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Daniel Greb, Stefan Kebekus, & Thomas Peternell
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PROJECTIVELY FLAT KLT VARIETIES

by Daniel Greb, Stefan Kebekus & Thomas Peternell

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Abstract. — In the context of uniformisation problems, we study projective varieties with klt singularities whose cotangent sheaf admits a projectively flat structure over the smooth locus. Generalising work of Jahnke-Radloff, we show that torus quotients are the only klt varieties with semistable cotangent sheaf and extremal Chern classes. An analogous result for varieties with nef normalised cotangent sheaves follows.

Résumé (Variétés klt projectivement plates). — Dans le cadre des problèmes d'uniformisation, nous étudions les variétés projectives avec singularités klt dont le faisceau cotangent admet une structure projective plate sur le lieu lisse. En généralisant le travail de Jahnke-Radloff, nous montrons que les quotients des tores sont les seules variétés klt avec un faisceau cotangent semi-stable et des classes de Chern extrémales. Un résultat analogue pour les variétés avec un faisceau cotangent normalisé nef s'ensuit.

Contents

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1. Introduction

1.1. Projective manifolds with projectively flat cotangent bundle. —

Let $\mathscr{E}$ be a locally free sheaf of rank $r \geqslant 1$ on a complex, projective manifold X of dimension $n \geqslant 2$. If $\mathscr{E}$ is semistable with respect to some ample divisor $H \in \mathrm{Div}(X)$, then the Bogomolov-Gieseker Inequality holds:

$$(1.0.1) \qquad \frac{r-1}{2r} \cdot c_1(\mathscr{E})^2 \cdot [H]^{n-2} \leqslant c_2(\mathscr{E}) \cdot [H]^{n-2}.$$

In case of equality, the sheaf $\mathscr{E}$ is known to be projectively flat. We refer the reader to Section 3.1 for a brief discussion of projective flatness.

Motivated by the structure theory of higher-dimensional projective manifolds, we are particularly interested in the case where $\mathscr{E}$ is the cotangent bundle Ω_X^1 of an n-dimensional manifold X. In this setup, the equality case of (1.0.1) reads

$$(1.0.2) \qquad \frac{n-1}{2n} \cdot c_1(X)^2 \cdot [H]^{n-2} = c_2(X) \cdot [H]^{n-2}.$$

While semistability of Ω_X^1 occurs in many relevant cases and has important geometric consequences, the Equality (1.0.2) poses severe restrictions on the geometry of X.

– In case K_X is ample, Equality (1.0.2) will never hold, owing to the stronger Miyaoka–Yau inequality.

– If $K_X \equiv 0$, then by Yau's theorem, X is an étale quotient of an Abelian variety.

– If X is Fano and Kähler-Einstein, again the equality (1.0.2) cannot occur, owing to the Chen–Oguie inequality.

In their remarkable paper [JR13], Jahnke and Radloff proved the following complete characterisation of manifolds with semistable cotangent bundle for which Equality (1.0.2) holds.

Theorem 1.1 (Characterisation of torus quotients, [JR13, Ths. 0.1 & 1.1])

Let X be a projective manifold of dimension n and assume that Ω_X^1 is H-semistable for some ample line bundle H. If Equality (1.0.2) holds, then X is a finite étale quotient of a torus. □

In particular, this deals with manifolds of intermediate Kodaira dimension, and moreover implies that in the Fano case Equality (1.0.2) can never happen, independent of the existence question for Kähler-Einstein metrics.

1.2. Main result of this paper. —

We have learned from numerous previous results, including [GKP16, LT18, GKT18, GKPT20, GKPT19a, GKP20], that the natural context for uniformisation results is that of minimal model theory. The aim of this paper is therefore to generalise the theorem of Jahnke–Radloff to the case when X has klt singularities.

Theorem 1.2 (Characterisation of quasi-Abelian varieties). — *Let X be a projective klt space of dimension $n \geqslant 2$ and let $H \in \mathrm{Div}(X)$ be ample. Assume that $\Omega_X^{[1]}$ is semistable with respect to H and that its $\mathbb{Q}$-Chern classes satisfy the equation*

$$(1.2.1) \qquad \frac{n-1}{2n} \cdot \widehat{c}_1\big(\Omega_X^{[1]}\big)^2 \cdot [H]^{n-2} = \widehat{c}_2\big(\Omega_X^{[1]}\big) \cdot [H]^{n-2}.$$

Then, X is quasi-Abelian and has at worst quotient singularities.

In Theorem 1.2, we say that a normal projective variety X is *quasi-Abelian* if there exists a quasi-étale cover $\widetilde{X} \to X$ from an Abelian variety $\widetilde{X}$ to X. The symbols $\widehat{c}_\bullet$ denote $\mathbb{Q}$-Chern classes on the klt variety X, as recalled in [GKPT19a, §3].

1.3. **Normalised cotangent sheaves.** — Work of Narasimhan, Seshadri and others, summarised for example in [JR13, Th. 1.1] and explained in detail by Nakayama in [Nak98, Th. A], can be used to reformulate Jahnke-Radloff's result in terms of positivity properties of natural tensor sheaves: a projective manifold X of dimension n is quasi-Abelian if and only if the *normalised cotangent bundle*, $\mathrm{Sym}^n \Omega_X^1 \otimes \mathscr{O}_X(-K_X)$, is nef.

While the arguments presented by Nakayama use intersection theory computations on the total space of the projectivised cotangent bundle that cannot immediately be carried over to singular varieties, we are nevertheless able to obtain the analogous result in our setup.

Theorem 1.3. — *Let X be a normal projective variety of dimension $n \geqslant 2$. Assume that X is klt and that the reflexive normalised cotangent sheaf*

$$\big(\mathrm{Sym}^n \Omega_X^1 \otimes \mathscr{O}_X(-K_X)\big)^{**}$$

is nef. Then, X is quasi-Abelian.

Definition 2.1 recalls the meaning of *nef* for a sheaf that is not necessarily locally free.

1.4. **Strategy of proof.** — While the general strategy of proof is similar to the one employed by Jahnke and Radloff, our argument introduces a number of new tools; this includes a detailed analysis of sheaves that are projectively flat on the smooth a klt variety, especially as their behaviour near the singularities is concerned. On the one hand, these tools allow us to deal with the (serious) complications arising from the singularities. On the other hand, they enable us to streamline parts of Jahnke-Radloff's proof, thereby clarifying the underlying geometric principles.

As an intermediate step, we obtain the following result, which might be of independent interest to some readers.

Theorem 1.4 (= Theorem 4.4 in Section 4.2). — *Let X be a normal, projective klt variety of dimension $n \geqslant 2$. Assume that the sheaf $\Omega_X^{[1]}$ of reflexive differentials is of the form $\Omega_X^{[1]} \cong \mathscr{L}^{\oplus n}$, where $\mathscr{L}$ is reflexive of rank one. Then, X is quasi-abelian.*

2. Conventions, notation, and variations of standard facts

2.1. Global conventions. — Throughout this paper, all schemes, varieties and morphisms will be defined over the complex number field. We follow the notation and conventions of Hartshorne's book [Har77]. In particular, varieties are always assumed to be irreducible. For all notation around Mori theory, such as klt spaces and klt pairs, we refer the reader to [KM98].

2.2. Varieties and complex spaces. — In order to keep notation simple, we will sometimes, when there is no danger of confusion, not distinguish between algebraic varieties and their underlying complex spaces. Along these lines, if X is a quasi-projective complex variety, we write $\pi_1(X)$ for the fundamental group of the associated complex space.

2.3. Reflexive sheaves. — As in most other papers on the subject, we will frequently consider reflexive sheaves and take reflexive hulls. Given a normal, quasi-projective variety (or normal, irreducible complex space) X, we write $\Omega_X^{[p]} := \left(\Omega_X^p\right)^{**}$ and refer to this sheaf as *the sheaf of reflexive differentials*. More generally, given any coherent sheaf $\mathscr{E}$ on X, write $\mathscr{E}^{[\otimes m]} := \left(\mathscr{E}^{\otimes m}\right)^{**}$ and $\det \mathscr{E} := \left(\wedge^{\operatorname{rank}\mathscr{E}}\mathscr{E}\right)^{**}$. Given any morphism $f : Y \to X$ of normal, quasi-projective varieties (or normal, irreducible, complex spaces), we write $f^{[*]}\mathscr{E} := (f^*\mathscr{E})^{**}$.

2.4. Nef sheaves. — We recall the notion of a nef sheaf in brief and collect basic properties.

Definition 2.1 (Nef and ample sheaves, [Anc82]). — Let X be a normal, projective variety and let $\mathscr{S} \neq 0$ be a non-trivial coherent sheaf on X. We call $\mathscr{S}$ *ample/nef* if the locally free sheaf $\mathscr{O}_{\mathbb{P}(\mathscr{S})}(1) \in \operatorname{Pic}(\mathbb{P}(\mathscr{S}))$ is ample/nef.

We refer the reader to [Gro61] for the definition of $\mathbb{P}(\mathscr{S})$, and to [Anc82, §2 & Th. 2.9] for a more detailed discussion of amplitude and for further references. We mention a few elementary facts without proof.

Fact 2.2 (Nef sheaves). — *Let X be a normal, projective variety.*

(2.2.1) *A direct sum of sheaves on X is nef iff every summand is nef.*
(2.2.2) *Pull-backs and quotients of nef sheaves are nef.*
(2.2.3) *A sheaf $\mathscr{E}$ is nef on X if and only if for every smooth curve C and every morphism $\gamma : C \to X$, the pull-back $\gamma^*\mathscr{E}$ is nef.* □

2.5. FLAT SHEAVES. — One key notion in our argument is that of a flat sheaf. We briefly recall the definition.

DEFINITION 2.3 (Flat sheaf, [GKP16, Def. 1.15]). — If X is any normal, irreducible complex space and $\mathscr{F}$ is any locally free coherent sheaf on X, we call $\mathscr{F}$ flat if it is defined by a (finite-dimensional) complex representation of the fundamental group $\pi_1(X)$. A locally free coherent sheaf on a quasi-projective variety is called flat if the associated analytic sheaf on the underlying complex space is flat.

REMARK 2.4 (Simple properties of flat sheaves). — Tensor powers, duals, symmetric products and wedge products of locally free, flat sheaves are locally free and flat. The pull-back of a locally free, flat sheaf under an arbitrary morphism is locally free and flat. If $\mathscr{F}$ is a locally free and flat sheaf on a normal, irreducible complex space X, then there exists a description of $\mathscr{F}$ in terms of a trivialising covering and transition functions where all transitions functions are constant.

The following lemma is a direct consequence of [Kob87, Chap. II, Prop. 3.1]. We leave the details to the reader.

LEMMA 2.5 (Chern class of flat sheaf). — *Every invertible, flat sheaf on a normal projective variety is numerically trivial.* □

2.6. COVERING MAPS AND QUASI-ÉTALE MORPHISMS. — A *cover* or *covering map* is a finite, surjective morphism $\gamma : X \to Y$ of normal, quasi-projective varieties (or normal, irreducible complex spaces). The covering map γ is called *Galois* if there exists a finite group $G \subset \operatorname{Aut}(X)$ such that γ is isomorphic to the quotient map.

A morphism $f : X \to Y$ between normal varieties (or normal, irreducible complex spaces) is called *quasi-étale* if f is of relative dimension zero and étale in codimension one. In other words, f is quasi-étale if $\dim X = \dim Y$ and if there exists a closed, subset $Z \subseteq X$ of codimension $\operatorname{codim}_X Z \geqslant 2$ such that $f|_{X \smallsetminus Z} : X \smallsetminus Z \to Y$ is étale.

2.7. MAXIMALLY QUASI-ÉTALE SPACES. — Let X be a normal, quasi-projective variety (or a normal, irreducible complex space). We say that X is *maximally quasi-étale* if the natural push-forward map of fundamental groups,

$$\pi_1(X_{\operatorname{reg}}) \xrightarrow{(\operatorname{incl})_*} \pi_1(X)$$

induces an isomorphism between the profinite completions, $\widehat{\pi}_1(X_{\operatorname{reg}}) \cong \widehat{\pi}_1(X)$.

REMARK 2.6 (Surjectivity of $(\operatorname{incl})_*$). — Recall from [FL81, 0.7.B on p. 33] that the natural push-forward map $(\operatorname{incl})_*$ is always surjective.

REMARK 2.7 (Existence of maximally quasi-étale covers). — If (X, Δ) is any quasi-projective klt pair, then X admits a quasi-étale cover $\gamma : \widehat{X} \to X$ where $\widehat{X}$ is maximally quasi-étale, [GKP16, Th. 1.14]. The pair $(\widehat{X}, \gamma^*\Delta)$ is again klt.

Remark 2.8 (Extension of representations and flat sheaves). — Assume that X is a normal, irreducible complex space that is maximally quasi-étale. Using Malcev's theorem, it has been shown in [GKP16, §8.1] that any representation $\rho : \pi_1(X_{\mathrm{reg}}) \to \mathrm{GL}(N, \mathbb{C})$ factorises via the fundamental group of X,

$$\pi_1(X_{\mathrm{reg}}) \xrightarrow[\mathrm{(incl)}_*]{\rho} \pi_1(X) \xrightarrow{\exists !} \mathrm{GL}(N, \mathbb{C}).$$

In particular, it follows that any flat bundle on X_{reg} extends to a flat bundle on X.

2.8. **Local fundamental groups of contraction morphisms.** — The behaviour of the fundamental group under extremal contractions and resolutions of singularities has been studied by Takayama in a series of papers. The following result is due to him.

Proposition 2.9 (Local fundamental groups of contraction morphisms)
   Let X be a projective klt variety and $f : X \to Y$ the contraction of a K_X-negative extremal ray. If $y \in Y$ is any point, then there exists a neighbourhood $U = U(y) \subseteq Y$, open in the analytic topology, such that $f^{-1}(U)$ is connected and simply connected.

Proof. — Takayama formulates a global version of the result in [Tak03, Th. 1.2]. The arguments of [Tak03, Proof of Th. 1.2 on p. 834] apply in our setup and give a full proof of Proposition 2.9.   □

2.9. **Abundance.** — We will later use the following special case of the abundance conjecture. Even though the proof uses only standard results from the literature, we found it worth to include it here in full, for the reader's convenience and for later referencing.

