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L^2-TYPE CONTRACTION FOR SYSTEMS OF CONSERVATION LAWS

by Denis Serre & Alexis F. Vasseur

Abstract. — The semi-group associated with the Cauchy problem for a scalar conservation law is known to be a contraction in L^1. However it is not a contraction in L^p for any $p > 1$. Leger showed in [20] that for a convex flux, it is however a contraction in L^2 up to a suitable shift. We investigate in this paper whether such a contraction may happen for systems. The method is based on the relative entropy method. Our general analysis leads us to the new geometrical notion of *Genuinely non-Temple* systems. We treat in details two examples: – the Keyfitz–Kranzer system with rotationally invariant flux, for which the L^2 contraction holds true, – the Euler system of gas dynamics, for which it does not.

Résumé (Contraction de type L^2 pour des systèmes de lois de conservation)

On sait que le semi-groupe associé au Problème de Cauchy pour une loi de conservation scalaire est contractant dans L^1, mais qu'il ne l'est pas dans L^p si $p > 1$. Leger a montré dans [20], pour un flux convexe, une propriété de contraction dans L^2 moyennant une translation. Nous examinons ici la possibilité d'une telle propriété pour les systèmes. Notre analyse nous conduit à la notion géométrique de système *Vraiment pas Temple*. Nous traitons en détail deux exemples : – le système de Keyfitz et Kranzer avec flux isotrope, pour lequel la contraction a lieu, – le système de la dynamique des gaz, où ce n'est pas le cas.

1. Introduction

Let us consider a strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \mathbb{R}^n. \tag{1}$$

We denote $\lambda_j(u), r_j(u), \ell_j(u)$ the jth eigen-(value, vector, form) of the differential $\mathrm{d}f(u)$. In particular, we have $\ell_j \cdot r_k \equiv 0$ for $k \neq j$.

When $n = 1$ (the scalar case), it is known that the semi-group associated with the Cauchy problem is L^1-contracting: if $v_0 - u_0 \in L^1(\mathbb{R})$, then the corresponding solutions u and v have the property that the difference $v(t) - u(t)$ remains space-integrable for every time $t > 0$, and $t \mapsto \|v(t) - u(t)\|_1$ is non-increasing. The Kruzhkov semi-group is not a contraction in L^p for $p > 1$, unless the equation is linear. However

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Leger proved recently [20] that if f is convex and if v is a pure shock wave, then the L^2-contraction is valid up to a suitable shift. Specifically, there exists a Lipschitz curve $t \mapsto h(t)$ such that $t \mapsto \|u(t) - \tau_{h(t)}v(t)\|_2$ is non-increasing.

The contraction part in Kruzkhov's analysis follows from the property that the function $(u, v) \mapsto |u - v|$ is a convex entropy with respect to either of the variables u or v. This is not true any more for $(u, v) \mapsto |u-v|^p$ if $p > 1$ and this is the reason why the semi-group is not L^p-contracting. When dealing with systems that are not linear, we don't have such bi-entropies (except for the useless affine functions), and therefore we don't expect a contraction property. Instead, several research papers make use of the so-called *relative entropies* to prove uniqueness and/or stability results.

It has been first used by Dafermos [13, 12] and DiPerna [15] to show the weak-strong uniqueness and stability of Lipschitz regular solution to conservation laws. (see also Dafermos' book [14]). The relative entropy method is also an important tool in the study of asymptotic limits to conservation laws. Applications of the relative entropy method in this context began with the work of Yau [33] and have been studied by Chen & Frid [7, 8, 9, 10] and many others. For incompressible limits, see Bardos, Golse, Levermore [1, 2], Lions and Masmoudi [22], Saint Raymond et al. [16, 26, 23, 25]. For the compressible limit, see Tzavaras [30] in the context of relaxation and [5, 4, 24] in the context of hydrodynamical limits. In all those papers, the method works as long as the limit solution is Lipschitz.

Relative entropies $\eta(u|v)$ are convex entropies of u, and dominate somehow the quantity $|u - v|^2$. However they loose the symmetry $u \longleftrightarrow v$, and previous results need special assumptions about v (typically Lipschitz regularity).

This paper is part of a general program initiated in [31] to apply this kind of method where v is a given shock. It is based on the uniqueness result of DiPerna [15] (see also Chen & Frid [9, 10] and Chen & Li [11] for asymptotic stability). Following the work of Leger for the scalar case [20], an application to the stability of extremal shocks for systems has been performed in [21] (see Texier and Zumbrun [29] and Barker, Freistühler and Zumbrun [3] for interesting comments on this result). Finally, a first application of the method to the study of asymptotic limit to a shock can be found in [18]. In this paper we are investigating systems for which shocks are not only stable, but also induce a contraction up to a shift.

In the present work, we shall assume that v is a pure shock, taking constant values u_ℓ, u_r on each side of the line $x = \sigma(u_\ell, u_r)t$.

We therefore assume that (1) admits a strongly convex entropy η, of class $\mathcal{C}^2$ with entropy flux q, the adverb *strongly* meaning that $\mathrm{D}^2\eta > 0_n$. We always assume that admissible solutions of (1) satisfy the *entropy inequality*

$$\partial_t\eta(u) + \partial_x q(u) \leqslant 0. \tag{2}$$

In particular, shocks (u_ℓ, u_r) of velocity σ satisfy both the Rankine–Hugoniot relation and an inequality

$$f(u_r) - f(u_\ell) = \sigma(u_r - u_\ell), \qquad q(u_r) - q(u_\ell) \leqslant \sigma(\eta(u_r) - \eta(u_\ell)).$$

As usual, the *relative entropy* is the expression

$$\eta(a|b) := \eta(a) - \eta(b) - \mathrm{d}\eta(b)\cdot(a-b),$$

and the relative entropy-flux is[1]

$$q(a|b) := q(a) - q(b) - \mathrm{d}\eta(b)\cdot(f(a)-f(b)).$$

While $\eta(a|b)$ is strictly positive for $a \neq b$, we have $\eta(a|b), q(a|b) = O(|a-b|^2)$ when $a \to b$.

Notice that admissible solutions satisfy

$$\partial_t \eta(u|a) + \partial_x q(u|a) \leqslant 0 \tag{3}$$

for every constant a.

We are interested in the stability of a shock wave (u_ℓ, u_r) with respect to η. Because we feel free to shift a solution at each time, we speak of *relative stability*. Let us give first a heuristic of our method. If u is a solution with the same values $u_{\ell,r}$ at infinity, we compute at each time the minimum of

$$h \mapsto E(u(t);h) := \int_{-\infty}^{h} \eta(u|u_\ell)dx + \int_h^{\infty} \eta(u|u_r)dx$$

and consider the evolution of this minimum as time increases.

Because $\eta(u_r|u_\ell) > 0$ and $\eta(u_\ell|u_r) > 0$, we have

$$\lim_{h\to\pm\infty} E(u(t);h) = +\infty.$$

Since $h \mapsto E(u(t);h)$ is continuous, the minimum is achieved at some finite $h = h(t)$, where we have

$$\left.\frac{d}{dh}\right|_{h(t)-0} E \leqslant 0 \leqslant \left.\frac{d}{dh}\right|_{h(t)+0} E.$$

These inequalities translate into

$$\eta(u_-|u_\ell) \leqslant \eta(u_-|u_r), \qquad \eta(u_+|u_r) \leqslant \eta(u_+|u_\ell),$$

where $u_\pm = u(t, h(t) \pm 0)$. The function $h(t)$ where the minimum is reached may be discontinuous and even non unique. For this reason, we shall construct the function h in a slightly different manner. We will show that it still verifies a slightly relaxed condition for almost every time $t > 0$:

$$\eta(u_-|u_\ell) - \eta(u_-|u_r) \ \text{ and } \ \eta(u_+|u_r) - \eta(u_+|u_\ell) \text{ have the same sign.} \tag{4}$$

In the sequel, we shall make use of the following notation: if F is a function of u, then

$$F_r = F(u_r), \quad F_\ell = F(u_\ell), \quad [F] = F_r - F_\ell, \quad F_\pm = F(u_\pm).$$

There are two regimes:

[1] We point out that the definition of $q(a|b)$ does not mimic that of $\eta(a|b)$.

