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MINIMAL RATIONAL CURVES ON WONDERFUL GROUP COMPACTIFICATIONS

by Michel Brion & Baohua Fu

Abstract. — Consider a simple algebraic group G of adjoint type, and its wonderful compactification X. We show that X admits a unique family of minimal rational curves, and we explicitly describe the subfamily consisting of curves through a general point. As an application, we show that X has the target rigidity property when G is not of type A_1 or C.

Résumé (Courbes rationnelles minimales sur les compactifications magnifiques des groupes)
Soient G un groupe algébrique simple et X sa compactification magnifique. Nous montrons que X possède une unique famille de courbes rationnelles minimales, et nous décrivons explicitement la sous-famille formée des courbes passant par un point général. Nous en déduisons une propriété de rigidité de X, lorsque G n'est pas de type A_1 ou C.

Contents

1. Introduction

Throughout this article, we consider algebraic varieties over the field of complex numbers. For a uniruled projective manifold X, let $\mathrm{RatCurves}^{\mathrm{n}}(X)$ denote the normalization of the space of rational curves on X (see [Kol96, II.2.11]). Every irreducible component $\mathscr{K}$ of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is a (normal) quasi-projective variety equipped with a quasi-finite morphism to the Chow variety of X; the image consists of the Chow points of irreducible, generically reduced rational curves. Also, there is a universal

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family $\mathscr{U}$ with projections $\rho : \mathscr{U} \to \mathscr{K}$, $\mu : \mathscr{U} \to X$, and ρ is a $\mathbb{P}^1$-bundle (for these results, see [Kol96, II.2.11, II.2.15]).

For any $x \in X$, let $\mathscr{U}_x := \mu^{-1}(x)$ and $\mathscr{K}_x := \rho(\mathscr{U}_x)$; then $\mathscr{K}$ is called a *family of minimal rational curves* if $\mathscr{K}_x$ is non-empty and projective for a general point x. There is a rational map $\tau : \mathscr{K}_x \dashrightarrow \mathbb{P}T_x(X)$ (the projective space of lines in the tangent space at x) that sends any curve which is smooth at x to its tangent direction. The closure of the image of τ is denoted by $\mathscr{C}_x$ and called the *variety of minimal rational tangents* (VMRT) at the point x. By [HM04a, Th. 1] and [Keb02, Th. 3.4], composing τ with the normalization map $\mathscr{K}_x^{\mathrm{n}} \to \mathscr{K}_x$ yields the normalization of $\mathscr{C}_x$. Also, $\mathscr{K}_x^{\mathrm{n}}$ is a union of components of the variety $\mathrm{RatCurves}^{\mathrm{n}}(x, X)$ defined in [Kol96, II.2.11.2], and hence is smooth for $x \in X$ very general by [Kol96, II.3.11.5].

The projective geometry of $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ encodes many geometric properties of X, and may be used effectively for solving a number of problems on uniruled varieties (see e.g. [Hwa01]). When X has Picard number one, the VMRT has been determined in many cases, including the homogeneous manifolds G/P, where G is a simple linear algebraic group and P a maximal parabolic subgroup (see [HM02, Prop. 1] for the case where P corresponds to a long simple root, and [HM04b, Prop. 3.5], [HM05, Prop. 3.2.1, Prop. 8.1.3] for the remaining cases; see also [LM03, Th. 4.8]). But very few examples with large Picard number are known. The case of complete toric manifolds is worked out in [CFH14], where it is shown that such a manifold may admit several families of minimal rational curves.

The aim of this article is to study minimal rational curves on wonderful compactifications of semisimple algebraic groups of adjoint type. For any such group G, De Concini and Procesi introduced in [DCP83] its wonderful compactification X. This is a projective manifold equipped with an action of $G \times G$ and containing G as an open orbit (where $G \times G$ acts on G by left and right multiplication). Moreover, the boundary $X \smallsetminus G$ is a union of ℓ irreducible divisors $D_1, \dots, D_\ell$ with smooth normal crossings, where ℓ is the rank of G, and the $G \times G$-orbit closures in X are exactly the partial intersections $D_{i_1} \cap \cdots \cap D_{i_s}$.

The manifold X is rational, since so is G; moreover, X is Fano with Picard number ℓ. Also, by homogeneity, the VMRT at every point x of the open orbit G is well-defined and independent of x. We may thus choose for x the neutral element of G, called the base point of X; then $T_x(X)$ is the Lie algebra $\mathfrak{g}$ of G. The isotropy group of x in $G \times G$ is G embedded diagonally; it acts on $T_x(X)$ via the adjoint representation, and the induced action on $\mathbb{P}\mathfrak{g}$ stabilizes the VMRT $\mathscr{C}_x$.

We may now state the main result of this article:

Theorem 1.1. — *Let X be the wonderful compactification of a simple algebraic group G of adjoint type. Then*

(i) *There exists a unique family of minimal rational curves $\mathscr{K}$ on X. Moreover, $\mathscr{K}_x$ is smooth and the rational map $\tau : \mathscr{K}_x \dashrightarrow \mathscr{C}_x$ is an isomorphism.*

(ii) *$\mathscr{C}_x$ is the closed G-orbit in $\mathbb{P}\mathfrak{g}$, if G is not of type A.*

(iii) *When G is of type A_ℓ ($\ell \geqslant 2$), so that $G = \mathrm{PGL}(V)$ for a vector space V of dimension $\ell+1$, the VMRT $\mathscr{C}_x$ is the image of $\mathbb{P}V \times \mathbb{P}V^*$ under the Segre embedding $\mathbb{P}V \times \mathbb{P}V^* \to \mathbb{P}\,\mathrm{End}(V)$, followed by the projection $\mathbb{P}\,\mathrm{End}(V) \dashrightarrow \mathbb{P}(\mathrm{End}(V)/\mathbb{C}\,\mathrm{id}) = \mathbb{P}\mathfrak{g}$ with center $x = [\mathrm{id}]$.*

The uniqueness of a family of minimal rational curves on X is somewhat surprising, since X can have an arbitrary large Picard number. The key point is the uniqueness of a B-stable irreducible curve through x, where B is a Borel subgroup of G acting on X by conjugation (see Proposition 2.6 for a more general result). When G is not of type A, it turns out that the VMRT has the same dimension as the closed orbit in $\mathbb{P}\mathfrak{g}$, which implies claim (ii). When $G = \mathrm{PGL}(V)$, we construct additional minimal rational curves through x by using multiplicative one-parameter subgroups. In contrast with the rational homogeneous manifolds, the wonderful compactification X is not covered by lines when $\ell \geqslant 2$ (see Remark 3.6; when $\ell = 1$, we have $G = \mathrm{PGL}_2$ and $X \cong \mathbb{P}^3$, hence $\mathscr{K}_x \cong \mathbb{P}^2$ and $\mathscr{C}_x \cong \mathbb{P}\mathfrak{g}$).

In the general case, where G is no longer assumed simple, we have a unique decomposition into simple factors,

$$G = G_1 \times \cdots \times G_m. \tag{1.1}$$

Then accordingly,

$$X = X_1 \times \cdots \times X_m, \tag{1.2}$$

where X_i denotes the wonderful compactification of G_i.

Proposition 1.2. — *With the above notation and assumptions, let $\mathscr{K}$ be a family of minimal rational curves on X. Then $\mathscr{K}$ consists of curves on a unique factor X_i. In particular, $\mathscr{C}_x$ is the VMRT of X_i at its base point.*

As an application, we shall show that the wonderful group compactification X has the Liouville property if G is simple and not of type A_1 or C. This implies the target rigidity property: for any such X and any projective variety Y, every deformation of a surjective morphism $f : Y \to X$ comes from automorphisms of X.

This article is organized as follows. In Section 2, we collect auxiliary results about rational curves on almost homogeneous manifolds. Some special rational curves are investigated in Section 3, namely, closures of multiplicative or additive one-parameter subgroups in the wonderful compactification. Section 4 contains the proofs of Theorem 1.1 and Proposition 1.2. Some geometric constructions of the minimal rational curves are presented in Section 5, and the above application in the final Section 6.

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2. Some auxiliary results

Throughout this section, X denotes a projective manifold on which a connected linear algebraic group G acts with an open orbit X^0, that is, X is almost homogeneous under G. We choose a base point $x \in X^0$ and denote by $H \subset G$ its isotropy group; the corresponding Lie algebras will be denoted by $\mathfrak{h} \subset \mathfrak{g}$. Thus, the orbit $X^0 = G \cdot x$ is identified with the homogeneous space G/H. Moreover, H acts on the tangent space $T_x(X)$ via the isotropy representation, identified to the quotient representation $\mathfrak{p} := \mathfrak{g}/\mathfrak{h}$. We begin with the following observation:

Lemma 2.1

(i) *Let $f : \mathbb{P}^1 \to X$ be a morphism which is birational over its image. If this image meets X^0, then $f^*(T_X)$ is generated by its global sections.*

(ii) *Let $\mathscr{K}$ be a covering family of rational curves on X. Then $\mathscr{K}_x$ is smooth; moreover, H acts on $\mathscr{K}_x$ and permutes transitively its components.*

Proof

(i) follows from [Kol96, II.3.11]. We give a direct proof in this particular case. The G-action on X yields a map of sheaves $\mathscr{O}_X \otimes \mathfrak{g} \to T_X$, which is surjective over X^0. So we obtain a generically surjective map $\mathscr{O}_{\mathbb{P}^1} \otimes \mathfrak{g} \to f^*(T_X)$. This yields the assertion by using the fact that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathscr{O}_{\mathbb{P}^1}(n)$, and the vanishing of $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n))$ for $n < 0$.