Proposition 2.10. — *Let $f : X \dashrightarrow Y$ be a dominant, almost holomorphic, rational map with connected fibres between projective varieties. Assume that*

(2.10.1) *The variety Y is smooth, positive-dimensional and of general type.*
(2.10.2) *The variety X is klt, and the canonical divisor K_X is nef.*
(2.10.3) *The variety X is smooth around general fibres of f.*
(2.10.4) *The general fibre is smooth and is a good minimal model.*

Then, X is a good minimal model. In other words, K_X is semiample.

Proof. — Blowing up X in the indeterminacy locus, we obtain a diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \pi\ } & X \\
\widetilde{f}\downarrow & & \vdots\,f \\
Y & =\!=\!= & Y
\end{array}
$$

where π is isomorphic around the general fibre $X_y \cong \widetilde{X}_y$ of f or $\widetilde{f}$, respectively. Next, consider the relative Iitaka fibration for $\pi^* K_X$ on $\widetilde{X}/Y$. As discussed in [BC15, §2.4], this is given by the (standard) Iitaka fibration for a line bundle of the form

$$\mathscr{L} := \pi^* K_X + \widetilde{f}^*(\text{sufficiently ample}).$$

Since sufficiently high powers of $\mathscr{L}$ have no base points along the general fibre $\widetilde{X}_y$, we may replace $\widetilde{X}$ by a further blow-up and obtain a diagram as follows,

$$
\begin{array}{ccccc}
\widetilde{X} & =\!=\!= & \widetilde{X} & \xrightarrow{\ \pi\ } & X \\
{\scriptstyle\text{rel. Iitaka}}\big\downarrow & & {\scriptstyle\widetilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Z & \longrightarrow & Y & =\!=\!= & Y
\end{array}
$$

where Z is smooth and where $\mathscr{L}$ is numerically trivial on the general fibre $\widetilde{X}_z$ of $\widetilde{X}$ over Z. The numerical dimension $\nu(X)$ and the Kodaira dimension $\kappa(X)$ are then estimated in terms of $\widetilde{X}_y$ and $\dim Y$ as follows,

$$
\begin{aligned}
\kappa(\widetilde{X}_y) + \dim Y = \dim Z & \qquad [\text{BC15, §2.4}] \\
= \nu(\pi^* K_X|_{\widetilde{X}_z}) + \dim Z & \qquad \text{since } \pi^* K_X|_{\widetilde{X}_z} \equiv 0 \\
\geqslant \nu(\pi^* K_X) = \nu(K_X) & \qquad [\text{Nak04, V. Lem. 2.3.(2)}] \\
\geqslant \kappa(K_X) & \qquad [\text{Kaw85b, Prop. 2.2}] \\
\geqslant \kappa(\widetilde{X}) & \qquad \text{because } \pi \text{ is birational} \\
\geqslant \kappa(\widetilde{X}_y) + \dim Y & \qquad \text{by } [\text{Vie82, Satz III}].
\end{aligned}
$$

It follows that $\kappa(X) = \nu(X)$, and hence by [KMM87, Cor. 6-1-13] that K_X is semi-ample, as desired. $\qquad\square$

2.10. **The negativity lemma.** — For later reference, we note the following two minor generalisations of the classic negativity lemma, as formulated for instance in [KM98, Lem. 3.39–3.41].

Lemma 2.11 (Negativity Lemma I). — *Let $f : X \to Y$ be a surjective morphism of normal, projective varieties. If $E \in \mathbb{Q}\operatorname{Div}(X)$ is non-zero and effective with $\operatorname{codim}_Y f(\operatorname{supp} E) \geqslant 2$, then E is not nef.*

Proof. — Choose a very ample line bundle $\mathscr{L}$ on X and a general tuple of sections $(H_1, \ldots, H_{\dim X - \dim Y}) \in |\mathscr{L}|^{\times(\dim X - \dim Y)}$. Set $S := H_1 \cap \cdots \cap H_{\dim X - \dim Y}$. This is a normal subvariety of X. The restricted morphism $f|_S$ is surjective and generically finite, and $E \cap S$ is a non-zero, effective, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S. The Stein factorisation of $f|_S$ is then birational and contracts $E \cap S$. The negativity lemma, [KM98, Lem. 3.39(1)] with $-B = E \cap S$, then applies to conclude that $E \cap S$ (and therefore E) are non-nef. $\qquad\square$

Lemma 2.12 (Negativity Lemma II). — *Let $f : X \to Y$ be a surjective morphism of normal, projective varieties, where Y is a curve. If $E \in \mathbb{Q}\operatorname{Div}(X)$ is non-zero, effective, and maps to a point in Y, then either $\kappa(E) = 1$ or E is not nef.*

Proof. — Write $y := f(E)$. If $\operatorname{supp}(E)$ equals the set-theoretic fibre $f^{-1}(y)$, then $\kappa(E) = 1$. We will therefore assume that $\operatorname{supp}(E)$ is a proper subset of $f^{-1}(y)$. Choose a very ample line bundle $\mathscr{L}$ on X and a general tuple of sections $(H_1, \ldots, H_{\dim X - 2}) \in |\mathscr{L}|^{\times(\dim X - \dim Y)}$. Set $S := H_1 \cap \cdots \cap H_{\dim X - 2}$. This is a normal surface in X. The

divisor $E \cap S$ is a non-zero, effective, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, supported on a proper subset of $(f|_S)^{-1}(y)$. Zariski's Lemma, [BHPVdV04, Chap. III, Lem. 8.2], will then show that $(E \cap S)^2 < 0$. It follows that $E \cap S$ (and therefore E) cannot be nef. $\square$

2.11. Abelian group schemes. — The proof of our main result, the characterisation of quasi-Abelian varieties in Theorem 1.2, uses a minor generalisation of Kollár's characterisation of étale quotients of Abelian group schemes, [Kol93, Th. 6.3]. We recall the relevant notions first.

Definition 2.13 (Abelian group scheme). — An *Abelian group scheme over a base B* is a smooth, proper morphism $a : A \to B$ between smooth varieties such that every fibre of a is an Abelian variety, and such that there exists a section $B \to A$.

Definition 2.14 (Generically large fundamental group, [Kol93, Defn. 6.1])
    Let X be a normal, projective variety and let $Y \subset X$ be a closed subvariety. We say that *X has a generically large fundamental group on Y* if for all very general points $y \in Y$ and for every closed and positive-dimensional subvariety $y \in Z \subset Y$ with normalisation $\overline{Z}$, the image of the natural morphism $\pi_1(\overline{Z}) \to \pi_1(X)$ is infinite.

    The generalisation of Kollár's result is then formulated as follows.

Proposition 2.15 (Characterisation of étale quotients of Abelian group schemes)
    Let $f : X \to Y$ be a surjective morphism with connected fibres between normal, projective varieties. Assume that there exists $\Delta \in \mathbb{Q}\operatorname{Div}(X)$ such that (X, Δ) is klt. Let $y \in Y$ be a very general point with fibre X_y and assume that the following holds.
    – *The fibre X_y has a finite étale cover that is birational to an Abelian variety.*
    – *The variety X has a generically large fundamental group on X_y.*
Then, there exists an étale cover $\gamma_X : \widehat{X} \to X$ such that the fibration $\widehat{a} : \widehat{A} \to \widehat{Y}$ obtained as the Stein factorisation of $(f \circ \gamma_X) : \widehat{X} \to Y$ is birational to an Abelian group scheme over a proper base.

Proof. — If X and Y are smooth, this is [Kol95, Th. 6.3]. If not, choose $\Delta \in \mathbb{Q}\operatorname{Div}(X)$ such that (X, Δ) is klt. We will consider resolutions of the singularities and construct a diagram of surjective morphisms between normal, projective varieties,

$$(2.15.1) \quad \begin{array}{ccccc} \check{X} & \xrightarrow{\gamma_{\widetilde{X}},\text{ finite and étale}} & \widetilde{X} & \xrightarrow{\pi_X,\text{ resolution}} & X \\ \downarrow{\check{a},\text{ conn. fibres}} & & \downarrow{\widetilde{f},\text{ conn. fibres}} & & \downarrow{f,\text{ conn. fibres}} \\ \check{Y} & \xrightarrow[\gamma_{\widetilde{Y}},\text{ finite}]{} & \widetilde{Y} & \xrightarrow[\pi_Y,\text{ resolution}]{} & Y \end{array}$$

where the morphism $\check{a}$ is birational to an Abelian group scheme over a proper base. For this, begin by choosing appropriate resolution morphisms π_X and π_Y to construct the right square in (2.15.1). Once this is done, choose a very general point $\widetilde{y} \in \widetilde{Y}$ with very general image point $y := \pi_Y(\widetilde{y})$. Using a result of Campana, [Cam91, Prop. 1.3], we see that the variety $\widetilde{X}$ has a generically large fundamental group on $\widetilde{X}_{\widetilde{y}}$. Kollár's result therefore applies to $\widetilde{f}$ and gives the left square.

There is more: using the assumption that (X, Δ) is klt, Takayama has shown in [Tak03, Th. 1.1] that the natural push-forward map $(\pi_X)_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ is isomorphic. The étale morphism $\gamma_{\widetilde{X}}$ therefore comes from an étale cover of X, say $\gamma_X : \widehat{X} \to X$. Stein factorising $\pi_X \circ \gamma_{\widetilde{X}}$ and $\pi_Y \circ \gamma_{\widetilde{Y}}$, we will therefore obtain a diagram of the following form,

$$(2.15.2)$$

where $\widehat{a} := \pi_{\widetilde{Y}} \circ \check{a} \circ \pi_{\widehat{X}}^{-1}$. This map makes (2.15.2) commute and is birational to $\check{a}$, which is in turn birational to an Abelian group scheme. To end the proof, it will therefore suffice to show that $\widehat{a}$ is actually a morphism. This is not so hard: if $\widehat{x} \in \widehat{X}$ is any point with preimage $Z := \pi_{\widehat{X}}^{-1}(\widehat{x})$, then

$$(\gamma_Y \circ \pi_{\widetilde{Y}} \circ \check{a})(Z) = f(\gamma_X(\widehat{x})) = \text{point in } Y$$

but since γ_Y is finite, this implies that $(\pi_{\widetilde{Y}} \circ \check{a})(Z)$ is a point in $\widehat{Y}$. The Rigidity Lemma in the form of [Deb01, Lem. 1.15] will then show that $\widehat{a}$ is well-defined at $\widehat{x}$. $\qquad\square$

3. PROJECTIVE FLATNESS

3.1. **PROJECTIVELY FLAT BUNDLES AND SHEAVES**. — *Projective flatness* is the core notion of this paper. We will only recall the most relevant definition here and refer the reader to [GKP20, §3] for a detailed discussion of projectively flat bundles and sheaves.

DEFINITION 3.1 (Projectively flat bundles and sheaves on complex spaces)
Let X be a normal and irreducible complex space, let $r \in \mathbb{N}$ be any number and let $\mathbb{P} \to X$ be a $\mathbb{P}^r$-bundle. We call the bundle $\mathbb{P} \to X$ *(holomorphically) projectively flat* if there exists a representation of the fundamental group, $\rho : \pi_1(X) \to \mathbb{P}\mathrm{GL}(r+1, \mathbb{C})$, and an isomorphism of complex spaces over X

$$\mathbb{P} \cong_X \widetilde{X} \times \mathbb{P}^r \big/ \pi_1(X),$$

where $\widetilde{X}$ is the universal cover of X and where the action $\pi_1(X) \circlearrowleft \widetilde{X} \times \mathbb{P}^r$ is the diagonal action. A locally free sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called *(holomorphically) projectively flat* if the associated bundle $\mathbb{P}(\mathscr{F})$ is projectively flat.

DEFINITION 3.2 (Projectively flat bundles and sheaves on complex varieties)
Let X be a connected, complex, quasi-projective variety and let $r \in \mathbb{N}$ be any number. A $\mathbb{P}^r$-bundle $\mathbb{P} \to X$ is called *projectively flat* if the associated analytic bundle $\mathbb{P}^{(\mathrm{an})} \to X^{(\mathrm{an})}$ is projectively flat. Ditto for coherent sheaves.

3.2. Projectively flat bundles induced by a solvable representation. — One way
to analyse a projectively flat sheaf $\mathscr{E}$ on a manifold or variety X is to look at the
Albanese map $X \to \mathrm{Alb}(X)$, and to try and understand the restriction $\mathscr{E}|_F$ to a
general fibre. The key insight here is that the Albanese map is a Shafarevich map for
the commutator subgroup, [Kol93, Prop. 4.3]. This allows to relate the geometry of the
Albanese map to the group theory of the representation that defines the projectively
flat structure on $\mathscr{E}$. The following proposition, which is a variation of material that
appears in [JR13], is key to our analysis.

Proposition 3.3 (Projectively flat bundles induced by a solvable representation)
    *Let X be a normal, irreducible complex space that is maximally quasi-étale. Let
$\mathscr{F}$ be a rank-n reflexive sheaf on X such that $\mathscr{F}|_{X_{\mathrm{reg}}}$ is locally free and projectively
flat, so that $\mathbb{P}(\mathscr{F}|_{X_{\mathrm{reg}}})$ is isomorphic to a projective bundle $\mathbb{P}_\rho$, for a representation
$\rho : \pi_1(X_{\mathrm{reg}}) \to \mathbb{P}\mathrm{GL}(n,\mathbb{C})$. If the image of ρ is contained in a connected, solvable,
algebraic subgroup $G \subset \mathbb{P}\mathrm{GL}(n,\mathbb{C})$, then there exists an isomorphism*

$$(3.3.1) \qquad\qquad\qquad \mathscr{F} \cong \mathscr{F}' \otimes \mathscr{A},$$

where $\mathscr{A}$ is Weil divisorial[1] and where $\mathscr{F}'$ is locally free and admits a filtration

$$(3.3.2) \qquad\qquad 0 = \mathscr{F}'_0 \subsetneq \mathscr{F}'_1 \subsetneq \cdots \subsetneq \mathscr{F}'_n = \mathscr{F}'$$

such that the following holds.

