proof. — Let V be a small tubular neighborhood of D, and let $R : V \to |D|$ be a deformation retract. Since the statement is local, from now on $\mathscr{H}$ will be seen as a Riccati foliation defined over V. Maybe passing to an étale covering $\widetilde{V} \to V$ of degree two, we may assume we are in the split case: the singular locus $\mathrm{sing}(\mathscr{H})$ consists in two disjoint sections $D \to P|_D$, and fibers over singular points of D.

If all the Stokes matrices along D are trivial, then Proposition 6.5 implies that we are in case (1) of the statement. If $(\mathscr{H})_\infty$ and D have distinct supports in V, this means that $(\mathscr{H})_\infty$ contains at least one logarithmic component, intersecting $|D|$ at some point p; the model at p is (3) in Theorem 5.6, and all Stokes matrices are therefore trivial along D.

Let us now assume that Stokes matrices are non trivial at a smooth point $x_0 \in D$. Let Σ be a germ of curve transverse to D at x_0. Let $\pi : P \to V$ be the natural projection. The restriction of $\mathscr{H}$ to $\pi^{-1}(\Sigma)$, is a Riccati foliation over Σ with an invariant fiber $\{x_0\} \times \mathbb{P}^1$ having two saddle-nodes over it. As explained in Section 5.2, if k is the order of D at x_0 then there are $2k$ closed sectors on Σ, such that over the interior of each of them, the Riccati foliation is analytically conjugated to $dz/z + dx/x^{k+1} + \lambda\, dx/x$ and the conjugation extends continuously to the boundary. Over each of these sectors there are exactly two leaves with distinguished topological behavior: the closure of each of these distinguished leaves intersect the central fiber $\{x_0\} \times \mathbb{P}^1$ at a unique point.

Let $\mathscr{S}_\Sigma$ be the interior of one of these sectors. The local triviality of $\mathscr{H}$ along the smooth part of D, and the local normal form of $\mathscr{H}$ at the singularities of D, allow us to construct an open set $\mathscr{S} \subset V \smallsetminus |D|$ which extends $\mathscr{S}_\Sigma$ and over which we never lose sight of the two distinguished leaves of the restriction of $\mathscr{H}$ to $\pi^{-1}(\mathscr{S}_\Sigma)$. This open subset can be indeed realized as the saturation of $\mathscr{S}_\Sigma$ by the fibers of the fibration $f|_{V \smallsetminus |D|} : V \smallsetminus |D| \to \mathbb{D}^*$ defined by Lemma 6.4. The intersection of the resulting open set $\mathscr{S}$ and the initial transversal Σ has $r = \#\psi(\pi_1(|D|))$ distinct connected components, where $\psi : \pi_1(|D|) \to \mathbb{C}^*$ is the unitary representation defined in Remark 6.3.
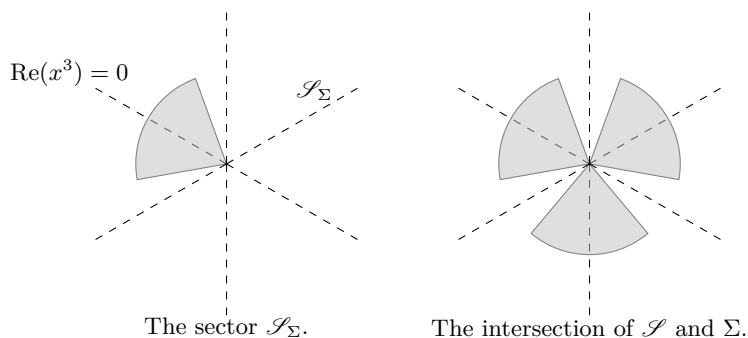
$$\text{Re}(x^3) = 0 \qquad \mathscr{S}_\Sigma$$

The sector $\mathscr{S}_\Sigma$.   The intersection of $\mathscr{S}$ and Σ.

Figure 6.1.

The restriction of the monodromy representation of $\mathscr{H}$ to $\mathscr{S}$ has its image (up to conjugation) contained in $\mathbb{C}^*$ as $\mathscr{H}$ over $\mathscr{S}$ has two distinguished leaves which are not permuted because of the assumption on $\text{sing}(\mathscr{H})$. By construction, the set $\mathscr{S}$ has the form $f^{-1}(\mathcal{S})$ with $\mathcal{S}$ a contractible open set of $\mathbb{D}^*$. Therefore, it has the same homotopy type as the general fiber. In particular, $\pi_1(\mathscr{S})$ is a normal subgroup of $\pi_1(V \smallsetminus |D|)$ thanks to the vanishing of $\pi_2(\mathbb{D}^*)$. If we do this construction choosing base points at two adjacent sectors with non trivial Stokes transition matrix, we conclude that indeed the monodromy of $\mathscr{H}$ on $\mathscr{S}$ is trivial since a nontrivial Stokes matrix at one hand conjugates the corresponding monodromy representations, and at the other hand it does not respect the fixed points of the two monodromy representations. We conclude that in the presence of a nontrivial Stokes matrix the monodromy factors through $f_* : \pi_1(V \smallsetminus |D|) \to \pi_1(\mathbb{D}^*)$ as wanted. $\qquad\square$

Lemma 6.8. — *Let $(P, \mathscr{H})$, D, and k be as in the statement of Proposition 6.7 and assume that we are not in case (1). If the normal bundle of D_{red} has torsion order $\geqslant k$, then the monodromy of $(P, \mathscr{H})$ is non-trivial.*

Proof. — Notation as in the proof of Proposition 6.7. Consider the unitary representation $\psi : \pi_1(|D|) \to \mathbb{C}^*$ defined in Remark 6.3; the torsion order r of the normal bundle of D satisfies $r = \#\psi(\pi_1(|D|))$. By Proposition 6.1, we know that r divides k (resp. $2k$) in the split (resp. unsplit) case. Since $r \geqslant k$, we have $r = k$ or $2k$.