(ii) Let f as in (i). Then f is free in the sense of [Kol96, II.3.1]; it follows that $\mathrm{Hom}(\mathbb{P}^1, X)$ is smooth at f, by [Kol96, II.3.5.4]. Thus, $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is smooth at its point corresponding to f, in view of [Kol96, II.2.15]. As a consequence, $\mathscr{K}$ is smooth at any curve meeting X^0.

Consider the universal family $\mathscr{U}$ with maps $\rho : \mathscr{U} \to \mathscr{K}$, $\mu : \mathscr{U} \to X$ as in the introduction. Then G acts on $\mathscr{U}$ and $\mathscr{K}$; moreover, ρ and μ are equivariant. Thus, $\mathscr{U}^0 := \mu^{-1}(X^0)$ (resp. $\mathscr{K}^0 := \rho(\mathscr{U}^0)$) is an open G-stable subset of $\mathscr{U}$ (resp. $\mathscr{K}$). Moreover, the restriction $\rho^0 : \mathscr{U}^0 \to \mathscr{K}^0$ is a $\mathbb{P}^1$-bundle. Since $\mathscr{K}^0$ is smooth, so is $\mathscr{U}^0$. But $\mathscr{U}^0$ is the associated fiber bundle $G \times^H \mathscr{U}_x$ over $X^0 = G/H$. Thus, $\mathscr{U}_x$ is smooth, and hence so is $\mathscr{K}_x$ since $\mathscr{U}_x \to \mathscr{K}_x$ is a $\mathbb{P}^1$-bundle. Also, $\mathscr{U}^0$ is irreducible; thus, H acts transitively on the set of components of $\mathscr{U}_x$, or equivalently of $\mathscr{K}_x$. $\square$

Next, we obtain a key technical result:

Lemma 2.2. — *Assume that H is reductive and the isotropy representation $\mathfrak{p}$ is a multiplicity-free sum of irreducible representations of $\mathfrak{h}$ with linearly independent highest weights (e.g., $\mathfrak{p}$ is irreducible as a representation of $\mathfrak{h}$). Let $C \subset X$ be an irreducible curve through x, stable under a Borel subgroup $B \subset H$, and set $C^0 := C \cap X^0$. Then*

(i) *C^0 is smooth and B-equivariantly isomorphic to its tangent line L at x, which is a highest weight line in $\mathfrak{p}$. Moreover, L determines C uniquely, and the stabilizer of C in the neutral component H^0 equals the stabilizer of L.*

(ii) *There exists an additive one-parameter subgroup $u : \mathbb{G}_a \to G$ such that $C = \overline{u(\mathbb{G}_a) \cdot x}$.*

Proof

(i) We first claim that C is not fixed pointwise by B. Indeed, the fixed point locus $(G/H)^B$ equals $(G/H)^{H^0}$, since H^0/B is complete and G/H is affine. Moreover, since H^0 is reductive, $(G/H)^{H^0}$ is smooth and $T_x(G/H)^{H^0} = (\mathfrak{g}/\mathfrak{h})^{H^0} = \mathfrak{p}^{\mathfrak{h}}$. The latter vanishes, since the highest weights of $\mathfrak{h}$ in $\mathfrak{p}$ are all nonzero. Thus, x is an isolated fixed point of $(G/H)^B$; this implies our claim.

By that claim, B has an open orbit in C^0, say $B \cdot y$. Thus, the isotropy subgroup $B_y \subset B$ has codimension 1.

Next, we choose a maximal torus $T \subset B$; then $B = TU$, where $U \subset B$ denotes the unipotent part. We claim that C is not fixed pointwise by T. Otherwise, $T \subset B_y$ and hence $B_y = TU_y$, where $U_y \subset U$ is a subgroup of codimension 1, normalized by T. Moreover, T acts trivially on $B/B_y \cong U/U_y$, and hence on its tangent space at the base point. It follows that T has a non-zero fixed point in a quotient of the Lie algebra of U. But this is impossible, since H is reductive.

Therefore, we may assume that $T \cdot y$ is open in C^0. Since $x \in C^0$ is T-fixed and C^0 is affine, it follows that $x \in \overline{T \cdot y}$; in particular, $x \in \overline{H^0 \cdot y}$. So C^0 is contained in the fiber at x of the geometric invariant theory quotient $G/H \to H^0\backslash\backslash G/H$. By [Lun73, II.1, III.1], this fiber is H^0-equivariantly isomorphic to the nilcone $\mathscr{N}$ of $\mathfrak{p}$ (the fiber at 0 of the quotient $\mathfrak{p} \to \mathfrak{p}/\!/H^0$). Thus, we may identify C^0 with a B-stable curve L in $\mathscr{N}$. Moreover, C and L have the same stabilizer in H.

If U fixes C^0 pointwise, then C^0 is contained in $\mathfrak{p}^U$. The latter is a multiplicity-free representation of T with linearly independent weights. It follows that C^0 is a highest weight line in $\mathfrak{p}$.

We may thus assume that U acts nontrivially on C; equivalently, B_y does not contain U. But B_y has codimension 1 in B, and hence contains a maximal torus of B; so we may assume that y is fixed by T. Then $x \notin \overline{T \cdot y}$, a contradiction.

(ii) Denote by λ the weight of B in L; then L is the weight space $\mathfrak{p}^U_\lambda$. Since H is reductive, the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} = \mathfrak{p}$ induces a surjective map $\mathfrak{g}^U_\lambda \to \mathfrak{p}^U_\lambda$. So we may choose $\xi \in \mathfrak{g}^U_\lambda$ whose projection spans L. Since $\lambda \neq 0$, we see that ξ is nilpotent, and hence defines an additive one-parameter subgroup $u : \mathbb{G}_a \to G$ such that $b\,u(t)\,b^{-1} = u(\lambda(b)t)$ for all $b \in B$ and $t \in \mathbb{G}_a$. Thus, $\overline{u(\mathbb{G}_a) \cdot x}$ is an irreducible B-stable curve with tangent line L at x. By uniqueness, this curve equals C. $\square$

Remark 2.3. — One may check that the assumptions of Lemma 2.2 hold for any symmetric space G/H of adjoint type, i.e., G is semisimple of adjoint type and H is the fixed point subgroup of an involutive automorphism of G. But these assumptions are not satisfied when X is a toric variety (then G is a torus and H is trivial).

Lemma 2.4. — *Consider a representation V of the additive group $\mathbb{G}_a$, and a point $x \in \mathbb{P}(V)$ which is not fixed by $\mathbb{G}_a$. Then the orbit map $\mathbb{G}_a \to X$, $t \mapsto t \cdot x$ extends to an isomorphism $f : \mathbb{P}^1 \to \overline{\mathbb{G}_a \cdot x}$.*

Proof. — Clearly, f is bijective, and smooth at every $t \in \mathbb{G}_a$. So it suffices to show that f is smooth at ∞ as well.

Choose a representative $v \in V$ of x. We may assume that V is spanned by the orbit $\mathbb{G}_a \cdot v$. Also, V is a direct sum of indecomposable representations of $\mathbb{G}_a$, and these are of the form $\mathbb{C}[x,y]_n$ (the space of homogeneous polynomials of degree n in x, y), where $\mathbb{G}_a$ acts via $(t \cdot f)(x,y) := f(x, tx + y)$. Using a suitable projection, we may assume that $V = V_n$; then $v = a_0 x^n + \cdots + a_n y^n$, where $a_n \neq 0$. So $t \cdot v = (a_n x^n) t^n + (n a_n x^{n-1} y + a_{n-1} x^n) t^{n-1} + \ldots$ In particular, the coefficients of t^n and t^{n-1} in $t \cdot v$ are both non-zero; this yields our assertion. □

Lemma 2.5. — *Let Y and Z be projective varieties equipped with an action of G, and let $f : Y \to Z$ be a finite G-equivariant morphism. Assume that Z contains a unique closed G-orbit $Z_{\min}$ and is equivariantly isomorphic to a subvariety of $\mathbb{P}V$ for some finite-dimensional representation V of G. Then Y contains a unique closed G-orbit $Y_{\min}$ as well. Moreover, $Y_{\min}$ is isomorphic to $Z_{\min}$ via f.*

Proof. — Choose a Borel subgroup $B \subset G$. Then the fixed point locus $Z^B \subset Z$ consists of a unique point, $z \in Z_{\min}$. Indeed, for any $x \in Z^B$, the stabilizer G_x is a parabolic subgroup of G, hence the orbit $G \cdot x$ is closed and $(G \cdot x)^B$ consists of the point x.