(3.3.3) *All sheaves $\mathscr{F}'_i$ are locally free and flat in the sense of Definition 2.3, with
$\mathrm{rank}(\mathscr{F}'_i)$ equal to i.*
(3.3.4) *The sheaf $\mathscr{F}'_1$ is trivial, $\mathscr{F}'_1 \cong \mathscr{O}_X$.*

Remark 3.4. — In the setting of Proposition 3.3, note that there exists an isomor-
phism of Weil divisorial sheaves, $\det \mathscr{F} \cong \mathscr{A}^{[\otimes n]} \otimes \det \mathscr{F}'$. Recall from Remark 2.4
that the invertible factor $\det \mathscr{F}'$ is flat and hence by Lemma 2.5 numerically trivial.

Proof of Proposition 3.3. — To begin, we claim that the projective representation ρ
lifts to a linear representation, which we will denote by σ. To formulate our result more
precisely, write $B(n,\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C})$ for the subgroup of upper-triangular matrices and
$B_1(n,\mathbb{C}) \subset B(n,\mathbb{C})$ for the subgroup of upper-triangular matrices whose top-left entry
equals one. In other words,

$$B_1(n,\mathbb{C}) := \left\{ \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & * & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & * \end{pmatrix} \Bigg| \text{ all } * \in \mathbb{C} \right\} = \Big\{ (b_{\bullet\bullet}) \in B(n,\mathbb{C}) \mid b_{11} = 1 \Big\}.$$

With this notation, we claim there exists a connected linear group $\widehat{G} \subseteq B_1(n,\mathbb{C}) \subset
\mathrm{GL}(n,\mathbb{C})$ such that the natural projection $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{P}\mathrm{GL}(n,\mathbb{C})$ induces an
isomorphism between $\widehat{G}$ and G. To this end, let $\widehat{G}' \subset \mathrm{SL}(n,\mathbb{C})$ be the maximal

---

[1]Weil divisorial sheaf = reflexive sheaf of rank one.

connected subgroup of $\pi^{-1}(G) \cap \mathrm{SL}(n, \mathbb{C})$. The restricted morphism $\pi|_{\widehat{G}'} : \widehat{G}' \to G$ is then finite and surjective, and induces an isomorphism of the Lie algebras $\mathrm{Lie}(\widehat{G}')$ and $\mathrm{Lie}(G)$. In particular, $\widehat{G}'$ is solvable. Therefore, choosing an appropriate basis of $\mathbb{C}^n$, Lie's Theorem allows us to assume without loss of generality that $\widehat{G}'$ is contained in the subgroup $B(n, \mathbb{C})$ of upper-triangular matrices. Finally, set $\widehat{G} := \pi^{-1}(G) \cap B_1(n, \mathbb{C})$. The obvious fact that any invertible upper triangular matrix $A \in \widehat{G}' \subseteq B(n, \mathbb{C})$ has a unique multiple $\lambda \cdot A$ that lies in $B_1(n, \mathbb{C})$ implies that the restriction $\pi|_{\widehat{G}} : \widehat{G} \to G$ is then isomorphic, as desired.

Remark 2.8 and the assumption that X is maximally quasi-étale allow us to identify representations of $\pi_1(X)$ and $\pi_1(X_{\mathrm{reg}})$. With these identifications made, the following commutative diagram summarises the situation,

$$
\begin{array}{ccccc}
& \widehat{G} & \longhookrightarrow & B_1(n, \mathbb{C}) & \longhookrightarrow & \mathrm{GL}(n, \mathbb{C}) \\
{\scriptstyle \sigma}\nearrow & \downarrow{\scriptstyle \cong} & & & \downarrow{\scriptstyle \pi,\ \text{nat. projection}} \\
\pi_1(X) \xrightarrow{\ \rho\ } & G & \longhookrightarrow & & \mathbb{PGL}(n, \mathbb{C})
\end{array}
$$

If $\mathscr{F}'$ denotes the locally free, flat sheaf on X associated with σ, then the factorisation of σ via the group $B_1(n, \mathbb{C})$ implies that $\mathscr{F}'$ admits a filtration as in (3.3.2) that satisfies conditions 3.3.3 and 3.3.4. By construction, we have $\mathbb{P}(\mathscr{F}|_{X_{\mathrm{reg}}}) \cong \mathbb{P}(\mathscr{F}'|_{X_{\mathrm{reg}}})$, and so there exists an invertible sheaf $\mathscr{A}_{\mathrm{reg}}$ on X_{reg} such that $\mathscr{F}|_{X_{\mathrm{reg}}} \cong \mathscr{F}'|_{X_{\mathrm{reg}}} \otimes \mathscr{A}_{\mathrm{reg}}$. A presentation of $\mathscr{F}$ as in Equation (3.3.1) therefore exists by taking $\mathscr{A} := \iota_* \mathscr{A}_{\mathrm{reg}}$, where $\iota : X_{\mathrm{reg}} \to X$ is the inclusion. $\qquad\square$

4. Varieties with splitting cotangent sheaves

One relevant case that keeps appearing in the discussion of varieties with projectively flat sheaf of reflexive differentials is the one where $\Omega_X^{[1]}$ decomposes as a direct sum of Weil divisorial sheaves. We discuss settings where there are local or global decompositions of this form. The results of this section will be used later in the proof of our main result, but might be of independent interest. In relation to [JR13], the local results are new, prompted by the appearance of singularities in our context, and the results on varieties with globally split cotangent sheaf allow us to replace some of the more involved arguments of loc. cit. by a clearer structure result.

4.1. **Varieties where $\Omega_X^{[1]}$ is locally split.** — Given a variety X where $\Omega_X^{[1]}$ is projectively flat, for instance because the flatness criterion of [GKP20, Th. 1.6] applies, the local description of projectively flat sheaves, [GKP20, Prop. 3.11], can often be used to reduce to the case where $\Omega_X^{[1]}$ decomposes locally. We will show that this already forces X to have quotient singularities.

Proposition 4.1. — *Let X be a normal complex space with klt singularities. Assume that the sheaf $\Omega_X^{[1]}$ of reflexive differentials is of the form*

$$(4.1.1) \qquad\qquad \Omega_X^{[1]} \cong \mathscr{L}^{\oplus \dim X},$$

where $\mathscr{L}$ is reflexive of rank one. Then, X has only isolated, cyclic quotient singularities.

We prove Proposition 4.1 in Section 4.3.

Remark 4.2. — If X is a quasi-projective variety with klt singularities where $\Omega_X^{[1]}$ is of the form $\mathscr{L}^{\oplus \dim X}$ for a Weil-divisorial sheaf $\mathscr{L}$, then the proof of Proposition 4.1 applies verbatim to show that X is Zariski-locally a quotient of a smooth variety. We do not use this fact however.

4.2. Varieties where $\Omega_X^{[1]}$ is globally split. — There are settings where $\Omega_X^{[1]}$ decomposes globally, for instance because it is projectively flat and X is known to be simply connected. This obviously puts strong conditions on the global geometry of X; see [Dru00] for further results in the smooth setup.

The following proposition, including the proof, is due to the referee. He also suggested Theorem 4.4, which is stronger than our original result that included a pseudo-effectivity assumption. Our previous arguments were of analytic nature and used a theorem of Demailly [Dem02].

Proposition 4.3. — *Let U be a connected complex manifold of dimension $n \geqslant 2$. Assume that $\Omega_U^1 \cong \mathscr{L}^{\oplus n}$, where $\mathscr{L}$ is invertible. Then, $c_1(\mathscr{L}) = 0 \in H^1(U, \Omega_U^1)$.*

Proof. — Consider the induced decomposition on cohomology

$$H^1(U, \Omega_U^1) \cong H^1(U, \mathscr{L})^{\oplus n}.$$

For reasons that will become apparent in a second, we are particularly interested in the coordinate hyperplanes

$$Z_i := \left\{ (\alpha_1, \ldots, \alpha_n) \in H^1(U, \mathscr{L})^{\oplus n} \mid \alpha_i = 0 \right\} \subsetneq H^1(U, \Omega_U^1).$$

To see how the Z_i come into play, restate the assumption as a decomposition of the tangent bundle: $\mathscr{T}_U \cong \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_n$. The summands $\mathscr{A}_\bullet \cong \mathscr{L}^*$ are regular rank-one foliations on U, with normal bundles

$$\mathscr{N}_i = \mathscr{A}_1 \oplus \cdots \oplus \widehat{\mathscr{A}_i} \oplus \cdots \oplus \mathscr{A}_n.$$

In this setting, a theorem of Baum-Bott, [BB70, Prop. 3.3 & Cor. 3.4] but see also [Bea00, Lem. 3.1], describes the first Chern class $c_1(\mathscr{N}_i)$ as lying in the subspace

$$Z_i = H^1(U, \mathscr{N}_i^*) \subsetneq H^1(U, \Omega_U^1).$$

But since $c_1(\mathscr{L})$ is proportional to any of the $c_1(\mathscr{N}_\bullet)$, we find that $c_1(\mathscr{L})$ is contained in $Z_1 \cap \cdots \cap Z_n = \{0\}$. The assertion is thus shown. $\qquad\square$

Applying Proposition 4.3 to the smooth locus of a klt variety, we obtain the following characterisation of quasi-abelian varieties.

Theorem 4.4. — *Let X be a normal, projective klt variety of dimension $n \geqslant 2$. Assume that the sheaf $\Omega_X^{[1]}$ of reflexive differentials is of the form $\Omega_X^{[1]} \cong \mathscr{L}^{\oplus n}$, where $\mathscr{L}$ is reflexive of rank one. Then, X is quasi-abelian.*

Proof. — We have seen in Proposition 4.3 that the $\mathbb{Q}$-Cartier sheaves $\mathscr{L}$ are numerically trivial, and then so is $\omega_X \cong \mathscr{L}^{[\otimes n]}$. Since X is klt, it follows from [Nak04, Prop. V.4.9] or [Amb05, Th. 4.2] that ω_X and $\mathscr{L}$ are torsion. As a consequence, we find a finite quasi-étale cover $\gamma : \widehat{X} \to X$ such that the reflexive pull-back $\gamma^{[*]}\mathscr{L}$ is trivial. The space $\widehat{X}$ is klt and its tangent sheaf $\mathscr{T}_{\widehat{X}} \cong \left(\gamma^{[*]}\mathscr{L}\right)^{\oplus n}$ is likewise trivial. By the solution of the Lipman-Zariski conjecture for spaces with klt singularities, [GKKP11, Th. 6], it follows that the covering space $\widehat{X}$ is smooth. In other words, $\widehat{X}$ is a parallelisable projective manifold. By a classic result of Wang, $\widehat{X}$ is an Abelian variety; see for example [Akh95, §3.9, Lem. 1]. □

4.3. **PROOF OF PROPOSITION 4.1.** — The proof of Proposition 4.1 uses the following lemma that might be of independent interest.

LEMMA 4.5. — *Let X be a normal, complex space with only cyclic quotient singularities. Let $S \subseteq X_{\mathrm{sing}}$ be any irreducible component of the singular locus. Then, there exists a non-empty, open set $S^\circ \subseteq S$ such that every $x \in S^\circ$ admits an open neighbourhood $U = U(x) \subseteq X$ of the form $U = W \times V$, where W is smooth and V has an isolated quotient singularity.*

Proof. — To begin, consider the special case where V is a complex vector space, $\rho : \mathbb{Z}_m \to \mathrm{GL}(V)$ is a linear representation of a cyclic group and where $X \subseteq V/\mathbb{Z}_m$ is an open neighbourhood of $[\vec{0}]$. Recall that ρ decomposes into direct a sum of one-dimensional representations. Identifying $\mathbb{Z}_m$ with roots of unity, we can thus find linear coordinates on V such that multiplication with $\xi \in \mathbb{Z}_m$ is given as

$$\xi \cdot (v_1, \ldots, v_m) = \left(\xi^{n_1} \cdot v_1, \ldots, \xi^{n_m} \cdot v_m\right).$$

An elementary computation in these coordinates shows that S° exists, and can in fact be chosen to be dense in S.

Returning to the general case where X is arbitrary, let $s \in S$ be any point. By assumption, there exists an open neighbourhood $U = U(x)$ and a cyclic cover $\gamma : \widetilde{U} \to U$, say with group G. Shrinking U and passing to a subgroup, if need be, we may assume without loss of generality that γ is totally branched over s, that is $\gamma^{-1}(s) = \{\widetilde{s}\}$. The cyclic group thus acts on $\widetilde{U}$ and fixes $\widetilde{s}$. The group G clearly acts on the vector space $T_{\widetilde{U}}|_{\widetilde{s}}$ and linearisation at the fixed point, [Car57, Proof of Th. 4] or [HO84, Cor. 2 on p. 13], shows the existence of a G-invariant open neighbourhood $\widetilde{U}'$ of $\vec{0} \in T_{\widetilde{U}}|_{\widetilde{s}}$ and an equivariant, biholomorphic map between $\widetilde{U}$ and $\widetilde{U}$. In other words, we are in the situation discussed in the first paragraph of this proof, where the claim has already been shown. □

Proof of Proposition 4.1. — As a first step in the proof of Proposition 4.1, we claim that $\mathscr{L}$ is $\mathbb{Q}$-Cartier. In fact, taking determinants of both sides in (4.1.1), we obtain that $\mathscr{L}^{[\otimes \dim X]} \cong \omega_X$, which is $\mathbb{Q}$-Cartier by assumption. Choose a minimal number $N \in \mathbb{N}^+$ such that $\mathscr{L}^{[\otimes N]}$ is locally free. The reflexive tensor product $\left(\Omega_U^{[1]}\right)^{[\otimes N]}$ is then likewise locally free.

As a second step, we show that X has cyclic quotient singularities. In fact, given any $x \in X$, we find an open neighbourhood $U = U(x)$ over which $\mathscr{L}^{[\otimes N]}$ is trivial. Chose a trivialisation and let $\gamma : \widetilde{U} \to U$ be the associated index-one cover, which is cyclic of order N. The complex space $\widetilde{U}$ has again klt singularities and $\gamma^{[*]}\mathscr{L} \cong \mathscr{O}_{\widetilde{U}}$. In particular, it follows that $\Omega_{\widetilde{U}}^{[1]} \cong \gamma^{[*]}\Omega_U^{[1]}$ and $\mathscr{T}_{\widetilde{U}}$ are both free. The solution of the Lipman-Zariski conjecture for spaces with klt singularities, [Dru14, Th. 3.8 & comments after Th. 1.1] or [KS21, Th. 1.13], then asserts that $\widetilde{U}$ is smooth. We conclude that X has cyclic quotient singularities only.

