Journal de l'École polytechnique
Mathématiques

Shu Shen & Jianqing Yu

Morse-Smale flow, Milnor metric, and dynamical zeta function

Tome 8 (2021), p. 585-607.

<http://jep.centre-mersenne.org/item/JEP_2021__8__585_0>

Journal de l'École polytechnique
Mathématiques
Tome 8, 2021, p. 585–607 DOI: 10.5802/jep.154

MORSE-SMALE FLOW, MILNOR METRIC, AND DYNAMICAL ZETA FUNCTION

by Shu Shen & Jianqing Yu

Abstract. — We introduce a Milnor metric on the determinant line of the cohomology of the underlying closed manifold with coefficients in a flat vector bundle, by means of interactions between the fixed points and the closed orbits of a Morse-Smale flow. This enables us to generalize the notion of absolute value at the zero point of the Ruelle dynamical zeta function, even in the case where this value is not well-defined in the classical sense. We give a formula relating the Milnor metric and the Ray-Singer metric. An essential ingredient of our proof is Bismut-Zhang's theorem.

Résumé (Flot de Morse-Smale, métrique de Milnor et fonction zêta dynamique)
À l'aide des interactions entre les points fixes et les orbites fermées d'un flot de Morse-Smale, nous introduisons une métrique de Milnor sur le déterminant de la cohomologie de la variété fermée sous-jacente à valeurs dans un fibré vectoriel plat. Ceci permet de généraliser la notion de valeur absolue au point zéro de la fonction zêta dynamique de Ruelle, même dans le cas où cette valeur n'est pas bien définie au sens classique. Nous donnons une formule reliant les métriques de Milnor et de Ray-Singer. Un ingrédient essentiel de notre preuve est le théorème de Bismut-Zhang.

Contents

Mathematical subject classification (2020). — 37D15, 37C30, 58J52, 57Q10.
Keywords. — Index theory and related fixed point theorems, analytic torsion, Selberg trace formula, dynamical zeta functions.

This paper was written while J.Y. was visiting the Institut de Mathématiques de Jussieu-Paris Rive Gauche in Spring 2018. The hospitality of the Institute and the partial financial support from NSFC under Grant No. 11771411 are gratefully acknowledged.

Introduction

The study of the relation between the combinatorial/analytic torsion of a flat vector bundle and the Morse-Smale flow was initiated by Fried [Fri87] and Sánchez-Morgado [SM96]. In this paper, we give a formula relating

– a spectral invariant: the Ray-Singer metric associated with a flat vector bundle with a Hermitian metric on a closed Riemannian manifold;

– a dynamical invariant: the Milnor metric which reflects the interactions between the fixed points and the closed orbits of the Morse-Smale flow, and generalizes the absolute value at zero point of the Ruelle dynamical zeta function;

– a transgressed Euler class: the Mathai-Quillen current.

0.1. Background. — Let X be a connected closed smooth manifold of dimension m. Let (F,∇^F) be a complex flat vector bundle of rank r on X with flat connection ∇^F. Let $\rho:\pi_1(X)\to \mathrm{GL}_r(\mathbf{C})$ be the holonomy representation of the fundamental group $\pi_1(X)$. Denote by $H^\bullet(X,F)$ the cohomology of the sheaf of locally constant sections of F, and by $\lambda=\bigotimes_{i=0}^m(\det H^i(X,F))^{(-1)^i}$ the determinant line of $H^\bullet(X,F)$.

Assume that $H^\bullet(X,F)=0$ and that F is equipped with a flat metric, which is equivalent to say that its holonomy representation ρ is unitary. The Reidemeister torsion [Rei35, Fra35, dR50] is a positive real number defined by means of of a triangulation on X. However, it does not depend on the triangulation and becomes a topological invariant. It is the first invariant that could distinguish closed manifolds such as lens spaces which are homotopy equivalent but not homeomorphic.

The analytic torsion was introduced by Ray and Singer [RS71] as an analytic counterpart of the Reidemeister torsion. In order to define the analytic torsion one has to choose a Riemannian metric on X. The analytic torsion is a certain weighted alternating product of regularized determinants of the Hodge Laplacians acting on the space of differential forms with values in F.

The celebrated Cheeger-Müller theorem [Che79, Mül78] tells us that the Ray-Singer analytic torsion coincides with the Reidemeister combinatorial torsion. Bismut-Zhang [BZ92] and Müller [Mül93] simultaneously considered generalizations of this result. Müller [Mül93] extended his result to the case where F is unimodular, i.e., $|\det\rho(\gamma)|=1$ for all $\gamma\in\pi_1(X)$. Bismut and Zhang [BZ92, Th. 0.2] generalized the original Cheeger-Müller theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics. There are also various extensions to the equivariant case by Lott-Rothenberg [LR91], Lück [Lüc93], and Bismut-Zhang [BZ94], to the family case by Bismut-Goette [BG01], and to manifolds with boundaries by Brüning-Ma [BM13].

Let us explain Bismut-Zhang's theorem [BZ92, Th. 0.2] in more detail. Indeed, to formulate their result in the case where the flat vector bundle is not necessarily acyclic or unitarily flat, Bismut and Zhang introduced the so-called Ray-Singer metric, which is a metric on λ defined as the product of the analytic torsion with an L^2-metric on λ. Also they introduced the Milnor metric on λ which is a combinatorial metric associated with a Morse-Smale gradient flow. It generalizes the Reidemeister torsion to the case where F is neither acyclic nor unitarily flat. In this way, they were able to extend the Cheeger-Müller theorem to a comparison theorem of two metrics on λ, one is analytic and the other one is combinatorial.

The study of the relation between the combinatorial/analytic torsion and the dynamical system can be traced back to Milnor [Mil68]. Fried [Fri86] showed that on hyperbolic manifolds the analytic torsion of an acyclic unitarily flat vector bundle is equal to the value at zero point of the Ruelle dynamical zeta function of the geodesic flow. He conjectured [Fri87, p. 66 Conj.] that similar results should hold true for more general flows. In [She18], following previous contributions by Moscovici-Stanton [MS91], using Bismut's orbital integral formula [Bis11], the author affirmed the Fried conjecture for geodesic flows on closed locally symmetric manifolds. In [SY17], the authors made a further generalization to closed locally symmetric orbifolds.

Besides the gradient flow, Morse-Smale flow is the simplest structurally stable dynamical system which has only two types of recurrent behaviors: closed orbits and fixed points [Pal68, PS70]. Fried [Fri87, Th. 3.1] proved his conjecture for the Morse-Smale flows without fixed points. When compared with Bismut-Zhang's theorem [BZ92, Th. 0.2], it seems natural to ask whether there is a relation between the torsion invariant (or more generally the Ray-Singer metric for non acyclic and non unitarily flat vector bundle) and a general Morse-Smale flow which has both fixed points and closed orbits.

This is one of the motivations of Sánchez-Morgado's work [SM96]. He showed that the heteroclinic orbits have a non trivial contribution in the torsion invariant, and in this way he constructed a counterexample to Fried's conjecture on Seifert manifolds.

In this paper, we introduce a new Milnor metric, which indeed contains the heteroclinic contributions and generalizes the absolute value at zero point of the Ruelle dynamical zeta function, and we give a comparison theorem for the Milnor and Ray-Singer metrics on λ. We believe that in this way we give a complete answer in the affirmative to the above question.

Let us mention that there is another interpretation of the Ruelle dynamical zeta function provided by Dang-Rivière [DR20c]. See also [DR19, DR20a, DR20b, DR21] for related works.

0.2. A new Milnor metric.

— A vector field V is called Morse-Smale if V generates a flow whose nonwandering set is the union of a finite set A of hyperbolic fixed points and a finite set B of hyperbolic closed orbits, and if the stable and unstable manifolds of the critical elements in $A \coprod B$ intersect transversally.

Let us take a Hermitian metric g^F on F. In Section 2.4, we construct on λ a Milnor type metric $\|\cdot\|^{\mathrm{M},2}_{\lambda,V}$ using long exact sequences associated with a Smale filtration of the Morse-Smale flow. Note that the long exact sequences encode the information about the interactions between the critical elements in $A \coprod B$. If V is a negative gradient of a Morse function, then our Milnor metric is just the classical one as defined in [BZ92, Def. 1.9], which generalizes [Mil66].

Our first result says that the Milnor metric $\|\cdot\|^{\mathrm{M},2}_{\lambda,V}$ is a generalization of the absolute value at zero point of the Ruelle dynamical zeta function. For a closed orbit $\gamma\in B$, let $\ell_\gamma\in\mathbf{R}^*_+$ be its minimal period, and let $\mathrm{ind}(\gamma)\in\mathbf{N}$ be its index (see (2.3)). Take $\Delta(\gamma)$ to be 1 if γ is untwist and -1 in the contrary case (see (2.4)). The Ruelle dynamical zeta function is defined for $s\in\mathbf{C}$ by

$$R_\rho(s)=\prod_{\gamma\in B}\det(1-\Delta(\gamma)\rho(\gamma)e^{-s\ell_\gamma})^{(-1)^{\mathrm{ind}(\gamma)}}.$$

Proposition 0.1. — *If V does not have any fixed points, and if none of $\Delta(\gamma)$ is an eigenvalue of $\rho(\gamma)$, then $H^\bullet(X,F)=0$, and the norm of the canonical section $\mathbf{1}\in\mathbf{C}=\lambda$ is given by*

$$\|\mathbf{1}\|^{\mathrm{M}}_{\lambda,V}=|R_\rho(0)|^{-1}.$$

0.3. The main result of the paper. — Let g^{TX} be a metric on TX. Let $\psi(TX,\nabla^{TX})$ be the Mathai-Quillen current associated with the Levi-Civita connection ∇^{TX} (see Section 3.2). It is a current of degree $m-1$ defined on the total space of the tangent bundle TX, which takes values in $o(TX)$, the orientation line bundle of TX. Let $\|\cdot\|^{\mathrm{RS},2}_{\lambda}$ be the Ray-Singer metric on λ associated with (g^{TX},g^F) (see Section 3.3). Set $\theta(F,g^F)=\mathrm{Tr}[(g^F)^{-1}\nabla^F g^F]\in\Omega^1(X)$. Our main result is the following.