Smooth case. — The solution $u(t)$ is continuous at $h(t)$, that is $u_- = u_+$, which we denote u below. Then (4) amounts to writing $\eta(u|u_\ell) = \eta(u|u_r)$. This is equivalent to the linear constraint

$$[\mathrm{d}\eta] \cdot u = [\mathrm{d}\eta \cdot u - \eta]. \tag{5}$$

Because of the strict convexity of η, we have

$$[\mathrm{d}\eta] \cdot [u] > 0 \tag{6}$$

and therefore (5) defines a hyperplane Π in the phase space, which separates strictly the points u_ℓ and u_r; for instance

$$[\mathrm{d}\eta] \cdot u_r - [\mathrm{d}\eta \cdot u - \eta] = \eta_r - \eta_\ell - \mathrm{d}\eta_\ell(u_r - u_\ell) \geqslant 0.$$

Sharp case. — On some time interval, the solution is discontinuous at $x = h(t)$. Then (4) is completed by the Rankine–Hugoniot condition

$$f(u_+) - f(u_-) = \sigma(u_-, u_+)(u_+ - u_-)$$

and the entropy inequality. The condition (4) rewrites

$$\min\left([\mathrm{d}\eta] \cdot u_-, [\mathrm{d}\eta] \cdot u_+\right) \leqslant [\mathrm{d}\eta \cdot u - \eta] \leqslant \max\left([\mathrm{d}\eta] \cdot u_-, [\mathrm{d}\eta] \cdot u_+\right). \tag{7}$$

This expresses that u_- and u_+ are separated by the hyperplane Π.

The dissipation rate. — Following [21], we consider the rate of dissipation of E for a given function h verifying (4):

$$\begin{aligned}
\frac{d}{dt}E(u(t);h(t)) &= \frac{d}{dt}\left(\int_{-\infty}^{h(t)} \eta(u|u_\ell)\,dx + \int_{h(t)}^{+\infty} \eta(u|u_r)\,dx\right) \\
&= \int_{-\infty}^{h(t)} \partial_t\eta(u|u_\ell)\,dx + \int_{h(t)}^{+\infty} \partial_t\eta(u|u_r)\,dx + \dot{h}\big(\eta(u_-|u_\ell) - \eta(u_+|u_r)\big) \\
&\leqslant -\int_{-\infty}^{h(t)} \partial_x q(u|u_\ell)\,dx - \int_{h(t)}^{+\infty} \partial_x q(u|u_r)\,dx + \dot{h}\big(\eta(u_-|u_\ell) - \eta(u_+|u_r)\big) \\
&\leqslant q(u_+|u_r) - q(u_-|u_\ell) - \dot{h}\big(\eta(u_+|u_r) - \eta(u_-|u_\ell)\big) =: D(u_{\ell,r}; u_\pm),
\end{aligned}$$

where we have used (3). Notice that the difference between $\frac{d}{dt}E(u(t);h(t))$ and $D(u_{\ell,r}; u_\pm)$ is only due to the entropy dissipation through shock waves in $u(t)$, away from $x = h(t)$. Because they may just not be present, we feel free to call $D(u_{\ell,r}; u_\pm)$ the *dissipation rate* of E.

In what we have called the *smooth* case, we have $u_- = u_+$ (denoted as u); because of (5), the rate D reduces to

$$D_{sm}(u_{\ell,r}; u) = q(u|u_r) - q(u|u_\ell) = [\mathrm{d}\eta \cdot f - q] - [\mathrm{d}\eta] \cdot f(u).$$

On the contrary, if $u_-(t) \neq u_+(t)$ on some time interval, then necessarily $\dot{h}(t) = \sigma(u_-, u_+)$. Then the dissipation rate becomes

$$D_{RH}(u_{\ell,r}; u_\pm) = q(u_+|u_r) - q(u_-|u_\ell) - \sigma(u_-, u_+)(\eta(u_+|u_r) - \eta(u_-|u_\ell)).$$

An alternative formula, which exploits Rankine–Hugoniot, is

$$D_{RH} = [\mathrm{d}\eta\cdot f - q] - \sigma[\mathrm{d}\eta\cdot u - \eta] + q_+ - q_- - \sigma(\eta_+ - \eta_-) - [\mathrm{d}\eta]\cdot(f-\sigma u)_\pm.$$

A DEFINITION. — We say that a given admissible discontinuity (u_ℓ, u_r) (a shock or a contact discontinuity) is *relative-entropy stable* if the dissipation rate D is always non-positive. This means on the one hand that $D_{sm}(u_{\ell,r}; u) \leqslant 0$ for every $u \in \Pi$ (i.e., satisfying (5); and on the other hand, that $D_{RH}(u_{\ell,r}; u_\pm) \leqslant 0$ for every admissible discontinuity (u_-, u_+) satisfying the constraint (7).

We show in the next section that if (u_ℓ, u_r) is relative-entropy stable, then the quantity $\inf_h E(u(t);h)$ remains smaller than $E(u(0);0)$. Indeed, we show the existence of a function $t \to h(t)$ such that $E(u(t);h)$ is non-increasing in time.

COMPATIBILITY WITH THE ENTROPY CONDITION. — We show here that the relative-entropy stability (in short, RES) contains a formulation of the entropy condition when the shock (u_ℓ, u_r) is weak. To do so, we employ the so-called *entropy variable* $z := \mathrm{d}\eta(u)$. Denoting η^* the convex conjugate function, we have $u = \mathrm{d}\eta^*(z)$ and $(\mathrm{D}^2\eta)^{-1} = \mathrm{D}^2\eta^*$. Finally, we know that the scalar function $M(z) := z\cdot f(u) - q(u)$ satisfies $f(u) = \mathrm{d}M(z)$. Then $D_{sm} = [M] - \mathrm{d}M\cdot[z]$, where the constraint is $\mathrm{d}\eta^*\cdot[z] = [\eta^*]$. This suggests to evaluate D_{sm} at the special point $\overline{u}\in\Pi$ given by the formula

$$\overline{u} = \int_0^1 \mathrm{d}\eta^*(\theta z_r + (1-\theta)z_\ell)\,d\theta.$$

Then the RES implies that $\mathrm{d}M(\overline{z})\cdot[z] \geqslant [M]$, where $\overline{z} = \mathrm{d}\eta(\overline{u})$.

When the shock strength is small, $\overline{z}$ is close to $z_{r,\ell}$. Developing $\mathrm{d}\eta^*(\theta z_r + (1-\theta)z_\ell)$ to the second order at $\overline{z}$, we find that

$$\overline{z} = \frac{1}{2}(z_r + z_\ell) + \frac{1}{24}(\mathrm{D}^2\eta^*)^{-1}\mathrm{D}^3\eta^*\cdot[z]^{\otimes 2} + O([z]^3).$$

Likewise, a Taylor expansion of M at $\overline{z}$ gives

$$\begin{aligned} D_{sm}(z_{\ell,r};\overline{z}) = \frac{1}{2}\mathrm{D}^2M_{\overline{z}}\left((z_r-\overline{z})^{\otimes 2} - (z_\ell - \overline{z})^{\otimes 2}\right)& \\ + \frac{1}{6}\mathrm{D}^3M_{\overline{z}}\left((z_r-\overline{z})^{\otimes 3} - (z_\ell-\overline{z})^{\otimes 3}\right) + O([z]^4)&. \end{aligned}$$

We now have

$$(z_r-\overline{z})^{\otimes 3} - (z_\ell-\overline{z})^{\otimes 3} \sim \frac{1}{4}[z]^{\otimes 3}$$

and

$$\begin{aligned} (z_r-\overline{z})^{\otimes 2} - (z_\ell-\overline{z})^{\otimes 2} &= [z]\otimes(z_r + z_\ell - 2\overline{z}) \\ &\sim -\frac{1}{12}[z]\otimes(\mathrm{D}^2\eta^*)^{-1}\mathrm{D}^3\eta^*\cdot[z]^{\otimes 2}. \end{aligned}$$