As a third and last step, we show that the singularities of X are isolated. Assume to the contrary and let $S \subseteq X_{\text{sing}}$ be a positive-dimensional, irreducible component. We have seen in Lemma 4.5 that there a non-empty, open subset $S^\circ \subseteq S$ such that every $x \in S^\circ$ admits an open neighbourhood $U = U(x) \subseteq X$ of product form $U = W \times V$, where W is smooth and V has an isolated quotient singularity. In particular, $\dim W = \dim S$ and

$$\Omega_U^{[1]} \cong \Omega_W^1 \oplus \Omega_V^{[1]}.$$

In particular, the reflexive tensor power $\big(\Omega_U^{[1]}\big)^{[\otimes N]}$ is written as a direct sum of reflexive sheaves with $\big(\Omega_W^{[1]}\big)^{[\otimes (N-1)]} \otimes \Omega_V^{[1]}$ as one of its direct summands. It follows that $\big(\Omega_U^{[1]}\big)^{[\otimes N]}$ is not locally free and accordingly, neither is $\big(\Omega_X^{[1]}\big)^{[\otimes N]}$. This contradicts the results obtained in the first paragraph of this proof, and therefore finishes the proof of Proposition 4.1. $\square$

5. Proof of Theorem 1.2

5.1. **Preparation for the proof.** — Using the criterion for projective flatness spelled out in [GKP20, Th. 1.6], the assumptions made in Theorem 1.2 imply that $\Omega_{X_{\text{reg}}}^1$ is projectively flat, at least after going to a quasi-étale cover. This allows to apply following lemma in various settings that appear throughout the proof of Theorem 1.2.

Lemma 5.1 (Consequences of projectively flat cotangent bundle). — *Let X be a smooth, quasi-projective variety of dimension n where Ω_X^1 is projectively flat.*

(5.1.1) *If K_X is nef, then X does not contain complete rational curves.*

(5.1.2) *Assume that there exists an immersion $\eta : F \to X$ where F is projective and smooth. If $\eta^* K_X$ is nef and if $K_F \equiv 0$, then F is quasi-Abelian.*

Proof. — We prove the statements separately. Both proofs use the fact that every locally free, projectively flat sheaf on a simply connected space is isomorphic to a direct sum of the form $\mathscr{L}^{\oplus \bullet}$ where $\mathscr{L}$ is invertible, cf. [GKP20, Prop. 3.11].

As to the first statement, let $\eta : \mathbb{P}_1 \to X$ be a non-constant map. Since $\eta^* \Omega_X^1$ is projectively flat, there is an integer m such that $\eta^* \Omega_X^1 \cong \mathscr{O}_{\mathbb{P}_1}(m)^{\oplus n}$. The canonical morphism $\eta^* \Omega_X^1 \to \Omega_{\mathbb{P}_1}^1$ then yields $m \leqslant -2$, contradicting the nefness of $\eta^* K_X$.

As to the second statement, applying the classic Decomposition Theorem [Bea83, Th. 1] and possibly passing to a finite étale cover, we may assume that F is of product form, $F \cong A \times Z$, where A is Abelian (possibly a point) and where Z is simply

connected with $K_Z = 0$. We aim to prove that Z is a point, argue by contradiction and assume that $m := \dim Z$ is positive. Let $\iota : Z \to F$ denote the inclusion morphism. The pull-back $(\eta \circ \iota)^* \Omega^1_X$ is projectively flat and hence of the form $\mathscr{L}^{\oplus n}$, for a suitable line bundle $\mathscr{L} \in \mathrm{Pic}(Z)$. The obvious surjection

$$\mathscr{L}^{\oplus n} \cong (\eta \circ \iota)^* \Omega^1_X \xrightarrow{\mathrm{d}(\eta \circ \iota)} \Omega^1_Z$$

induces a non-trivial map $\mathscr{L}^{\otimes m} \to \omega_Z \cong \mathscr{O}_Z$. But then the assumption that $\eta^* K_X$ and hence also $\mathscr{L}$ are nef shows that $\mathscr{L}^{\otimes m}$ cannot be a proper ideal sheaf, so $\mathscr{L}^{\otimes m} \cong \mathscr{O}_Z$. It follows that either $\mathscr{L}$ is torsion or that $H^0\big(Z, \Omega^1_Z\big) \neq 0$, both contradicting the simple connectedness of Z. $\qquad\square$

5.2. **Proof of Theorem 1.2**. — We maintain notation and assumptions of Theorem 1.2 throughout the present Section 5.2. The proof follows the strategy of [JR13] closely, but has to overcome a fair number of technical difficulties arising from the presence of singularities. It is fairly long and therefore subdivided into numerous steps. The main idea is to show abundance for X, and then to analyse the Iitaka fibration. Abundance is shown in Steps 4, 5 and 6, with an inductive argument using repeated covers, fibrations, and restrictions to fibres.

– Steps 1–3 set the stage, prove minimality of X and put limits on its possible numerical dimension and Kodaira dimension.

– Step 4 fibres X using a Shafarevich map construction. The general fibre is called F.

– Step 5, which is the longest step of this proof, considers a suitable cover of $\widehat{F}$ of F and takes a general fibre of its Albanese map. An analysis of this fibre will then give abundance for F.

– Step 6 uses abundance for F to prove abundance for X.

– Steps 7–8 end the proof by showing that the Iitaka fibration of a suitable cover of X is birational (and then isomorphic) to an Abelian group scheme over a proper base.

Step 1: Simplification. — Since assumption and conclusion of Theorem 1.2 are stable under quasi-étale covers, we may apply [GKP16, Th. 1.5], pass to a maximal quasi-étale cover and assume that the following holds in addition.

Assumption w.l.o.g. 5.2 (X is maximally quasi-étale). — The variety X is maximally quasi-étale. In other words, the algebraic fundamental groups $\widehat{\pi}_1(X_{\mathrm{reg}})$ and $\widehat{\pi}_1(X)$ agree.

With this assumption in place, the criterion for projective flatness, [GKP20, Th. 1.6], implies that $\Omega^1_{X_{\mathrm{reg}}}$ is projectively flat. As we have seen, this has a number of interesting consequences.

Consequence 5.3 (Extension of projectively flat bundles). — *The extension result for projectively flat bundles,* [GKP20, Prop. 3.10]*, allows to find a projectively flat* $\mathbb{P}^{n-1}$*-bundle* $\mathbb{P} \to X$ *and an isomorphism of* X*-schemes,* $\mathbb{P}|_{X_{\mathrm{reg}}} \cong_X \mathbb{P}\big(\Omega^1_{X_{\mathrm{reg}}}\big)$.

Consequence 5.4 (Quotient singularities). — *The local description of projectively flat sheaves,* [GKP20, Prop. 3.11]*, and Proposition* 4.1 *imply that* X *has only isolated cyclic quotient singularities.*

Notation 5.5. — We chose a projectively flat bundle $\mathbb{P} \to X$ as in Consequence 5.3 and maintain this choice throughout. We write $\tau : \pi_1(X) \to \mathbb{P}\mathrm{GL}(n, \mathbb{C})$ for the representation that defines the projectively flat structure on $\mathbb{P}$.

If K_X is numerically trivial, then the Chern class equality (1.2.1) reduces to $\widehat{c}_2(X) \cdot [H]^{n-2} = 0$, and we may apply [LT18, Th. 1.2] to conclude that $\widehat{X}$ is quasi-Abelian. This allows to make the following assumption.

Assumption w.l.o.g. 5.6 ($K_X \not\equiv 0$). — The canonical class K_X is not numerically trivial.

Step 2: Minimality of X. — Next, we show that the variety X is minimal.

Claim 5.7 (Minimality of X). — *The canonical divisor* K_X *is nef.*

Proof of Claim 5.7. — Choose an integer m such that $\omega_X^{[\otimes m]}$ is invertible. We argue by contradiction and assume that there exists a contraction of a K_X-negative extremal ray. Choosing a resolution of X, we find a sequence of morphisms

$$\widetilde{X} \xrightarrow{\ \pi,\ \text{resolution}\ } X \xrightarrow{\ \phi,\ \text{contraction}\ } Y.$$

If $y \in Y$ is any point, Takayama's result on the local fundamental groups of ϕ, Proposition 2.9, allows to find a neighbourhood $U = U(y) \subseteq Y^{(\mathrm{an})}$, open in the analytic topology, such that $V := \phi^{-1}(U) \subseteq X^{(\mathrm{an})}$ is connected and simply connected. Using simple connectedness, the local description of projectively flat sheaves, [GKP20, Prop. 3.11], provides us with a reflexive, rank-one coherent analytic sheaf $\mathscr{L}$ on V and an isomorphism $\Omega_V^{[1]} \cong \mathscr{L}^{\oplus n}$. We are done once we show that the first Chern class of the invertible sheaf $\mathscr{L}^{[\otimes n \cdot m]}$,

$$c_1(\mathscr{L}^{[\otimes n \cdot m]}) \in H^1\big(V, \Omega_V^1\big),$$

intersects all curves trivially, as this will contradict negativity of the contracted extremal ray. To this end, write $\widetilde{V} := \pi^{-1}(V)$ and consider the following commutative diagram,

$$(5.7.1) \quad
\begin{array}{ccccccc}
\mathrm{Pic}(V) & \xrightarrow{\ c_1\ } & H^1\big(V, \Omega_V^1\big) & \longrightarrow & H^1\big(V, \Omega_V^{[1]}\big) & \xrightarrow{\ \alpha\ } & H^1\big(V_{\mathrm{reg}}, \Omega^1_{V_{\mathrm{reg}}}\big) \\
{\scriptstyle \pi^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{d}\,\pi} & & \big\downarrow{\scriptstyle \beta} & & \\
\mathrm{Pic}(\widetilde{V}) & \xrightarrow{\ c_1\ } & H^1\big(\widetilde{V}, \Omega^1_{\widetilde{V}}\big) & =\!=\!= & H^1\big(\widetilde{V}, \Omega^1_{\widetilde{V}}\big) & &
\end{array}$$

where β is induced by the pull-back map for reflexive differentials, and where c_1 maps a bundle with transition functions f_{ij} to $\{d\log f_{ij}\}_{i,j}$. We have seen in Proposition 4.3 that the image of $c_1(\mathscr{L}^{[\otimes n \cdot m]})$ in $H^1\big(V_{\mathrm{reg}}, \Omega^1_{V_{\mathrm{reg}}}\big)$ vanishes. As $\Omega^{[1]}_X$ is reflexive, the morphism α is injective by [ST71, Th. 1.14] and [BS76, Cor. II.3.15]; see also [BS76, Th. II.3.6]. By commutativity of (5.7.1), the Chern class $c_1(\pi^*\mathscr{L}^{[\otimes n \cdot m]})$ of the pullback via π vanishes. The claim follows from the projection formula for intersection numbers. $\qquad\square$

STEP 3: KODAIRA DIMENSION. — We aim to show that the minimal variety X is quasi-Abelian, which implies in particular that $\kappa(X) = 0$. As a first step, we show that X is at least not of general type.

CLAIM 5.8. — *The variety X is not of general type.*

Proof of Claim 5.8. — Suppose to the contrary that X *is* of general type. Using that X is also minimal, recall from [GKPT19a, Th. 1.1] that the Chern classes of X satisfy a $\mathbb{Q}$-Miyaoka-Yau inequality,

$$(5.8.1) \qquad \frac{n}{2(n+1)} \cdot [K_X]^n = \frac{n}{2(n+1)} \cdot \widehat{c}_1\big(\Omega^{[1]}_X\big)^2 \cdot [K_X]^{n-2} \leqslant \widehat{c}_2\big(\Omega^{[1]}_X\big) \cdot [K_X]^{n-2}.$$

On the other hand, we assume that Equation (1.2.1) holds. An elementary computation, using [Kob87, (1.14) on p. 34] and [GKPT19a, §3.8] shows that Equation (1.2.1) is equivalent to the assertion that the semistable sheaf $\mathscr{E}nd(\Omega^{[1]}_X)$ has vanishing $\mathbb{Q}$-Chern classes with respect to H, in the sense of [GKPT20, Defn. 6.1]. But then we have seen in [GKPT20, Fact and Defn. 6.5] that $\mathscr{E}nd(\Omega^{[1]}_X)$ has vanishing $\mathbb{Q}$-Chern classes with respect any ample bundle, and hence also with respect to any nef bundle. We obtain that

$$(5.8.2) \qquad \frac{n-1}{2n} \cdot [K_X]^n = \frac{n-1}{2n} \cdot \widehat{c}_1\big(\Omega^{[1]}_X\big)^2 \cdot [K_X]^{n-2} = \widehat{c}_2\big(\Omega^{[1]}_X\big) \cdot [K_X]^{n-2}.$$

But we know that K_X is big and nef, so that $[K_X]^n > 0$. Putting (in)equalities (5.8.1) and (5.8.2) together hence produces a contradiction. $\qquad\square$ (Claim 5.8)

STEP 4: THE SHAFAREVICH MAP. — Recall Notation 5.5 and consider the representation $\tau : \pi_1(X) \to \mathbb{P}\mathrm{GL}(n, \mathbb{C})$ that defines the projectively flat structure on $\mathbb{P}$. Write

$$G_X = \overline{\mathrm{img}(\tau)} \subseteq \mathbb{P}\mathrm{GL}(n, \mathbb{C})$$

for the algebraic Zariski closure of its image. This is a linear algebraic group, which has finitely many components. Applying Selberg's Lemma[2] and passing to an appropriate étale cover of X, we may hence assume without loss of generality that the following holds.

ASSUMPTION W.L.O.G. 5.9 (Connectivity and torsion-freeness). — The linear algebraic group G_X is connected. Writing $\mathrm{Rad}(G_X) \lhd G_X$ for the solvable radical of G_X,

---

[2]see [Alp87] for a brief account of this classic result.

the image of the composed group homomorphism to the linear algebraic group $G_X/\mathrm{Rad}(G_X)$ is torsion-free,

$$(5.9.1) \qquad\qquad \pi_1(X) \xrightarrow{\ \tau\ } G_X \longrightarrow {}^{G_X}\!/_{\mathrm{Rad}(G_X)}.$$

If G_X reduces to a point, we have $\Omega_X^{[1]} \cong \mathscr{L}^{\oplus n}$ for a suitable Weil-divisorial sheaf $\mathscr{L}$. We may then apply Theorem 4.4 to conclude that X is quasi-Abelian. As a consequence, we may assume without loss of generality that the following holds.