Consider first the split case: this obviously implies $r = k$. Let $\gamma \in |D|$ a loop such that $\psi(\gamma) = \exp(2\pi i/k)$ We can lift γ to a path $\widetilde{\gamma}$ in a region $\mathscr{S}$ such that the initial and final points lie in two consecutive small sectors. Moreover, since $\mathscr{H}$ is not given by a closed meromorphic 1-form we can assume that at least one of the two distinguished leaves over the initial sector is not a distinguished leaf for the final sector. Closing the path $\widetilde{\gamma}$ using an arc on the transversal Σ, we obtain a path in $V \smallsetminus |D|$ with non-trivial monodromy.

Consider now the unsplit case. Going back to the proof of Proposition 6.1, we see that there must be some loop $\gamma \in |D|$ such that $\psi(\gamma)^k = -1$. This shows that $r = k$

is impossible, thus $r = 2k$ in this case. After a two-fold étale covering $\widetilde{V} \to V$, we are in the previous split case with normal bundle having torsion k coinciding with irregularity index. We thus obtain a loop $\widetilde{\gamma}$ with non trivial monodromy on $\widetilde{V}$; its projection on V has the same property. □

7. Structure

7.1. Proof of Theorem C. — Let $(P, \mathscr{H})$ be a Riccati foliation over the projective manifold X with irregular singular points. Assuming that it is not defined by a closed 1-form after a two-fold covering, our aim is to prove that it factors through a curve.

By Proposition 6.6, we may assume that $f^*(P, \mathscr{H})$ is not defined by a closed rational 1-form for any generically finite dominant rational map $f : X' \dashrightarrow X$. According to Proposition 4.11, it suffices to prove the factorization of $(P, \mathscr{H})$ in restriction to a surface $S \subset X$ obtained as an intersection of general hyperplane sections. According to Proposition 4.12 (and an obvious induction on the dimension), we may assume that the restriction is not defined by a closed rational 1-form after any generically finite dominant rational map $f : S' \dashrightarrow S$. Furthermore, Proposition 4.8 allows us to assume that the singularities of $(P, \mathscr{H})$ are as in the conclusion of Theorem 5.6.

Let D be a connected component of the irregular divisor I: we have $D = kD_{\mathrm{red}}$. It follows from Proposition 6.1 that D has torsion normal bundle, and from Proposition 6.7 that the monodromy of $(P, \mathscr{H})$ is virtually abelian at the neighborhood V of D.

First case: the global monodromy of $(P, \mathscr{H})$ on S is not virtually abelian. — In particular, it is strictly larger than the local monodromy around D:

$$\rho(\pi_1(S \smallsetminus |(\mathscr{H})_\infty|)) \ \neq \ \rho(\pi_1(V \smallsetminus |(\mathscr{H})_\infty|)).$$

Then arguments used in the proof of Theorem 3.1 show that D is the fiber of a fibration $f : S \to C$ (we use extra topology in $S - V$ to construct a ramified cover with several disjoint copies of D and then apply Theorem 2.2). Let U a dense open Zariski subset of $S \smallsetminus |(\mathscr{H})_\infty|$ such that $f_{|U}$ is a topologically a locally trivial fibration on its image[3]. As in the proof of Theorem 3.1, the monodromy of a general fiber of $f_{|U}$ is a normal subgroup of the global monodromy group and is therefore trivial. Thus, the monodromy representation factors through f. But the general fiber of f does not intersect the irregular divisor and we can apply Proposition 4.9 to conclude that the Riccati foliation factors as well.

We can now assume that the global monodromy is virtually abelian, and after passing to a finite covering, that it is torsion free. Thus the global monodromy either is abelian and infinite, or trivial.

---

[3] Beware that the general fiber of $f_{|U}$ is not necessarily compact, especially when D intersect a logarithmic component of the polar locus $(\mathscr{H})_\infty$.

Second case: the global monodromy of $(P, \mathscr{H})$ *on* S *is abelian and infinite.* — We first aim to prove that D is a fiber of a fibration $S \to C$ (up to finite cover). If the global monodromy is strictly larger than the local monodromy around D, we can argue as in the first case: we can use the extra monodromy to produce a finite cover on which we have several copies of D providing a fibration. Assume now

$$\rho(\pi_1(S \smallsetminus |(\mathscr{H})_\infty|)) \ = \ \rho(\pi_1(V \smallsetminus |(\mathscr{H})_\infty|)).$$

If the Riccati foliation $\mathscr{H}$ nearby D satisfies conclusion (1) of Proposition 6.7, then there is a 2-section $V \dashrightarrow P|_V$ which is invariant by $\mathscr{H}$, and in particular by the monodromy. This 2-section therefore extends outside the extra poles of $\mathscr{H}$. If the Riccati foliation has only logarithmic poles outside of D, then the 2-section extends on the whole of S, proving that $\mathscr{H}$ is given by a closed 1-form on a two-fold cover (see Lemma 4.6), contradiction. If, on the contrary, $\mathscr{H}$ has an extra irregular pole, say D', disjoint from D, both of them having torsion normal bundle by Proposition 6.1; then, we get a fibration by Theorem 2.2.