This yields

$$24D_{sm}(z_{\ell,r};\overline{z}) \sim \mathrm{D}^3M_{\overline{z}}[z]^{\otimes 3} - \mathrm{D}^2M_{\overline{z}}([z],(\mathrm{D}^2\eta^*)^{-1}\mathrm{D}^3\eta^*\cdot[z]^{\otimes 2}).$$

We come back to the original variable u. In the course of the computation, we use the fact that $\mathrm{d}f$ is $\mathrm{D}^2\eta$-symmetric, and we obtain

$$24D_{sm}(z_{\ell,r};\overline{z}) \sim \mathrm{D}^2\eta_{\overline{u}}([u],\mathrm{D}^2 f_{\overline{u}}[u]^{\otimes 2}).$$

According to Lax [19], we know that $[u] \sim \epsilon r_k(\overline{u})$ for some index k and a small ϵ. Then we derive

$$24D_{sm}(z_{\ell,r};\overline{z}) \sim \epsilon^3\ \mathrm{D}^2\eta(r_k,\mathrm{D}^2 f r_k^{\otimes 2})\big|_{\overline{u}} = \epsilon^3 \mathrm{d}\lambda_k\cdot r_k,$$

with the normalization $\mathrm{D}^2\eta(r_k,r_k) = 1 = \ell_k\cdot r_k$. Thus the RES tells us that $\epsilon\mathrm{d}\lambda_k\cdot r_k \leqslant 0$, which is clearly compatible with the entropy condition. For instance, if the kth field is GNL at $\overline{u}$, the entropy condition is equivalent for the small shock to $\epsilon\mathrm{d}\lambda_k\cdot r_k < 0$.

The rest of the article is organized as follows. In Section 2, we show that RES ensures that the infimum of $E(u(t);h)$ over h is a non-increasing function of time (see Theorem 2.1). In Section 3, we recall Leger's result that scalar shocks are RES if the flux is either convex or concave (see Proposition 3.1); the relative stability fails otherwise. Section 4 is devoted to the Keyfitz-Kranzer system with rotationally symmetric flux $\phi(|u|)u$; we show that shocks are RES if and only if $\rho\phi$ is convex (or concave) and ϕ is decreasing (resp. increasing) (see Theorem 4.1). However, the RES of contact discontinuities needs only strict hyperbolicity (Theorem 4.2). Section 5 concerns general strictly hyperbolic systems. We focus on whether the rate D_{sm} achieves a non-positive maximum over the hyperplane of constraints; we did not analyze that deeply the rate D_{RH}, which behaves in a more non-linear way because of the Rankine–Hugoniot constraint over the pair (u_-,u_+). Whether a characteristic field belongs to the Temple class or not turns out to be crucial. We are led to the apparently new notion of Genuinely Non Temple field (Proposition 5.4). In the case where the characteristic field associated with the shock admits a Riemann invariant w (the field is *rich*), we show that GNT amounts to saying that a level set of w has a non-degenerate curvature (Proposition 5.5). In the case of an extreme shock, we even find that the maximality D_{sm} is equivalent to a convexity property of this level set (Proposition 5.9). Section 6 begins with the observation that RES is intrinsic, in the sense that it is invariant under an Euler–Lagrange-type transformation. When dealing with examples taken from continuum mechanics, this allows us to perform calculations in Lagrangian coordinates, where the system looks simpler. Sadly, we find that shocks are not RES in the cases of p-system or full gas dynamics. Finally, the appendix provides a detailed proof of Lemma 1.1.

We point out that the drift chosen here is not the only possible one. For instance, Leger [20] used a different one. Also, we need a different choice to conclude in Section 2 below. But the one that minimizes $E(u(t):h)$ is optimal in the sense that if some drift $\widehat{h}(t)$ makes $t\mapsto E(u(t);\widehat{h}(t))$ decay, then $t\mapsto \inf_h E(u(t);h)$ decays too. Thus the non-positivity of D is also a necessary condition for the relative stability, whence the terminology RES.

2. Construction of the drift h

This section is devoted to the following theorem.

Theorem 2.1. — *We consider (u_ℓ, u_r) a relative-entropy stable discontinuity. Then for any $u \in BV_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$ solution of* (1), (2) *with*

$$E(u(t=0), 0) < \infty,$$

there exists a Lipschitz function $t \to h(t)$ such that E(u(t),h(t)) is a non-increasing function.

Note that the result does not depend on the L^∞ norm of u, nor on the BV_{loc} norm of u. The boundedness of u only ensures that h is Lipschitz, and the BV_{loc} norm ensures that u_- and u_+ are well-defined. This condition can be relaxed by imposing some strong traces on the solution u (see [21]).

Proof. — Recall that $\Pi = \{\eta(u|u_r) - \eta(u|u_\ell) = 0\}$. If $u \notin \Pi$, we define

$$V_\epsilon(u) = \frac{[q(u|u_r) - q(u|u_\ell) - \epsilon]_+}{\eta(u|u_r) - \eta(u|u_\ell)},$$

where $[\cdot]_+ = \max(0, \cdot)$. If instead $u \in \Pi$, then we set $V_\epsilon(u) = 0$. Using that (u_ℓ, u_r) is relative-entropy stable, we have that for any $\epsilon > 0$, $V_\epsilon \in C^\infty(\mathbb{R})$. Indeed, the function is smooth outside of Π. And on Π, $q(u|u_r) - q(u|u_\ell) - \epsilon \leqslant -\epsilon$. Hence, $V_\epsilon(u) = 0$ on a neighborhood of Π.[(2)]

Consider now $u \in BV_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$ solution to (1), (2), such that

$$E(u(t=0), 0) < \infty.$$

We construct (in the Filippov sense) a solution to

$$\begin{aligned} &\dot{h}_\epsilon = V_\epsilon(u(t, h_\epsilon)), \\ &h(0) = 0. \end{aligned}$$

We have the following lemma (see [21]):

Lemma 2.2. — *There exists a Lipschitz function h_ϵ such that:*

$$\begin{aligned} &h_\epsilon(0) = 0, \\ &\|\dot{h}_\epsilon\|_{L^\infty} \leqslant \|V_\epsilon\|_{L^\infty}, \\ &\dot{h}_\epsilon \in I(V_\epsilon(u(t, h_\epsilon(t)-)), V_\epsilon(u(t, h_\epsilon(t)+))), \qquad \text{for almost any } t > 0, \end{aligned}$$

where $I(a, b)$ is the interval with endpoints a and b. Moreover, for almost every $t > 0$,

$$\begin{aligned} f(u_+) - f(u_-) &= \dot{h}_\epsilon(u_+ - u_-), \\ q(u_+) - q(u_-) &\leqslant \dot{h}_\epsilon(\eta_+ - \eta_-), \end{aligned}$$

(2) If (u_ℓ, u_r) is strictly relative-entropy stable, that is inequalities are strict in the definition, then we can directly take $\epsilon = 0$.

which means that for almost every $t > 0$, either $(u_-, u_+, \dot{h}_\epsilon)$ is an admissible entropic discontinuity or $u_- = u_+$.

The proof of this lemma can be found in [21]. It is based on the Filippov flows and was already used by Dafermos. We give a version of it in the appendix for the reader's convenience.