Denote by $U \subset B$ the unipotent part, and consider the fixed point locus Z^U. Then Z^U is connected by [Hor69, Th. 6.2]; it is also B-stable, and B acts on Z^U via its quotient torus $T = B/U$. Moreover, $(Z^U)^T = Z^B$ consists of z only. As Z^U is T-equivariantly isomorphic to a subvariety of $\mathbb{P}V$, it follows that Z^U is just the point z (indeed, for any $x \in \mathbb{P}V$ which is not fixed by T, the orbit closure $\overline{T \cdot x}$ contains at least 2 fixed points of T).

Next, consider Y^U: it is also connected, and finite since so is f. Thus, Y^U is a unique point, say y. So Y^B is the point y as well, and Y contains a unique closed G-orbit, namely, $Y_{\min} = G \cdot y$. Moreover, f restricts to a finite equivariant map $Y_{\min} \to Z_{\min} = G \cdot z$. Since G_z (a parabolic subgroup of G) is connected, this implies the assertion. □

We now come to the main result of this section:

Proposition 2.6. — *Assume that H is reductive, and the representation of $\mathfrak{h}$ in $\mathfrak{p}$ is irreducible and non-trivial. Then*

(i) *There exists a unique family of minimal rational curves $\mathscr{K}$ on X.*

(ii) *$\mathscr{K}_x$ contains a unique closed orbit of H^0; it is isomorphic to the orbit of the highest weight line in $\mathfrak{p}$.*

(iii) *$\mathscr{K}_x$ is smooth and connected.*

(iv) *$\mu \times \rho : \mathscr{U} \to X \times \mathscr{K}$ is a closed immersion (i.e., $\mathscr{K}$ consists of embedded curves).*

Proof

(i) and (ii) Let $\mathscr{K} \subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ be a family of minimal rational curves. Consider the action of a Borel subgroup $B \subset H$ on the projective variety $\mathscr{K}_x$. By construction, $\mathscr{K}_x$ has an equivariant finite morphism to the Chow variety of X. In view of Lemma 2.2 and Borel's fixed point theorem, the image of $\mathscr{K}_x$ contains a unique closed H^0-orbit: that of the Chow point of the irreducible B-stable curve C that corresponds to the highest weight line L in $\mathfrak{p}$. Applying Lemma 2.5, it follows that $\mathscr{K}_x$ also contains a unique closed H^0-orbit, isomorphic to that of L. This implies that $\mathscr{K}$ is the family of deformations of C, and hence is unique.

(iii) The smoothness of $\mathscr{K}_x$ follows from Lemma 2.1, and the connectedness from the fact that $\mathscr{K}_x$ contains a unique B-fixed point.

(iv) Let $p_2 : X \times \mathscr{K} \to \mathscr{K}$ denote the projection. Then $p_2 \circ (\mu \times \rho) = \rho$ is proper and flat. Applying [Kol96, I.1.10.1] together with Borel's fixed point theorem again, it suffices to show that the restriction of μ to $\rho^{-1}(C)$ is a closed immersion, where C denotes, as above, the irreducible B-stable curve through x. But this follows from [Kol96, II.3.5.4] together with Lemmas 2.2 and 2.4, since C is the image of a closed immersion $f : \mathbb{P}^1 \to X$ which is a free morphism. $\square$

For later use, we also record the following easy result:

Lemma 2.7. — *Assume that H is reductive and the boundary $\partial X := X \smallsetminus X^0$ is a unique G-orbit. Let Y be another G-equivariant compactification of G/H, where Y is complete (possibly singular). Then there is a bijective morphism $f : X \to Y$ which restricts to the identity on G/H.*

Proof. — Note that G/H is affine; thus, ∂X and ∂Y have pure codimension 1. Since X is smooth and Y is complete, the identity map of G/H, viewed as a rational map $f : X \dashrightarrow Y$, is defined in codimension 1. As ∂X is homogeneous, it follows that f is a morphism. Thus, ∂Y is a unique G-orbit and f restricts to a finite equivariant map $g : \partial X \to \partial Y$, hence g is an isomorphism. This implies that f is bijective. $\square$

3. Closures of one-parameter subgroups

From now on, G denotes a semisimple linear algebraic group of adjoint type and rank $\ell \geqslant 2$, and $\mathfrak{g}$ the corresponding (semisimple) Lie algebra.

We first introduce some notation: we choose a Borel subgroup $B \subset G$ as well as a maximal torus $T \subset B$. Then $B = TU$, where $U \subset B$ denotes the unipotent part. We denote by R the root system of (G, T) and by $R^+ \subset R$ the subset of positive roots consisting of roots of B. The corresponding set of simple roots is denoted by $S = \{\alpha_1, \ldots, \alpha_\ell\}$; we use the ordering of simple roots as in [Bou07]. The half-sum of positive roots is denoted by ρ. Also, we denote by W the Weyl group, and by $w_0 \in W$ the unique element that sends R^+ to $-R^+$; then $w_0^2 = 1$ and $-w_0$ stabilizes S.

The coroot of any $\alpha \in R$ is denoted by $\alpha^\vee$; this is a one-parameter subgroup of T. The coroots form the dual root system $R^\vee$. The pairing between characters and one-parameter subgroups is denoted by $\langle , \rangle$; we have $\langle \alpha, \alpha^\vee \rangle = 2$ for any $\alpha \in R$.

We denote by Λ the weight lattice, with the submonoid Λ^+ of dominant weights, and the fundamental weights $\varpi_1, \ldots, \varpi_\ell$. For any $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the irreducible representation of the simply-connected cover of G having highest weight λ.

We then have a projective representation

$$\varphi_\lambda : G \longrightarrow \mathrm{PGL}(V(\lambda)).$$

Moreover, the G-orbit of the highest weight line in $V(\lambda)$ yields the unique closed orbit in the projectivization $\mathbb{P}V(\lambda)$; it is isomorphic to G/P_λ, where the parabolic subgroup P_λ only depends on the type of λ, i..e., the set of simple roots that are orthogonal to that weight.

The closure of the image of φ_λ in the projective space $\mathbb{P}\operatorname{End} V(\lambda)$ will be denoted by X_λ. This is a projective variety on which $G\times G$ acts via its action on $\mathbb{P}\operatorname{End} V(\lambda)$ by left and right multiplication. Moreover, X_λ contains a unique closed orbit of $G\times G$; it is isomorphic to $G/Q_\lambda \times G/P_\lambda$, where Q_λ denotes the parabolic subgroup containing T and opposite to P_λ.

When the dominant weight λ is regular, X_λ turns out to be smooth and independent of the choice of λ; this defines the wonderful compactification X of G. The boundary $X \smallsetminus G$ is a union of ℓ irreducible divisors $D_1, \dots, D_\ell$ with smooth normal crossings. The $G\times G$-orbits in X are indexed by the subsets of $\{1,\dots,\ell\}$, by assigning to each such subset $I = \{i_1, \dots, i_s\}$ the unique open orbit $\mathscr{O}_I$ in the partial intersection $D_{i_1} \cap \dots \cap D_{i_s}$. In particular, the open orbit X^0 is $\mathscr{O}_\varnothing = (G\times G)/\operatorname{diag}(G)$, and the closed orbit is $\mathscr{O}_{\{1,\dots,\ell\}}$. Each orbit closure is equipped with a $G\times G$-equivariant fibration

$$f_I : \overline{\mathscr{O}_I} \longrightarrow G/Q_I \times G/P_I,$$

where P_I denotes the parabolic subgroup associated with the dominant weight $\sum_{i\in I} \varpi_i$, and Q_I stands for the opposite parabolic subgroup. The fiber of f_I at the base point of $G/Q_I \times G/P_I$ is isomorphic to the wonderful compactification of the adjoint group of $L_I := P_I \cap Q_I$ (a Levi subgroup of both). In particular, the closed orbit $\mathscr{O}_{1,\dots,\ell}$ is isomorphic to $G/B \times G/B$. These results are due to De Concini and Procesi (see [DCP83]) in the more general setting of symmetric spaces of adjoint type.

For an arbitrary λ, the variety X_λ is usually singular; its normalization X_λ^{n} only depends on the type of λ (criteria for normality and smoothness of X_λ are obtained in [BGMR11, Th. A, Th. B]). The homomorphism φ_λ extends to a $G\times G$-equivariant morphism $X \to X_\lambda$ that we shall still denote by φ_λ. The pull-backs $\mathscr{L}_X(\lambda) := \varphi_\lambda^* \mathscr{O}_{\mathbb{P}\operatorname{End} V(\lambda)}(1)$, $\lambda \in \Lambda^+$, are exactly the globally generated line bundles on X; moreover, $\mathscr{L}_X(\lambda)$ is ample if and only if λ is regular. In particular, X admits a unique minimal ample line bundle, namely, $\mathscr{L}_X(\rho)$. The assignment $\lambda \mapsto \mathscr{L}_X(\lambda)$ extends to an isomorphism $\Lambda \xrightarrow{\cong} \operatorname{Pic}(X)$ (see e.g. [BK05, Prop. 6.1.11]).

We shall index the boundary divisors so that $\mathscr{O}_X(D_i) = \mathscr{L}_X(\alpha_i)$ for $i = 1, \dots, \ell$. Thus, D_i comes with a homogeneous fibration over $G/Q_i \times G/P_i$, where P_i denotes the maximal parabolic subgroup associated with the fundamental weight ϖ_i, and Q_i denotes its opposite.