Theorem 0.2. — *We have*

$$\log\big(\|\cdot\|^{\mathrm{RS},2}_{\lambda}/\|\cdot\|^{\mathrm{M},2}_{\lambda,V}\big)=-\int_X\theta(F,g^F)(-V)^*\psi(TX,\nabla^{TX}).\tag{0.1}$$

If V does not have any closed orbits, Theorem 0.2 reduces to [BZ92, Th. 0.2]. Note also that if F is unitarily flat, then the right-hand side of (0.1) varnishes. Therefore, if V does not have any fixed points and if F is unitarily flat, by Proposition 0.1, our theorem corresponds to [Fri87, Th. 3.1].

Our proof of Theorem 0.2 is based on a result of Franks [Fra79, Prop. 5.1], who constructed a gradient flow by destroying the closed orbits of the Morse-Smale flow. In Section 2.5, we first establish a comparison formula between our Milnor metric associated with the original Morse-Smale flow and the classical one associated with Franks' gradient flow. In Section 3, to obtain Theorem 0.2, we apply Bismut-Zhang's formula [BZ92, Th. 0.2], which compares the Ray-Singer metric with the Milnor metric for Franks' gradient flow.

Recall that F is said to be unimodular, if its holonomy representation ρ is unimodular, i.e., $|\det\rho(\gamma)|=1$ for all $\gamma\in\pi_1(X)$. This is equivalent to the fact that there is a Hermitian metric g^F such that $\theta(F,g^F)=0$. By Theorem 0.2, we get

Corollary 0.3. — *If (F,g^F) is unimodular, then*

$$\|\cdot\|^{\mathrm{RS},2}_{\lambda}=\|\cdot\|^{\mathrm{M},2}_{\lambda,V}.$$

0.4. Organization of the paper. — In Section 1, we introduce some conventions on the determinant line, the cohomology of a circle, and also a long exact sequence associated with three manifolds $Y_1 \subset Y_2 \subset Y_3$. In Section 2, we recall some background on Morse-Smale flows. We also introduce the Milnor type metric, and we show Proposition 0.1. In Section 3, we recall the constructions of Mathai-Quillen current and Ray-Singer metric. We show our main result. We use the convention $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{R}_+^* = (0, \infty)$.

Acknowledgements. — We are indebted to Xiaolong Han and Xiaonan Ma for reading a preliminary version of this paper and for useful suggestions. S.S. would like to thank Nguyen Viet Dang and Gabriel Rivière for fruitful discussions on Morse-Smale flows.

1. Preliminaries

This section is organized as follows. In Section 1.1, we introduce our convention on the determinant line. In Section 1.2, we give a metric on the determinant line of the cohomology of $\mathbb{S}^1$. This is our model case near the closed orbits of a flow. In Section 1.3, we explain a long exact sequence associated with a triple of manifolds $Y_1 \subset Y_2 \subset Y_3$.

1.1. The determinant line. — Let W be a complex finite dimensional vector space. We denote by W^* the dual space. If $\dim W = 1$, we write $W^{-1} = W^*$. Take $\Lambda^\bullet(W) = \bigoplus_{j=0}^{\dim W} \Lambda^j(W)$ to be the exterior algebra. Set

$$\det W = \Lambda^{\dim W}(W).$$

Clearly, $\det W$ is a complex line. If $W = \{0\}$, then

$$\det W = \mathbf{C}.$$

Let $E^\bullet = \bigoplus_{i\in\mathbf{Z}} E^i$ be a finite dimensional $\mathbf{Z}$-graded space. Put

$$\det E^\bullet = \bigotimes_{i\in\mathbf{Z}} (\det E^i)^{(-1)^i}.$$

For $m \in \mathbf{N}$, let

$$(C^\bullet, d) : 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^m \longrightarrow 0$$

be a complex of finite dimensional vector spaces. By [KM76] or [BGS88, (1.5)], we have the canonical isomorphism of lines

$$\tau_{C^\bullet} : \det C^\bullet \simeq \det H^\bullet(C^\bullet, d). \tag{1.1}$$

If $s_j^k \in C^k$ such that $\{s_j^k\}_{j=1}^{k_j}$ projects to a basis of $C^k/\ker(d_{|C^k})$, if $\mu_j^k \in \ker(d_{|C^k})$ such that $\{\mu_j^k\}_{j=1}^{k'_j}$ projects to a basis of $H^k(C^\bullet, d)$, then

$$(\wedge_j s_j^0 \otimes \wedge_j \mu_j^0) \otimes (\wedge_j (ds_j^0) \otimes \wedge_j s_j^1 \otimes \wedge_j \mu_j^1)^{-1} \otimes \cdots \otimes (\wedge_j (ds_j^{m-1}) \otimes \wedge_j \mu_j^m)^{(-1)^m} \tag{1.2}$$

defines a canonical element of $\det C^\bullet$. If $\overline{\mu}_j^k$ denotes the image of μ_j^k in $H^k(C^\bullet, d)$, under (1.1), the element (1.2) maps to

$$(\wedge_j \overline{\mu}_j^0) \otimes (\wedge_j \overline{\mu}_j^1)^{-1} \otimes \cdots \otimes (\wedge_j \overline{\mu}_j^m)^{(-1)^m} \in \det H^\bullet(C^\bullet, d). \tag{1.3}$$

1.2. The cohomology of $\mathbb{S}^1$. — Let $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ be an oriented circle. Let F be a flat vector bundle of rank r on $\mathbb{S}^1$. Let $\rho : \pi_1(\mathbb{S}^1) \to \mathrm{GL}_r(\mathbf{C})$ be the holonomy[(1)] of F. Let $a_0 \in \pi_1(\mathbb{S}^1)$ be the generator of $\pi_1(\mathbb{S}^1)$, which is compatible with the orientation on $\mathbb{S}^1$. Set $A = \rho(a_0) \in \mathrm{GL}_r(\mathbf{C})$.

Consider the canonical triangulation on $\mathbb{S}^1$ induced by one 0-simplex σ_0 and one 1-simplex σ_1 as in Figure 1.1. It induces a complex of simplicial cochains with values

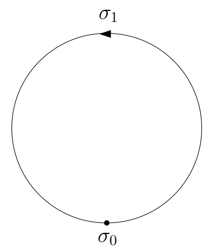

Figure 1.1. A triangulation on $\mathbb{S}^1$.

in F given by

$$(C^\bullet(\mathbb{S}^1, F), d) : 0 \longrightarrow \mathbf{C}^r \xrightarrow{\ A-1\ } \mathbf{C}^r \longrightarrow 0.$$

By (1.1), the canonical element $\mathbf{1} \in \mathbf{C} = \det C^\bullet(\mathbb{S}^1, F)$ defines an element

$$\sigma_A \in \det H^\bullet(\mathbb{S}^1, F). \tag{1.4}$$

We equip $\det H^\bullet(\mathbb{S}^1, F)$ with a metric $\|\cdot\|^2_{\det H^\bullet(\mathbb{S}^1,F)}$ such that

$$\|\sigma_A\|_{\det H^\bullet(\mathbb{S}^1,F)} = 1. \tag{1.5}$$

If 1 is not an eigenvalue of A, then $H^\bullet(\mathbb{S}^1, F) = 0$. By (1.2), the norm of the canonical element $\mathbf{1} \in \mathbf{C} = \det H^\bullet(\mathbb{S}^1, F)$ is given by

$$\|\mathbf{1}\|_{\det H^\bullet(\mathbb{S}^1,F)} = |\det(1-A)|^{-1}. \tag{1.6}$$

Remark 1.1. — Equation (1.6) is just Proposition 0.1 for the rotation flow on $\mathbb{S}^1$.

Remark 1.2. — Since the flat vector bundle is not necessarily unimodular, i.e., $|\det(A)|$ is not necessarily equal to 1, the choice of the orientation on $\mathbb{S}^1$ is very important.

---

(1) For any flat vector bundle F on a manifold X, the holonomy is a representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbf{C})$ of the fundamental group $\pi_1(X)$ such that $F = \pi_1(X) \backslash (\widetilde{X} \times \mathbf{C}^r)$, where $\widetilde{X}$ is the universal cover of X and $\pi_1(X)$ acts on the left on $\widetilde{X}$ by the deck transformation and on $\mathbf{C}^r$ by ρ.

1.3. A fusion principle. — Let $Y_1 \subset Y_2 \subset Y_3$ be three compact smooth manifolds with boundaries of the same dimension such that $Y_1 \subset \mathring{Y}_2$ and $Y_2 \subset \mathring{Y}_3$. Let F be a flat vector bundle on Y_3 of rank r. Denote by $H^\bullet(Y_3, Y_2, F)$, $H^\bullet(Y_3, F), \dots,$ the corresponding relative or absolute cohomologies with coefficients in F.