For almost every time $t > 0$ such that $u_- = u_+$, we have

$$\dot{h}_\epsilon = V_\epsilon(u_\pm).$$

For those times t, thanks to the definition of V_ϵ, we have

$$D(u_{\ell,r}, u_\pm) \leqslant \epsilon.$$

Now, for almost every time $t > 0$ such that $u_- \neq u_+$, we have two possibilities. Either u_- and u_+ are separated by Π. In this case $D(u_{\ell,r}, u_\pm) \leqslant 0$, thanks to the lemma and the definition of relative stability. Or, u_- and u_+ are not separated by Π. In this case, $V_\epsilon(u_-)$, $V_\epsilon(u_+)$, $\eta(u_-|u_\ell) - \eta(u_-|u_r)$, and $\eta(u_+|u_\ell) - \eta(u_+|u_r)$ have the same sign. And so, $\dot{h}_\epsilon \in I(V_\epsilon(u_-), V_\epsilon(u_+))$ has also the same sign. Then, using that $(u_-, u_+, \dot{h}_\epsilon)$ is an entropic discontinuity, we have for both $v = u_-$ and $v = u_+$

$$\begin{aligned} D(u_{\ell,r}, u_\pm) &\leqslant q(v|u_r) - q(v, u_\ell) - \dot{h}_\epsilon(\eta(v|u_r) - \eta(v|u_\ell)), \\ &= q(v|u_r) - q(v, u_\ell) - |\dot{h}_\epsilon|\ |\eta(v|u_r) - \eta(v|u_\ell)|. \end{aligned}$$

Consider this inequality for the value of v such that $|V_\epsilon(v)| = \inf(|V_\epsilon(u_-)|, |V_\epsilon(u_+)|)$. For this value v we have

$$|\dot{h}_\epsilon| \geqslant |V_\epsilon(v)|,$$

and so

$$\begin{aligned} D(u_{\ell,r}, u_\pm) &\leqslant q(v|u_r) - q(v, u_\ell) - |V_\epsilon(v)|\ |\eta(v|u_r) - \eta(v|u_\ell)|, \\ &= q(v|u_r) - q(v, u_\ell) - V_\epsilon(v)(\eta(v|u_r) - \eta(v|u_\ell)), \\ &= \epsilon - [q(v|u_r) - q(v, u_\ell) - \epsilon]_- \leqslant \epsilon. \end{aligned}$$

Therefore, for every $t > s > 0$

$$E(u(t), h_\epsilon(t)) \leqslant t\epsilon + E(u(s), h_\epsilon(s)).$$

Note that $\|V_\epsilon\|_{L^\infty}$ is uniformly bounded with respect to ϵ. Hence, up to a subsequence, h_ϵ converges, uniformly on bounded sets, to a Lipschitz function h, and $E(u(t), h_\epsilon(t))$ converges to $E(u(t), h(t))$. At the limit, we have

$$E(u(t), h(t)) \leqslant E(u(s), h(s)),$$

whenever $t > s > 0$. □

3. The scalar case

The case of a scalar equation has been treated by Leger [20]. Even if we don't pretend to originality, we provide (for the sake of completeness) a proof that the dissipation rate is non-positive under natural assumptions.

Without loss of generality, we may assume that $u_r < u_\ell$. We limit ourselves to the solutions given by Kruzkhov's theory, and therefore we have the Oleĭnik inequality that the graph of f lies below its chord between u_r and u_ℓ.

The smooth case. — We ask ourselves whether

$$D_{sm} := [\eta' f - q] - [\eta'] f(u).$$

is non-positive whenever $u \in \Pi$, that is when

$$u = \frac{[u\eta' - \eta]}{[\eta']}. \tag{8}$$

In other words, we ask whether

$$f\left(\frac{[u\eta' - \eta]}{[\eta']}\right) \leqslant \frac{[f\eta' - q]}{[\eta']} \quad ?$$

Since every convex function η is an entropy (in the scalar case), it is natural to ask for a relative stability for every such η. Because $\eta''(u)\mathrm{d}u$ may be any non-negative measure, the above inequality amounts to saying that

$$f\bigg(\int u\, d\nu\bigg) \leqslant \int f(u)\, d\nu$$

for every probability measure ν over $[u_r, u_\ell]$. This Jensen-type inequality is equivalent to saying that f is convex over $[u_r, u_\ell]$.

The discontinuous case. — We therefore assume for the rest of this section that f is convex, not only over (u_r, u_ℓ), but globally. If $u_+ \neq u_-$, we thus have $u_+ < u_-$ and $[\eta'] < 0$. So, the constraint (7) is that

$$u_+ \leqslant \frac{[u\eta' - \eta]}{[\eta']} \leqslant u_-.$$

The velocity of the shock (u_-, u_+) is given by

$$\sigma = \frac{f_+ - f_-}{u_+ - u_-}\,,$$

and we have

$$D_{RH} = q_+ - q_- - \sigma(\eta_+ - \eta_-) + [\eta' f - q] - \sigma[u\eta' - \eta] - [\eta'](f - \sigma u)_\pm.$$

Up to the use of a moving frame, we may assume $\sigma = 0$, that is $f_+ = f_-$, which we denote $\overline{f}$. This amounts to replacing $f - \sigma u$ by f and $q - \sigma\eta$ by q. We then have

$$D_{RH} = q_+ - q_- + [\eta' f - q] - [\eta']\overline{f} = \left(\int_{u_+}^{u_-} - \int_{u_r}^{u_\ell}\right) f\eta'' du + (\eta'_\ell - \eta'_r - \eta'_- + \eta'_+)\overline{f}.$$

This rewrites as $D_{RH} = \epsilon(I)A(I) + \epsilon(J)A(J)$ where I, J are disjoint intervals such that
$$I \cup J = \big((u_+, u_-) \cup (u_r, u_\ell)\big) \smallsetminus \big((u_+, u_-) \cap (u_r, u_\ell)\big).$$
The sign $\epsilon(I)$ is $+1$ if $I \subset (u_+, u_-)$ and -1 otherwise. Finally,
$$A(I) = \int_I f\eta'' du - \overline{f} \int_I \eta''(u) du.$$
Because f is convex, $A(I)$ is negative if $I \subset (u_+, u_-)$ and positive otherwise. In all cases, $\epsilon(I)A(I) \leqslant 0$ and we receive $D \leqslant 0$.

In conclusion, we have the

Proposition 3.1 (Leger [20]). — *Let us assume that f is a convex flux, and η is a convex entropy. Let (u_ℓ, u_r) be an admissible shock of the conservation law $u_t + (f(u))_x = 0$. Then the dissipation rate D is non-positive:*

- *when u is given by* (8)*, then $D_{sm} \leqslant 0$,*
- *when (u_-, u_+) is another admissible shock, then $D_{RH} \leqslant 0$.*

Of course, the proposition remains true if f is concave instead. This amounts to changing x into $-x$.

4. The Keyfitz–Kranzer system with a symmetric flux

Let $\phi : \mathcal{U} \to \mathbb{R}$ be a smooth function over a planar domain. The Keyfitz–Kranzer system writes

$$\partial_t u + \partial_x(\phi(u)u) = 0. \tag{9}$$

We concentrate here on the case where the flux $f(u) = \phi(u)u$ is rotationally symmetric:
$$\phi = \phi(\rho), \qquad u = \rho e^{i\theta},$$
and we choose a half-space domain $\mathcal{U}$, for instance that defined by $u_1 > 0$. We denote $g(\rho) := \rho f(\rho)$.

The following facts are well-known (see Keyfitz & Kranzer [17])

- The wave velocities are $\mu = \phi$ and $\lambda = \rho\phi' + \phi$. They are associated with Riemann invariants θ and ρ, respectively. The μ-field is linearly degenerate, with contact discontinuities satisfying $[\rho] = 0$. The λ-field is genuinely nonlinear whenever g'' does not vanish; the λ-shocks satisfy $[\theta] = 0$, together with $\rho_r < \rho_\ell$ in the convex case.

We point out that the system is strictly hyperbolic if ϕ' does not vanish, an assumption that we make from now on.

- Each sub-domain of the form
$$\{u \,|\, \theta \in [\theta_1, \theta_2] \text{ and } \rho \in [\rho_1, \rho_2]\}$$
is invariant under the Riemann solver[3]. It is therefore invariant for the semi-group $(S_t)_{t \geqslant 0}$ constructed through the Glimm scheme.

[3] The lack of convexity of such domains is compensated by the linear degeneracy of the μ-field.

• The Riemann solver does not increase the total variation of θ and ρ. Therefore $(S_t)_{t\geqslant 0}$ is TVD in terms of these coordinates. This has two important consequences: – on the one hand the semi-group is globally defined for data of arbitrary large total variation, – on the other hand, we may apply Bressan & coll.'s theory, which tells us that $(S_t)_{t\geqslant 0}$ is unique among the TVD semi-groups; see [6].

• The system decouples formally into a scalar equation

$$\partial_t\rho + \partial_x(g(\rho)) = 0, \tag{10}$$

and a transport equation

$$\partial_t\theta + \phi(\rho)\partial_x\theta = 0. \tag{11}$$

Actually, it is known that when u is an admissible solution of (9), then its modulus ρ is an admissible solution of (10).