In conclusion (2) of Proposition 6.7, the monodromy factors, around D, through the C^∞-fibration $f : V \smallsetminus D \to \mathbb{D}^*$: the monodromy is infinite cyclic. If the representation takes values in $\mathbb{G}_a$ we compose a suitable multiple of it with an exponential so that we can assume that it takes values in $\mathbb{G}_m$ and is still infinite. Consider the Deligne logarithmic flat connection (L, ∇) realizing this representation. By construction, the residue of ∇ along an irreducible component E_i of the polar divisor takes the form $k_i \lambda + n_i$, where λ is an irrational number (the monodromy is infinite) and k_i, n_i are integers. The residue formula gives

$$0 = \sum_i \mathrm{Res}_{E_i} c_1(E_i) \ + \ c_1(L) = \left(\sum_i k_i c_1(E_i) \right) \lambda \ + \ \left(\sum_i n_i c_1(E_i) \ + \ c_1(L) \right).$$

Since λ is irrational, each term in parenthesis is zero and we get in particular

$$0 = \sum_i k_i c_1(E_i) = c_1(D) - c_1(E),$$

where D is our connected component of the irregular divisor, and E is a divisor disjoint from D and contained in $|(\mathscr{H})_\infty|$. After splitting $E = E_+ - E_-$ with E_+ and E_- effective, we know that E_+ and E_- have non positive self-intersection by Hodge index Theorem. On the other hand, the equality

$$0 = D \cdot D = E \cdot E = E_+ \cdot E_+ - 2E_+ \cdot E_- + E_- \cdot E_-$$

forces each term, a priori $\leqslant 0$, to be zero. Finally, again by Hodge index Theorem, E_+ and E_- must be disjoint and have Chern classes proportional to the Chern class of D. We can apply Theorem 2.2 to produce a fibration.

We have just proved, in the case monodromy is abelian and infinite, that D is fiber of a holomorphic fibration $f : S \to C$ (we keep the same notation as for the C^∞-fibration). It remains to show that the Riccati foliation $\mathscr{H}$ factors. When the monodromy is trivial along a generic fiber of f, this clearly follows from Proposition 4.9. If not, we have infinite monodromy along fibers and, by looking at a fiber

close to D, we must be in case (1) of Proposition 6.7. In particular, there is a 2-section of the $\mathbb{P}^1$-bundle P that are invariant by the Riccati foliation $\mathscr{H}$. In fact, since the monodromy is torsion-free, we are in the split case: the 2-section splits into two disjoint sections. These two sections are tangent to the Riccati foliation $\mathscr{H}$ and also to the pull-back to P of the foliation defined by the fibration. We obtain two curves in the Hilbert scheme of P. Since tangency to $\mathscr{H}$ imposes a closed condition on the Hilbert scheme of P, these two curves have Zariski closure of dimension one (recall that over each fiber of f we have exactly two sections of P). Thus they spread two surfaces, sections of P, which are invariant by $\mathscr{H}$. By Lemma 4.6, $\mathscr{H}$ is defined by a closed 1-form, contradiction.

Third case: the monodromy of the Riccati foliation $\mathscr{H}$ is trivial. — Like in the previous case, if $\mathscr{H}$ is defined by a closed 1-form at the neighborhood of D, then it extends as a global 1-form, except if there is another irregular polar component, in which case we get two disjoint divisors with torsion normal bundle (see Proposition 6.1) and therefore a fibration (see Theorem 2.2) through which $\mathscr{H}$ factors (see Proposition 4.9). On the other hand, if we are in case (2) of Proposition 6.7, then Lemma 6.8 implies that $r < k$, i.e., the order of the normal bundle of D_{red} is strictly smaller than the multiplicity of the irregular divisor. The existence of a fibration with a fiber supported on $|D|$ follows from Theorem 2.3, and the Riccati foliation factors according to Proposition 4.9. □

7.2. **Proof of Theorem E.** — Let (E, ∇) be a flat meromorphic $\mathfrak{sl}_2$-connection on X. In the case (E, ∇) is regular, then the conclusion of Theorem E directly follows from Corollary B. Indeed, the Riemann-Hilbert correspondence establishes a one-to-one correspondence between representations up to conjugacy and regular connections up to birational bundle transformations (see [17]). For instance, if the monodromy is virtually abelian, i.e., abelian after a finite cover, then it is either diagonal, or unipotent, and can be realized by one of the two models of case (1) in Theorem E; the Riemann-Hilbert correspondence provides the birational equivalence. Similarly, cases (1) and (2) of Corollary B for the monodromy respectively imply cases (2) and (3) of Theorem E for the regular connection. Let us now assume (E, ∇) irregular.