As in [BM13, (0.16)], we have a long exact sequence

$$
\begin{multline}
\tag{1.7} \cdots \longrightarrow H^i(Y_3, Y_2, F) \longrightarrow H^i(Y_3, Y_1, F) \longrightarrow H^i(Y_2, Y_1, F)\\
\longrightarrow H^{i+1}(Y_3, Y_2, F) \longrightarrow \cdots .
\end{multline}
$$

By (1.1) and (1.7), we get an isomorphism of lines

$$
\tag{1.8} f_{21,32} : \det H^\bullet(Y_2, Y_1, F) \otimes \det H^\bullet(Y_3, Y_2, F) \simeq \det H^\bullet(Y_3, Y_1, F).
$$

Using the other triples $(\varnothing, Y_1, Y_2)$ and $(\varnothing, Y_1, Y_3)$, we get similar isomorphisms

$$
\tag{1.9}
\begin{aligned}
f_{1,21} &: \det H^\bullet(Y_1, F) \otimes \det H^\bullet(Y_2, Y_1, F) \simeq \det H^\bullet(Y_2, F),\\
f_{1,31} &: \det H^\bullet(Y_1, F) \otimes \det H^\bullet(Y_3, Y_1, F) \simeq \det H^\bullet(Y_3, F).
\end{aligned}
$$

By (1.8) and (1.9), we see that $f_{2,32} \circ (f_{1,21} \otimes \mathrm{id})$ and $f_{1,31} \circ (\mathrm{id} \otimes f_{21,32})$ define two isomorphisms

$$
\tag{1.10} \det H^\bullet(Y_1, F) \otimes \det H^\bullet(Y_2, Y_1, F) \otimes \det H^\bullet(Y_3, Y_2, F) \simeq \det H^\bullet(Y_3, F).
$$

Proposition 1.3. — *There is $\mu = 1$ or -1[2] such that*

$$
\tag{1.11} f_{2,32} \circ (f_{1,21} \otimes \mathrm{id}) = \mu \times f_{1,31} \circ (\mathrm{id} \otimes f_{21,32}).
$$

Proof. — As in [BM13, (0.15)], let us take a smooth triangulation of Y_3 such that it induces also smooth triangulations on Y_1 and Y_2. Denote by

$$
(C^\bullet(Y_1, F), d),\ (C^\bullet(Y_2, Y_1, F), d), \dots,
$$

the complexes of simplicial cochains with coefficients in F. Then we have an exact sequence of complexes

$$
\tag{1.12} 0 \longrightarrow (C^\bullet(Y_2, Y_1, F), d) \longrightarrow (C^\bullet(Y_2, F), d) \longrightarrow (C^\bullet(Y_1, F), d) \longrightarrow 0.
$$

By (1.1) and (1.12), we get an isomorphism of lines

$$
f^C_{1,21} : \det C^\bullet(Y_1, F) \otimes \det C^\bullet(Y_2, Y_1, F) \longrightarrow \det C^\bullet(Y_2, F).
$$

We can define $f^C_{2,32}$, $f^C_{1,31}$ and $f^C_{21,32}$ in a similar way. By an easy calculation, there is $\mu = 1$ or -1 such that

$$
\tag{1.13} f^C_{2,32} \circ (f^C_{1,21} \otimes \mathrm{id}) = \mu \times f^C_{1,31} \circ (\mathrm{id} \otimes f^C_{21,32}).
$$

[2] See [BGS88, Rem. 1.2] or [KM76] for the detail about the sign.

By (1.2) and (1.3), there is $\mu = 1$ or -1 such that the diagram

$$\begin{tikzcd}
\det C^\bullet(Y_1,F)\otimes \det C^\bullet(Y_2,Y_1,F) \arrow[r, "f^C_{1,21}"] \arrow[d, "\tau_{C^\bullet(Y_1,F)}\otimes\tau_{C^\bullet(Y_2,Y_1,F)}"] & \det C^\bullet(Y_2,F) \arrow[d, "\mu\tau_{C^\bullet(Y_2,F)}"] \\
\det H^\bullet(Y_1,F)\otimes \det H^\bullet(Y_2,Y_1,F) \arrow[r, "f_{1,21}"] & \det H^\bullet(Y_2,F)
\end{tikzcd} \tag{1.14}$$

commutes. Tensoring each vertical line of (1.14) by the isomorphism

$$\tau_{C^\bullet(Y_3,Y_2,F)} : \det C^\bullet(Y_3,Y_2,F) \simeq \det H^\bullet(Y_3,Y_2,F),$$

and using (1.14) again for the pair (Y_2,Y_3), we find that there is $\mu = 1$ or -1 such that the diagram

(1.15)

$$\begin{tikzcd}
\det C^\bullet(Y_1,F)\otimes \det C^\bullet(Y_2,Y_1,F)\otimes \det C^\bullet(Y_3,Y_2,F) \arrow[r, "f^C_{2,32}\circ(f^C_{1,21}\otimes\mathrm{id})"] \arrow[d, "\tau_{C^\bullet(Y_1,F)}\otimes\tau_{C^\bullet(Y_2,Y_1,F)}\otimes\tau_{C^\bullet(Y_3,Y_2,F)}"] & \det C^\bullet(Y_3,F) \arrow[d, "\mu\tau_{C^\bullet(Y_3,F)}"] \\
\det H^\bullet(Y_1,F)\otimes \det H^\bullet(Y_2,Y_1,F)\otimes \det H^\bullet(Y_3,Y_2,F) \arrow[r, "f_{2,32}\circ(f_{1,21}\otimes\mathrm{id})"] & \det H^\bullet(Y_3,F)
\end{tikzcd}$$

commutes. In (1.15), if we replace the horizontal morphisms by $f^C_{1,31}\circ(\mathrm{id}\otimes f^C_{21,32})$ and $f_{1,31}\circ(\mathrm{id}\otimes f_{21,32})$, the corresponding diagram still commutes. By (1.13), we get (1.11). $\square$

2. Milnor metric

This section is organized as follows. In Sections 2.1 and 2.2, we recall the definitions of Morse-Smale flow and the associated Ruelle dynamical zeta function. In Section 2.3, we recall some results due to Franks [Fra79, Fra82] on the construction of a new gradient flow by destroying the closed orbits of the original Morse-Smale flow. In Section 2.4, using the Smale filtration, we introduce the Milnor metric. In Section 2.5, we establish a comparison formula for the two Milnor metrics, one is associated with the Morse-Smale flow and the other is associated with the gradient flow constructed by Franks.

We refer the reader to the classical textbook of Palis and de Melo [PdM82] for the basic notion on dynamical system.

2.1. Morse-Smale flow.

— Let X be a connected closed smooth manifold of dimension m. Let V be a vector field on X. Consider the differential equation

$$\frac{dx}{dt} = V(x). \tag{2.1}$$

Equation (2.1) defines a group of diffeomorphism $(\phi_t)_{t\in\mathbf{R}}$ of X.

If $x\in X$, an orbit of x is defined by the image $t\in\mathbf{R}\mapsto\phi_t(x)\in X$. We call $x\in X$ is a fixed point, if its orbit reduces to a point, i.e, for all $t\in\mathbf{R}$,

$$\phi_t(x) = x.$$

Clearly, $x \in X$ is a fixed point if and only if $V(x) = 0$. We call an orbit is closed if it is diffeomorphic to $\mathbb{S}^1$. Denote by A the set of fixed points and by B the set of closed orbits.

Definition 2.1. — A fixed point $x \in X$ of the flow $\phi.$ is called hyperbolic if there is a ϕ_t-invariant splitting

$$T_x X = E^{\mathrm{u}}_x \oplus E^{\mathrm{s}}_x,$$

and there exist $C > 0, \theta > 0$ and a Riemannian metric g^{TX} on X such that for $v \in E^{\mathrm{u}}_x$, $v' \in E^{\mathrm{s}}_x$, and $t > 0$, we have

$$|\phi_{-t,*} v| \leqslant C e^{-\theta t} |v|, \quad |\phi_{t,*} v'| \leqslant C e^{-\theta t} |v'|.$$

The unstable and stable manifolds of the hyperbolic fixed point x are defined by

$$W^{\mathrm{u}}_x = \Big\{ y \in X : \lim_{t \to -\infty} d_X(\phi_t(y), x) = 0 \Big\}, \quad W^{\mathrm{s}}_x = \Big\{ y \in X : \lim_{t \to \infty} d_X(\phi_t(y), x) = 0 \Big\},$$

where d_X denotes the Riemannian distance on (X, g^{TX}). The index $\mathrm{ind}(x) \in \mathbf{N}$ of x is defined by

$$\mathrm{ind}(x) = \dim E^{\mathrm{u}}_x. \tag{2.2}$$

Note that if $V = -\nabla f$ is a negative gradient of a Morse function f with respect to some Riemannian metric, then the index $\mathrm{ind}(x)$ of the critical point x is just the Morse index of f at x.

Definition 2.2. — A closed orbit γ of the flow $\phi.$ is called hyperbolic, if there is a ϕ_t-invariant continuous splitting

$$TX|_\gamma = \mathbf{R}V \oplus E^{\mathrm{u}}_\gamma \oplus E^{\mathrm{s}}_\gamma,$$

of C^0-vector bundles over γ such that (2.1) holds. The associated unstable and stable manifolds are defined by

$$\begin{aligned} W^{\mathrm{u}}_\gamma &= \bigcup_{x \in \gamma} \big\{ y \in X : \lim_{t \to -\infty} d_X(\phi_t(y), \phi_t(x)) = 0 \big\}, \\ W^{\mathrm{s}}_\gamma &= \bigcup_{x \in \gamma} \big\{ y \in X : \lim_{t \to +\infty} d_X(\phi_t(y), \phi_t(x)) = 0 \big\}. \end{aligned}$$

The index $\mathrm{ind}(\gamma) \in \mathbf{N}$ of γ is defined by

$$\mathrm{ind}(\gamma) = \mathrm{rk}\, E^{\mathrm{u}}_\gamma. \tag{2.3}$$

Definition 2.3. — The nonwandering set of $\phi.$ is defined by

$$\big\{ x \in X : \forall \text{ open neighborhood } U \text{ of } x, \forall\, T > 0,\ U \cap \textstyle\bigcup_{t \geqslant T} \phi_t(U) \neq \varnothing \big\}.$$

Clearly, $A \cup \bigcup_{\gamma \in B} \gamma$ is contained in the nonwandering set.