We now investigate the closures in X of the images of the multiplicative one-parameter subgroups $\eta : \mathbb{G}_m \to G$, and determine the degrees of the line bundles $\mathscr{L}_X(\lambda)$ on these rational curves. Recall that every such η is conjugate in G to a dominant one-parameter subgroup of T. Since $\mathscr{L}_X(\lambda)$ is invariant under conjugation, we may assume that $\eta : \mathbb{G}_m \to T$ is dominant. We may further assume that η is indivisible in the lattice of one-parameter subgroups of T; equivalently, η is an isomorphism over its image.

Lemma 3.1. — *Let $\eta : \mathbb{G}_m \to T$ be a dominant indivisible one-parameter subgroup, and denote again by $\eta : \mathbb{P}^1 \to X$ the corresponding morphism, with image C_η. Then*

(i) *$\eta(0) \in \mathscr{O}_I$, where $I := \{i \mid 1 \leqslant i \leqslant \ell, \langle \alpha_i, \eta\rangle \neq 0\}$. Also, $\eta(\infty) \in \mathscr{O}_J$, where $J := \{j \mid 1 \leqslant j \leqslant \ell, \langle w_0\alpha_j, \eta\rangle \neq 0\}$.*

(ii) *C_η is smooth if and only if there exists i such that $\langle \alpha_i, \eta\rangle = 1$.*

(iii) *$\deg \eta^*\mathscr{L}_X(\lambda) = \langle \lambda - w_0\lambda, \eta\rangle$ for any $\lambda \in \Lambda$.*

Proof. — Recall that the closure $\overline{T}$ of T in X contains an open affine $T\times T$-subset $\overline{T}_0$, isomorphic to the affine space $\mathbb{A}^\ell$ on which $T \times T$ acts linearly with weights $(\alpha_1, -\alpha_1), \ldots, (\alpha_\ell, -\alpha_\ell)$. Moreover, $\overline{T}$ is the union of the conjugates $w\overline{T}_0w^{-1}$, where $w \in W$, and the pull-back of $\mathscr{L}_X(\lambda)$ to $w\overline{T}_0w^{-1}$ has a trivializing section (as a $T\times T$-linearized line bundle) of weight $(w\lambda, -w\lambda)$; see [BK05, Lem. 6.1.6, Prop. 6.2.3] for these results.

Since η is dominant, it extends to a morphism $\mathbb{A}^1 \to \overline{T}_0$, $t \mapsto (t^{\langle\alpha_1,\eta\rangle}, \ldots, t^{\langle\alpha_\ell,\eta\rangle})$, where $\overline{T}_0$ is identified with $\mathbb{A}^\ell$ as above. In particular, $\eta(0) \in \overline{T}_0$; moreover, C_η is smooth at $\eta(0)$ if and only if there exists i such that $\langle \alpha_i, \eta\rangle = 1$. Also, the one-parameter subgroup $-w_0\eta : \mathbb{G}_m \to T$, $t \mapsto w_0(t^{-1})$ is dominant, and hence extends to a morphism $\mathbb{A}^1 \to \overline{T}_0$, $t \mapsto (t^{\langle -w_0\alpha_1),\eta\rangle}, \ldots, t^{\langle -w_0\alpha_\ell,\eta\rangle})$. Since $-w_0$ permutes the simple roots, it follows that $\eta(\infty) = (-\eta)(0) \in w_0\overline{T}_0w_0$; moreover, C_η is smooth at $\eta(\infty)$ if and only if there exists i such that $\langle \alpha_i, \eta\rangle = 1$. This proves (i) and (ii).

For (iii), note that $\eta^*\mathscr{L}_X(\lambda)$ is a $\mathbb{G}_m$-linearized line bundle on $\mathbb{P}^1$ with weights $\langle \lambda, \eta\rangle$ at 0 and $\langle w_0\lambda, \eta\rangle$ at ∞. Since the degree of such a line bundle is the difference of its weights, this yields our assertion. □

Next, we consider the curves obtained as closures of the additive one-parameter subgroups associated with the roots; these curves are clearly rational, and smooth by Lemma 2.4. For any $\alpha \in R$, we denote by U_α the corresponding root subgroup of G (with Lie algebra the root subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$) and by C_α the closure of U_α in X. Since α is conjugate in W to a dominant root, we may assume α dominant; then α is either the highest root or the highest short root of an irreducible component of the root system R. The highest roots are of special interest in view of the following:

Lemma 3.2. — *The irreducible B-stable curves through x (the neutral element of G) are exactly $C_{\theta_1}, \ldots, C_{\theta_m}$, where $\theta_1, \ldots, \theta_m$ denote the highest roots of R.*

Proof. — The isotropy representation of the homogeneous space $(G \times G)/\operatorname{diag}(G)$ is the adjoint representation of G in $\mathfrak{g}$; this is the direct sum of irreducible representations with highest weights $\theta_1, \ldots, \theta_m$. Since these weights are linearly independent, the assertion follows from Lemma 2.2. $\square$

To analyze further these curves associated with the highest roots, we may and shall assume that G is simple, by using the decompositions (1.1) and (1.2); we denote the highest root by θ. We shall need the following observation:

Lemma 3.3

(i) *We have* $\langle \alpha, \theta^\vee \rangle = 0$ *or* 1 *for any* $\alpha \in R^+$ *such that* $\alpha \neq \theta$.

(ii) *If* G *is of type* A_ℓ, *then* $\langle \alpha_i, \theta^\vee \rangle = 1$ *if and only if* $i = 1$ *or* $i = \ell$. *If* G *is not of type* A, *then there is a unique simple root* α_{i_0} *such that* $\langle \alpha_{i_0}, \theta^\vee \rangle = 1$.

Proof

(i) follows from [Bou07, VI.1.8, Prop. 25].

(ii) is checked by inspection (see also [CP11, Table I]). $\square$

We now obtain an analogue of Lemma 3.1 for the curve C_θ:

Lemma 3.4

(i) *For any* $\lambda \in \Lambda$, *we have* $\mathscr{L}_X(\lambda) \cdot C_\theta = \langle \lambda, \theta^\vee \rangle$.

(ii) *Let* y *denote the unique point of* $C_\theta \setminus U_\theta$ *(the point at infinity of* C_θ*). Then* $y \in \mathscr{O}_{1,\ell}$ *if* G *is of type* A_ℓ, *and* $y \in \mathscr{O}_{i_0}$ *otherwise.*

Proof

(i) The coroot $\theta^\vee$ is a dominant one-parameter subgroup of T, and satisfies $w_0\theta^\vee = -\theta^\vee$. Thus, $\deg(\theta^\vee)^*\mathscr{L}_X(\lambda) = 2\langle \lambda, \theta^\vee \rangle$ by Lemma 3.1. We now relate C_θ with the curve $C_{\theta^\vee}$, the image of $\theta^\vee : \mathbb{P}^1 \to X$. For this, we consider the subgroup $G_\theta \subset G$ generated by U_θ and $U_{-\theta}$. Then G_θ is the image of a homomorphism $h : \mathrm{SL}_2 \to G$; in particular, G_θ is semisimple with maximal torus the image of $\theta^\vee$ (viewed as a one-parameter subgroup of T). Recall from Lemma 3.3 that $\langle \alpha_i, \theta^\vee \rangle = 1$ for some i. It follows that $\theta^\vee$ is indivisible, and (using Lemma 3.1 again) $C_{\theta^\vee}$ is smooth. Also, $\theta^\vee(-1) = h(-\mathrm{id})$ is a central element of G_θ, and is non-trivial since $\theta^\vee$ is indivisible. So G_θ is isomorphic to SL_2 via h, and the closure $\overline{G_\theta} \subset X$ may be viewed as an $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariant compactification of SL_2. Using the local structure theorem as in [Tim03, Sec. 6], one obtains that $\overline{G_\theta}$ is smooth.

Consider the standard compactification $\overline{\mathrm{SL}_2}$, the closure of $\mathrm{SL}_2 \subset \mathrm{End}(\mathbb{C}^2)$ in $\mathbb{P}(\mathrm{End}(\mathbb{C}^2) \oplus \mathbb{C})$. This is a smooth quadric with equation $ad - bc = z^2$, and the boundary $\overline{SL_2} \setminus \mathrm{SL}_2$ is the quadric $(ad - bc = z = 0)$, an irreducible divisor on which $\mathrm{SL}_2 \times \mathrm{SL}_2$ acts transitively with isotropy group a Borel subgroup. In view of Lemma 2.7, it follows that the equivariant rational map $\overline{\mathrm{SL}_2} \dashrightarrow \overline{G_\theta}$ is a bijective morphism, and hence an isomorphism. The closure in $\overline{\mathrm{SL}_2}$ of the diagonal torus is the conic $(ad = z^2, b = c = 0)$, while the closure of the upper triangular unipotent subgroup is the line $(a = d = z, b = 0)$. Thus, $2C_\theta$ is rationally equivalent to $C_{\theta^\vee}$ in $\overline{G_\theta}$; this implies our formula.