Let us consider $P := \mathbb{P}(E)$ the $\mathbb{P}^1$-bundle associated to E; horizontal section of ∇ induce a Riccati foliation $\mathscr{H}$ on $\pi : P \to X$ which is irregular by assumption. We can apply Theorem C to the projective connection/Riccati foliation $(P, \mathscr{H})$ and discuss the two possible conclusions.

Assume first that $\mathscr{H}$ is defined by a closed 1-form (maybe passing to a finite covering of X). After birational bundle transformation, we can assume $P_0 = X \times \mathbb{P}^1$ and $\mathscr{H}_0$ defined by
$$\Omega_0 = \frac{dz}{z} + 2\omega \quad \text{or} \quad \Omega_0 = dz + \omega,$$
with ω a closed 1-form on X (see Proposition 4.7). These Riccati foliations are induced by those explicit connections of Theorem E (1):
$$\nabla_0 = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad \nabla_0 = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

on the trivial bundle $E_0 := \mathscr{O}_X \oplus \mathscr{O}_X$. There is a birational bundle trivialization $E \dashrightarrow E_0$ making commutative the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\;\phi\;} & E_0 \\
\downarrow & & \downarrow \\
P & \dashrightarrow & P_0
\end{array}
$$

Obviously, $\phi_*\nabla$ is projectively equivalent to one of the models ∇_0: $\phi_*\nabla = \nabla_0 \otimes \zeta$ with $(\mathscr{O}_X, \zeta)$ a flat rank one connection over X, birationally equivalent to the trivial connection by construction. This means that one can write $\zeta = d + df/f$ and, maybe tensoring by the logarithmic connection $(\mathscr{O}_X, d + \frac{1}{2}\frac{df}{f})$ (whose square has trivial monodromy), we get equality $\phi_*\nabla = \nabla_0$. Note that, passing to the 2-fold covering defined by $z^2 = f$, the connection ∇ is birationally gauge equivalent to ∇_0 (without tensoring).

Assume now that $(P, \mathscr{H})$ is birationally gauge equivalent to the pull-back $f^*(P_0, \mathscr{H}_0)$ of a Riccati foliation over a curve, $f : X \dashrightarrow C$ with $P_0 = C \times \mathbb{P}^1$. Denote by ∇_0 the unique $\mathfrak{sl}_2$-connection on the trivial bundle E_0 over C inducing the projective connection $(P_0, \mathscr{H}_0)$. Then (E, ∇) is birationally equivalent to $f^*(E_0, \nabla_0) \otimes (\mathscr{O}_X, \zeta)$ with ζ logarithmic rank one connection having monodromy in the center of $\mathrm{SL}_2(\mathbb{C})$. $\qquad\square$

8. Transversely projective foliations

8.1. Basic Lemma. — Let $\mathscr{F}$ be a transversely projective foliation on a projective manifold X defined by a triple $(\omega_0, \omega_1, \omega_2)$. The Riccati foliation $\mathscr{H}$ defined on $X \times \mathbb{P}^1$ by

$$ dz + \omega_0 + z\omega_1 + z^2\omega_2 = 0 $$

is integrable (4.1):

$$
\begin{cases}
d\omega_0 = \omega_0 \wedge \omega_1 \\
d\omega_1 = 2\omega_0 \wedge \omega_2 \\
d\omega_2 = \omega_1 \wedge \omega_2
\end{cases}
$$

and the foliation $\mathscr{F}$ is defined by restricting $\mathscr{H}$ to the section $\sigma : X \to P$ given by $z = 0$. Another projective triple $(\omega_0', \omega_1', \omega_2')$ defines the same transversely projective foliation, i.e., the same foliation $\mathscr{F}$ with the same collection of local first integrals at a general point of X if, and only if, there are rational functions a, b on X such that $a \not\equiv 0$ and

$$
(8.1) \qquad
\begin{cases}
\omega_0' = a\omega_0 \\
\omega_1' = \omega_1 - \dfrac{da}{a} + 2b\omega_0 \\
\omega_2' = \dfrac{1}{a}\left(\omega_2 + b\omega_1 + b^2\omega_0 - db\right)
\end{cases}
$$

(see [39]). This exactly means that the Riccati foliation $\mathscr{H}'$ defined by

$$ \Omega' = d\widetilde{z} + \omega_0' + \widetilde{z}\omega_1' + \widetilde{z}^2\omega_2' $$

is derived by gauge transformation $1/z = a/\widetilde{z} + b$. The following is well known (cf. [39, Chap. II, Prop. 2.1, p. 193] and [9, Lem. 2.20]).