Definition 2.4. — A vector field V or a flow $\phi.$ is called Morse-Smale if

– the sets A and B are finite and contain only hyperbolic elements;
– the nonwandering set of $\phi.$ is equal to $A \cup \bigcup_{\gamma \in B} \gamma$;
– the stable and unstable manifold of any critical element in $A \coprod B$ intersect transversally.

In the sequel, we assume that V is a Morse-Smale vector field.

2.2. Ruelle dynamical zeta function. — For $\gamma \in B$, denote by $\ell_\gamma \in \mathbf{R}_+^*$ its minimal period. A closed orbit $\gamma \in B$ is called untwist if E_γ^{u} is orientable along γ, and is called twist otherwise. Put

$$\Delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is untwist,} \\ -1 & \text{if } \gamma \text{ is twist.} \end{cases} \tag{2.4}$$

Let $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbf{C})$ be a representation of the fundamental group of X. If $\gamma \in B$, denote by $\rho(\gamma)$ the holonomy along γ. Clearly, $\rho(\gamma)$ is well-defined up to a conjugation.

Definition 2.5. — The twist Ruelle dynamical zeta function is a meromorphic function on $\mathbf{C}$ defined for $s \in \mathbf{C}$ by

$$R_\rho(s) = \prod_{\gamma \in B} \det(1 - \Delta(\gamma)\rho(\gamma)e^{-s\ell_\gamma})^{(-1)^{\mathrm{ind}(\gamma)}}. \tag{2.5}$$

2.3. Franks' Morse function. — We follow [Fra79, §1]. Let $\mathbb{D}^r$ be the r-dimensional open unit ball of center $0 \in \mathbf{R}^r$. A fixed point $x \in A$ of index p is said to be of standard form if there is a system of coordinates $(y_1, \ldots, y_m) \in \mathbb{D}^m$ on a neighborhood of x such that x is represented by 0 and

$$V = \sum_{i=1}^{p} y_i \frac{\partial}{\partial y_i} - \sum_{i=p+1}^{m} y_i \frac{\partial}{\partial y_i}.$$

For closed orbits we must distinguish the following four cases in establishing the standard forms. Assume $\gamma \in B$ such that $\mathrm{ind}(\gamma) = p$.

Case 1. — Suppose that $TX|_\gamma$ is orientable and that γ is untwist. In this case, γ is said to be of standard form, if there is a system of coordinates $(t, y_1, \ldots, y_{m-1}) \in \mathbb{S}^1 \times \mathbb{D}^{m-1}$ on a tubular neighborhood U_γ of γ such that γ is represented by $(t, 0) \in \mathbb{S}^1 \times \mathbb{D}^{m-1}$ and

$$V = \frac{\partial}{\partial t} + \sum_{i=1}^{p} y_i \frac{\partial}{\partial y_i} - \sum_{i=p+1}^{m-1} y_i \frac{\partial}{\partial y_i}.$$

Case 2. — Suppose that $TX|_\gamma$ is orientable and that γ is twist. In this case, γ is said to be of standard form, if U_γ and V can be obtained from Case 1 by the identification

$$(t, x_1, \ldots, x_{m-1}) \sim (t + 1/2, -x_1, x_2, \ldots, x_p, -x_{p+1}, x_{p+2}, \ldots, x_{m-1}).$$

Case 3. — Suppose that $TX|_\gamma$ is not orientable and that γ is untwist. In this case, γ is said to be of standard form, if U_γ and V can be obtained from Case 1 by the identification

$$(t, x_1, \ldots, x_{m-1}) \sim (t + 1/2, x_1, \cdots, x_p, -x_{p+1}, x_{p+1}, \ldots, x_{m-1}).$$

Case 4. — Suppose that $TX|_\gamma$ is not orientable and that γ is twist. In this case, γ is said to be of standard form, if U_γ and V can be obtained from Case 1 by the identification

$$(t, x_1, \ldots, x_{m-1}) \sim (t + 1/2, -x_1, x_2, \cdots, x_{m-1}).$$

Note that in [Fra79, §1] the author assumed that X is orientable, so only the first two cases appear.

The following three propositions are [Fra79, Prop. 1.6, Th. 2.2, Prop. 5.1]. Their proofs can be generalized to the non orientable case with some evident modifications. We omit the details.

Proposition 2.6. — *For any Morse-Smale vector field V, there is a smooth family of Morse-Smale vector fields $(V_\ell)_{0\leqslant\ell\leqslant1}$ such that $V_0 = V$ and that the critical elements of V_1 are all of standard forms and are precisely the same as the critical elements of V. Moreover, V_0 and V_1 are topologically conjugated, i.e., there is a homeomorphism carrying the orbits of V_0 to those of V_1 and preserving their orientations.*

Remark 2.7. — The second part of Proposition 2.6 is a consequence of the Structural stability of the Morse-Smale flow [Pal68, PS70].

Remark 2.8. — Following the proof of [Fra79, Prop. 1.6] given by Franks, we can choose the family $(V_\ell)_{0\leqslant\ell\leqslant1}$ such that the critical elements are preserved under the deformation. However, in the proof of our main result Theorem 0.2 given in Section 3, we need only choose a family such that all the set of the fixed points of V_ℓ are in a small neighbourhood of the set of the fixed points of V.

The relation between the Morse-Smale flow and the Morse function is summarized in the following two propositions. The first one is due to Smale [Sma61, Th. B].

Proposition 2.9. — *If V is a Morse-Smale vector field whose flow has fixed points in standard form and no closed orbits, then V is a negative gradient of a certain Morse function with respect to some Riemannian metric.*

To state the following proposition, let us introduce some notation. For $x, y \in A$ such that $\operatorname{ind}(y) = \operatorname{ind}(x) - 1$, then $W^{\mathrm{u}}_x \cap W^{\mathrm{s}}_y$ consists of a finite set $\Gamma(x, y) = \{a_\bullet\}$ of integral curves of V such that $a_{-\infty} = x$ and $a_\infty = y$ along which W^{u}_x and W^{s}_y intersect transversally. Let us fix an orientation on each W^{u}_x with $x \in A$. Define $n(a) = \pm 1$ as in [BZ92, (1.28)], whose definition does not require the manifold to be orientable.

Proposition 2.10. — *For some small neighborhood $U = \bigcup_{\gamma\in B} U_\gamma$ of closed orbits $\bigcup_{\gamma\in B}\gamma$, there is a Morse function f on X whose gradient vector field ∇f with respect to a certain Riemannian metric is Morse-Smale, such that*

– *on $X \smallsetminus U$, we have*

$$-\nabla f = V, \tag{2.6}$$

– *on each U_γ, the Morse function f has only two critical points x_γ, x'_γ of index* $\operatorname{ind}(\gamma) + 1$ *and* $\operatorname{ind}(\gamma)$ *respectively.*

Also, $\Gamma(x_\gamma, x'_\gamma)$ *consists of two integral curves* a_γ, a'_γ *(see Figure* 2.1*) such that their composition* $a_\gamma \circ (a'_\gamma)^{-1}$ *and the closed orbit* γ *lie in the same freely homotopy class of loops on* X *and that*(3)

$$n(a_\gamma)n(a'_\gamma) = -\Delta(\gamma).$$

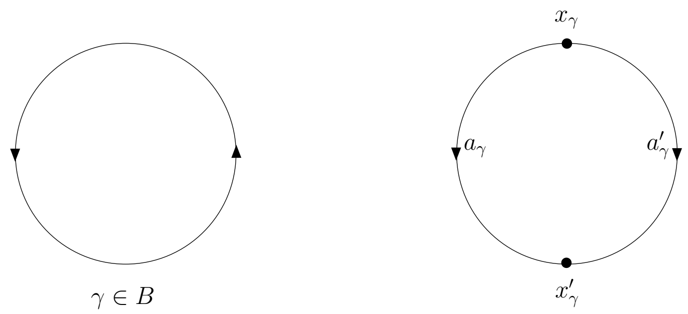

Figure 2.1. A closed orbit and integral curves

Remark 2.11. — We recall the essential step in the construction of the gradient ∇f given by Franks [Fra82, Prop. 8.5]. For simplicity, assume that there is only one closed orbit γ and it is of index p and in standard form of Case 1. For $\delta > 0$ small enough, let $\rho \in C_c^\infty(\mathbb{D}^{m-1}, [0,1])$ be a cutoff function, which is equal to 1 on $|y| < \delta$ and to 0 on $|y| > 2\delta$. Put

$$V_1 = \begin{cases} \big((1-\rho) + \rho \sin(2\pi t)\big)\dfrac{\partial}{\partial t} + \displaystyle\sum_{i=1}^{p} y_i \frac{\partial}{\partial y_i} - \sum_{i=p+1}^{m-1} y_i \frac{\partial}{\partial y_i}, & \text{in } U_\gamma, \\ V, & \text{outside } U_\gamma. \end{cases}$$

It is easy to see that the nonwandering set of V_1 in U_γ consists of two points $(1,0), (1/2,0) \in \mathbb{S}^1 \times \mathbb{D}^{m-1}$. Then, by a small perturbation on V_1, we get a Morse-Smale gradient vector field $-\nabla f$ which has the desired transversality and other properties.

We remark that by the above construction, we can find a family of vector fields $(V_\varepsilon)_{0 \leqslant \varepsilon \leqslant 1}$ connecting V and $-\nabla f$ such that near $\{1/4\} \times \mathbb{D}^{m-1}$, for any $\varepsilon \in [0,1]$, V_ε does not vanish. Similar remark holds for γ in standard forms of Cases 2–4. In Section 3.5, we will use this fact to simplify the proof of our main theorem.