(ii) In view of (i) and Lemma 3.3, we have $D_{i_0} \cdot C_\theta = 1$ and $D_i \cdot C_\theta = 0$ for $i \neq i_0$, when G is not of type A. It follows that y sits in D_{i_0} and in no other boundary divisor. Also, $D_1 \cdot C_\theta = D_\ell \cdot C_\theta = 1$ and $D_i \cdot C_\theta = 0$ for $1 < i < \ell$, when G is of type A. In that case, y sits in $D_1 \cap D_\ell$ and in no other boundary divisor. This yields our assertion by using the fact that $\mathscr{O}_I = (\bigcap_{i\in I} D_i) \smallsetminus (\bigcup_{j\notin I} D_j)$ for any $I \subset \{1\ldots,\ell\}$. □

Remark 3.5. — We still assume that G is simple, and in addition that it has two root lengths; we denote by θ_s the highest short root. Then we have $\mathscr{L}_X(\lambda)\cdot C_{\theta_s} = \langle \lambda, \theta_s^\vee\rangle$ for any $\lambda \in \Lambda$. Moreover, the point at infinity y_s of C_{θ_s} sits in a $G\times G$-orbit $\mathscr{O}_i$ of codimension 1.

Indeed, the root system R is of type B_ℓ, C_ℓ, F_4 or G_2. In type B_ℓ, where $\ell \geqslant 3$, one has $\langle \alpha_1, \theta_s^\vee\rangle = 2$ and $\langle \alpha_i, \theta_s^\vee\rangle = 0$ for all $i \geqslant 2$. In particular, $\theta_s^\vee = 2\eta$ for an indivisible one-parameter subgroup $\eta : \mathbb{G}_m \to T$. In that case, the subgroup $G_{\theta_s} \subset G$ generated by U_{θ_s} and $U_{-\theta_s}$ is isomorphic to PGL_2, and one obtains that C_{θ_s} and C_η are lines in $\overline{G_{\theta_s}} \cong \mathbb{P}^3$. Then one concludes by using Lemma 3.1. Note that $D_1\cdot C_{\theta_s} = 2$ and $D_i \cdot C_{\theta_s} = 0$ for all $i \geqslant 2$; thus, $y_s \in \mathscr{O}_1$.

In type C_ℓ ($\ell \geqslant 3$), one has $\langle \alpha_2, \theta_s^\vee\rangle = 1$ and $\langle \alpha_i, \theta_s^\vee\rangle = 0$ for all $i \neq 2$. In that case, $\theta_s^\vee$ is indivisible and the assertion follows by arguing as in the proof of Lemma 3.4; here $y_s \in \mathscr{O}_2$.

The cases of F_4 and G_2 are similar; in the former one, one obtains $\langle \alpha_4, \theta_s^\vee\rangle = 1$ and $\langle \alpha_i, \theta_s^\vee\rangle = 0$ for all other i, so that $y_s \in \mathscr{O}_4$. For G_2, one has $\langle \alpha_1, \theta_s^\vee\rangle = 1$ and $\langle \alpha_2, \theta_s^\vee\rangle = 0$, so that $y_s \in \mathscr{O}_1$.

Remark 3.6. — We have $L\cdot C \geqslant \ell$ for any ample line bundle L on X and any curve C through x. In particular, X is not covered by lines when $\ell \geqslant 2$.

Indeed, note first that C is rationally equivalent to an effective 1-cycle stable by B whose support contains x. To see this, consider the Chow variety Chow_x of effective 1-cycles through x, which has a natural B-action. The closure of the B-orbit of $[C] \in \mathrm{Chow}_x$ is then a rational variety, on which B has a fixed point $[\gamma]$. Thus, the curve C is rationally equivalent to the effective 1-cycle γ.

Next, by Lemma 3.2 and the description of ample line bundles on X, it suffices to check that $\mathscr{L}_X(\rho)\cdot C_\theta \geqslant \ell$, i.e., $\langle \rho, \theta^\vee\rangle \geqslant \ell$ in view of Lemma 3.4. But this follows from the fact that $\theta^\vee = \sum_{i=1}^{\ell} m_i \alpha_i^\vee$, where the m_i are positive integers.

Yet X is covered by strict transforms of lines under some morphism $\varphi_\lambda : X \to X_\lambda$, unless G is of type E_8 (recall that these morphisms yield all contractions from X). Indeed, by the above argument, X_λ is covered by lines if and only if $\langle \lambda, \theta^\vee\rangle = 1$. In turn, this is equivalent to $\lambda = \varpi_i$, where $m_i = 1$. The list of these fundamental weights according to the type of G is as follows:

$$A_\ell : \text{all} \qquad B_\ell : \varpi_1, \varpi_\ell \qquad C_\ell : \text{all} \qquad D_\ell : \varpi_1, \varpi_{\ell-1}, \varpi_\ell$$
$$E_6 : \varpi_1, \varpi_6 \qquad E_7 : \varpi_7 \qquad E_8 : \text{none}$$
$$F_4 : \varpi_4 \qquad G_2 : \varpi_1.$$

By [BGMR11, Th. A], X_λ is normal in all these cases, except for (B_ℓ, ϖ_1) and (C_ℓ, ϖ_ℓ). Also, X_λ is smooth in the cases (A_ℓ, ϖ_1), (A_ℓ, ϖ_ℓ), (B_ℓ, ϖ_ℓ) and (G_2, ϖ_1) only, as follows from [BGMR11, Th. B].

The weights in the above list include the minuscule weights, i.e., those fundamental weights ϖ_i such that $\alpha_i^\vee$ has coefficient 1 in the highest coroot, $\theta_s^\vee$. When G is simply laced (i.e., all roots have the same length), one has $\theta_s^\vee = \theta$ and hence one gets exactly the minuscule weights. In the non-simply laced case, one obtains additional weights for each type.

4. Proofs of Theorem 1.1 and Proposition 1.2

Proof of Theorem 1.1. — By Proposition 2.6 and Lemma 3.2, $\mathscr{K}$ is unique and $\mathscr{K}_x$ is smooth and connected. The assertion that τ is an isomorphism will be checked separately in types other than A (case (ii)) and in type A (case (iii)).

By Proposition 2.6 again, G has a unique closed orbit in $\mathscr{K}_x$, namely, that of C_θ. By Lemma 3.2, the isotropy group of C_θ in G is the parabolic subgroup P_θ, the stabilizer of the highest weight line $[\mathfrak{g}_\theta] \in \mathbb{P}\mathfrak{g}$. Thus, $G \cdot C_\theta \cong G/P_\theta$ is the adjoint variety in the sense of [CP11]; it may be viewed as the projectivization $\mathbb{P}\mathscr{O}_{\min}$, where $\mathscr{O}_{\min}$ denotes the minimal (non-zero) nilpotent orbit, i.e., the G-orbit of any non-zero point in $\mathfrak{g}_\theta$. By Lemma 3.3, P_θ is the maximal parabolic subgroup P_{i_0} associated with the simple root α_{i_0} when G is not of type A; in type A_ℓ, we have $P_\theta = P_1 \cap P_\ell$.

The dimension of $\mathscr{K}_x$ is given by $-K_X \cdot C_\theta - 2$ by [Hwa01, Prop. 2.3]. In view of [BK05, Prop. 6.1.11], we have $\mathscr{O}_X(-K_X) = \mathscr{L}_X(\kappa)$, where

$$\kappa := 2\rho + \sum_{i=1}^{\ell} \alpha_i.$$

By Lemma 3.4, it follows that

$$\dim \mathscr{K}_x = \langle \kappa, \theta^\vee \rangle - 2.$$

Lemma 4.1

(i) *If $\mathfrak{g}$ is of type A_ℓ ($\ell \geqslant 2$), then* $\langle \kappa, \theta^\vee \rangle = \dim \mathbb{P}\mathscr{O}_{\min} + 3$.

(ii) *If $\mathfrak{g}$ is not of type A, then* $\langle \kappa, \theta^\vee \rangle = \dim \mathbb{P}\mathscr{O}_{\min} + 2$.

Proof. — The dimension of G/P_θ is the number of positive roots non-orthogonal to θ. In view of Lemma 3.3 (i), this implies that

$$\dim \mathbb{P}\mathscr{O}_{\min} = \Big(\sum_{\alpha \in R^+} \langle \alpha, \theta^\vee \rangle \Big) - 1 = \langle 2\rho, \theta^\vee \rangle - 1.$$

Also, by Lemma 3.3 (ii), we have $\big\langle \sum_i \alpha_i, \theta^\vee \big\rangle = 2$ for G of type A_ℓ ($\ell \geqslant 2$), and $\big\langle \sum_i \alpha_i, \theta^\vee \big\rangle = 1$ otherwise. $\square$

When G is not of type A, Lemma 4.1 yields that $\dim \mathscr{C}_x = \dim \mathscr{K}_x = \dim \mathbb{P}\mathscr{O}_{\min}$. Since $\mathbb{P}\mathscr{O}_{\min}$ is the closed orbit in $\mathbb{P}\mathfrak{g}$, it follows that $\mathscr{C}_x = \mathbb{P}\mathscr{O}_{\min}$. As the rational map $\tau : \mathscr{K}_x \dashrightarrow \mathscr{C}_x$ is G-equivariant, it must be an isomorphism. This completes the proof of Theorem 1.1 in that case.