LEMMA 8.1. — *If $\mathscr{F}$ is defined by a closed 1-form ω, $d\omega = 0$, then up to gauge transformation (8.1), the Riccati foliation $\mathscr{H}$ is defined by*

$$\Omega = dz + \omega(1 + \phi z^2) \quad \text{where } \phi \text{ is a first integral for } \mathscr{F}.$$

Proof. — One can first write $\omega = a\omega_0$ for some rational function a and use gauge transformation (8.1) to set $(\omega_0, \omega_1, \omega_2) = (\omega, 0, \omega_2')$. Indeed, once $\omega_0 = \omega$ is closed, we deduce from (4.1) that $\omega_0 \wedge \omega_1 = 0$, and therefore $\omega_1 = b\omega_0$ for some rational function b; consequently, ω_1 can be killed by gauge transformation. Finally, integrability condition gives $d\omega_2' = \omega \wedge \omega_2' = 0$, which means that $\omega_2' = f\omega$ and $d(f\omega) = 0$ for some rational function f. The latter condition, together with closedness of ω, gives $df \wedge \omega = 0$, i.e., f is a first integral for $\mathscr{F}$. □

COROLLARY 8.2. — *If $\mathscr{F}$ is defined by a closed 1-form, then $\mathscr{H}$ is also defined by a closed 1-form, or factors through a curve. If $\mathscr{F}$ admits a rational first integral, then $\mathscr{H}$ factors through a curve.*

Proof. — If $\mathscr{F}$ admits a rational first integral f, then after resolution of indeterminacy points of f by blowing-ups $\widetilde{X} \to X$, Stein Factorization gives a fibration $\widetilde{f} : \widetilde{X} \to C$ over a curve with connected fibers coinciding generically with leaves of $\widetilde{\mathscr{F}}$. Applying Lemma 8.1 to $\omega = df$, we get that $\phi = \phi(\widetilde{f})$ and $\mathscr{H}$ actually factors through $\widetilde{f}$. If $\mathscr{F}$ is now defined by a closed 1-form ω, but does not admit a rational first integral, then applying Lemma 8.1, we get that $\phi = c \in \mathbb{C}$ is a constant, and $\mathscr{H}$ is defined by $dz/(1 + cz^2) + \omega$ which is closed. □

REMARK 8.3. — Statements similar to Lemma 6.8 and Corollary 8.2 hold with the very same proofs in the local setting, replacing rational functions and 1-forms by their meromorphic analogues.

Proof of Lemma 5.3. — At a generic point $p \in \pi^{-1}(H)$, the foliation $\mathscr{H}$ is smooth and vertical: let $p_0 = \pi(p)$ be the projection and $\sigma : (X, p_0) \to P$ be a germ of section transverse to $\mathscr{H}$. The induced transversely projective foliation is regular and therefore admits a holomorphic first integral. By the local version of Corollary 8.2, $\mathscr{H}$ locally factors through a curve. □

8.2. PROOF OF THEOREM D. — Under previous notations, we now apply to $(P, \mathscr{H})$ our results on projective connections (Corollary B and Theorem C, or equivalently Theorem E). There are three cases.

First case. — There is a generically finite morphism $f : Y \to X$ such that $f^*\mathscr{H}$ is defined by a closed 1-form Ω. The pull-back $f^*\mathscr{F}$ on Y is still defined by restricting the Riccati foliation to the pull-back section $\widetilde{\sigma} : Y \to f^*P$; it is therefore also defined by restricting the closed 1-form Ω to the section.

Second case. — There is a map $f : X \dashrightarrow C$ to a curve and a Riccati $\mathscr{H}_0$ on $P_0 = C \times \mathbb{P}^1$ such that $(P, \mathscr{H})$ is equivalent to $f^*(P_0, \mathscr{H}_0)$ by birational bundle transformation. We thus deduce a map $\Phi : P \dashrightarrow P_0$ such that $\mathscr{H} = \Phi^* \mathscr{H}_0$. Consider the composition $\Phi \circ \sigma : X \dashrightarrow P_0$. Then either it is dominant and $\mathscr{F}$ is the corresponding pull-back of $\mathscr{H}_0$ (we are in case (2) of the statement), or the image is a curve and fibers force the leaves of $\mathscr{F}$ to be algebraic (we are in case (1)).

Third case. — There exists a rational map $f : X \dashrightarrow \mathfrak{H}$ to a polydisk Shimura modular orbifold such that $(P, \mathscr{H})$ is equivalent to the pull-back of one the tautological Riccati foliations $(P_\rho := \mathfrak{H} \times_\rho \mathbb{P}^1, \mathscr{H}_\rho)$ by birational bundle transformation. Again, we deduce a map $\Phi : P \dashrightarrow P_\rho$ such that $\mathscr{H} = \Phi^* \mathscr{H}_\rho$, and considering $\Phi \circ \sigma : X \to P_\rho$, we can conclude as before: we are in the case (1) or (3) of the statement depending whether the image is a curve or higher dimensional (not necessarily dominant). □

8.3. **Theorem D implies Theorem E.** — Although we have followed the other direction, it is interesting to notice that our results on projective (or $\mathfrak{sl}_2$) connections and transversely projective foliations are actually equivalent. Indeed, given say a Riccati foliation $(\pi : P \to S, \mathscr{H})$, with P birationally trivial, we can take a general rational section $\sigma : S \to P$ and consider the induced transversely projective foliation $\mathscr{F} := \sigma^* \mathscr{H}$. Applying Theorem D to $\mathscr{F}$ gives the following possibilities.

First case. — The foliation $\mathscr{F}$ has a rational first integral. By Corollary 8.2, we deduce that $\mathscr{H}$ factors through a curve.