2.4. Smale filtration and Milnor metric.

— Following [Fra82, Def. 9.10], let

$$\varnothing = X^0 \subset X^1 \subset \cdots \subset X^N = X \tag{2.7}$$

be a Smale filtration on X associated with V. Note that each $X^p \subset X$ is a submanifold with boundary, and can be constructed by the sublevel set of a smooth Lyapunov function. Also, we have

(3)This requires a choice of the orientations on the unstable manifolds of x_γ, x'_γ. Such choice is irrelevant.

– on each ∂X^p, V does not vanish and points toward the inside of X^p;
– there is only one critical element $c \in A \coprod B$ contained in $X^{p+1} \smallsetminus X^p$ and

$$\{c\} = \bigcap_{t \in \mathbf{R}} \phi_t(X^{p+1} \smallsetminus X^p).$$

Let (F, ∇^F) be a flat vector bundle on X induced by the representation ρ. Let $H^\bullet(X, F)$ be the cohomology of the sheaf of locally constant sections of F. Put

$$\lambda = \det H^\bullet(X, F). \tag{2.8}$$

We use the notation in Section 1.3. By (1.10), we get an isomorphism

$$\sigma_V : \bigotimes_{p=0}^{N-1} \det H^\bullet(X^{p+1}, X^p, F) \simeq \lambda. \tag{2.9}$$

By Proposition 1.3, up to a sign, the morphism σ_V does not depend on the way that the cohomologies are fused.

By [Fra82, Th. 9.11] (see also [SM96, §2]), if the critical element $c \in X^{p+1} \smallsetminus X^p$ is a fixed point, then

$$H^q(X^{p+1}, X^p, F) = \begin{cases} F_c, & q = \operatorname{ind}(c), \\ 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

and if the critical element $c \in X^{p+1} \smallsetminus X^p$ is a closed orbit, then

$$H^q(X^{p+1}, X^p, F) = \begin{cases} H^{q-\operatorname{ind}(c)}\big(c, o(E_c^{\mathrm{u}}) \otimes F|_c\big), & q - \operatorname{ind}(c) = 0 \text{ or } 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.11}$$

where $o(E_c^{\mathrm{u}})$ is the orientation line bundle of E_c^{u} along the closed orbit c.

We equip $\det H^\bullet(\gamma, o(E_\gamma^{\mathrm{u}}) \otimes F|_\gamma)$ with the metric $\|\cdot\|^2_{\det H^\bullet(\gamma, o(E_\gamma^{\mathrm{u}}) \otimes F|_\gamma)}$ defined in (1.5). Let g^F be a Hermitian metric on F. By (2.9)–(2.11), the restriction $g^F|_A$ and the metric $\|\cdot\|^2_{\det H^\bullet(\gamma, o(E_\gamma^{\mathrm{u}}) \otimes F|_\gamma)}$ induce a metric $\|\cdot\|^{\mathrm{M},2}_{\lambda,V}$ on λ. By Proposition 1.3, this metric does not depend on the way that the cohomologies are fused.

Definition 2.12. — The metric $\|\cdot\|^{\mathrm{M},2}_{\lambda,V}$ on λ is called the Milnor metric associated with V.

Remark 2.13. — If $V = -\nabla f$ is a negative gradient of a Morse function f, then $\|\cdot\|^{\mathrm{M},2}_{\lambda,V}$ coincides with the one constructed by Bismut-Zhang [BZ92, Def. 1.9]. In fact, there is a small difference with Bismut-Zhang's construction, where they used a filtration [BZ92, (1.37)] induced by sublevel sets of a nice Morse function. Using Proposition 1.3, we can deduce that the two constructions coincide.

Remark 2.14. — For two topologically conjugated Morse-Smale vector fields whose critical elements coincide, we can choose the same Smale filtration. From our construction, the corresponding Milnor metrics coincide.

Remark 2.15. — The Milnor metric for general Morse-Smale flow does not depend on the Smale filtration (2.7). We will not give a direct proof since it is a consequence of our main Theorem 0.2.

Let us restate and prove Proposition 0.1.

Proposition 2.16. — *If V does not have any fixed points, and if none of $\Delta(\gamma)$ is an eigenvalue of $\rho(\gamma)$, then $H^\bullet(X,F)=0$, and the norm of the canonical section $\mathbf{1}\in\mathbf{C}=\det H^\bullet(X,F)$ is given by*

$$\|\mathbf{1}\|^{\mathrm{M}}_{\lambda,V}=|R_\rho(0)|^{-1}. \tag{2.12}$$

Proof. — For $\gamma\in B$, the holonomy of $o(E^{\mathrm{u}}_\gamma)\otimes F|_\gamma$ along γ is $\Delta(\gamma)\rho(\gamma)$. By our assumption,

$$H^\bullet(\gamma,o(E^{\mathrm{u}}_\gamma)\otimes F_\gamma)=0. \tag{2.13}$$

By (1.7), (2.10), (2.11), and (2.13), we can deduce that $H^\bullet(X,F)=0$. By (1.6), (2.5), and (2.9), we get (2.12). $\square$

2.5. A comparison formula for Milnor metrics. — In this section, we assume that all the critical elements of V are in standard forms, and that f is chosen as in Proposition 2.10.

Let $\det\tau(a'_\gamma)\in\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}$ be the canonical element induced by the parallel transport with respect to the flat connection along the integral curve a'_γ (see Figure 2.1). Let $\|\cdot\|^2_{\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}}$ be the metric on $\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}$ induced by $g^F_{x_\gamma}$ and $g^F_{x'_\gamma}$.

Proposition 2.17. — *The following identity holds,*

$$\log\big(\|\cdot\|^{\mathrm{M},2}_{\lambda,-\nabla f}/\|\cdot\|^{\mathrm{M},2}_{\lambda,V}\big)=\sum_{\gamma\in B}(-1)^{\mathrm{ind}(\gamma)}\log\big\|\det\tau(a'_\gamma)\big\|^2_{\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}}. \tag{2.14}$$

Proof. — We refine the filtration (2.7) by the new critical points of f. By Propositions 1.3 and 2.10, and by (2.9), the following diagram commutes

$$\begin{tikzcd} \bigotimes_{x\in A}\big(\det F_x\big)^{(-1)^{\mathrm{ind}(x)}}\bigotimes_{\gamma\in B}\Big\{\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}\Big\}^{(-1)^{\mathrm{ind}(\gamma)}} \arrow[r, "\sigma_{-\nabla f}"] \arrow[d] & \lambda \arrow[d] \\ \bigotimes_{x\in A}\big(\det F_x\big)^{(-1)^{\mathrm{ind}(x)}}\bigotimes_{\gamma\in B}\Big\{\det H^\bullet(\gamma,o(E^{\mathrm{u}}_\gamma)\otimes F|_\gamma)\Big\}^{(-1)^{\mathrm{ind}(\gamma)}} \arrow[r, "\sigma_V"] & \lambda \end{tikzcd} \tag{2.15}$$

where the first vertical arrow is induced by (1.4) and the second vertical arrow is a multiplication by ±1. The Milnor metric $\|\cdot\|^{\mathrm{M},2}_{\lambda,-\nabla f}$ is obtained from the metric $g^F|_{A\cup\{x_\gamma,x'_\gamma:\gamma\in B\}}$ via $\sigma_{-\nabla f}$. By (2.15), we get (2.14). $\square$

3. An extension of Bismut-Zhang's theorem to Morse-Smale flow

This section is organized as follows. In Sections 3.1-3.4, following [BZ92], we recall the constructions of the Berezin integral, the Mathai-Quillen current, and the Ray-Singer metric. In Section 3.5, we restate and prove our main theorem.

3.1. Berezin integral. — Let E be a real Euclidean space of dimension n with the scalar product $\langle , \rangle$, and let W be a real vector space of finite dimension. We use the supersymmetric formalism of Quillen [Qui85]. Denote by $\widehat{\otimes}$ the tensor product of super algebras.

Suppose temporarily that E is oriented and that $e_1, \ldots, e_n$ is an oriented orthonormal basis of E. Let $e^1, \ldots, e^n$ be the corresponding dual basis of E^*. We define $\int^B$ to be the linear map from $\Lambda^\bullet(W^*) \widehat{\otimes} \Lambda^\bullet(E^*)$ into $\Lambda^\bullet(W^*)$, such that if $\alpha \in \Lambda^\bullet(W^*)$, $\beta \in \Lambda^\bullet(E^*)$,

$$\int^B \alpha\beta = 0, \quad \text{if } \deg \beta < n,$$
$$\int^B \alpha e^1 \wedge \cdots \wedge e^n = \frac{(-1)^{n(n+1)/2}}{\pi^{n/2}} \alpha.$$

More generally, if $o(E)$ is the orientation line of E, then $\int^B$ defines a linear map from $\Lambda^\bullet(W^*) \widehat{\otimes} \Lambda^\bullet(E^*)$ into $\Lambda^\bullet(W^*) \widehat{\otimes} o(E)$, which is called a Berezin integral.

Let $A \in \mathrm{End}^{\mathrm{anti}}(E)$ be an antisymmetric endomorphism of E. We identify A with

$$\dot{A} = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} \langle e_i, Ae_j \rangle e^i \wedge e^j \in \Lambda^2(E^*). \tag{3.1}$$

By definition, the Pfaffian $\mathrm{Pf}[A] \in o(E)$ of A is given by

$$\mathrm{Pf}[A/2\pi] = \int^B \exp(-\dot{A}/2).$$

Clearly, $\mathrm{Pf}[A]$ vanishes if n is odd.