Assumption w.l.o.g. 5.10 (Representation has infinite image). — The group G_X is of positive dimension.

We now follow the strategy of [JR13] and use Assumption 5.10 to consider the Shafarevich map for the solvable radical of G_X. We refer the reader to [Kol95, §3] for more on Shafarevich maps.

Construction 5.11. — Consider the composed morphism (5.9.1) of Assumption 5.9, let $K \lhd \pi_1(X)$ be its kernel and consider an associated K-Shafarevich map, $\mathrm{sha}^K(X) : X \dashrightarrow \mathrm{Sha}^K(X)$, where $\mathrm{sha}^K(X)$ is dominant and $\mathrm{Sha}^K(X)$ is a smooth, projective variety.

The following two properties of $\mathrm{sha}^K(X)$ will be most relevant in the sequel.

Fact 5.12 (Shafarevich maps are almost holomorphic, [Kol95, Th. 3.6])
The rational map $\mathrm{sha}^K(X)$ is almost holomorphic. In other words, there exists a Zariski open set $X^\circ \subseteq X$ such that $\mathrm{sha}^K(X)|_{X^\circ}$ is well-defined and proper. The fibres of $\mathrm{sha}^K(X)|_{X^\circ}$ are connected. □

Fact 5.13 (Shafarevich maps and fundamental groups, [Kol95, Th. 3.6])
Let $x \in X^\circ$ be a very general point and let $Z \subset X$ be a subvariety through x, with normalisation $\eta : \widetilde{Z} \to Z$. Then, the rational map $\mathrm{sha}^K(X)$ maps Z to a point if and only if the composed morphism

$$\pi_1(\widetilde{Z}) \xrightarrow{\ \eta_*\ } \pi_1(X) \xrightarrow{\ \tau\ } G_X \longrightarrow {}^{G_X}\!/_{\mathrm{Rad}(G_X)}$$

has finite image. □

Claim 5.14 (The base of the Shafarevich map). — *The (smooth) variety $\mathrm{Sha}^K(X)$ is of general type.*

Proof of Claim 5.14. — Choose a resolution $\pi : \widetilde{X} \to X$, recall from [Tak03, Th. 1.1] that the natural push-forward map $\pi_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ is isomorphic and observe that the composed map $\mathrm{sha}^K(X) \circ \pi : \widetilde{X} \dashrightarrow \mathrm{Sha}^K(X)$ is therefore a Shafarevich map for $K \lhd \pi_1(\widetilde{X})$. Using the assumption that the image of the composed morphism (5.9.1) is torsion-free, the claim then follows from [CCE15, Th. 1]. □ (Claim 5.14)

Claims 5.8 and 5.14 together imply in particular that the fibres of $\mathrm{sha}^K(X)$ are positive-dimensional. These will be investigated next.

Step 5: Fibres of the Shafarevich map. — Throughout the present step, choose a general fibre $F \subset X$ of $\mathrm{sha}^K(X)$. There are two cases to consider: either $\mathrm{Sha}^K(X)$ is a point and $F = X$ is potentially singular, or F is a proper subvariety of X, and then it avoids the (finitely many!) singularities of X, cf. Consequence 5.4, and must hence be smooth. Either way, we will show in this step that F is quasi-Abelian. The proof is somewhat long and therefore divided into sub-steps.

Notation 5.15. — Given an étale cover $\gamma : \widehat{F} \to F$, consider the composed group morphism

(5.15.1) $$\pi_1\big(\widehat{F}\big) \longrightarrow \pi_1(F) \longrightarrow \pi_1(X) \overset{\tau}{\longrightarrow} \mathbb{PGL}(n, \mathbb{C})$$

and let $G_\gamma \subseteq \mathbb{PGL}(n, \mathbb{C})$ be the Zariski closure of its image.

Remark 5.16. — Given an étale cover $\gamma : \widehat{F} \to F$, the group G_γ is an algebraic subgroup of $\mathbb{PGL}(n, \mathbb{C})$ with finitely many connected components. Fact 5.13 implies that the maximal connected subgroup of G_γ is solvable.

Step 5-1: Irregularity. — To begin our analysis of the fibres of the Shafarevich map, we show that the fibres have positive irregularity, at least once we pass to a suitable étale cover. This will later allow to analyse the fibres by means of their Albanese map.

Claim 5.17. — *There exists an étale cover $\gamma : \widehat{F} \to F$ such that $h^0\big(\widehat{F}, \Omega_{\widehat{F}}^{[1]}\big) > 0$.*

Proof of Claim 5.17. — Passing to a first étale cover of F, we may assume without loss of generality that G_γ is connected. If the composed map (5.15.1) has infinite image, then by Remark 5.16 this defines a solvable representation of $\pi_1\big(\widehat{F}\big)$ with infinite image. It follows that $H^1\big(\widehat{F}, \mathbb{Z}\big)$ has positive rank and consequently, that $\dim_{\mathbb{C}} H^1\big(\widehat{F}, \mathbb{C}\big) > 0$. The description of the natural Hodge structure on this space, [Sch16, Th. 1], then implies that $h^0\big(\widehat{F}, \Omega_{\widehat{F}}^{[1]}\big) > 0$, as desired.

It therefore remains to consider the case where the composed map (5.15.1) has finite image. By Assumption 5.10, this implies that F is a proper subvariety of X, entirely contained in X_{reg}. We may then choose γ such that the composed map (5.15.1) is trivial, implying as before that the pull-back of the projectively flat sheaf $\Omega_X^{[1]}$ is a direct sum of the form, $\gamma^* \Omega_X^1 \cong \mathscr{L}^{\oplus n}$, for some $\mathscr{L} \in \mathrm{Pic}(\widehat{F})$. The pull-back of the conormal bundle sequence, which is non-trivial for reasons of dimension, will then read as follows,

$$0 \longrightarrow \underbrace{\gamma^* N_{F/X}^*}_{\text{trivial bundle}} \longrightarrow \underbrace{\gamma^* \Omega_X^1}_{\cong \mathscr{L}^{\oplus n}} \longrightarrow \Omega_{\widehat{F}}^1 \longrightarrow 0.$$

Hence $H^0\big(\widehat{F}, \mathscr{L}\big) \neq \{0\}$, and consequently $H^0\big(\widehat{F}, \Omega_{\widehat{F}}^1\big) \neq \{0\}$.   □ (Claim 5.17)

Notation 5.18. — For the remainder of this proof, fix one particular Galois cover $\gamma : \widehat{F} \to F$ such that G_γ is connected, and where $h^0\big(\widehat{F}, \Omega_{\widehat{F}}^{[1]}\big) > 0$. Recalling Remark 5.16, we may apply Proposition 3.3 and using its notation we write

(5.18.1) $$\gamma^* \Omega_X^{[1]} \cong \mathscr{F} \otimes \mathscr{A},$$

where $\mathscr{A}$ is Weil divisorial with numerical class $[\mathscr{A}] \equiv [\gamma^* K_X]$, and where $\mathscr{F}$ is locally free and admits a filtration

$$(5.18.2) \qquad 0 = \mathscr{F}_0 \subsetneqq \mathscr{F}_1 \subsetneqq \cdots \subsetneqq \mathscr{F}_n = \mathscr{F}$$

such that all sheaves $\mathscr{F}_i$ are locally free and flat with $\mathrm{rank}(\mathscr{F}_i) = i$ and such that the sheaf $\mathscr{F}_1$ is trivial, $\mathscr{F}_1 \cong \mathscr{O}_X$. In particular, there exists an inclusion

$$(5.18.3) \qquad \mathscr{A} = \mathscr{A} \otimes \mathscr{F}_1 \hookrightarrow \gamma^* \Omega_X^{[1]}.$$

The pull-back of the normal bundle sequence (which is trivial if $F = X$) reads

$$(5.18.4) \qquad 0 \longrightarrow \underbrace{\gamma^* N_{F/X}^*}_{\text{trivial bundle}} \longrightarrow \underbrace{\gamma^* \Omega_X^{[1]}}_{=\mathscr{F} \otimes \mathscr{A}} \longrightarrow \Omega_{\widehat{F}}^{[1]} \longrightarrow 0.$$

Step 5-2: Numerical dimension and Kodaira dimension. — Claim 5.17 has fairly direct consequences for the Kodaira dimension of the fibres, which will turn out to be either zero or one, the second case disappearing eventually.

Claim 5.19 (Bounding the Kodaira dimension from below). — *Maintaining Notation 5.18, if $\pi : \widetilde{F} \to \widehat{F}$ is any resolution of singularities, then the Kodaira dimension of $\widetilde{F}$ is non-negative. In particular, it follows that $\kappa(\widehat{F}) \geqslant 0$.*

Proof of Claim 5.19. — The assumption that $h^0\big(\widehat{F}, \Omega_{\widehat{F}}^{[1]}\big) > 0$, together with the Isomorphism (5.18.1) and the Filtration (5.18.2) allows to find an index i such that

$$(5.19.1) \qquad h^0\big(\widehat{F}, \mathscr{A} \otimes \mathscr{F}_i \big/ \mathscr{F}_{i-1}\big) \neq 0.$$

Writing $\mathscr{L}_i := \mathscr{F}_i \big/ \mathscr{F}_{i-1}$ and recalling that $\mathscr{L}_i$ is invertible and flat, hence numerically trivial by Lemma 2.5, Equation (5.19.1) implies in particular that the sheaf

$$\mathscr{A}^{[n]} \otimes \mathscr{L}_i^{\otimes n} = \underbrace{\mathscr{A}^{[n]} \otimes \det \mathscr{F}}_{=\gamma^* \omega_X = \omega_{\widehat{F}}} \quad \otimes \quad \underbrace{(\det \mathscr{F})^* \otimes \mathscr{L}_i^{\otimes n}}_{=:\mathscr{L}', \text{ num. trivial by Rem. 3.4}}$$

also has a non-trivial section. In this setting, the extension theorem for differential forms, [GKKP11, Th. 4.3] yields a non-trivial morphism $\pi^{[*]}\omega_{\widehat{F}} \hookrightarrow \omega_{\widetilde{F}}$ and hence a non-trivial section of $\omega_{\widetilde{F}} \otimes \pi^* \mathscr{L}'$, where again $\pi^* \mathscr{L}' \in \mathrm{Pic}(\widetilde{F})$ is numerically trivial. The claim thus follows from the "numerical character of the effectivity of adjoint line bundles", [CKP12, Th. 0.1], applied to the smooth space $\widetilde{F}$. $\square$ (Claim 5.19)

Claim 5.20 (Bounding the numerical dimension from above). — *The numerical dimension of $\widehat{F}$ satisfies $\nu(\widehat{F}) \leqslant 1$.*

Proof of Claim 5.20. — To begin, recall from (5.18.1) that the (nef!) numerical classes $n \cdot [\mathscr{A}]$ and $[\gamma^* K_X] = [K_{\widehat{F}}]$ agree. It follows that $\nu(\widehat{F}) = \nu(\mathscr{A})$, and so it suffices to consider $\nu(\mathscr{A})$ and to show that $\nu(\mathscr{A}) \leqslant 1$. In other words, given any very ample divisor H on $\widehat{F}$, we need to show that

$$(5.20.1) \qquad [\mathscr{A}]^2 \cdot [H]^{m-2} = 0, \quad \text{where } m := \dim \widehat{F}.$$

Recall from (5.18.3) that there exists an embedding $\mathscr{A} \hookrightarrow \gamma^* \Omega_X^{[1]}$. Combined with Sequence (5.18.4), we obtain a sheaf morphism

$$(5.20.2) \qquad\qquad \mathscr{A} \longrightarrow \Omega_{\widehat{F}}^{[1]}.$$

If this morphism is constantly zero, then $\mathscr{A}$ maps into the trivial sheaf $\gamma^* N_{F/X}^*$, the nef sheaf $\mathscr{A}$ must thus be trivial, and the claim is shown. We will therefore continue this proof under the assumption that (5.20.2) is an embedding.

Next, choose a general tuple $(H_1, \ldots, H_{m-2}) \in |H| \times \cdots \times |H|$ and consider the associated complete intersection surface $S := H_1 \cap \cdots \cap H_{m-2}$. Then, S has klt singularities[3] and $\mathscr{A}|_S$ is reflexive. The kernel of the restriction morphism for reflexive differentials is

$$N_{S/\widehat{F}}^* = \ker\left(\Omega_{\widehat{F}}^{[1]}\big|_S \to \Omega_S^{[1]}\right) \cong \mathscr{O}_{\widehat{F}}(-H)^{\oplus(m-2)},$$

so in particular anti-ample. But since the nef sheaf $\mathscr{A}$ never maps to an anti-ample, we find that the composed sheaf morphism

$$\mathscr{A}|_S \longrightarrow \Omega_{\widehat{F}}^{[1]}\big|_S \longrightarrow \Omega_S^{[1]}$$

cannot vanish. The Bogomolov-Sommese vanishing theorem for the potentially singular surface S, [GKK10, Th. 8.3], implies that $\mathscr{A}|_S$ is not big. Since it is nef, we conclude that

$$0 = [\mathscr{A}|_S]^2 = [\mathscr{A}]^2 \cdot [H]^{m-2},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □ (Claim 5.20)

Claim 5.21 (Kodaira dimension and numerical dimension of F). — *The Kodaira-dimension and the numerical dimension of F satisfy the inequalities*

$$(5.21.1) \qquad\qquad 0 \leqslant \kappa(F) \leqslant \nu(F) \leqslant 1.$$

Proof of Claim 5.21. — Given that $\gamma : \widehat{F} \to F$ is étale, Claim 5.20 immediately implies that $\nu(F) \leqslant 1$. Since $K_F = K_X|_F$ is nef, [Kaw85b, Prop. 2.2] gives the two rightmost inequalities in (5.21.1), and so it remains to show that $\kappa(F) \geqslant 0$. But we already know from Claim 5.19 that $\kappa(\widehat{F}) \geqslant 0$, so that there exists a number $p \in \mathbb{N}$ and a non-trivial section $\sigma \in H^0\left(\widehat{F}, \omega_{\widehat{F}}^{\otimes p}\right)$. But then

$$\bigotimes_{g \in \mathrm{Gal}(\gamma)} g^* \sigma \in H^0\left(\widehat{F}, \omega_{\widehat{F}}^{\otimes p \cdot \# \mathrm{Gal}(\gamma)}\right)$$

is a non-trivial Galois-invariant pluri-form on $\widehat{F}$, which hence descends to a non-trivial pluri-form on F. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □ (Claim 5.21)

---

[3]The surface S will in fact almost always be smooth, except perhaps when $X = F$ and $\dim X = 2$.