• The entropies of the system are the functions of the form $u \mapsto e(\rho)+\rho j(\theta)$. Keeping track of the rotational invariance, it makes sense to choose an entropy depending upon ρ only. Then it is convex if and only if $e' \geqslant 0$ and $e'' \geqslant 0$.

In this section, we study the relative stability of an admissible discontinuity for (10), which may be a shock or a contact. In both cases, we shall prove that the dissipation rate D is always non-positive. For both situations, this requires studying three positions: the "smooth" one and the "discontinuous" one when (u_-, u_+) is either a shock or a contact.

For the sake of simplicity, we choose the entropy $\eta(u) = \frac{1}{2}|u|^2 = \frac{1}{2}\rho^2$, for which

$$\mathrm{d}\eta\cdot u - \eta = \frac{1}{2}\rho^2, \qquad \mathrm{d}\eta\cdot f(u) - q = \rho g(\rho) - q(\rho), \qquad q' = \rho g'(\rho).$$

We leave the reader verifying that the conclusions hold the same when $\eta = \eta(\rho)$ is another convex entropy.

4.1. Relative stability of a shock wave.

— If shocks are going to be relatively stable in the sense that $D \leqslant 0$ in all situations, then in particular they must be relatively stable when the initial perturbation is purely longitudinal, meaning that $\theta \equiv$ cst at initial time. But then θ remains constant for all time and our system reduces to the equation (10). We have seen in the previous section that this relative stability is equivalent to the global convexity (or global concavity) of g.

We therefore assume that g is a convex function. For a shock wave we have $\theta_r = \theta_\ell$ and $0 < \rho_r < \rho_\ell$.

A necessary condition. — In the smooth case, we have

$$D_{sm} = [\rho g - q] - \phi(\rho)[u]\cdot u,$$

where u obeys to the constraint

$$[u]\cdot u = \Big[\frac{1}{2}\rho^2\Big]. \tag{12}$$

This gives us

$$D_{sm} = [\rho g - q] - \phi(\rho)\Big[\frac{1}{2}\rho^2\Big],$$

where ρ is any value larger than or equal to $\langle\rho\rangle := \frac{1}{2}(\rho_\ell + \rho_r)$ (apply Cauchy–Schwarz to (12)).

In order to find a necessary condition upon the flux, we assume that every shock is relatively stable. Since $[\rho^2/2] < 0$, this tells us that

$$\phi(\rho) \leqslant \frac{[\rho g - q]}{[\rho^2/2]}\,.$$

Let us fix a number $\overline{\rho} > 0$ and write that $D_{sm} \leqslant 0$ for every (ρ_ℓ, ρ_r) satisfying $\langle\rho\rangle = \overline{\rho}$, and for every $\rho \geqslant \langle\rho\rangle$. Passing to the limit when the shock strength vanishes, and using $(\rho g - q)' = g$ as well as $[\rho^2/2] = \langle\rho\rangle[\rho]$, we obtain that

$$\phi(\rho) \leqslant \phi(\overline{\rho}),$$

whenever $\rho \geqslant \overline{\rho}$. In other words, it is necessary that ϕ be decreasing ($\phi' < 0$, to ensure strict hyperbolicity) in order that all shocks be relatively stable.

We therefore make the assumption in the remaining part of this stability analysis(4), that $\phi' < 0$. Then the smooth case is easy: the constraint ensures that $\phi(\rho) \leqslant \phi(\langle\rho\rangle)$. Because of $[\rho^2] \leqslant 0$, there follows

$$D_{sm} \leqslant [\rho g - q] - g(\langle\rho\rangle)[\rho],$$

where the right-hand side is non-positive thanks to the Jensen inequality:

$$g(\langle\rho\rangle) = g\left(\frac{1}{\rho_\ell - \rho_r}\int_{\rho_r}^{\rho_\ell} \rho d\rho\right) \leqslant \frac{1}{\rho_\ell - \rho_r}\int_{\rho_r}^{\rho_\ell} g(\rho)d\rho = \frac{[\rho g - q]}{[\rho]}\,.$$

Finally, when $\phi' < 0$ and g is convex, D_{sm} is non-positive.

When (u_-, u_+) *is a shock*. — We turn now to the first discontinuous case, when the auxiliary discontinuity is also a shock. The dissipation rate

$$D_{RH} = D_0 + cD_1$$

is linear in $c := \cos(\theta_\pm - \theta_{r,\ell})$, with $D_1 = -[\rho](g - \sigma\rho)_\pm$. We point out that, because all states belong to the same half-space $\mathcal{U}$, we have $c \in (0, 1]$.

To determine the sign of D_1, we observe that because of the Lax shock inequality $\sigma < g'_-$, we have

$$(g - \sigma\rho)_\pm = g_- - \sigma\rho_- > g_- - \rho_- g'_- = -\rho_-^2\phi'_- > 0.$$

Therefore $D_1 > 0$ and $D_{RH} \leqslant D_0 + D_1$.

Because the latter value $D_0 + D_1$ is that obtained for $\theta_\pm = \theta_{r,\ell}$, it corresponds to the scalar case, which we know is relatively stable (see Section 3). Therefore $D_0 + D_1 \leqslant 0$ and there follows $D_{RH} \leqslant 0$. This rules out the shock-shock case.

(4)If we had assumed g concave, then the condition would be $\phi' > 0$.

When (u_-,u_+) *is a contact.* — There remains the case where the auxiliary discontinuity is a contact. Then $\rho_+=\rho_-$ (denoted ρ). Let us denote now $c_\pm=\cos(\theta_\pm-\theta_{r,\ell})$, which belong to $(0,1]$. The constraint

$$\min([u]\cdot u_-,[u]\cdot u_+)\leqslant\left[\frac{1}{2}\rho^2\right]\leqslant\max([u]\cdot u_-,[u]\cdot u_+)$$

recasts as

$$\min(c_-,c_+)\leqslant\frac{\langle\rho\rangle}{\rho}\leqslant\max(c_-,c_+)\,.$$

In particular, we have $\rho\geqslant\langle\rho\rangle$.

Thanks to $\rho_+=\rho_-$, we have

$$D_{RH}=[\rho g-q]-\sigma\left[\frac{1}{2}\rho^2\right]=[\rho g-q]-\phi(\rho)\left[\frac{1}{2}\rho^2\right].$$

Because of $\rho\geqslant\langle\rho\rangle$, $\phi'<0$, and $[\rho^2]<0$, there follows

$$D_{RH}\leqslant[\rho g-q]-\phi(\langle\rho\rangle)\left[\frac{1}{2}\rho^2\right]=[\rho g-q]-[\rho]g(\langle\rho\rangle),$$

where we have seen that the right-hand side is non-positive. We deduce again that $D_{RH}\leqslant 0$.

In conclusion, we may state the

Theorem 4.1. — *Let us assume that g is convex and $\phi'<0$ (or as well g is concave and $\phi'>0$). Then the shocks (u_ℓ,u_r) are relative-entropy stable, in the sense that the dissipation rates D_{sm} / D_{RH} are non-positive.*

4.2. Relative stability of a contact discontinuity. — We now assume that (u_ℓ,u_r) is a contact discontinuity, that is $[\rho]=0$. It turns out that the strict hyperbolicity suffices to carry out the calculations; in particular, we don't need genuine nonlinearity.

In the smooth case, the constraint is

$$[u]\cdot u=\left[\frac{1}{2}\rho^2\right]=0,$$

which means that $\theta=\langle\theta\rangle$. Then

$$D_{sm}=[\mathrm{d}\eta\cdot f-q]-[\mathrm{d}\eta]\cdot f(u)=[\rho g-q]-[u]\cdot\phi(\rho)u=0-0=0.$$

When (u_-,u_+) *is another contact,* we have

$$\begin{aligned}D_{RH}&=q_+-q_--\sigma(\eta_+-\eta_-)+[\mathrm{d}\eta\cdot f-q]-\sigma[\mathrm{d}\eta\cdot u-\eta]-[\mathrm{d}\eta]\cdot(f-\sigma u)_\pm\\&=0+[\rho g-q]-\sigma\left[\frac{1}{2}\rho^2\right]-[u]((\phi-\sigma)u)_\pm\\&=0+0-0-0=0,\end{aligned}$$

because of $\sigma=\phi_\pm$.