3.2. Mathai-Quillen formalism. — Recall that X is a connected closed smooth manifold of dimension m. Let E be a Euclidean vector bundle of rank n on X with a Euclidean metric g^E and a metric connection ∇^E. Let

$$R^E = (\nabla^E)^2 \in \Omega^2(X, \mathrm{End}^{\mathrm{anti}}(E))$$

be its curvature. Denote by $o(E)$ the orientation line bundle of E. The Euler form of (E, ∇^E) is given by

$$e(E, \nabla^E) = \mathrm{Pf}[R^E/2\pi] \in \Omega^n(X, o(E)).$$

Clearly, $e(E, \nabla^E) = 0$, if n is odd.

Let $\mathscr{E}$ be the total space of E, and let $\pi : \mathscr{E} \to X$ be the natural projection. We will use the formalism of the Berezin integral developed in Section 3.1 with $W = T\mathscr{E}$. If ω is a smooth section of $\Lambda^\bullet(T^*\mathscr{E}) \otimes \pi^*\Lambda^\bullet(E^*)$ over $\mathscr{E}$, then $\int^B \omega$ is a smooth section of $\Lambda^\bullet(T^*\mathscr{E}) \otimes \pi^* o(E)$ over $\mathscr{E}$.

Let $T^V\mathscr{E} \subset T\mathscr{E}$ be the vertical subbundle of $T\mathscr{E}$, and let $T^H\mathscr{E} \subset T\mathscr{E}$ be the horizontal subbundle of $T\mathscr{E}$ with respect to ∇^E, so that

$$T\mathscr{E} = T^H\mathscr{E} \oplus T^V\mathscr{E}. \tag{3.2}$$

By the identification $T^V\mathscr{E}\simeq\pi^*E$, the vertical projection with respect to the splitting (3.2) induces a section $P^E\in C^\infty(\mathscr{E},T^*\mathscr{E}\otimes\pi^*E)$. Using the metric g^E, we identify P^E with $\dot{P}^E\in C^\infty(\mathscr{E},T^*\mathscr{E}\otimes\pi^*E^*)$. Let $Y\in C^\infty(\mathscr{E},\pi^*E)$ be the tautological section. Write $\widehat{Y}$ the corresponding section in $C^\infty(\mathscr{E},\pi^*E^*)$ induced by g^E. Recall that $\dot{R}^E\in C^\infty(X,\Lambda^2(T^*X)\otimes\Lambda^2(E^*))$ is defined in (3.1).

Definition 3.1. — For $T\geqslant 0$, set

$$A_T=\frac{1}{2}\,\pi^*\dot{R}^E+\sqrt{T}\,\dot{P}^E+T|Y|^2\in C^\infty\big(\mathscr{E},\Lambda^\bullet(T^*\mathscr{E})\otimes\pi^*\Lambda^\bullet(E^*)\big).$$

Let $(\alpha_T)_{T\geqslant 0}$ and $(\beta_T)_{T>0}$ be families of forms on $\mathscr{E}$ defined by

$$\begin{aligned}\alpha_T&=\int^B\exp(-A_T)\in\Omega^n(\mathscr{E},\pi^*o(E)),\\ \beta_T&=\int^B\frac{\widehat{Y}}{2\sqrt{T}}\exp(-A_T)\in\Omega^{n-1}(\mathscr{E},\pi^*o(E)).\end{aligned}\tag{3.3}$$

Clearly,

$$\alpha_0=\pi^*e(E,\nabla^E).$$

Let us recall [BZ92, Th. 3.4, 3.5, & 3.7].

Theorem 3.2. — *For $T\geqslant 0$, the form α_T is closed whose cohomology class does not depend on T. For $T>0$, α_T represents the Thom class of E so that $\pi_*\alpha_T=1$, and we have*

$$\frac{\partial\alpha_T}{\partial T}=-d\beta_T.$$

We identify X as a submanifold of $\mathscr{E}$ by the zero section. The normal bundle to X in $\mathscr{E}$ is exactly E and the conormal bundle is E^*. Let δ_X be the current on $\mathscr{E}$ defined by the integration on X. If μ is a smooth compactly supported form on $\mathscr{E}$ with values in $\pi^*o(TX)$, then

$$\int_{\mathscr{E}}\mu\delta_X=\int_X\mu.$$

For a current v on $\mathscr{E}$, denote by $\mathrm{WF}(v)\subset T^*\mathscr{E}$ its wave front set [Hör90, §8.1].

Theorem 3.3. — *Let $K\subset\mathscr{E}$ be a compact subset of $\mathscr{E}$. There is $C_K>0$ such that for any $\mu\in\Omega^\bullet(\mathscr{E},\pi^*o(TX))$ whose support is contained in K and for any $T\geqslant 1$, we have*

$$\Big|\int_{\mathscr{E}}\mu(\alpha_T-\delta_X)\Big|\leqslant\frac{C_K}{\sqrt{T}}\,\|\mu\|_{C^1},\quad\Big|\int_{\mathscr{E}}\mu\beta_T\Big|\leqslant\frac{C_K}{T^{3/2}}\,\|\mu\|_{C^1},\tag{3.4}$$

where $\|\cdot\|_{C^1}$ denotes the C^1-norm.

In view of (3.3) and (3.4),

$$\psi(E,\nabla^E)=\int_0^\infty\beta_T dT$$

is a well-defined current of degree $n-1$ on $\mathscr{E}$ with values in $\pi^*o(E)$.

Theorem 3.4. — *The current $\psi(E,\nabla^E)$ is locally integrable such that*

$$\mathrm{WF}\left(\psi(E,\nabla^E)\right) \subset E^*.$$

The following identity of currents on $\mathscr{E}$ holds,

$$d\psi(E,\nabla^E) = \pi^* e(E,\nabla^E) - \delta_X.$$

Definition 3.5. — The current $\psi(E,\nabla^E)$ is called the Mathai-Quillen current.

Remark 3.6. — The restriction of $\psi(E,\nabla^E)$ to the sphere bundle of E was first constructed in Mathai-Quillen [MQ86, §7]. If $E = TX$, this restriction coincides with the transgressed Euler class defined by Chern [Che45].

Assume now $n \leqslant m$. Let $s \in C^\infty(X,E)$ be a smooth section of E. Set

$$X' = \{x \in X : s(x) = 0\}. \tag{3.5}$$

Suppose that over X', the differential of s has maximal rank. By transversality, X' is a smooth submanifold of X of dimension $m-n$. Let $N_{X'/X}$ be the normal bundle to X' in X. Using [Hör90, Th. 8.2.4], Bismut and Zhang have shown the following proposition in [BZ92, Rem. 3.8].

Proposition 3.7. — *The pull-back currents $s^*\psi(E,\nabla^E), s^*\delta_X$ on X are well-defined and satisfy*

$$\mathrm{WF}(s^*\psi(E,\nabla^E)) \subset N^*_{X'/X}, \quad \mathrm{WF}(s^*\delta_X) \subset N^*_{X'/X}. \tag{3.6}$$

The following identity of currents on X holds,

$$d(s^*\psi(E,\nabla^E)) = e(E,\nabla^E) - s^*\delta_X. \tag{3.7}$$

Remark 3.8. — If $U \in C^\infty(X,TX)$ is a vector field on X which has only isolated non degenerated zeros, i.e., in a neighbourhood of a zero x of U there is a system of coordinates $y = (y_1,\dots,y_m)$ and a matrix A with $\det A \neq 0$ such that x is represented by $y = 0$ and

$$U(y) = Ay + \mathscr{O}(|y|^2).$$

By Proposition 3.7, $U^*\psi(TX,\nabla^{TX})$ is a well defined current. Moreover, if $\varepsilon_x = \operatorname{sgn}\det(A)$ is the Poincaré-Hopf index[4] at x, then $U^*\delta_X$ is a Radon measure on X defined for $\mu \in C^\infty(X)$ by

$$\int_X \mu\, U^*\delta_X = \sum_{x\text{: zero of } U} \varepsilon_x\, \mu(x). \tag{3.8}$$

[4] If U is Morse-Smale, then $(-1)^m\varepsilon_x = (-1)^{\mathrm{ind}(x)}$. See the discussion after (2.2) about the sign $(-1)^m$.

3.3. Ray-Singer metric. — We use the notation in Section 2. Recall that (F,∇^F) is a flat vector bundle on X. Let $(\Omega^\bullet(X,F),d^X)$ be the de Rham complex of smooth forms on X with values in F. By de Rham's theory, its cohomology is $H^\bullet(X,F)$.

Take metrics g^{TX} and g^F on TX and F. Let $\langle,\rangle_{\Lambda^\bullet(T^*X)\otimes F}$ be the induced metric on $\Lambda^\bullet(T^*X)\otimes F$. Let $dv_X\in\Omega^m(X,o(TX))$ be the Riemannian volume form on X. For $s_1,s_2\in\Omega^\bullet(X,F)$, set

$$\langle s_1,s_2\rangle_{\Omega^\bullet(X,F)}=\int_{x\in X}\langle s_1(x),s_2(x)\rangle_{\Lambda^\bullet(T^*X)\otimes F}dv_X. \tag{3.9}$$

Then (3.9) defines an L^2-metric on $\Omega^\bullet(X,F)$.

Let d^{X*} be the formal adjoint of d^X with respect to the L^2-metric $\langle\cdot,\cdot\rangle_{\Omega^\bullet(X,F)}$. Put

$$\Box^X=d^Xd^{X*}+d^{X*}d^X.$$

Then $\Box^X$ is a formally self-adjoint second order elliptic differential operator acting on $\Omega^\bullet(X,F)$. By Hodge theory, we have

$$\ker\Box^X\simeq H^\bullet(X,F). \tag{3.10}$$

By (2.8) and (3.10), the restriction of the L^2-metric $\langle\cdot,\cdot\rangle_{\Omega^\bullet(X,F)}$ to $\ker\Box^X$ induces a metric $|\cdot|^{\mathrm{RS},2}_\lambda$ on λ.