Step 5-3: Fibres of the Albanese map. — We pointed out above that the Albanese map of F is not trivial, at least once we pass to a suitable cover. Using the fact that the Albanese yields a Shafarevich map for the commutator subgroup of the fundamental group, there is much that we can say about its fibres, which will eventually turn out to be abundant.
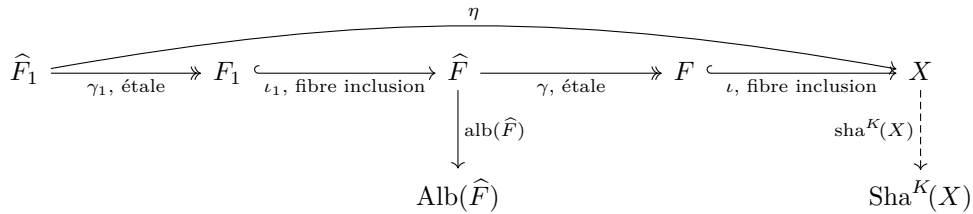
Construction 5.22. — Maintaining Notation 5.18, consider the Albanese map $\mathrm{alb}(\widehat{F}) : \widehat{F} \to \mathrm{Alb}(\widehat{F})$ and recall from Claim 5.17 and Notation 5.18 that this map is non-trivial, which is to say that $\dim \mathrm{Alb}(\widehat{F}) > 0$. Let $F_1 \subsetneq \widehat{F}$ be a general fibre component of $\mathrm{alb}(\widehat{F})$. This might be a point if $\widehat{F}$ is of maximal Albanese dimension. Since $\widehat{F}$ has at worst isolated singularities, F_1 is necessarily smooth. Now, we use the fact that Albanese maps are Shafarevich maps by recalling from [Kol95, §0.1.3] that the push-forward morphism of Abelianised fundamental groups,

$$(\iota_1)_*^{(\mathrm{ab})} : \pi_1(F_1)^{(\mathrm{ab})} \longrightarrow \pi_1\big(\widehat{F}\,\big)^{(\mathrm{ab})}$$

has finite image. We can thus find an étale cover $\gamma_1 : \widehat{F}_1 \to F_1$ such that the composed push-forward

$$(\iota_1 \circ \gamma_1)_*^{(\mathrm{ab})} : \pi_1(\widehat{F}_1)^{(\mathrm{ab})} \longrightarrow \pi_1\big(\widehat{F}\,\big)^{(\mathrm{ab})}$$

is trivial. The following diagram summarises the morphisms that we have discussed so far,

$$
\begin{array}{ccccccccc}
& & & & & \eta & & & \\
\widehat{F}_1 & \xrightarrow[\gamma_1,\ \text{étale}]{} & F_1 & \xhookrightarrow[\iota_1,\ \text{fibre inclusion}]{} & \widehat{F} & \xrightarrow[\gamma,\ \text{étale}]{} & F & \xhookrightarrow[\iota,\ \text{fibre inclusion}]{} & X \\
& & & & \downarrow{\scriptstyle \mathrm{alb}(\widehat{F})} & & & & \downarrow{\scriptstyle \mathrm{sha}^K(X)} \\
& & & & \mathrm{Alb}(\widehat{F}) & & & & \mathrm{Sha}^K(X)
\end{array}
$$

Observing that $\mathrm{img}(\eta) \subsetneq X_{\mathrm{reg}}$, we consider the exact sequence of differentials,

$$(5.22.1) \qquad 0 \longrightarrow \underbrace{\Omega^1_{\widehat{F}_1/X}}_{\text{trivial, non-zero}} \longrightarrow \eta^*\Omega^1_X \xrightarrow{\mathrm{d}\,\eta} \Omega^1_{\widehat{F}_1} \longrightarrow 0.$$

Claim 5.23 (Numerical invariants of F_1 and $\widehat{F}_1$). — *Assume the setting of Construction 5.22. Then, $0 \leqslant \kappa(F_1) \leqslant \nu(F_1) \leqslant 1$ and $0 \leqslant \kappa(\widehat{F}_1) \leqslant \nu(\widehat{F}_1) \leqslant 1$.*

Proof of Claim 5.23. — In order to lighten notation a little bit, we will suppress ι_1 in the following discussion. Given that the morphism γ_1 is étale, it suffices to consider the variety $\widehat{F}_1$ only. We will also assume that $\widehat{F}_1$ is not a point, or else there is little to show, cf. Claim 5.21. On the one hand, using that $\mathrm{img}(\eta) \subsetneq X_{\mathrm{reg}}$, we obtain a description of the canonical bundle,

$$\omega_{\widehat{F}_1} \overset{(5.22.1)}{\cong} \eta^*(\omega_X) \overset{(5.18.1)}{\cong} \gamma_1^*\big(\mathscr{A}^{[\otimes n]} \otimes \det \mathscr{F}\big).$$

Observing that $\gamma_1^* \mathscr{A}$ is invertible and that $\gamma_1^* \mathscr{F}$ is flat, this gives a numerical equivalence $[\omega_{\widehat{F}_1}] \equiv n \cdot [\gamma_1^* \mathscr{A}]$. On the other hand, (5.18.3) and (5.22.1) combine to give a morphism

$$\alpha : \gamma_1^* \mathscr{A} \longrightarrow \Omega^1_{\widehat{F}_1}.$$

There are two cases to consider. If the morphism α is not zero, Claim 5.23 follows from [JR13, Lem. 3.1]. Otherwise, if the morphism α is zero, we obtain an embedding of the nef line bundle $\gamma_1^* \mathscr{A}$ into the trivial sheaf $\Omega^1_{\widehat{F}_1/X}$. It follows that $\gamma_1^* \mathscr{A}$ and hence $\omega_{\widehat{F}_1}$ are numerically trivial, so $\nu(\widehat{F}_1) = 0$. By [Kaw85a, Th. 8.2], this implies $\kappa(\widehat{F}_1) = 0$. Either way, Claim 5.23 follows. □ (Claim 5.23)

CLAIM 5.24 (Semiampleness of $K_{\widehat{F}_1}$ and K_{F_1}). — *Assume the setting of Construction 5.22. Then, the canonical bundles $K_{\widehat{F}_1}$ and K_{F_1} are semiample.*

Proof of Claim 5.24. — As before, we consider the variety $\widehat{F}_1$ only and assume that it is not a point. We also consider the following commutative diagram of group morphisms,

$$
\begin{array}{ccccc}
\pi_1(\widehat{F}_1) & \xrightarrow{\;\tau \circ \eta_*\;} & G_X & \xhookrightarrow{\;\text{subgroup}\;} & \mathbb{PGL}(n, \mathbb{C}) \\
{\scriptstyle (\iota_1 \circ \gamma_1)_*} \Big\downarrow {\scriptstyle \text{trivial after Abelianisation}} & & \Big\uparrow {\scriptstyle \text{subgroup}} & & \\
\pi_1(\widehat{F}) & \xrightarrow{\;\tau \circ (\iota \circ \gamma)_*\;} & G_\gamma & &
\end{array}
$$

The representation $\tau \circ \eta_*$ is solvable because it factors via the solvable group G_γ. Better still, the representation $\tau \circ \eta_*$ is in fact trivial, or else there would be a non-trivial Abelian subrepresentation, contradicting triviality of the Abelianised push-forward $(\iota_1 \circ \gamma_1)_*^{(\text{ab})}$. As a consequence, we find that $\eta^* \Omega^1_X$ is isomorphic to a direct sum of line bundles,

$$\exists \, \mathscr{L} \in \operatorname{Pic}(\widehat{F}_1) : \eta^* \Omega^1_X \cong \mathscr{L}^{\oplus n}.$$

Recalling from Sequence (5.22.1) that $\eta^* \Omega^1_X$ contains the trivial, non-zero subbundle $\Omega^1_{\widehat{F}_1/X}$, we find that $\mathscr{L}$ admits a section. But then $\eta^* \Omega^1_X$ is generically generated, and so is its quotient $\Omega^1_{\widehat{F}_1}$. By [Fuj13, Rem. 2.2], the manifold $\widehat{F}_1$ is then of maximal Albanese dimension. But then [Fuj13, Th. 4.2] implies that the nef canonical bundle $K_{\widehat{F}_1}$ is semiample, as claimed. □ (Claim 5.24)

STEP 5-4: ABUNDANCE AND DESCRIPTION OF F. — In this step, we finally prove abundance for F and describe its Iitaka fibration. According to Claim 5.21, there are only two cases, $\kappa(F) = 0$ and $\kappa(F) = 1$, which we consider separately.

CLAIM 5.25. — *If $\kappa(F) = 0$, then F is quasi-Abelian.*

Proof of Claim 5.25. — To keep this proof readable, we consider three cases separately.

(5.25.1) The divisor K_F is numerically trivial and $F = X$.

(5.25.2) The divisor K_F is numerically trivial and $F \subsetneq X$.

(5.25.3) The divisor K_F is not numerically trivial.

Case 5.25.1 is easiest. If K_X is numerically trivial, then Equation (1.2.1) guarantees that

$$\widehat{c}_2\big(\Omega_X^{[1]}\big) \cdot [H]^{n-2} = 0.$$

Then, X is quasi-Abelian by [LT18, Th. 1.2] and we are done. In Case 5.25.2, apply Item 5.1.2 of Lemma 5.1 to the inclusion $F \subset X_{\mathrm{reg}}$ to find again that F is quasi-Abelian.

We consider Case 5.25.3 for the remainder of the proof, with the aim of producing a contradiction. To begin, we consider a strong log resolution of $\widehat{F}$ and the Stein factorisation of the associated Albanese morphism. We obtain a commutative diagram as follows

$$
\begin{array}{ccccc}
\widetilde{F} & \xrightarrow{\ a,\ \mathrm{conn.\ fibres}\ } & S & \xrightarrow{\ b,\ \mathrm{finite}\ } & \mathrm{Alb}(\widetilde{F}) \\
{\scriptstyle \pi,\ \mathrm{strong\ log\ res.}}\downarrow & & & & \downarrow{\scriptstyle \cong} \\
F \xleftarrow{\ \gamma,\ \mathrm{\acute{e}tale}\ } \widehat{F} & & \xrightarrow{\qquad\qquad \mathrm{alb}_{\widehat{F}} \qquad\qquad} & & \mathrm{Alb}(\widehat{F})
\end{array}
$$

By Claim 5.17, $\mathrm{Alb}(\widehat{F})$ is not a point. A general fibre $\widetilde{F}_1 \subsetneq \widetilde{F}$ of a is therefore smooth.

Observe that the image $F_1 = \pi(\widetilde{F}_1)$ is a connected component of a general fibre of $\mathrm{alb}_{\widehat{F}}$, which avoids the (finitely many) singularities of $\widehat{F}$. The restriction $\pi|_{\widetilde{F}_1} : \widetilde{F}_1 \to F_1$ is thus isomorphic, and Claim 5.24 applies to show that $\widetilde{F}_1$ is a good minimal model. More is true. Since $\kappa(F) = 0$ by assumption, we deduce from Claim 5.19 that $\kappa(\widetilde{F}) = 0$. Hence it follows from [Lai11, Th. 4.2] that $\widetilde{F}$ has a good minimal model $\widehat{F}_{\min}$, which has terminal singularities and numerically trivial canonical class.

We use the existence of $\widehat{F}_{\min}$ to describe $K_{\widehat{F}}$ in more detail. To this end, resolve the singularities of the rational map $\widehat{F}_{\min} \dashrightarrow \widehat{F}$ and obtain morphisms as follows,

$$
\begin{array}{ccccc}
\widehat{F}_{\min} & \xleftarrow[\mathrm{birational}]{\ \pi_2\ } & Y & \xrightarrow[\mathrm{birational}]{\ \pi_1\ } & \widehat{F} \\
{\scriptstyle \mathrm{alb}_{\widehat{F}_{\min}}}\downarrow & & \downarrow{\scriptstyle \mathrm{alb}_Y} & & \downarrow{\scriptstyle \mathrm{alb}_{\widehat{F}}} \\
\mathrm{Alb}(\widehat{F}_{\min}) & =\!=\!= & \mathrm{Alb}(Y) & =\!=\!= & \mathrm{Alb}(\widehat{F})
\end{array}
$$

The fact that $\widehat{F}_{\min}$ has terminal singularities implies that $K_Y \equiv \widehat{E}$, where $\widehat{E} \in \mathbb{Q}\,\mathrm{Div}(Y)$ is effective with π_2-exceptional support. Recalling that every fibre of π_2 is rationally connected by [HM07, Cor. 1.5], we find that every component of $\mathrm{supp}\,\widehat{E}$ is uniruled. The canonical class $K_{\widehat{F}}$ is then numerically equivalent to $E := (\pi_1)_*\widehat{E}$. The $\mathbb{Q}$-divisor E is effective, nef and not trivial (because $K_{\widehat{F}}$ is not numerically trivial), and its components are again uniruled. But the Albanese map $\mathrm{alb}_{\widehat{F}}$ contracts all rational curves in $\mathrm{supp}\,E$! We claim that the image set $\mathrm{alb}_{\widehat{F}}(\mathrm{supp}\,E)$ cannot contain any isolated points.

– If $\dim \mathrm{alb}_{\widehat{F}}(\widehat{F}) \geqslant 2$, this follows from nefness of $E \equiv K_{\widehat{F}} = \gamma^* K_X$ and from the Negativity Lemma 2.11.

– If $\dim \mathrm{alb}_{\widehat{F}}(\widehat{F}) = 1$, this follows from nefness of $E \equiv K_{\widehat{F}} = \gamma^* K_X$, from the assumption that $0 = \kappa(F) = \kappa(\widehat{F})$ and from the Negativity Lemma 2.12.