When G is of type A_ℓ, we may assume that $G = \mathrm{PGL}_{\ell+1}$ and T (resp. B) is the image in G of the group of diagonal (resp. upper triangular) invertible matrices. Let $\eta : \mathbb{G}_m \to T$ denote the image of the one-parameter subgroup $t \mapsto \operatorname{diag}(t, 1, \ldots, 1)$ of $\mathrm{GL}_{\ell+1}$. Then η is dominant and $\eta - w_0\eta = \theta^\vee$. So Lemma 3.4 yields that $\deg \eta^* \mathscr{L}_X(\lambda) = \langle \lambda, \theta^\vee \rangle = \mathscr{L}_X(\lambda) \cdot C_\theta$. As a consequence, the image C_η of η in $\mathrm{RatCurves}^{\mathrm{n}}(X)$ has minimal degrees. Since C_η meets X^0, it sits in a covering family of rational curves of minimal degrees. But every such family is minimal, since its image in the Chow variety is closed (see the proof of Theorem IV.2.4 in [Kol96] for details). Thus, $C_\eta \in \mathscr{K}_x$ by the uniqueness of $\mathscr{K}$. The stabilizer of C_η under the action of G is the normalizer of the image of η, that is, the image in $\mathrm{PGL}_{\ell+1}$ of $\mathrm{GL}_\ell \subset \mathrm{GL}_{\ell+1}$. Thus, the G-orbit of C_η has dimension $2\ell = \langle \kappa, \theta^\vee \rangle - 2 = \dim \mathscr{K}_x$. Since $\mathscr{K}_x$ is irreducible, it is the G-orbit closure of C_η, an equivariant compactification of $\mathrm{PGL}_{\ell+1} / \mathrm{GL}_\ell$. This homogeneous variety has another equivariant compactification, $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$, with homogeneous and irreducible boundary: the incidence variety. By Lemma 2.7 and the smoothness of $\mathscr{K}_x$, it follows that $\mathscr{K}_x = \mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$. Moreover, the tangent line of C_η at x is spanned by the differential of η at 1, which is the image in $\mathfrak{g} = \operatorname{End}(\mathbb{C}^{\ell+1})/\mathbb{C}\,\mathrm{id}$ of the projection of $\mathbb{C}^{\ell+1}$ onto the first coordinate line. The orbit of that line is open in $\mathscr{C}_x$, which is thus the image of $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$ (the variety of projections onto lines) in $\mathbb{P}\mathfrak{g}$ under the rational map $\mathbb{P}\operatorname{End}(\mathbb{C}^{\ell+1}) \dashrightarrow \mathbb{P}(\operatorname{End}(\mathbb{C}^{\ell+1})/\mathbb{C}\,\mathrm{id}) = \mathbb{P}\mathfrak{g}$ (the projection with center [id]). Since $\ell \geqslant 2$, the secant variety of $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$ does not contain [id]; hence $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$ is sent isomorphically to its image in $\mathbb{P}\mathfrak{g}$. Using Lemma 2.7 again, we conclude that $\tau : \mathscr{K}_x \dashrightarrow \mathscr{C}_x$ is an isomorphism. □

Proof of Proposition 1.2. — Let $\mathscr{K}$ be a family of minimal rational curves on X. Then $\mathscr{K}_x$ contains some irreducible B-stable curve C_{θ_i} by Lemma 3.2. As a consequence, every curve C in $\mathscr{K}_x$ satisfies $L \cdot C = L \cdot C_{\theta_i}$ for any line bundle L on X. Also, recall the decomposition (1.2), where $C_{\theta_i} \subset X_i$. Thus, $L \cdot C = 0$ whenever L is the pull-back of a ample line bundle on some factor X_j with $j \neq i$. Thus, C is contracted by the corresponding projection $X \to X_j$, i.e., C is contained in X_i. □

5. Geometric constructions of the minimal rational curves

Consider first a simple group G of adjoint type A_ℓ, where $\ell \geqslant 2$; then $G = \mathrm{PGL}(V)$, where $\dim(V) = \ell + 1$. The wonderful compactification admits a $G \times G$-equivariant morphism $f : X \to \mathbb{P}\operatorname{End}(V)$, which restricts to an isomorphism over the open orbit $\mathrm{PGL}(V)$. We recall a classical description of f as a sequence of blow-ups with smooth centers, see e.g. [Vai84, Th. 1].

The $G \times G$-orbit closures in $\mathbb{P}\operatorname{End}(V)$ are exactly the loci Y_i of endomorphisms of rank $\leqslant i$, where $i = 1, \ldots, \ell + 1$; we have $Y_1 \subset \cdots \subset Y_\ell \subset Y_{\ell+1} = \mathbb{P}\operatorname{End}(V)$ and Y_i is the singular locus of Y_{i+1}. Put $X_0 = \mathbb{P}\operatorname{End}(V)$ (so that $X_0 = X_{\varpi_1}$) and let $\pi_i : X_i \to X_{i-1}$, $i = 1, \ldots, \ell - 1$, be the blow-up of X_{i-1} along the strict transform of Y_i. Then $X = X_{\ell-1}$ and f is the composition $\pi_{\ell-1} \circ \cdots \circ \pi_1$.

In particular, Y_1 is the closed $G \times G$-orbit in $\mathbb{P}\operatorname{End}(V)$, the image of the Segre embedding $\mathbb{P}V \times \mathbb{P}V^* \to \mathbb{P}(V \otimes V^*) = \mathbb{P}\operatorname{End}(V)$. Moreover, the closed orbit $\mathbb{P}\mathscr{O}_{\min}$ is the incidence variety in $\mathbb{P}V \times \mathbb{P}V^*$, and $Y_1 \smallsetminus \mathbb{P}\mathscr{O}_{\min}$ is a unique orbit of $G \times G$.

Proposition 5.1. — *With the above notation, the family of minimal rational curves $\mathscr{K}_x$ consists of the strict transforms in X of the lines in $\mathbb{P}\operatorname{End}(V)$ through x that intersect Y_1.*

Proof. — Consider the lines in $\mathbb{P}\operatorname{End}(V)$ that intersect both the open orbit $\operatorname{PGL}(V)$ and the closed orbit Y_1. Since $\ell \geqslant 2$, every such line meets Y_1 at a unique point. By [FH12, Prop. 9.7], the strict transforms of these lines form a family of minimal rational curves $\mathscr{K}_1$ on $X_1 = \operatorname{Bl}_{Y_1}(\mathbb{P}\operatorname{End}(V))$, whose VMRT at the point x is the image of $Y_1 \subset \mathbb{P}\operatorname{End}(V)$ under the linear projection $\mathbb{P}\operatorname{End}(V) \dashrightarrow \mathbb{P}(\operatorname{End}(V)/[\mathrm{id}])$. Also, this projection maps Y_1 isomorphically to its image.

Since the join variety $J(Y_1, Y_i)$ is contained in Y_{i+1}, the lines in $\mathbb{P}\operatorname{End}(V)$ intersecting $\operatorname{PGL}(V)$ and Y_1 are disjoint from $Y_i \smallsetminus Y_1$ for all $i = 2, \ldots, \ell - 1$. Hence the strict transforms of these lines are disjoint from the strict transforms of Y_i, $i = 2, \ldots, \ell - 1$. It follows that the strict transforms of these lines in X form a family $\mathscr{K}$ of minimal rational curves with the same VMRT as that on X_1. This concludes the proof by the uniqueness of $\mathscr{K}$ (Proposition 2.6). $\square$

Next, consider a simple group H of adjoint type C_ℓ, i.e., $H = \operatorname{PSp}(V)$, where $\dim(V) = 2\ell$. Then H is the fixed point subgroup of an involutive automorphism σ of $\operatorname{PGL}(V) =: G$. Choose a maximal torus $T_H \subset H$; then the centralizer of T_H in G is a maximal torus, T. Moreover, T is contained in a σ-stable Borel subgroup $B \subset G$, and $B_H := B \cap H$ is a Borel subgroup of H; the roots of (H, T_H) are exactly the restrictions of the roots of (G, T) (see [Bou07, VIII.13.3]). Consider the wonderful compactifications $H \subset Y$ and $G \subset X$. Then σ extends to a unique involution of X, still denoted by σ. Moreover, Y may be identified with a component of the fixed locus X^σ, in view of [Kan99, Th. 4.7]; this identifies the base point $x \in X$ with that of Y.

Proposition 5.2. — *Under the above identification, the family of minimal rational curves $\mathscr{K}_{Y,x}$ is identified with $\mathscr{K}^\sigma_{X,x}$. The latter is the image of the closed immersion $\iota : \mathbb{P}V \to \mathbb{P}V \times \mathbb{P}V^*$, $[v] \mapsto ([v], [v^*])$, where $v \mapsto v^*$ denotes the isomorphism $V \to V^*$ associated with the symplectic form defining* $\operatorname{PSp}(V)$.