When (u_-,u_+) *is a shock wave,* the constraint is

$$\min([u]\cdot u_-,[u]\cdot u_+)\leqslant\left[\frac{1}{2}\rho^2\right]=0\leqslant\max([u]\cdot u_-,[u]\cdot u_+).$$

Because u_+ and u_- are colinear with the same orientation, we deduce again that $[u]\cdot u_\pm = 0$. There follows

$$\begin{aligned} D_{RH} &= q_+ - q_- - \sigma(\eta_+ - \eta_-) + [\mathrm{d}\eta\cdot f - q] - \sigma[\mathrm{d}\eta\cdot u - \eta] - [\mathrm{d}\eta]\cdot(f-\sigma u)_\pm \\ &= q_+ - q_- - \sigma(\eta_+ - \eta_-) + [\rho g - q] - \sigma\Big[\frac{1}{2}\rho^2\Big] - [u]((\phi-\sigma)u)_\pm \\ &= q_+ - q_- - \sigma(\eta_+ - \eta_-) + 0 - 0 + 0 \leqslant 0, \end{aligned}$$

where we have used the Lax entropy condition. In conclusion, we have

Theorem 4.2. — *Let us assume that ϕ' does not vanish (strict hyperbolicity). Then the contact discontinuities (u_ℓ, u_r) are relatively stable, in the sense that the dissipation rates D_{sm} / D_{RH} are non-positive.*

5. General systems; a study of D_{sm}

We go back to a strictly hyperbolic system of the general form (1). The analysis of D_{sm} leads us to maximise

$$u \mapsto [\mathrm{d}\eta\cdot f - q] - [\mathrm{d}\eta]\cdot f(u)$$

over the hyperplane Π defined by $[\mathrm{d}\eta]\cdot u = [\mathrm{d}\eta\cdot u - \eta]$. We distinguish two situations, whether this function attains its supremum or not. In the latter case, the supremum is obtained as $u\in\Pi$ tends to the boundary of $\mathcal{U}$; because $\partial\mathcal{U}$ plays the role of infinity, it is unlikely that D_{sm} remain bounded, in particular be non-positive.

We therefore focus onto the first situation: let $\overline{u}\in\Pi$ be a maximum of D_{sm} over Π. Then $[\mathrm{d}\eta]\mathrm{d}f(\overline{u})$ is parallel to $[\mathrm{d}\eta]$, meaning that $[\mathrm{d}\eta]$ is an eigenform of $\mathrm{d}f(\overline{u})$. When (u_ℓ, u_r) is a k-shock of small amplitude, we expect that $\overline{u}$ be close to $u_{r,\ell}$ (see Proposition 5.4 below); then $[u]\sim\epsilon r_k(\overline{u})$ with $|\epsilon|\ll 1$, and $[\mathrm{d}\eta]\sim\epsilon \mathrm{D}^2\eta_{\overline{u}} r_k(\overline{u}) = \epsilon\ell_k(\overline{u})$, where we have normalized $\mathrm{D}^2\eta(r_k, r_k) = \ell_k r_k$. The separation between the eigen-directions thus implies that $\ell_k(\overline{u})$ is the eigenform parallel to $[\mathrm{d}\eta]$.

The case of a Temple field. — We anticipate that the search of a (local) maximum of D_{sm} over Π is better done under the assumption that the kth characteristic field is *genuinely not Temple*. To make this evident, let us consider the opposite case, where this field is of Temple class. This terminology means that ℓ_k is parallel to the differential $\mathrm{d}w_k$ of some function w_k whose level sets are hyperplanes. Then w_k is a called Riemann invariant, and it satisfies formally $\partial_t w_k + \lambda_k\partial_x w_k = 0$.

When the restriction of D_{sm} to Π admits a critical point $\overline{u}$, the hyperplanes Π and $\{u\,|\,w_k(u) = w_k(\overline{u})\}$ both contain $\overline{u}$ and have the same normal $\ell_k(\overline{u})$ at this point. Thus they coincide. Now, at every other point $u\in\Pi$, the normal remains the same, namely $[\mathrm{d}\eta]$. But because Π is a level set of w_k, the normal has to be colinear to $\mathrm{d}w_k(u)$, or to $\ell_k(u)$, and therefore $[\mathrm{d}\eta]$ is an eigenform of $\mathrm{d}f(u)$ for every $u\in\Pi$. This implies that D_{sm} remains constant over Π!

PROPOSITION 5.1. — *Suppose that the kth characteristic field is of Temple class, and that (u_ℓ, u_r) is an admissible discontinuity of the kth family. Then*

- *either D_{sm} is constant over Π (non-generic),*
- *or it does not have any critical point over Π (generic).*

5.1. THE NON-GENERIC CASE IN A TEMPLE SYSTEM. — Suppose that $n = 2$ and both characteristic fields are Temple (we say that the system is of Temple class). The characteristic curves are lines. Assuming that they are not parallel to fixed directions, each line L_a has an equation $u_1 + au_2 = h(a)$. Conversely, when $u \in \mathcal{U}$, the equation $h(a) - u_1 - au_2 = 0$ has two roots $w(u) < z(u)$ which are the Riemann invariants.

Following Chapter 13 of [28], the convex entropies of the system have the form

$$\eta(u) = \int_{w(u)}^{z(u)} (u_1 + au_2 - h(a))\, \mathrm{d}\mu(a),$$

where μ is any non-negative measure. Denoting R, S and T functions of a such that $\mathrm{d}R = \mathrm{d}\mu$, $\mathrm{d}S = a\mathrm{d}\mu$ and $\mathrm{d}T = h(a)\mathrm{d}\mu$, η is given in closed form by the formula

$$\eta(u) = (R(z) - R(w))u_1 + (S(z) - S(w))u_2 - T(z) + T(w)$$

and its differential is

$$\mathrm{d}\eta = (R(z) - R(w))du_1 + (S(z) - S(w))du_2.$$

We have $\mathrm{d}\eta \cdot u - \eta = T(z) - T(w)$. For a k-shock, the opposite Riemann invariant is constant and therefore (up to a constant sign)

$$[\mathrm{d}\eta] = [R(w_k)]du_1 + [S(w_k)]du_2, \qquad [\mathrm{d}\eta \cdot u - \eta] = [T(w_k)].$$

The equation of the line of constraint Π is therefore

$$[R(w_k)]u_1 + [S(w_k)]u_2 = [T(w_k)].$$

In the non-generic case of Proposition 5.1, this line is characteristic, which means

$$\frac{[T(w_k)]}{[R(w_k)]} = h\left(\frac{[S(w_k)]}{[R(w_k)]}\right). \tag{13}$$

The equation (13) amounts to writing (say that $w_k = z$)

$$\frac{1}{\int_{z_g}^{z_d} \mathrm{d}\mu(a)} \int_{z_g}^{z_d} h(a)\mathrm{d}\mu(a) = h\biggl(\frac{1}{\int_{z_g}^{z_d} \mathrm{d}\mu(a)} \int_{z_g}^{z_d} a\mathrm{d}\mu(a)\biggr).$$

If this is going to be true for every convex entropy, that is for every positive measure μ, then h has to be affine. This amounts to saying that all the lines L_a intersect at some point; this point may lie at infinity, in which case the lines are parallel.