Either way, we find that E contains rational curves that stay away from the finitely many singular points of $\widehat{F}$. Since K_X is nef, this contradicts Item 5.1.1 of Lemma 5.1.
$\square$ (Claim 5.25)

Claim 5.26 (Description of F if $\kappa(F) = 1$). — *If $\kappa(F) = 1$, then K_F is semiample and the general fibre of the Iitaka fibration is quasi-Abelian.*

Proof of Claim 5.26. — If $\kappa(F) = 1$, then $\nu(F) = \kappa(F)$ by Claim 5.21. It follows that K_F is semiample, [KMM87, Cor. 6-1-13]. As F has isolated singularities at worst, the general fibre A of the associated Iitaka fibration $F \to B$ is smooth, contained in X_{reg}, and has numerical trivial canonical class. As before, use Item 5.1.2 of Lemma 5.1 to find that A is quasi-Abelian.
$\square$ (Claim 5.26)

Step 6: Abundance for X, description of the Iitaka fibration. — We have seen in the previous step that the general fibres of the Shafarevich map are abundant. As we will see now, this implies Abundance for X.

Claim 5.27 (Abundance for X). — *The canonical bundle K_X is semiample.*

Proof of Claim 5.27. — If $\mathrm{Sha}^K(X)$ is a point, then $F = X$ and we have seen in Claim 5.21 that $0 \leqslant \kappa(F) = \kappa(X) \leqslant 1$. We know that abundance holds because the two cases have been described in Claims 5.25 and 5.26, respectively. If $\mathrm{Sha}^K(X)$ is positive-dimensional, then we can apply Proposition 2.10 to the Shafarevich map.

– Assumption 2.10.1 is satisfied by Claim 5.14
– Assumption 2.10.2 is satisfied as nefness of K_X has been shown in Claim 5.7.
– Assumption 2.10.3 holds by Consequence 5.4.
– Assumption 2.10.4 has been shown in Claims 5.25 and 5.26, respectively.

The claim thus follows.
$\square$ (Claim 5.27)

Remark 5.28. — Claim 5.27 implies in particular that

– the Iitaka fibration has a positive-dimensional base by Assumption 5.6, and
– the fibres of the Iitaka fibration are quasi-Abelian by Item 5.1.2 of Lemma 5.1.

We denote the Iitaka fibration by $\mathrm{iit}(X) : X \to \mathrm{Iit}(X)$.

Remark 5.29 (Ambro's canonical bundle formula). — Ambro's canonical bundle formula for projective klt pairs, [Amb05, Th. 4.1] or [FG12, Th. 3.1], provides us with an effective $\mathbb{Q}$-divisor Δ on $\mathrm{Iit}(X)$ that makes the pair $\big(\mathrm{Iit}(X), \Delta\big)$ klt.

Step 7: Abelian group schemes. — Following the ideas of Jahnke-Radloff, [JR13], we use Kollár's characterisation of étale quotients of Abelian group schemes to show that the Iitaka fibration of a suitable étale cover of X is birational to an Abelian group scheme.

Claim 5.30. — *There exists an étale cover $\widehat{X} \to X$ whose Iitaka fibration is birational to an Abelian group scheme over a smooth projective base admitting a level three structure. More precisely, there exists a commutative diagram*

$$(5.30.1) \qquad \begin{array}{ccccc} A & \dashrightarrow^{\phi,\ birational} & \widehat{X} & \xrightarrow{\ étale\ } & X \\ {\scriptstyle\sigma}\uparrow\!\downarrow{\scriptstyle\alpha} & & {\scriptstyle iit(\widehat{X})}\downarrow & & \\ S & \xrightarrow[\psi,\ birational]{} & \mathrm{Iit}(\widehat{X}) & & \end{array}$$

where S is smooth and projective, and α is a family of polarised Abelian varieties admitting a level three structure.

Proof of Claim 5.30. — We construct from right to left a diagram as follows,

$$(5.30.2) \qquad \begin{array}{ccccccc} A & \xrightarrow{\ étale\ } & \breve{A} & \dashrightarrow^{birational} & \breve{X} & \xrightarrow{\varepsilon,\ étale} & X \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\breve{\alpha}}\downarrow & & {\scriptstyle\breve{a}}\downarrow & & \downarrow{\scriptstyle iit(X)} \\ S & \xrightarrow[\varepsilon_S,\ étale]{} & \breve{S} & \xrightarrow[\rho,\ birational]{} & \breve{Y} & \xrightarrow[\beta,\ finite]{} & \mathrm{Iit}(X) \end{array}$$

where $\breve{a}$ has connected fibres, where α and $\breve{\alpha}$ are Abelian group schemes and where α has the additional structure of a family of Abelian varieties with level three structure. To begin, we claim that the variety X has generically large fundamental group along the general fibre X_y of the Iitaka fibration. In case where X is smooth, this claim has been shown in [JR13, Prop. 5.1]. We leave it to the reader to check that in our case, where X has only finitely many singularities, the proof of [JR13, Prop. 5.1] still applies verbatim. Given that the fibres of the Iitaka fibration are quasi-Abelian, Proposition 2.15 will therefore apply to yield the right and middle square of Diagram (5.30.2) — strictly speaking, Proposition 2.15 gives ρ only as a rational map, but we can always blow up to make it a morphism, and then pull back $\breve{A}$. Once this is done, we can find a suitable étale cover $S \to \breve{S}$ such that the pull-back $A := \breve{A} \times_{\breve{S}} S$ is an Abelian group scheme with level three structure, and a result of Grothendieck allows to equip A/S with a polarisation, cf. [Ray70, Th. XI.1.4]. For later reference, we note that the following property holds by construction.

(5.30.3) If $s \in S$ is any general point, with image $y \in \mathrm{Iit}(X)$, then the induced map between fibres, $A_s \dashrightarrow X_y$ is birational.

We still need to construct $\widehat{X}$. We begin its construction with the observation that $K_X^{[m]} = iit(X)^*(H)$ for a suitable number $m \in \mathbb{N}^+$ and a suitable ample $H \in \mathrm{Div}\big(\mathrm{Iit}(X)\big)$. But then,

$$(5.30.4) \qquad K_{\breve{X}}^{[m]} = \varepsilon^* K_X^{[m]} = \varepsilon^*\, iit(X)^*(H) = \breve{a}^*(\beta^* H),$$

where $\beta^* H$ is ample on $\widehat{Y}$; this shows that $\breve{a}$ is the Iitaka fibration for $\breve{X}$ and allows to apply Ambro's canonical bundle formula, Remark 5.29, which shows that $\breve{Y}$ is the underlying space of a klt pair. We are interested in this, because [Tak03, Th. 1.1] then asserts that $\pi_1(\breve{S}) = \pi_1(\breve{Y})$. As a consequence, we find that the Stein factorisation of the morphism $S \to \breve{Y}$ factors via an étale cover of $\breve{Y}$ and therefore gives a diagram,

$$
\begin{array}{ccccc}
A & \dashrightarrow & \widehat{X} & \xrightarrow{\text{étale}} & \breve{X} \\
\alpha \downarrow & \phi & \downarrow & & \downarrow \breve{a} \\
S & \xrightarrow{\psi,\text{ birational}} & \widehat{Y} & \xrightarrow{\varepsilon_Y,\text{ étale}} & \breve{Y}
\end{array}
$$
$$
\rho \circ \varepsilon_S
$$

where $\widehat{X} := \breve{X} \times_{\breve{Y}} \widehat{Y}$ and where the map ϕ is induced by the universal property of the fibre product. Observe, exactly as in (5.30.4), that the natural projection map $\widehat{X} \to \widehat{Y}$ is the Iitaka fibration of $\widehat{X}$, $\mathrm{iit}(\widehat{X}) : \widehat{X} \to \mathrm{Iit}(\widehat{X}) = \widehat{Y}$. Assertion 5.30.3 implies that ϕ is birational. □ (Claim 5.30)

Claim 5.31. — *There exists an étale cover $\widehat{X} \to X$ whose Iitaka fibration is birational to an Abelian group scheme over $\mathrm{Iit}(\widehat{X})$ with level three structure. More precisely, there exists a commutative diagram*

(5.31.1)
$$
\begin{array}{ccccc}
A & \xrightarrow{\phi,\ birational} & \widehat{X} & \xrightarrow{\text{étale}} & X \\
\sigma \uparrow \downarrow \alpha & & \downarrow \mathrm{iit}(\widehat{X}) & & \\
\mathrm{Iit}(\widehat{X}) & =\!=\!=\!= & \mathrm{Iit}(\widehat{X}) & &
\end{array}
$$

where α is a family of polarised Abelian varieties with level three structure.

Proof of Claim 5.31. — Consider Diagram (5.30.1) and recall that polarised Abelian varieties with level three structure admit a fine moduli space $\mathscr{A}_3$ with universal family $\mathscr{U}_3 \to \mathscr{A}_3$, and that moreover $\mathscr{A}_3$ does not contain any rational curve [Kol93, Lem. 5.9.3]. On the other hand, while observing Remark 5.29 recall from [HM07, Cor. 1.5] that the fibres of the morphism ψ are rationally chain connected. It follows that the moduli map $S \to \mathscr{A}_3$ factors via ψ to give a morphism $\mathrm{Iit}(\widehat{X}) \to \mathscr{A}_3$. To conclude, set $A := \mathscr{U}_3 \times_{\mathscr{A}_3} \mathrm{Iit}(\widehat{X})$. □ (Claim 5.31)

Step 8: End of proof. — We end the proof by showing that $\widehat{X}$ itself is an Abelian group scheme over a smooth base with ample canonical bundle.

Claim 5.32. — *In the setting of Claim 5.31, the rational map $\delta := \phi^{-1} : \widehat{X} \dashrightarrow A$ is a morphism.*

Proof of Claim 5.32. — Given that $\widehat{X}$ is klt and that the fibres of α do not contain any rational curves, the claim follows from [HM07, Cor. 1.6]. □ (Claim 5.32)

Claim 5.33. — *The base variety $\mathrm{Iit}(\widehat{X})$ is smooth. In particular, A is smooth.*

Proof of Claim 5.33. — Assume not. Let $s \in \mathrm{Iit}(\widehat{X})_{\mathrm{sing}}$ be any singular point, and let $a \in \alpha^{-1}(s)$ be a general point of the fibre. Note that the morphism α is smooth, so a will be a singular point of A. On the other hand, note that the general choice of a guarantees that $\delta^{-1}(a)$ does not contain any of the (finitely many) singular points of X. It follows that δ resolves the singularity at a, and that the fibre $\delta^{-1}(a)$ is thus necessarily positive-dimensional.

There is more that we can say. Recalling from Remark 5.29 that $\big(\mathrm{Iit}(X), \Delta\big)$ is klt for a certain $\mathbb{Q}$-divisor Δ it follows from smoothness of α that $(A, \alpha^*\Delta)$ is klt as well. But then $\delta^{-1}(a) \subsetneq \mathrm{iit}(\widehat{X})^{-1}(s) \cap \widehat{X}_{\mathrm{reg}}$ is covered by rational curves, violating Item 5.1.1 of Lemma 5.1 from above. □ (Claim 5.33)

Claim 5.34. — *The morphism δ is an isomorphism and $\widehat{X}$ is therefore smooth.*

Proof of Claim 5.34. — If $a \in A$ is any (smooth!) point where δ is not isomorphic, then $\delta^{-1}(a)$ is positive-dimensional, and every section of $K_{\widehat{X}}$ will necessarily vanish along $\delta^{-1}(a)$. That contradicts the semiampleness of $K_{\widehat{X}}$, Claim 5.27. □ (Claim 5.34)

In the current, now nonsingular setup, the following has already been observed in [JR13, Proof of Lem. 5.1].

Claim 5.35. — *The canonical bundle of $\mathrm{Iit}(\widehat{X})$ is ample.*

Proof of Claim 5.35. — The smooth variety $\mathrm{Iit}(\widehat{X})$ is of general type, while the existence of σ and minimality of X and hence $\widehat{X}$ together imply that its canonical bundle is nef, hence abundant. Fibres of the Iitaka fibration for $\mathrm{Iit}(\widehat{X})$ are covered by rational curves, which do not exist by the existence of σ and the fact that $\widehat{X}$ (as an étale cover of X) does not contain rational curves by Item 5.1.1 of Lemma 5.1. □

Now that we know that $\widehat{X}$ is an Abelian group scheme over a smooth projective base with ample canonical bundle, the final step of the argument of Jahnke-Radloff, [JR13, Th. 6.1], applies to conclude also our proof. □ (Theorem 1.2)

6. Proof of Theorem 1.3

Step 1: Preparations. — In the setup of Theorem 1.3, we set

$$\mathscr{F} := \big(\mathrm{Sym}^n \Omega_X^1 \otimes \mathscr{O}_X(-K_X)\big)^{**}.$$

Let H be a very ample divisor on X and let $(D_1, \ldots, D_{n-1}) \in |H|^{\times(n-1)}$ be a general tuple of divisors, with associated general complete intersection curve $C := D_1 \cap \cdots \cap D_{n-1}$. The curve C is then smooth and entirely contained in X_{reg}. The restriction $\mathscr{F}|_C$ is locally free and nef, with $c_1(\mathscr{F}|_C) = 0$. It follows that $\mathscr{F}|_C$ is semistable and therefore that $\mathscr{F}$ is semistable with respect to H. But then, $\Omega_X^{[1]}$ will likewise be semistable with respect to H.

STEP 2: LOCAL FREENESS OF $\mathscr{F}$. — Next, choose a resolution of singularities $\pi : \widetilde{X} \to X$ such that the quotient sheaf

$$\widetilde{\mathscr{F}} := \pi^*(\mathscr{F}) \big/ \mathrm{tor}$$

is locally free; such a resolution exists by [Ros68, Th. 3.5]. Recall from Fact 2.2 that since $\mathscr{F}$ is nef, then so are $\pi^*(\mathscr{F})$, $\widetilde{\mathscr{F}}$ and $\det \widetilde{\mathscr{F}}$. There is more that we can say. Since the determinant of $\mathscr{F}$ is trivial by construction, $\det \mathscr{F} \cong \mathscr{O}_X$, we may write

$$\det \widetilde{\mathscr{F}} = \mathscr{O}_{\widetilde{X}}\Big(\sum a_i \cdot E_i\Big) \quad \text{where } E_i \text{ are } \pi\text{-exceptional and } a_i \in \mathbb{Z}.$$

The divisor $\sum a_i \cdot E_i$ is nef. But then the Negativity Lemma, [KM98, Lem. 3.39(1)], asserts that $a_i \leqslant 0$ for all i, which of course means that all $a_i = 0$. It follows that $\det \widetilde{\mathscr{F}} = \mathscr{O}_{\widetilde{X}}$ and thus that $\widetilde{\mathscr{F}}$ is numerically flat. This has two consequences.