Proof. — By construction, σ acts on the root system R and fixes the highest root θ. One checks that $U_\theta \subset H$, and hence $C_\theta \subset Y$ with the notation of Lemma 3.2. It follows that $\mathscr{K}_{Y,x} \subset \mathscr{K}_{X,x}$. Also, σ acts on $\mathscr{K}_{X,x}$ and fixes $\mathscr{K}_{Y,x}$ pointwise. Under the isomorphism $\mathscr{K}_{X,x} \cong \mathbb{P}V \times \mathbb{P}V^*$, the action of σ is identified with $([v], [f]) \mapsto ([f^*], [v^*])$. Thus, $\mathscr{K}_{Y,x}$ is contained in the image of ι. Since H acts transitively on $\mathbb{P}V$, we must have $\mathscr{K}_{Y,x} = \iota(\mathbb{P}V)$. $\square$

Remark 5.3. — We may consider a similar construction in type B_ℓ, $\ell \geqslant 2$, that is, $H = \mathrm{SO}(V) \subset \mathrm{PGL}(V) = G$, where $\dim(V) = 2\ell + 1$. Then the wonderful compactification Y is still contained in X as a component of the fixed locus of an involution σ, by [Kan99, Th. 4.7] again. But one may check that $\mathscr{C}_{Y,x}$ is not contained in $\mathscr{C}_{X,x}$ in that case.

We now obtain an alternative description of the minimal rational curves, for any simple group G of type $\neq A$. Recall that P_θ is then the maximal parabolic subgroup P_{i_0}. Moreover, the boundary divisor D_{i_0} admits a $G \times G$-equivariant morphism to $G/Q_{i_0} \times G/P_{i_0}$, where Q_{i_0} denotes the maximal parabolic subgroup opposite to P_{i_0} (note that $Q_{i_0} = w_0 P_{i_0} w_0$, since $w_0(\alpha_{i_0}) = -\alpha_{i_0}$). We denote by

$$p : \mathscr{O}_{i_0} \longrightarrow G/P_{i_0} = G/P_\theta$$

the resulting projection; then p is G-equivariant for the right action of G on D_{i_0}.

Proposition 5.4

(i) *Every $C \in \mathscr{K}_x$ intersects $\mathscr{O}_{i_0}$ transversally at a unique point, denoted by $f(C)$.*

(ii) *The assignment $C \mapsto p(f(C))$ yields a G-equivariant isomorphism $\mathscr{K}_x \xrightarrow{\cong} G/P_\theta$.*

(iii) *The morphism $\varphi_\theta : X \to X_\theta \subset \mathbb{P}\,\mathrm{End}(\mathfrak{g})$ contracts D_{i_0} to the closed $G \times G$-orbit, isomorphic to $G/P_\theta \times G/P_\theta$.*

(iv) *For any $C \in \mathscr{K}_x$, the image $\varphi_\theta(C)$ is a smooth conic in $\mathbb{P}\,\mathrm{End}(\mathfrak{g})$, which intersects $G/P_\theta \times G/P_\theta$ at a unique point, $f_\theta(C) \in \mathrm{diag}(G/P_\theta)$. Moreover, the assignment $C \mapsto f_\theta(C)$, $\mathscr{K}_x \to G/P_\theta$ gives back the isomorphism $p \circ f$.*

Proof

(i) We have $D_i \cdot C = D_i \cdot C_\theta$ for $i = 1, \ldots, \ell$. This implies our assertion by arguing as in the proof of Lemma 3.4.

(ii) We claim that the assignment $C \mapsto f(C)$ is a morphism. To see this, consider the subvariety Y of the universal family $\mathscr{U}_x \subset X \times \mathscr{K}_x$ consisting of those pairs (y, C) such that $y \in \mathscr{O}_{i_0}$. Then the projection $\rho : \mathscr{U}_x \to \mathscr{K}_x$ restricts to a bijective morphism $Y \to \mathscr{K}_x$ in view of (i). Since $\mathscr{K}_x$ is smooth, this yields our claim.

By that claim, the assignment $C \mapsto p(f(C))$ yields a morphism, which is clearly equivariant. Since $\mathscr{K}_x \cong G/P_\theta$ by Theorem 1.1, we thus obtain an equivariant endomorphism of G/P_θ. But the identity is the unique such endomorphism, since P_θ is its own normalizer in G.

(iii) Recall that the orbit $\mathscr{O}_{i_0}$ contains a point x_{i_0} fixed by $R_u(P_\theta) \times R_u(P_\theta)$, where R_u stands for the unipotent radical. Thus, $\varphi_\theta(x_{i_0}) \in \mathbb{P}\,\mathrm{End}(\mathfrak{g}) \cong \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g})$ is also fixed by $R_u(P_\theta) \times R_u(P_\theta)$. But we have

$$\mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g})^{R_u(P_\theta) \times R_u(P_\theta)} = \mathbb{P}((\mathfrak{g}^*)^{R_u(P_\theta)}) \times \mathbb{P}(\mathfrak{g}^{R_u(P_\theta)}).$$

Moreover, $\mathfrak{g}^{R_u(P_\theta)}$ is a representation of the Levi subgroup L_θ with a unique highest weight line: $\mathfrak{g}_\theta$, stable by L_θ. It follows that $\mathfrak{g}^{R_u(P_\theta)}$ is just the highest weight line $\mathfrak{g}_\theta$; likewise, $(\mathfrak{g}^*)^{R_u(P_\theta)}$ is the highest weight line. Thus, $\varphi_\theta(x_{i_0})$ sits in the closed $G \times G$-orbit; this yields our assertion.

(iv) We have $\mathscr{L}_X(\theta)\cdot C = \mathscr{L}_X(\theta)\cdot C_\theta = \langle\theta,\theta^\vee\rangle = 2$. Thus, $\varphi_\theta(C)$ is a conic. Moreover, $\varphi_\theta(C_\theta)$ is smooth by Lemma 2.4, and intersects the closed $G\times G$-orbit at a unique point, fixed by $\mathrm{diag}(B)$; thus, this point is identified with the base point of $G/P_\theta \times G/P_\theta$. The remaining assertions follow by using (i) and Theorem 1.1. $\square$

Remark 5.5. — By [BGMR11, Th. A, Th. B] again, X_θ is smooth in types A_ℓ and G_2; normal and singular in types C_ℓ ($\ell \geqslant 3$), D_ℓ, E_6, E_7 and E_8; non-normal in the remaining types, B_ℓ and F_4.

Also, the above constructions can be adapted to the type A_ℓ: then every $C\in\mathscr{K}_x$ intersects transversally the boundary divisor D_1 at a unique point, and likewise for D_ℓ. Using the projections $D_1 \to G/P_1 \cong \mathbb{P}^\ell$ and $D_\ell \to G/P_\ell \cong (\mathbb{P}^\ell)^*$, one obtains a $\mathrm{PGL}_{\ell+1}$-equivariant morphism $f:\mathscr{K}_x \to \mathbb{P}^\ell\times(\mathbb{P}^\ell)^*$, which is thus identified with an equivariant endomorphism of $\mathbb{P}^\ell\times(\mathbb{P}^\ell)^*$. One concludes as above that f is an isomorphism, by using the fact that the isotropy group of the open orbit (the image of GL_ℓ in $\mathrm{PGL}_{\ell+1}$) is its own normalizer. But this construction just yields a variant of that considered in Proposition 5.1.

6. An application

We return to the general setup of the introduction, where X is a projective uniruled manifold and $\mathscr{K}\subset \mathrm{RatCurves}^{\mathrm{n}}(X)$ a family of minimal rational curves. Let $\mathscr{C}$ be the subvariety of the projectivized tangent bundle $\mathbb{P}T(X)$ formed by VMRTs of $\mathscr{K}$, namely the fiber of $\mathscr{C}\to X$ at a general point $x\in X$ is the VMRT $\mathscr{C}_x\subset\mathbb{P}T_x(X)$. A germ of holomorphic vector field v at x is said to *preserve* $\mathscr{C}$ if the local one-parameter family of biholomorphisms integrating v lifts to local biholomorphisms of $\mathbb{P}T(X)$ preserving $\mathscr{C}$. The set of all such germs forms a Lie algebra, called the *Lie algebra of infinitesimal automorphisms of the VMRT structure* $\mathscr{C}$ at x, to be denoted by $\mathfrak{aut}(\mathscr{C},x)$. The manifold X is said to have the *Liouville property with respect to* $\mathscr{K}$ if every infinitesimal automorphism of $\mathscr{C}$ at a general point $x\in X$ extends to a global holomorphic vector field on some Zariski open subset of X. In particular, if $\mathfrak{aut}(\mathscr{C},x)=\mathfrak{aut}(X)$ (the Lie algebra of global holomorphic vector fields), then X has the Liouville property.

As an application of Theorem 1.1, we shall prove:

Proposition 6.1. — *Let X be the wonderful compactification of a simple algebraic group G of adjoint type. Assume that G is not of type A_1 or C. Then X has the Liouville property.*

Proof. — Assume first that G is of type A_ℓ ($\ell\geqslant 2$). Let $Y_1=\mathbb{P}V\times\mathbb{P}V^*\subset\mathbb{P}\,\mathrm{End}(V)$ be the Segre embedding. Then the blow-up $\mathrm{Bl}_{Y_1}(\mathbb{P}\,\mathrm{End}(V))$ has the Liouville property by [FH12, Th. 9.6]. It follows then from [FH12, Prop. 9.4] that X has the Liouville property as well.