In conclusion, the non-generic case of a Temple system, the one for which D_{sm} is constant over the line of constraint Π, happens precisely when the characteristic lines L_a of one family are converging or are parallel. This rules out the so-called Leroux system

$$\partial_t u_1 + \partial_x(u_1 u_2) = 0, \qquad \partial_t u_2 + \partial_x(u_1 + u_2^2) = 0,$$

but it is consistent with the system of electrophoresis (where actually $n \geqslant 2$)

$$\partial_t u_i + \partial_x \frac{a_i u_i}{m} = 0, \quad m = \sum_i u_i, \qquad u_i(x,t) > 0.$$

When an $n \times n$ Temple system is non-generic in the above sense, then $D_{sm} \equiv D_{sm}(\overline{u})$ over Π, where $\overline{u}$ is the intersection point of Π with the segment $[u_\ell, u_r]$. Because this segment is contained in a characteristic line L, and this line is an invariant subset, the calculation of $D_{sm}(\overline{u})$ actually occurs in the relative stability analysis of the shock, when considering disturbances that take values only along L. This is nothing but the relative stability of the shock, as a solution of a scalar equation (the system restricted to L), which we know is true when the corresponding field is genuinely nonlinear. Under this GNL assumption, we deduce that $D_{sm}(\overline{u}) \leqslant 0$, and therefore $D_{sm} \leqslant 0$ over Π. We summarize this analysis into

Proposition 5.2. — *Suppose that the system is of Temple class. Suppose that the kth characteristic field is genuinely nonlinear and non-generic as described in Proposition* 5.1. *Then given a k-shock (u_ℓ, u_r), we have $D_{sm} \leqslant 0$ for every $u \in \Pi$.*

Let us instead consider the rate D_{RH} when the shocks (u_ℓ, u_r) and (u_-, u_+) correspond to distinct families. Then $w_k(u_-) = w_k(u_+)$; but because Π is a level set of w_k that separates u_- from u_+, we are actually in the limit situation, where both $u_\pm$ belong to Π: $[\mathrm{d}\eta] \cdot u_\pm = [\mathrm{d}\eta \cdot u - \eta]$. We then have

$$\begin{aligned} D_{RH} &= q_+ - q_- - \sigma(\eta_+ - \eta_-) + [\mathrm{d}\eta \cdot f - q] - \sigma[\mathrm{d}\eta \cdot u - \eta] - [\mathrm{d}\eta] \cdot (f - \sigma u)_\pm \\ &= q_+ - q_- - \sigma(\eta_+ - \eta_-) + [\mathrm{d}\eta \cdot f - q] - [\mathrm{d}\eta] \cdot f(u_\pm) \\ &\leqslant D_{sm}(u_\pm) \leqslant 0, \end{aligned}$$

where we have used the Lax entropy inequality and then Proposition 5.2. In conclusion, we state

Proposition 5.3. — *Suppose that the system is of Temple class. Suppose that the kth characteristic field is genuinely nonlinear and non-generic as described in Proposition* 5.1. *Then given a k-shock (u_ℓ, u_r) and a j-shock (u_-, u_+) with $j \neq k$, we have $D_{RH}(u_\ell, u_r; u_-, u_+) \leqslant 0$.*

5.2. Critical points of D_{sm} over Π; genuinely non-Temple fields. — We are therefore interested in critical points u of D_{sm} over Π. As explained above, this means that

$$[\mathrm{d}\eta] \cdot u = [\mathrm{d}\eta \cdot u - \eta], \qquad [\mathrm{d}\eta] \text{ is an eigenform of } \mathrm{d}f(u). \tag{14}$$

We perform a local analysis, which covers the case where the shock strength is small and u is close to $u_{\ell,r}$. Recall that in this situation, the strict hyperbolicity implies that the second part of (14) is that $[\mathrm{d}\eta]$ is parallel to $\ell_k(u)$. We thus rewrites (14) as

$$[\mathrm{d}\eta] \cdot u = [\mathrm{d}\eta \cdot u - \eta] \quad \text{and} \quad [\mathrm{d}\eta] \cdot r_j(u) = 0, \quad \forall j \neq k. \tag{15}$$

Let us recall the local description of the kth Hugoniot curve $H_k(\overline{u})$. The Rankine–Hugoniot equation $f(v) - f(\overline{u}) = \sigma(v - \overline{u})$ can be recast as an eigenvalue problem:

$$(A(v,\overline{u}) - \sigma)(v - \overline{u}) = 0, \qquad A(v,\overline{u}) := \int_0^1 \mathrm{d}f(sv + (1-s)\overline{u})\, ds.$$

The matrix $A(v,\overline{u})$ is a smooth function of its arguments. At $v = \overline{u}$, it coincides with $\mathrm{d}f(\overline{u})$. Because the latter has real, simple eigenvalues, this is also true for $A(v,\overline{u})$ as long as v remains in some neighbourhood $\mathcal{V}(\overline{u})$ then the eigenvalues/-vectors $\lambda_j(v,\overline{u})$ / $r_j(v,\overline{u})$ are smooth functions(5). Then the Rankine–Hugoniot relation amounts to saying that σ is an eigenvalue $\lambda_j(v,\overline{u})$ and $v - \overline{u}$ is colinear to $r_j(v,\overline{u})$. The curve $H_k(\overline{u})$ is thus defined implicitly as a parametrized curve $\epsilon \mapsto v(\epsilon)$ by

$$v = \overline{u} + \epsilon r_k(v,\overline{u}).$$

Thank to the implicit function theorem, $v(\epsilon)$ is well-defined for ϵ small enough, with

$$v(0) = \overline{u}, \qquad \left.\frac{dv}{d\epsilon}\right|_{\epsilon=0} = r_k(\overline{u}).$$

Let us now rewrite (15), using the same trick as for Rankine–Hugoniot, where $u_\ell = \overline{u}$ and $u_r = v(\epsilon)$. For instance,

$$[\mathrm{d}\eta] = [u]^T \Sigma(u_\ell, u_r), \qquad \Sigma(u_\ell, u_r) := \int_0^1 \mathrm{D}^2\eta_{su_\ell + (1-s)u_r}\, ds, \tag{16}$$

and

$$[\mathrm{d}\eta \cdot u - \eta] = m(u_r, u_\ell) \cdot [u], \qquad m(u_r, u_\ell) := \int_0^1 (su_\ell + (1-s)u_r)^T \mathrm{D}^2\eta_{su_\ell + (1-s)u_r}\, ds.$$

Then (15) can be recast as

$$(r_k^T \Sigma)_{u_\ell, u_r} u = (m \cdot r_k)_{u_\ell, u_r} \quad \text{and} \quad (r_k^T \Sigma)_{u_\ell, u_r} r_j(u) = 0, \quad \forall j \neq k. \tag{17}$$

After these preliminaries, we may define a non-linear map

$$(\epsilon, u) \mapsto \mathcal{N}(\epsilon, u) := \left.\begin{pmatrix} r_k^T \Sigma u - m \cdot r_k \\ r_k^T \Sigma r_j(u), \quad \forall j \neq k \end{pmatrix}\right|_{u_\ell = v(\epsilon), u_r = \overline{u}},$$

where the arguments of Σ, r_k and m, the quantities that do not depend explicitly on u, are $(v(\epsilon), \overline{u})$. Then (15) is equivalent to

$$\mathcal{N}(\epsilon, u) = 0. \tag{18}$$

When $\epsilon = 0$, we know that $\Sigma = \mathrm{D}^2\eta_{\overline{u}}$ and $r_k = r_k(\overline{u})$; because the eigenbasis of $\mathrm{d}f$ is orthogonal relatively to $\mathrm{D}^2\eta$, we deduce that $r_k^T \Sigma$ reduces to a kth eigenform of $\mathrm{d}f(\overline{u})$, say $\ell_k(\overline{u})$. Likewise, m reduces to $\overline{u}^T \mathrm{D}^2\eta_{\overline{u}}$. Therefore

$$\mathcal{N}(0, u) = \begin{pmatrix} \ell_k(\overline{u}) \cdot (u - \overline{u}) \\ \ell_k(\overline{u}) \cdot r_j(u), \quad \forall j \neq k \end{pmatrix}.$$

(5)Here we need a choice of the eigenfield. It can be specified by a normalization, say that $r_j^T \Sigma r_j = 1$ where $\Sigma(v, \overline{u})$ is given as in (16) below.