Let $(\ker\Box^X)^\perp$ be the orthogonal space to $\ker\Box^X$ in $\Omega^\bullet(X,F)$. Then $\Box^X$ acts as an invertible operator on $(\ker\Box^X)^\perp$. Let $(\Box^X)^{-1}$ be the inverse of $\Box^X$ acting on $(\ker\Box^X)^\perp$. Let $N^{\Lambda^\bullet(T^*X)}$ be the number operator on $\Lambda^\bullet(T^*X)$, which is multiplication by p on $\Lambda^p(T^*X)$. For $s\in\mathbf{C}$ such that $\mathrm{Re}(s)>m/2$, set

$$\zeta(s)=-\operatorname{Tr}[(-1)^{N^{\Lambda^\bullet(T^*X)}}N^{\Lambda^\bullet(T^*X)}(\Box^X)^{-s}].$$

By a result of Seeley [See67] or by [BZ92, Th. 7.10], $\zeta(s)$ extends to a meromorphic function of $s\in\mathbf{C}$, which is holomorphic at $s=0$.

Definition 3.9. — The Ray-Singer metric $\|\cdot\|^{\mathrm{RS},2}_\lambda$ on λ is defined by

$$\|\cdot\|^{\mathrm{RS},2}_\lambda=|\cdot|^{\mathrm{RS},2}_\lambda\exp\big(\zeta'(0)\big).$$

Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}). Let $\psi(TX,\nabla^{TX})$ be the Mathai-Quillen current. By (2.1) and by Proposition 3.7, for any Morse-Smale vector field V, the pull-back $V^*\psi(TX,\nabla^{TX})$ is a well-defined current of degree $m-1$ on X with values in $o(TX)$. Set

$$\theta(F,g^F)=\operatorname{Tr}[(g^F)^{-1}\nabla^Fg^F]\in\Omega^1(X).$$

Then, $\theta(F,g^F)$ is a closed 1-form and its cohomology class $\theta(F)=[\theta(F,g^F)]\in H^1(X)$ does not depend on the metric g^F. Up to a normalization, the class $\theta(F)$ coincides with the first Kamber-Tondeur class [KT74].

The main result of Bismut-Zhang [BZ92, Th. 0.2] is the following.

Theorem 3.10. — *Suppose that f is a Morse function, whose gradient ∇f with respect to some Riemannian metric is Morse-Smale. The following identity holds,*

$$\log\big(\|\cdot\|^{\mathrm{RS},2}_{\lambda}/\|\cdot\|^{\mathrm{M},2}_{\lambda,-\nabla f}\big) = -\int_X \theta(F,g^F)(\nabla f)^*\psi(TX,\nabla^{TX}).$$

3.4. A variation formula for certain characteristic form. — Let us follow [BZ92, §VI.a) and VI.b)]. Let $(U_\ell)_{0\leqslant\ell\leqslant 1}$ be a smooth family of vector fields on X, such that each U_ℓ has only isolated non degenerated zeros. By Proposition 3.7 and Remark 3.8, the integral

$$\int_X \theta(F,g^F)U_\ell^*\psi(TX,\nabla^{TX})$$

is well-defined. Let us study its variation with respect to $\ell\in[0,1]$.

Let $q:[0,1]\times X\to X$ be the obvious projection. Consider a smooth section $U\in C^\infty([0,1]\times X, q^*(TX))$ defined by

$$U:(\ell,x)\in[0,1]\times X\longrightarrow U_\ell(x)\in T_xX.$$

By the consideration after (3.5), the zero set of U is a manifold of dimension 1. Therefore, if $x_{1,0},\dots,x_{N,0}$ are the zeros of U_0, we can parametrize the zeros of U_ℓ by $x_{1,\ell},\dots,x_{N,\ell}$ such that all the maps $\ell\in[0,1]\to x_{i,\ell}\in X$ are smooth. Also, the Poincaré-Hopf index of U_ℓ at $x_{i,\ell}$ does not depend on ℓ and will be denoted by $\varepsilon_i\in\{\pm1\}$. The following proposition is a generalization of [BZ92, Prop. 6.4]. Since we will use this proposition several times in Section 3.5, let us give a detailed proof.

Proposition 3.11. — *The following identity holds:*

(3.11)
$$\int_X \theta(F,g^F)\Big(U_1^*\psi(TX,\nabla^{TX})-U_0^*\psi(TX,\nabla^{TX})\Big)=\sum_{i=1}^N\varepsilon_i\int_0^1\theta(F,g^F)(\dot{x}_{i,\ell})d\ell.$$

Proof. — Let us follow the proof of [BZ92, Prop. 6.1, Prop. 6.4]. Equip the pull-back vector bundle $q^*(TX)$ over $[0,1]\times X$ with the pull-back metric and the pullback metric connection $\nabla^{q^*(TX)}$. Let $\psi(q^*(TX),\nabla^{q^*(TX)})$ be the corresponding Mathai-Quillen current. By Proposition 3.7, $U^*\psi(q^*(TX),\nabla^{q^*(TX)})$ and $U^*\delta_{[0,1]\times X}$ are well-defined currents such that

$$d^{[0,1]\times X}(U^*\psi(q^*(TX),\nabla^{q^*(TX)}))=e(q^*(TX),\nabla^{q^*(TX)})-U^*\delta_{[0,1]\times X}. \tag{3.12}$$

By our construction,

$$e(q^*(TX),\nabla^{q^*(TX)})=q^*e(TX,\nabla^{TX}). \tag{3.13}$$

Since $\theta(F,g^F)$ is a closed 1-form on X, by (3.12) and (3.13), we have

$$d^{[0,1]\times X}(q^*\theta(F,g^F)\wedge U^*\psi(q^*(TX),\nabla^{q^*(TX)}))=q^*\theta(F,g^F)\wedge U^*\delta_{[0,1]\times X}.$$

Integrating the above formula over $[0,1]\times X$, by the Stokes formula, we get (3.11). □

3.5. Proof of the main result. — We restate our main result Theorem 0.2, which is an extension of Theorem 3.10.

Theorem 3.12. — *Suppose that V is a Morse-Smale vector field. The following identity holds,*

$$\log\big(\|\cdot\|_{\lambda}^{\mathrm{RS},2}/\|\cdot\|_{\lambda,V}^{\mathrm{M},2}\big) = -\int_X \theta(F,g^F)(-V)^*\psi(TX,\nabla^{TX}).$$

Proof. — Take $(V_\ell)_{0\leqslant \ell\leqslant 1}$ as in Proposition 2.6. By Remark 2.14, we have

$$\|\cdot\|_{\lambda,V}^{\mathrm{M},2} = \|\cdot\|_{\lambda,V_1}^{\mathrm{M},2}. \tag{3.14}$$

Since the critical elements of V and V_1 coincide, the fixed points of V_ℓ form smooth loops on X. By Remark 2.8, we can assume that the fixed points of V_ℓ are in a small neighbourhood of the fixed points set of V. In particular, the above loops are contractible. By Proposition 3.11 and by the closedness of $\theta(F,g^F)$, we have

$$\int_X \theta(F,g^F)(-V)^*\psi(TX,\nabla^{TX}) = \int_X \theta(F,g^F)(-V_1)^*\psi(TX,\nabla^{TX}). \tag{3.15}$$

By (3.14) and (3.15), it is enough to show our theorem for the Morse-Smale vector field V whose critical elements are all of standard forms.

Take f as in Proposition 2.10. By Proposition 2.17 and Theorem 3.10, we have

$$\begin{aligned}\log\big(\|\cdot\|_{\lambda}^{\mathrm{RS},2}/\|\cdot\|_{\lambda,V}^{\mathrm{M},2}\big) = &-\int_X \theta(F,g^F)(\nabla f)^*\psi(TX,\nabla^{TX})\\ &+\sum_{\gamma\in B}(-1)^{\mathrm{ind}(\gamma)}\log\big\|\det\tau(a'_\gamma)\big\|^2_{\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}}.\end{aligned} \tag{3.16}$$

By (3.16), it remains to show

$$\begin{aligned}\int_X \theta(F,g^F)(\nabla f)^*\psi(TX,\nabla^{TX}) &- \int_X \theta(F,g^F)(-V)^*\psi(TX,\nabla^{TX})\\ &= \sum_{\gamma\in B}(-1)^{\mathrm{ind}(\gamma)}\log\big\|\det\tau(a'_\gamma)\big\|^2_{\det F_{x'_\gamma}\otimes(\det F_{x_\gamma})^{-1}}.\end{aligned} \tag{3.17}$$

We assume that all the closed orbits are in standard form of Case 1. Cases 2–4 can be dealt similarly.

Following [BZ92, §IV.c)], choose a smooth triangulation K of X such that $A\cap K^{m-1}=\varnothing$, and such that on $\overline{U}_\gamma\simeq \mathbb{S}^1\times\overline{\mathbb{D}}^{m-1}$ the triangulation is given by the m-simplex $\sigma^m_\gamma = (\mathbb{S}^1-\{1/4\})\times\overline{\mathbb{D}}^{m-1}$ and $(m-1)$-simplex $\sigma^{m-1}_\gamma=\{1/4\}\times\overline{\mathbb{D}}^{m-1}$.