– First, it follows from the descent theorem for vector bundles on resolutions of klt spaces, [GKPT19b, Th. 1.2], that $\mathscr{F}$ is locally free, numerically flat, and that $\widetilde{\mathscr{F}} = \pi^*(\mathscr{F})$.

– Second, it follows from [GKP20, Prop. 3.7] that $\Omega^1_{X_{\mathrm{reg}}}$ is projectively flat.

STEP 3: SINGULARITIES OF X. — If $\widehat{X} \to X$ is any maximally quasi-étale cover, then local freeness of $\mathscr{F}$ immediately implies local freeness and nefness for the reflexive normalised cotangent sheaf of $\widehat{X}$. The covering space $\widehat{X}$ therefore reproduces the assumptions of Theorem 1.3, which allows us to assume without loss of generality that X is itself maximally quasi-étale. Together with projective flatness of $\Omega^1_{X_{\mathrm{reg}}}$, this assumption allows to apply the local description of projectively flat sheaves found in [GKP20, Prop. 3.11]: every singular point $x \in X^{(\mathrm{an})}$ admit a neighbourhood U, open in the analytic topology, such that

$$\Omega^{[1]}_U \simeq \mathscr{L}_U^{\oplus n}$$

with some Weil-divisorial sheaf $\mathscr{L}_U$ on U. It follows from Proposition 4.1 that X has at worst isolated singularities.

STEP 4: END OF PROOF IN CASE WHERE X IS HIGHER-DIMENSIONAL. — If $n \geqslant 3$, consider the general complete intersection surface $S := D_1 \cap \cdots \cap D_{n-2}$. Using that X has isolated singularities, we find that S is smooth and contained in X_{reg}, so that $\Omega^1_X|_S$ is locally free, H-semistable and projectively flat. But by [Kob87, Prop. 3.1.b on p. 42] this implies

$$\frac{n-1}{2n} \cdot c_1\big(\Omega^1_X|_S\big)^2 = c_2\big(\Omega^1_X|_S\big)$$

and therefore

$$\frac{n-1}{2n} \cdot \widehat{c}_1(X)^2 \cdot [H]^{n-2} = \widehat{c}_2(X) \cdot [H]^{n-2}.$$

The claim thus follows from the semistability of Ω^1_X and Theorem 1.2.

Step 5: End of proof in case where X is a surface. — It remains to consider the case where $\dim X = 2$, so that X is a surface with klt quotient singularities. We have seen above that

$$\mathrm{Sym}^{[2]} \Omega_X^1 = \mathscr{F} \otimes \mathscr{O}_X(K_X),$$

where $\mathscr{F}$ is a locally free numerically flat sheaf of rank three. The second $\mathbb{Q}$-Chern class is therefore computed as

$$\widehat{c}_2\big(\mathrm{Sym}^{[2]} \Omega_X^{[1]}\big) = \widehat{c}_2\big(\mathscr{F} \otimes \mathscr{O}_X(K_X)\big) = \widehat{c}_2\big(\mathscr{O}_X(K_X)^{\oplus 3}\big) = 3 \cdot [K_X]^2.$$

On the other hand, the standard formula for the second Chern class of a symmetric product gives

$$\widehat{c}_2\big(\mathrm{Sym}^{[2]} \Omega_X^{[1]}\big) = 2 \cdot [K_X]^2 + 4 \cdot \widehat{c}_2(X).$$

Comparing these two equations, we find $\frac{1}{4} \cdot [K_X]^2 = \widehat{c}_2(X)$. Thanks to the semistability of $\Omega_X^{[1]}$, we may again apply Theorem 1.2 and end the proof. $\qquad\square$

References

[Akh95]   D. N. Akhiezer – *Lie group actions in complex analysis*, Aspects of Mathematics, vol. E27, Friedr. Vieweg & Sohn, Braunschweig, 1995.

[Alp87]   R. C. Alperin – "An elementary account of Selberg's lemma", *Enseign. Math. (2)* **33** (1987), no. 3-4, p. 269–273.

[Amb05]   F. Ambro – "The moduli *b*-divisor of an lc-trivial fibration", *Compositio Math.* **141** (2005), no. 2, p. 385–403.

[Anc82]   V. Ancona – "Faisceaux amples sur les espaces analytiques", *Trans. Amer. Math. Soc.* **274** (1982), no. 1, p. 89–100.

[BS76]   C. Bănică & O. Stănăşilă – *Algebraic methods in the global theory of complex spaces*, Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976.

[BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters & A. Van de Ven – *Compact complex surfaces*, second ed., Ergeb. Math. Grenzgeb. (3), vol. 4, Springer-Verlag, Berlin, 2004.

[BB70]   P. F. Baum & R. Bott – "On the zeros of meromorphic vector-fields", in *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, Springer, New York, 1970, p. 29–47.

[Bea83]   A. Beauville – "Variétés kähleriennes dont la première classe de Chern est nulle", *J. Differential Geom.* **18** (1983), no. 4, p. 755–782.

[Bea00]   _____, "Complex manifolds with split tangent bundle", in *Complex analysis and algebraic geometry*, de Gruyter, Berlin, 2000, p. 61–70.

[BC15]   C. Birkar & J. A. Chen – "Varieties fibred over abelian varieties with fibres of log general type", *Adv. Math.* **270** (2015), p. 206–222.

[Cam91]   F. Campana – "On twistor spaces of the class $\mathscr{C}$", *J. Differential Geom.* **33** (1991), no. 2, p. 541–549.

[CCE15]   F. Campana, B. Claudon & P. Eyssidieux – "Représentations linéaires des groupes kählériens: factorisations et conjecture de Shafarevich linéaire", *Compositio Math.* **151** (2015), no. 2, p. 351–376.

[CKP12]   F. Campana, V. Koziarz & M. Păun – "Numerical character of the effectivity of adjoint line bundles", *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 1, p. 107–119.

[Car57]   H. Cartan – "Quotient d'un espace analytique par un groupe d'automorphismes", in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, Princeton University Press, Princeton, NJ, 1957, p. 90–102.

[Deb01]   O. Debarre – *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[Dem02]   J.-P. Demailly – "On the Frobenius integrability of certain holomorphic *p*-forms", in *Complex geometry (Göttingen, 2000)*, Springer, Berlin, 2002, p. 93–98.

[Dru00]   S. Druel – "Variétés algébriques dont le fibré tangent est totalement décomposé", *J. reine angew. Math.* **522** (2000), p. 161–171.

[Dru14]   _____, "The Zariski-Lipman conjecture for log canonical spaces", *Bull. London Math. Soc.* **46** (2014), no. 4, p. 827–835.

[Fuj13]   O. Fujino – "On maximal Albanese dimensional varieties", *Proc. Japan Acad. Ser. A Math. Sci.* **89** (2013), no. 8, p. 92–95.

[FG12]   O. Fujino & Y. Gongyo – "On canonical bundle formulas and subadjunctions", *Michigan Math. J.* **61** (2012), no. 2, p. 255–264.

[FL81]   W. Fulton & R. Lazarsfeld – "Connectivity and its applications in algebraic geometry", in *Algebraic geometry (Chicago, Ill., 1980)*, Lect. Notes in Math., vol. 862, Springer, Berlin, 1981, p. 26–92.

[GKK10]   D. Greb, S. Kebekus & S. J. Kovács – "Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties", *Compositio Math.* **146** (2010), p. 193–219, A slightly extended version is available as `arXiv:0808.3647`.

[GKKP11]   D. Greb, S. Kebekus, S. J. Kovács & T. Peternell – "Differential forms on log canonical spaces", *Publ. Math. Inst. Hautes Études Sci.* **114** (2011), no. 1, p. 87–169, An extended version with additional graphics is available as `arXiv:1003.2913`.

[GKP16]   D. Greb, S. Kebekus & T. Peternell – "Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties", *Duke Math. J.* **165** (2016), no. 10, p. 1965–2004.

[GKP20]   _____, "Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor", 2020, `arXiv:2006.08769`.

[GKPT19a] D. Greb, S. Kebekus, T. Peternell & B. Taji – "The Miyaoka-Yau inequality and uniformisation of canonical models", *Ann. Sci. École Norm. Sup. (4)* **52** (2019), no. 6, p. 1487–1535.

[GKPT19b] _____, "Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles", *Compositio Math.* **155** (2019), no. 2, p. 289–323.

[GKPT20]   _____, "Harmonic metrics on Higgs sheaves and uniformization of varieties of general type", *Math. Ann.* **378** (2020), no. 3-4, p. 1061–1094.

[GKT18]   D. Greb, S. Kebekus & B. Taji – "Uniformisation of higher-dimensional varieties", in *Algebraic Geometry (Salt Lake City, 2015)* (T. de Fernex, B. Hassett, M. Mustaţă, M. Olsson, M. Popa & R. Thomas, eds.), Proc. Sympos. Pure Math., vol. 97, American Mathematical Society, Providence, RI, 2018, p. 277–308.

[Gro61]   A. Grothendieck – "Techniques de construction en géométrie analytique. V. Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux", in *Séminaire Henri Cartan*, vol. 13, no. 1, Secrétariat mathématique, Paris, 1960–1961, Exp. no. 12, p. 1–15.

[HM07]   C. D. Hacon & J. McKernan – "On Shokurov's rational connectedness conjecture", *Duke Math. J.* **138** (2007), no. 1, p. 119–136.

[Har77]   R. Hartshorne – *Algebraic geometry*, Graduate Texts in Math., vol. 52, Springer-Verlag, New York, 1977.

[HO84]   A. Huckleberry & E. Oeljeklaus – *Classification theorems for almost homogeneous spaces*, vol. 9, Université de Nancy, Institut Élie Cartan, Nancy, 1984.

[JR13]   P. Jahnke & I. Radloff – "Semistability of restricted tangent bundles and a question of I. Biswas", *Internat. J. Math.* **24** (2013), no. 1, article no. 1250122 (15 pages).

[Kaw85a]   Y. Kawamata – "Minimal models and the Kodaira dimension of algebraic fiber spaces", *J. reine angew. Math.* **363** (1985), p. 1–46.

[Kaw85b]   _____, "Pluricanonical systems on minimal algebraic varieties", *Invent. Math.* **79** (1985), no. 3, p. 567–588.

[KMM87]   Y. Kawamata, K. Matsuda & K. Matsuki – "Introduction to the minimal model problem", in *Algebraic geometry (Sendai, 1985)*, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, p. 283–360.

[KS21]   S. Kebekus & C. Schnell – "Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities", *J. Amer. Math. Soc.* **34** (2021), no. 2, p. 315—368.

[Kob87]   S. Kobayashi – *Differential geometry of complex vector bundles*, Publications of the
            Math. Society of Japan, vol. 15, Iwanami Shoten and Princeton University Press,
            Princeton, NJ, 1987.

[Kol93]   J. Kollár – "Shafarevich maps and plurigenera of algebraic varieties", *Invent. Math.*
            113 (1993), no. 1, p. 177–215.

[Kol95]   ———, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton
            University Press, Princeton, NJ, 1995.

[KM98]   J. Kollár & S. Mori – *Birational geometry of algebraic varieties*, Cambridge Tracts in
            Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[Lai11]   C.-J. Lai – "Varieties fibered by good minimal models", *Math. Ann.* **350** (2011), no. 3,
            p. 533–547.

[LT18]   S. Lu & B. Taji – "A characterization of finite quotients of abelian varieties", *Internat.
            Math. Res. Notices* (2018), no. 1, p. 292–319.

[Nak98]   N. Nakayama – "Normalized tautological divisors of semi-stable vector bundles", RIMS
            preprint 1214, available at `https://www.kurims.kyoto-u.ac.jp/preprint/preprint_`
            `y1998.html`, 1998.

[Nak04]   ———, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical
            Society of Japan, Tokyo, 2004.

[Ray70]   M. Raynaud – *Faisceaux amples sur les schémas en groupes et les espaces homogènes*,
            Lect. Notes in Math., vol. 119, Springer-Verlag, Berlin-New York, 1970.

[Ros68]   H. Rossi – "Picard variety of an isolated singular point", *Rice Univ. Studies* **54** (1968),
            no. 4, p. 63–73.

[Sch16]   M. Schwald – "Low degree Hodge theory for klt varieties", 2016, `arXiv:1612.01919`.

[ST71]   Y.-T. Siu & G. Trautmann – *Gap-sheaves and extension of coherent analytic subsheaves*,
            Lect. Notes in Math., vol. 172, Springer-Verlag, Berlin, 1971.

[Tak03]   S. Takayama – "Local simple connectedness of resolutions of log-terminal singularities",
            Internat. J. Math. **14** (2003), no. 8, p. 825–836.

[Vie82]   E. Viehweg – "Die Additivität der Kodaira Dimension für projektive Faserräume über
            Varietäten des allgemeinen Typs", *J. reine angew. Math.* **330** (1982), p. 132–142.

Daniel Greb, Essener Seminar für Algebraische Geometrie und Arithmetik, Fakultät für Mathematik,
Universität Duisburg–Essen
45117 Essen, Germany
E-mail : `daniel.greb@uni-due.de`
Url : `https://www.esaga.uni-due.de/daniel.greb`

Stefan Kebekus, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg
Ernst-Zermelo-Straße 1, 79104 Freiburg im Breisgau, Germany
& Freiburg Institute for Advanced Studies (FRIAS)
Freiburg im Breisgau, Germany
E-mail : `stefan.kebekus@math.uni-freiburg.de`
Url : `https://cplx.vm.uni-freiburg.de`

Thomas Peternell, Mathematisches Institut, Universität Bayreuth
95440 Bayreuth, Germany
E-mail : `thomas.peternell@uni-bayreuth.de`
Url : `https://www.komplexe-analysis.uni-bayreuth.de`