Second case. — There is a generically finite morphism $f : Y \to X$ such that $f^* \mathscr{F}$ is defined by a closed 1-form ω, but $\mathscr{F}$ does not admit a rational first integral. By Corollary 8.2, we deduce that $f^* \mathscr{H}$ is also defined by a closed 1-form.

Third case. — Maybe after blowing-up S, there is a morphism $f : S \to P_0 := C \times \mathbb{P}^1$ and a Riccati foliation $\mathscr{H}_0$ on P_0 such that $\mathscr{F} = f^* \mathscr{H}_0$. Then, considering now the pull-back $\phi^* \mathscr{H}_0$ the fiber product:

$$
\begin{array}{ccc}
P & \xrightarrow{\ \phi\ } & P_0 \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
X & \xrightarrow{\ f\ } & C
\end{array}
$$

we get another transversely projective structure for $\mathscr{F}$. If $\mathscr{F}$ is not defined by a closed 1-form up to finite cover, then its projective structure is unique, and $\mathscr{H}$ is birational to $\mathscr{H}_0$, thus pull-back from a curve.

Fourth case. — There exists a polydisk Shimura modular orbifold $\mathfrak{H}$ and a rational map $f : X \dashrightarrow P_\rho := \mathfrak{H} \times_\rho \mathbb{P}^1$ towards one of its tautological Riccati foliations such that $\mathscr{F} = f^* \mathscr{H}_\rho$. Like in the previous case, we can prove that $\mathscr{F}$ is defined by a closed 1-form after a finite cover (and go back to the first two cases), or $\mathscr{H}$ is birationally equivalent to the pull-back of $(P_\rho, \mathscr{H}_\rho)$ by f. □

9. Examples

We will now present some examples which show that our results are sharp.

Example 9.1. — Let Y be a projective manifold and consider a representation $\rho : \pi_1(Y) \to (\mathbb{C}, +)$. It determines a cohomology class $[\rho] \in H^1(Y, \mathbb{C})$. If its image under the natural morphism $H^1(Y, \mathbb{C}) \to H^1(Y, \mathscr{O}_Y)$ is non zero then it determines a non trivial extension $0 \to \mathscr{O}_Y \to E \to \mathscr{O}_Y \to 0$, endowed with a flat connection with monodromy given by

$$\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}.$$

The projectivization $X = \mathbb{P}(E)$ is a $\mathbb{P}^1$-bundle over Y with a Riccati foliation defined by a closed rational 1-form ω with polar divisor equal to 2Δ, where Δ is image of the unique section $Y \to \mathbb{P}(E)$. If P is the trivial $\mathbb{P}^1$-bundle P over X then we have a family of Riccati foliations $\mathscr{H}_\lambda$, $\lambda \in \mathbb{C}$, on it defined by

$$dz + \omega(1 + \lambda z^2).$$

The Riccati foliation $\mathscr{H}_\lambda$ is irregular for $\lambda \neq 0$, does not factor through a curve, and its irregular divisor is not a fiber of a fibration. If we take Y equal to an elliptic curve, then $X \smallsetminus |\Delta|$ is nothing but Serre's example of Stein quasi-projective surface which is not affine.

Example 9.2. — Let $n \geqslant 2$ and let X be the quotient of $\mathbb{H}^n$ by cocompact torsion-free irreducible lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^n$. The natural projections $\mathbb{H}^n \to \mathbb{H}$ define n codimension one smooth foliations on X which are transversely projective (indeed transversely hyperbolic). Contrary to what have been stated by the second author and Mendes in [30, Th. 1], countably many leaves of these foliations may have nontrivial topology (with fundamental groups isomorphic to isotropy groups of the action of Γ on the corresponding one dimensional factor of $\mathbb{H}^n$), but the very general leaf is biholomorphic to $\mathbb{H}^{n-1}$. The maximal principle tells us that the general leaf cannot contain positive dimensional sub-varieties, and consequently the foliations are not pull-backs from lower dimensional manifolds.

Again the assumptions on Γ can be considerably weakened. All we have to ask is that Γ is an irreducible lattice of $\mathrm{PSL}_2(\mathbb{R})^n$ for some $n \geqslant 2$. Notice that the rigidity theorem of Margulis (resp. the classification of representations by Corlette and Simpson) implies that all these lattices are commensurable to (resp. conjugated to a subgroup of) arithmetic lattices of the form $U(P, \Phi)/\pm \mathrm{Id}$ for some totally imaginary quadratic extension L of a totally real number field F, some rank two projective $\mathscr{O}_L$-module P and some skew Hermitian form Φ. Besides the n representations coming from the n projections $\pi_1^{\mathrm{orb}}(X) \simeq \Gamma \subset \mathrm{PSL}_2(\mathbb{R})^n \to \mathrm{PSL}_2(\mathbb{R})$, we also have $[L : \mathbb{Q}] - 2n$ representations of $\pi_1^{\mathrm{orb}}(X)$ with values in $\mathrm{PSL}_2(\mathbb{C})$ which do not factor through lower dimensional projective manifolds. The associated $\mathbb{P}^1$-bundles are birationally trivial (since the underlying representation is a Galois conjugate of the representations coming from the transversely projective foliations on X defined by

the submersions $\mathbb{H}^n \to \mathbb{H}$), and by taking a rational section we can produce further examples of transversely projective foliations on X which do not factor. Although the underlying representations are Galois conjugate to the representations in $\mathrm{PSL}_2(\mathbb{R})$, the topology of the Riccati foliations over X associated to embeddings $\sigma : L \to \mathbb{C}$ for which $\sqrt{-1}\,\Phi$ is definite is quite different. In the former case the Riccati foliation leaves invariant two open subsets, corresponding to the complement of $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ in $\mathbb{P}^1$, while in the latter case the Riccati foliation is quasi-minimal: all the leaves not contained in $\pi^{-1}((\mathscr{H})_\infty)$ are dense in the corresponding $\mathbb{P}^1$-bundle.