Next, assume G is not of type A nor of type C. By Theorem 1.1, the VMRT $\mathscr{C}_x\subset\mathbb{P}T_x(X)$ of X at a general point x is $\mathbb{P}\mathscr{O}_{\min}=G/P_\theta\subset\mathbb{P}\mathfrak{g}$. Using [Dem77, Th. 1], it follows that the natural map $\mathfrak{g}\to\mathfrak{aut}(\mathscr{C}_x)$ is an isomorphism. Consider the affine cone $\widehat{\mathscr{C}_x} \subset T_x(X)$ and its Lie algebra of linear infinitesimal automorphisms, $\mathfrak{aut}(\widehat{\mathscr{C}_x})$; then we obtain $\mathfrak{aut}(\widehat{\mathscr{C}_x}) \cong \mathfrak{g} \oplus \mathbb{C}$.

We now use a theorem of Cartan and Kobayashi-Nagano (cf. [FH12, Th. 1.1]); to state it, we need the notion of prolongation. Let V be a finite-dimensional vector space and $\mathfrak{l} \subset \mathrm{End}(V)$ a Lie subalgebra. The *k-th prolongation* (denoted by $\mathfrak{l}^{(k)}$) of $\mathfrak{g}$ is defined by $\mathfrak{l}^{(k)} = \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V) \cap \mathrm{Hom}(\mathrm{Sym}^k V, \mathfrak{l})$, where the intersection is taken in $(V^*)^{\otimes(k+1)} \otimes V = (V^*)^{\otimes k} \otimes \mathrm{End}(V)$. The theorem asserts that if V is an irreducible representation of $\mathfrak{l}$, and if $\mathfrak{l}^{(2)} \neq 0$, then $\mathfrak{l} = \mathfrak{gl}(V), \mathfrak{sl}(V), \mathfrak{sp}(V)$ or $\mathfrak{csp}(V)$. Moreover, if $\mathfrak{l}^{(2)} = 0$ but $\mathfrak{l}^{(1)} \neq 0$, then $\mathfrak{l} \subset \mathrm{End}(V)$ is the isotropy representation of an irreducible Hermitian symmetric space of compact type, different from the projective space.

Applying this result to $\mathfrak{aut}(\widehat{\mathscr{C}_x}) \subset \mathrm{End}(\mathfrak{g})$, we obtain that $\mathfrak{aut}(\widehat{\mathscr{C}_x})^{(1)} = 0$. In view of [FH12, Def. 5.3, Prop. 5.10], it follows that

$$\dim \mathfrak{aut}(\mathscr{C}, x) \leqslant \dim(G) + \dim \mathfrak{aut}(\widehat{\mathscr{C}_x}) = 2n + 1, \tag{6.1}$$

where $n := \dim(G) = \dim(X)$. On the other hand, we have $\mathfrak{aut}(X) \subset \mathfrak{aut}(\mathscr{C}, x)$, since the action of $\mathrm{Aut}(X)$ preserves the VMRT structure $\mathscr{C}$ on X. Now recall from [Bri07, Ex. 2.4.5] that the natural map $\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{aut}(X)$ is an isomorphism (in all types except A_1). This gives the inequalities $2n = \dim \mathfrak{aut}(X) \leqslant \dim(\mathfrak{aut}(\mathscr{C}, x) \leqslant 2n + 1$.

If $\dim \mathfrak{aut}(\mathscr{C}, x) = 2n$, then X has the Liouville property. So we may assume that $\dim \mathfrak{aut}(\mathscr{C}, x) = 2n + 1$. Then the inequality in (6.1) is an equality, which implies by [FH12, Cor. 5.13] that the VMRT structure $\mathscr{C}$ is locally flat, that is, for a general point $x \in X$, there exists an analytic open subset $U \subset X$, an analytic open subset $W \subset \mathbb{C}^n$ and a biholomorphic map $\phi : U \to W$ such that the induced map $d\phi : \mathbb{P}T(U) \to \mathbb{P}T(W)$ sends $\mathscr{C}|_U$ to $W \times \mathscr{C}_x$.

By [FH12, Prop. 5.14], it follows that $\mathfrak{aut}(\mathscr{C}, x) = V \oplus \mathfrak{aut}(\widehat{\mathscr{C}_x})$, where V denotes the abelian Lie algebra of dimension n. Since $\mathfrak{aut}(\widehat{\mathscr{C}_x}, x) = \mathfrak{g} \oplus \mathbb{C}$, this gives an inclusion of Lie algebras: $\mathfrak{aut}(X) = \mathfrak{g} \oplus \mathfrak{g} \subset V \oplus \mathfrak{g} \oplus \mathbb{C}$, which is not possible since $\mathfrak{g}$ is simple. ☐

Combining Proposition 6.1 with [FH12, Prop. 9.5], we obtain the following:

Corollary 6.2. — *Assume that G is not of type A_1 or C. Then for any projective variety Y and any family of morphisms $\{f_t : Y \to X, |t| < 1\}$, where f_0 is surjective, there exists a family of automorphisms $\sigma_t : X \to X$ such that $f_t = \sigma_t \circ f_0$.*

Remark 6.3. — When G is of type A_1, the variety X is isomorphic to $\mathbb{P}^3$, which does not have the Liouville property. The statement in Corollary 6.2 also fails in this case.

References

[Bou07] N. Bourbaki – *Éléments de mathématique. Groupes et algèbres de Lie*, Springer, Berlin, 2006–2007.

[BGMR11] P. Bravi, J. Gandini, A. Maffei & A. Ruzzi – "Normality and non-normality of group compactifications in simple projective spaces", *Ann. Inst. Fourier (Grenoble)* **61** (2011), no. 6, p. 2435–2461 (2012).

[Bri07] M. Brion – "The total coordinate ring of a wonderful variety", *J. Algebra* **313** (2007), no. 1, p. 61–99.

[BK05] M. Brion & S. Kumar – *Frobenius splitting methods in geometry and representation theory*, Progress in Math., vol. 231, Birkhäuser Boston, Inc., Boston, MA, 2005.

[CP11] P. E. Chaput & N. Perrin – "On the quantum cohomology of adjoint varieties", *Proc. London Math. Soc. (3)* **103** (2011), no. 2, p. 294–330.

[CFH14] Y. Chen, B. Fu & J.-M. Hwang – "Minimal rational curves on complete toric manifolds and applications", *Proc. Edinburgh Math. Soc. (2)* **57** (2014), no. 1, p. 111–123.

[DCP83] C. De Concini & C. Procesi – "Complete symmetric varieties", in *Invariant theory (Montecatini, 1982)*, Lect. Notes in Math., vol. 996, Springer, Berlin, 1983, p. 1–44.

[Dem77] M. Demazure – "Automorphismes et déformations des variétés de Borel", *Invent. Math.* **39** (1977), no. 2, p. 179–186.

[FH12] B. Fu & J.-M. Hwang – "Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity", *Invent. Math.* **189** (2012), no. 2, p. 457–513.

[Hor69] G. Horrocks – "Fixed point schemes of additive group actions", *Topology* **8** (1969), p. 233–242.

[Hwa01] J.-M. Hwang – "Geometry of minimal rational curves on Fano manifolds", in *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, p. 335–393.

[HM02] J.-M. Hwang & N. Mok – "Deformation rigidity of the rational homogeneous space associated to a long simple root", *Ann. Sci. École Norm. Sup. (4)* **35** (2002), no. 2, p. 173–184.

[HM04a] ———, "Birationality of the tangent map for minimal rational curves", *Asian J. Math.* **8** (2004), no. 1, p. 51–63.

[HM04b] ———, "Deformation rigidity of the 20-dimensional F_4-homogeneous space associated to a short root", in *Algebraic transformation groups and algebraic varieties*, Encyclopaedia Math. Sci., vol. 132, Springer, Berlin, 2004, p. 37–58.

[HM05] ———, "Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation", *Invent. Math.* **160** (2005), no. 3, p. 591–645.

[Kan99] Senthamarai S. Kannan – "Remarks on the wonderful compactification of semisimple algebraic groups", *Proc. Indian Acad. Sci. Math. Sci.* **109** (1999), no. 3, p. 241–256.

[Keb02] S. Kebekus – "Families of singular rational curves", *J. Algebraic Geom.* **11** (2002), no. 2, p. 245–256.

[Kol96] J. Kollár – *Rational curves on algebraic varieties*, Ergeb. Math. Grenzgeb. (3), vol. 32, Springer-Verlag, Berlin, 1996.

[LM03] J. M. Landsberg & L. Manivel – "On the projective geometry of rational homogeneous varieties", *Comment. Math. Helv.* **78** (2003), no. 1, p. 65–100.

[Lun73] D. Luna – "Slices étales", in *Sur les groupes algébriques*, Mém. Soc. Math. France (N.S.), vol. 33, Société Mathématique de France, Paris, 1973, p. 81–105.

[Tim03] D. A. Timashëv – "Equivariant compactifications of reductive groups", *Mat. Sb.* **194** (2003), no. 4, p. 119–146.

[Vai84] I. Vainsencher – "Complete collineations and blowing up determinantal ideals", *Math. Ann.* **267** (1984), no. 3, p. 417–432.

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