On each simplex $\sigma\in K^m\smallsetminus K^{m-1}$ of maximal degree, choose a primitive $W_{0,\sigma}\in C^\infty(\sigma)$ of $\theta(F,g^F)$, such that on σ we have

$$dW_{0,\sigma}=\theta(F,g^F).$$

Let W_0 be the locally integrable current on X, such that for each $\sigma\in K^m\smallsetminus K^{m-1}$ we have

$$W_0|_\sigma = W_{0,\sigma}.$$

By our construction of K, for $\gamma \in B$, the two points x_γ, x'_γ, and the integral curve a'_γ are in the same simplex $\sigma^m_\gamma \in K^m$, so that

$$W_0(x'_\gamma) - W_0(x_\gamma) = \log \left\|\det \tau(a'_\gamma)\right\|^2_{\det F_{x'_\gamma} \otimes (\det F_{x_\gamma})^{-1}}. \tag{3.18}$$

Set

$$W_1 = \theta(F, g^F) - dW_0. \tag{3.19}$$

Then W_1 is a closed current of degree 1 on X such that $\mathrm{Supp}(W_1) \subset K^{m-1}$. By (3.6) and by $A \cap K^{m-1} = \varnothing$, $(-V)^*\psi(TX, \nabla^{TX})$ is smooth in the neighbourhood of the support of W_1, so that

$$W_1 \wedge (-V)^*\psi(TX, \nabla^{TX})$$

is a well-defined current on X. By (3.7), (3.8), and (3.19), we have

$$\begin{multline} -\int_X \theta(F, g^F)(-V)^*\psi(TX, \nabla^{TX}) \\ = \int_X W_0 \; e(TX, \nabla^{TX}) - \sum_{x \in A} (-1)^{\mathrm{ind}(x)} W_0(x) - \int_X W_1 \wedge (-V)^*\psi(TX, \nabla^{TX}). \end{multline} \tag{3.20}$$

Similar when $-V$ is replaced by ∇f, we get

$$\begin{multline} -\int_X \theta(F, g^F)(\nabla f)^*\psi(TX, \nabla^{TX}) \\ = \int_X W_0 \; e(TX, \nabla^{TX}) - \sum_{x \in A} (-1)^{\mathrm{ind}(x)} W_0(x) \\ + \sum_{\gamma \in B} (-1)^{\mathrm{ind}(\gamma)} (W_0(x_\gamma) - W_0(x'_\gamma)) - \int_X W_1 \wedge (-V)^*\psi(TX, \nabla^{TX}). \end{multline} \tag{3.21}$$

By (3.18), (3.20), and (3.21), we see that (3.17) is equivalent to

$$\int_X W_1 \wedge (\nabla f)^*\psi(TX, \nabla^{TX}) = \int_X W_1 \wedge (-V)^*\psi(TX, \nabla^{TX}). \tag{3.22}$$

By (2.6), on any simplex in K^{m-1} other than σ^{m-1}_γ, we have $\nabla f = -V$. By Remark 2.11, near σ^{m-1}_γ, ∇f and $-V$ can be connected by a family of vector fields without zero. Using the fact that $\mathrm{Supp}(W_1) \subset K^{m-1}$, and by a version of Proposition 3.11 where $\theta(F, g^F)$ is replaced by the closed current W_1, we get (3.22). The proof of our theorem is completed. □

References

[Bis11] J.-M. Bismut – *Hypoelliptic Laplacian and orbital integrals*, Annals of Math. Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011.

[BGS88] J.-M. Bismut, H. Gillet & C. Soulé – "Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion", *Comm. Math. Phys.* **115** (1988), no. 1, p. 49–78.

[BG01] J.-M. Bismut & S. Goette – *Families torsion and Morse functions*, Astérisque, vol. 275, Société Mathématique de France, Paris, 2001.

[BZ92] J.-M. Bismut & W. Zhang – *An extension of a theorem by Cheeger and Müller*, Astérisque, vol. 205, Société Mathématique de France, Paris, 1992, With an appendix by François Laudenbach.

[BZ94] ———, "Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle", *Geom. Funct. Anal.* **4** (1994), no. 2, p. 136–212.

[BM13] J. Brüning & X. Ma – "On the gluing formula for the analytic torsion", *Math. Z.* **273** (2013), no. 3-4, p. 1085–1117.

[Che79] J. Cheeger – "Analytic torsion and the heat equation", *Ann. of Math. (2)* **109** (1979), no. 2, p. 259–322.

[Che45] S. S. Chern – "On the curvatura integra in a Riemannian manifold", *Ann. of Math. (2)* **46** (1945), p. 674–684.

[DR19] N. V. Dang & G. Rivière – "Spectral analysis of Morse-Smale gradient flows", *Ann. Sci. École Norm. Sup. (4)* **52** (2019), no. 6, p. 1403–1458.

[DR20a] ———, "Spectral analysis of Morse-Smale flows I: construction of the anisotropic spaces", *J. Inst. Math. Jussieu* **19** (2020), no. 5, p. 1409–1465.

[DR20b] ———, "Spectral analysis of Morse-Smale flows, II: Resonances and resonant states", *Amer. J. Math.* **142** (2020), no. 2, p. 547–593.

[DR20c] ———, "Topology of Pollicott-Ruelle resonant states", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **XXI** (2020), p. 827–871, special issue.

[DR21] ———, "Pollicott-Ruelle spectrum and Witten Laplacians", *J. Eur. Math. Soc. (JEMS)* (2021), online first.

[Fra79] J. M. Franks – "Morse-Smale flows and homotopy theory", *Topology* **18** (1979), no. 3, p. 199–215.

[Fra82] ———, *Homology and dynamical systems*, CBMS Regional Conference Series in Math., vol. 49, American Mathematical Society, Providence, RI, 1982.

[Fra35] W. Franz – "Über die Torsion einer Überdeckung", *J. reine angew. Math.* **173** (1935), p. 245–254.

[Fri86] D. Fried – "Analytic torsion and closed geodesics on hyperbolic manifolds", *Invent. Math.* **84** (1986), no. 3, p. 523–540.

[Fri87] ———, "Lefschetz formulas for flows", in *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, Contemp. Math., vol. 58, American Mathematical Society, Providence, RI, 1987, p. 19–69.

[Hör90] L. Hörmander – *The analysis of linear partial differential operators. I*, 2nd ed., Grundlehren Math. Wiss., vol. 256, Springer-Verlag, Berlin, 1990.

[KT74] F. W. Kamber & P. Tondeur – "Characteristic invariants of foliated bundles", *Manuscripta Math.* **11** (1974), p. 51–89.

[KM76] F. F. Knudsen & D. Mumford – "The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"", *Math. Scand.* **39** (1976), no. 1, p. 19–55.

[LR91] J. Lott & M. Rothenberg – "Analytic torsion for group actions", *J. Differential Geom.* **34** (1991), no. 2, p. 431–481.

[Lüc93] W. Lück – "Analytic and topological torsion for manifolds with boundary and symmetry", *J. Differential Geom.* **37** (1993), no. 2, p. 263–322.

[MQ86] V. Mathai & D. Quillen – "Superconnections, Thom classes, and equivariant differential forms", *Topology* **25** (1986), no. 1, p. 85–110.

[Mil66] J. Milnor – "Whitehead torsion", *Bull. Amer. Math. Soc.* **72** (1966), p. 358–426.

[Mil68] ———, "Infinite cyclic coverings", in *Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967)*, Prindle, Weber & Schmidt, Boston, Mass., 1968, p. 115–133.

[MS91] H. Moscovici & R. J. Stanton – "R-torsion and zeta functions for locally symmetric manifolds", *Invent. Math.* **105** (1991), no. 1, p. 185–216.

[Mül78] W. Müller – "Analytic torsion and R-torsion of Riemannian manifolds", *Adv. in Math.* **28** (1978), no. 3, p. 233–305.

[Mül93] ———, "Analytic torsion and R-torsion for unimodular representations", *J. Amer. Math. Soc.* **6** (1993), no. 3, p. 721–753.

[Pal68] J. Palis – "On Morse-Smale dynamical systems", *Topology* **8** (1968), p. 385–404.

[PdM82] J. Palis & W. de Melo – *Geometric theory of dynamical systems*, Springer-Verlag, New York-Berlin, 1982.

[PS70] J. Palis & S. Smale – "Structural stability theorems", in *Global Analysis (Berkeley, Calif., 1968)*, Proc. Sympos. Pure Math., vol. XIV, American Mathematical Society, Providence, RI, 1970, p. 223–231.

[Qui85] D. Quillen – "Superconnections and the Chern character", *Topology* **24** (1985), no. 1, p. 89–95.

[RS71] D. B. Ray & I. M. Singer – "R-torsion and the Laplacian on Riemannian manifolds", *Adv. in Math.* **7** (1971), p. 145–210.

[Rei35] K. Reidemeister – "Homotopieringe und Linsenräume", *Abh. Math. Sem. Univ. Hamburg* **11** (1935), no. 1, p. 102–109.

[dR50] G. de Rham – "Complexes à automorphismes et homéomorphie différentiable", *Ann. Inst. Fourier (Grenoble)* **2** (1950), p. 51–67.

[SM96] H. Sánchez-Morgado – "Reidemeister torsion and Morse-Smale flows", *Ergodic Theory Dynam. Systems* **16** (1996), no. 2, p. 405–414.

[See67] R. T. Seeley – "Complex powers of an elliptic operator", in *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, American Mathematical Society, Providence, RI, 1967, p. 288–307.

[She18] S. Shen – "Analytic torsion, dynamical zeta functions, and the Fried conjecture", *Anal. PDE* **11** (2018), no. 1, p. 1–74.

[SY17] S. Shen & J. Yu – "Flat vector bundles and analytic torsion on orbifolds", 2017, to appear in *Comm. Anal. Geom.*, `arXiv:1704.08369`.

[Sma61] S. Smale – "On gradient dynamical systems", *Ann. of Math. (2)* **74** (1961), p. 199–206.

Manuscript received 12th June 2020
accepted 8th February 2021

Shu Shen, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Sorbonne Université
4 place Jussieu, 75252 Paris Cedex 05, France
E-mail : `shu.shen@imj-prg.fr`
Url : `https://webusers.imj-prg.fr/~shu.shen/`

Jianqing Yu, School of Mathematical Sciences, University of Science and Technology of China
96 Jinzhai Road, Hefei, Anhui 230026, P. R. China
E-mail : `jianqing@ustc.edu.cn`