Explicit examples of foliations on $\mathbb{P}^2$ defined by the submersions $\mathbb{H}^2 \to \mathbb{H}$ with $\Gamma \subset \mathrm{PSL}_2(2,\mathbb{R})^2$ and Γ isomorphic to $\mathrm{PSL}_2(\mathscr{O}_K)$ (and certain subgroups) have been determined by the second author and Mendes in [30] ($K = \mathbb{Q}(\sqrt{5})$) using the work of Hirzebruch on the description of these surfaces, and by Cousin in [15], see also [14], ($K = \mathbb{Q}(\sqrt{3})$) using an algebraic solution of Painlevé VI equation.

Example 9.3. — The degree of the generically finite morphism in Assertion (1) of Theorem D cannot be bounded, even if we restrict to transversely projective foliations on a rational surface. Let $C_d = \{x^d + y^d + z^d = 0\} \subset \mathbb{P}^2$ be the Fermat curve of degree $d \geqslant 3$. On $S_d = C_d \times C_d$ consider the action of $\mathbb{Z}/d\mathbb{Z}$ given by $\varphi(x,y) = (\xi_d x, \xi_d y)$ where ξ_d is a primitive d-th root of the unity. Let $\omega \in H^0(S_d, \Omega^1_{S_d})$ be a general holomorphic 1-form satisfying $\varphi^*\omega = \xi_d\omega$. The induced foliation is invariant by the action of $\mathbb{Z}/d\mathbb{Z}$, but the 1-form ω is not. The quotient of S_d by $\mathbb{Z}/d\mathbb{Z}$ is a rational surface R and the foliation induced by ω on R is transversely projective (indeed transversely affine). The monodromy group is an extension of the group of d-th of unities by an infinite subgroup of $(\mathbb{C}, +)$. Explicit equations for birational models of these foliations on $\mathbb{P}^2$ can be found in [36, Ex. 3.1].

Appendix. Reduction of singularities of Riccati foliations

This appendix is devoted to the proof of Theorem 5.6. Let $(\pi : P \to S, \mathscr{H})$ be a Riccati foliation over a projective surface S. One can first blow-up S until we get a simple normal crossing polar divisor $(\mathscr{H})_\infty$, and then apply elementary transformations over irreducible components of $(\mathscr{H})_\infty$ until we minimize order of poles on each component. This is explained in [24]. Then all components of $(\mathscr{H})_\infty$ satisfy assumptions of Lemma 5.3 and, outside of special π-fibers, the Riccati foliation locally factors into one dimensional models of Proposition 5.1 or $Irreg^{ram}$. In particular, we can define the irregularity divisor as $I = \sum_i k_i D_i$ where D_i run over irreducible components of $(\mathscr{H})_\infty$, and $k_i \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ is the irregularity index.

To get rid of special points, we use the fact that $\pi : P \to S$ is birationally trivial and choose a rational section $\sigma : S \dashrightarrow P$ which is not invariant by $\mathscr{H}$. The foliation $\mathscr{F} = \sigma^*\mathscr{H}$ is transversely projective; indeed, after birational trivialization $z : P \to \mathbb{P}^1$ of the bundle, we may assume σ given by $z = 0$: the foliation $\mathscr{H}$ defined by $dz + \omega_0 + z\omega_1 + z^2\omega_2 = 0$ and $(\omega_0, \omega_1, \omega_2)$ is a projective triple for $\mathscr{F}$. Now, apply Seidenberg's Theorem: maybe blowing-up S, we can now assume that $\mathscr{F}$ has only reduced singular

points. Following the classification of Berthier and the third author [3, 42], reduced singular points of transversely projective foliations fall into one of the following types:

(1) $\mathscr{F}$ admits the holomorphic first integral:
  – *First integral: $\omega = d(x^p y^q)$ with $p, q \in \mathbb{Z}_{>0}$;*

(2) $\mathscr{F}$ is defined by a closed 1-form ω (but without first integral):
  – *Linear: $\omega = dx/x + \lambda\, dy/y$ with $\lambda \in \mathbb{C} - \mathbb{Q}$;*
  – *Saddle-node: $\omega = dy/y + dx/x^{k+1} + \lambda\, dx/x$ with $\lambda \in \mathbb{C}$;*
  – *Resonant saddle: $\omega = dy/y + df/f^{k+1} + \lambda\, df/f$ with $f = x^p y^q$ and $\lambda \in \mathbb{C}$;*

(3) $\mathscr{F}$ is the pull-back of a singular point of a Riccati foliation by a ramified cover:
  – *Riccati saddle-node: $\omega = dy + \big(b(x)y^2 + (1/x^{k+1} + \lambda/x)y + c(x)\big)\, dx$;*
  – *Bernoulli saddle-node: $\omega = dz + \big(b(x)z^2 + (1/x^{k+1} + \lambda/x)z\big)\, dx$ with $z = y^\nu$;*
  – *Resonant saddle: $\omega = dz + \big(b(f)z^2 + (1/f^{k+1} + \lambda/f)z + c(f)\big)\, df$ with $f = x^p y^q$ and $z = y^\nu$;*

In case (1), the local version of Corollary 8.2 (see Remark 8.3) tells us that $\mathscr{H}$ factors through f, and we can reduce it to the models given by Proposition 5.1.

In case (2), where $\mathscr{F}$ is defined by a closed 1-form ω, we can similarly reduce $\mathscr{H}$ to either $\Omega = dz + \omega$, or $dz/z + \lambda\omega$ with $\lambda \in \mathbb{C}^*$.

Finally, in case (3), we again see that $\mathscr{H}$ factors through $f = x$ or $f = x^p y^q$, reducing to the model

$$\Omega = dz + \Big(b(f)z^2 + \Big(\frac{1}{f^{k+1}} + \frac{\lambda}{f}\Big)z + c(f)\Big)df,$$

and $\mathscr{F}$ is defined by $z = y$ or $z = y^\nu$.

We summarize all the possibilities studied above in the next result.

Theorem A.1. — *Let $(\pi : P \to S, \mathscr{H})$ be a Riccati foliation over a projective surface S, with P a birationally trivial bundle. There exists a birational morphism $\phi : \widetilde{S} \to S$ (with $\widetilde{S}$ smooth) and a birational bundle modification of the pull-back bundle $\phi^* P$ such that the pull-back Riccati foliation has normal crossing divisor D, and is locally defined over any point $p \in |D|$ by a Riccati 1-form having one of the following types:*

(1) $\mathrm{Log}^{\mathbb{G}_m}$: $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \lambda_x \dfrac{dx}{x}$, *or* $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \lambda_x \dfrac{dx}{x} + \lambda_y \dfrac{dy}{y}$;

(2) $\mathrm{Log}^{\mathbb{G}_a}$: $\Omega = dz + \dfrac{dx}{x}$, *or* $\Omega = dz + \dfrac{dx}{x} + \lambda \dfrac{dy}{y}$;

(3) Irreg^0: $\dfrac{\Omega}{z} = \dfrac{dz}{z} + \dfrac{df}{f^{k+1}} + \lambda_x \dfrac{dx}{x} + \lambda_y \dfrac{dy}{y}$, *where $f = x$ or $x^p y^q$;*

(4) Irreg: $\Omega = dz + \Big(\dfrac{df}{f^{k+1}} + \lambda \dfrac{df}{f}\Big)z + \big(b(f)z^2 + c(f)\big)\, df$, *where $f = x$ or $x^p y^q$;*

(5) $\mathrm{Irreg}^{\mathrm{ram}}$: $\Omega = dz + (z^2 + x^{\epsilon_1} y^{\epsilon_2} \phi(f))\dfrac{df}{x^{\widetilde{p}} y^{\widetilde{q}} f^k} - \Big(\widetilde{p}\dfrac{dx}{x} + \widetilde{q}\dfrac{dy}{y} + k\dfrac{df}{f}\Big)$ *with f and ϕ holomorphic, $\phi(0) \neq 0$, $\epsilon_1, \epsilon_2 = 0, 1$ and $(p, q) = (2\widetilde{p} - \epsilon_1, 2\widetilde{q} - \epsilon_2)$.*

We note that the meromorphic gauge reduction process of Proposition 5.1 which is a priori local can be done globally along irreducible components of the polar divisor. Indeed, it can be checked on the proof of Proposition 5.1 that when the polar divisor is not minimal along an irreducible component D_i, then the non vertical part of the singular set of $\mathscr{H}$ is a (smooth) section; after an elementary transformation along it (blowing-up the section and then contracting the strict transform of $\pi^{-1}(D_i)$), the polar order decreases. After a finite number of steps, the polar order is minimal, and all local models reduce to those of Proposition 5.1 by local biholomorphic bundle trivialization. For details, see the proof of the main result of [24].

The last step towards Theorem 5.6 consists in passing to use a ramified covering in order to kill ramifications. Going back to the irregular divisor $I = \sum_i k_i D_i$, we can choose positive integers m_i such that $m_i k_i = k$ for some fixed positive integer k (a common multiple of all k_i's). We now consider a Kawamata covering ([20, Prop. 4.1.12]): there exists a ramified cover $f : \widetilde{S} \to S$ with S smooth such that $f^* D_i = m_i \widetilde{D}_i$ for some smooth reduced divisors $\widetilde{D}_i$ on $\widetilde{S}$, and $\sum_i D_i$ has simple normal crossings. One easily checks from models of Theorem A.1 that they are stable under ramified covers in variables x or y, and that ramifying at order m_i along D_i multiplies the irregularity index by m_i. Finally, after covering, the irregularity index is k all along the irregular divisor. In particular, there are no more ramified components, and we have no more to consider the model $Irreg^{ram}$ of Theorem A.1. □

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