By construction, we have $\mathcal{N}(0,\overline{u})=0$. Differentiating at this point, there comes

$$(\mathrm{D}_u\mathcal{N}_{0,\overline{u}})h=\begin{pmatrix}\ell_k(\overline{u})\cdot h\\ \ell_k(\overline{u})\cdot(\mathrm{d}_{\overline{u}}r_j\cdot h),\quad \forall j\neq k\end{pmatrix}.$$

In order to apply the Implicit Function Theorem, we make the assumption

(GNT) $\mathrm{D}_u\mathcal{N}_{0,\overline{u}}$ *is non-singular.*

When (GNT) is fulfilled, the IFT tells us that the equation (18) is locally uniquely solvable as a smooth function $\epsilon\mapsto u(\epsilon)$. This $u(\epsilon)$ is therefore the unique critical point of D_{sm} over Π, close to $\overline{u}$, when the shock is given by $u_\ell=\overline{u}$ and $u_r=v(\epsilon)$.

Proposition 5.4. — *Suppose that the system* (1) *is strictly hyperbolic. Denote $H_k(u_\ell)$ the local Hugoniot curve, tangent at u_ℓ to $r_k(u_\ell)$.*

If (GNT) *is satisfied at u_ℓ, then for every $u_r\in H_k(u_\ell)\cap\mathcal{V}$ ($\mathcal{V}$ a suitable neighbourhood of u_ℓ), there exists a unique point $u\in\mathcal{W}$ ($\mathcal{W}$ a suitable neighbourhood of u_ℓ) such that*

- $u\in\Pi$, *where* Π *is the constrained hyperplane defined by* (5),
- *the restriction of D_{sm} to Π is critical at u.*

This point u is a smooth function of u_r along $H(u_\ell)$.

Geometrical interpretation of (GNT). — A vector h is in the kernel of $\mathrm{D}_u\mathcal{N}_{0,\overline{u}}$ if $h\in\ell_k^\perp$ and $\ell_k\cdot(\mathrm{d}r_j\cdot h)=0$ for every $j\neq k$. Because $\ell_k\cdot r_j\equiv 0$, we have

$$\ell_k\cdot(\mathrm{d}r_j\cdot h)=-(\mathrm{D}\ell_k\cdot h)\cdot r_j.$$

Therefore, the second part of the kernel condition is that $\mathrm{D}\ell_k\cdot h$ is parallel to ℓ_k.

The situation is especially clear when the kth characteristic field is *rich.* This means (see Chapter 12 of [28]) that ℓ_k derives from a Riemann Invariant w_k, say that $\ell_k=\alpha\,\mathrm{d}w_k$ where α is a positive function. For instance, every 2×2 system is rich. Then h belongs to the kernel if and only if $\mathrm{d}w_k\cdot h=0$ and $\mathrm{D}^2w_k(h,r)=0$ for every linear combination r of the r_j's, that is for every r such that $\mathrm{d}w_k\cdot r=0$. In other words, h belongs to the kernel of the quadratic form

$$\mathrm{D}^2w_k\big|_{\ker\mathrm{d}w_k}.$$

The condition (GNT) thus expresses that this kernel is trivial. This amounts to saying that the second fundamental form of the level set $\{w_k=w_k(\overline{u})\}$ at $\overline{u}$ is non-degenerate. Because the Temple property would be that this second fundamental form vanish identically, we say that the system is *Genuinely Non Temple* at $\overline{u}$.

Proposition 5.5. — *Suppose that the system* (1) *is strictly hyperbolic, and that the kth characteristic fields admits a Riemann invariant w_k in the strong sense (that is $\mathrm{d}w_k$ is an eigenform of $\mathrm{d}f$ associated with λ_k). Then* (GNT) *is equivalent to the property that the second fundamental form of the level set $\{w_k=w_k(\overline{u})\}$ is non-degenerate at $\overline{u}$.*

Comments. — It is remarkable that this condition, which originally was associated with a prescribed convex entropy (because Π and D_{sm} do depend upon the choice of η), actually depends only on the geometry of the characteristic fields. – When the kth field is not rich, the form ℓ_k defines a non-integrable field of hyperplanes. When $n = 3$, this is just a *contact structure*. If there is an ambient Riemannian metric, this is a so-called a *CR manifold* (though it is not a manifold!), to which a curvature can be attributed. Then (GNT) amounts to saying that this curvature is non-degenerate. Although we do have a Riemannian structure, that inherited from $\mathrm{D}^2\eta$, the one under consideration here is the flat metric of $\mathbb{R}^n$. Mind that this flat metric is defined up to a linear change of coordinates, so that the curvature is not well-defined; but its (non)-degeneracy is intrinsic in that it does not depend upon the reference frame.

5.3. LOCAL MAXIMALITY AT THE CRITICAL POINT. — Recall that among the critical points of D_{sm} over Π, we are really interested in maxima. We have seen in Proposition 5.4 that under (GNT), and if the shock strength is weak enough, then D_{sm} has a privileged critical point, which we denote U. A natural question is thus whether U is a local maximum. For this we calculate the Hessian of D_{sm} at U in the direction of Π. The global Hessian of D_{sm} is $-[\mathrm{d}\eta]\mathrm{D}^2 f_U$.

Let us assume genuine nonlinearity, so that $[u] \sim -\epsilon r_k$ where $\epsilon > 0$ and $\mathrm{d}\lambda_k \cdot r_k > 0$. Then $-[\mathrm{d}\eta] \sim \epsilon \mathrm{D}^2\eta r_k = \epsilon\ell_k$. Because U is the critical point, we know that $[\mathrm{d}\eta]$ is colinear to $\ell_k(U)$. Therefore we do have

$$-[\mathrm{d}\eta] = (\epsilon + O(\epsilon^2))\ell_k(U).$$

This shows that the Hessian of D_{sm} at U is positively proportional to

$$Q_U := \ell_k \mathrm{D}^2 f_U.$$

Now, we are interested in the restriction of Q_U to the subspace $\ell_k(U)^\perp$, the direction of Π.

LEMMA 5.6. — *Under* (GNT), *the restriction of the quadratic form Q_U to $\ell_k(U)^\perp$ is non-degenerate.*

Proof. — Starting from the identity $\ell_k(\mathrm{d}f - \lambda_k) = 0$, we have

$$\ell_k \mathrm{D}^2 f + \mathrm{D}\ell_k(\mathrm{d}f - \lambda_k) = \ell_k \otimes \mathrm{d}\lambda_k.$$

If $h, k \in \ell_k^\perp$, this gives

$$\ell_k \mathrm{D}^2 f(h, k) + (\mathrm{D}\ell_k \cdot h)(\mathrm{d}f - \lambda_k)k = 0.$$

Suppose now that $h \in \ker Q_U$, that is $\ell_k \mathrm{D}^2 f(h, k) = 0$ (with $u = U$) for every $k \in \ell_k^\perp$. Then we find $(\mathrm{D}\ell_k \cdot h)(\mathrm{d}f - \lambda_k)k = 0$ for every such k. By strict hyperbolicity, $\mathrm{d}f - \lambda_k$ is an automorphism of $\ell_k^\perp$ and therefore this tells us that $\mathrm{D}\ell_k \cdot h$ is parallel to ℓ_k. By (GNT), this implies $h = 0$. □

Thanks to Lemma 5.6, D_{sm} achieves a local maximum if and only if the restriction of Q_U is negative definite, which amounts to saying that the restriction to $\ell_k^\perp$ of the quadratic form

$$R_U(h) := (\mathrm{D}\ell_k \cdot h)(\mathrm{d}f - \lambda_k)h$$

is positive definite.

About normalizations. — When carrying calculations about some specific system, it may be boring to follow all the normalizations of the eigenfields. This is not needed actually, if we express the final result in terms that are invariant under the flips $r_k \longleftrightarrow -r_k$ or $\ell_k \longleftrightarrow -\ell_k$. When the kth field is GNL and GNT, the necessary condition for D_{sm} to achieve a local maximum at U is that the quadratic form

$$X \mapsto (\ell_k \cdot r_k)(\mathrm{d}\lambda_k \cdot r_k)\ell_k \mathrm{D}^2 f_U X \otimes X$$

be negative definite over $\ker \ell_k$.

5.3.1. *The rich case.* — When the kth characteristic field admits a Riemann invariant (in the strong sense) w_k, we may replace ℓ_k by $\mathrm{d}w_k$. We point out that because $[w_k] \sim -\epsilon \mathrm{d}w_k \cdot r_k$, and $\mathrm{d}w_k$ has the same orientation as ℓ_k, we have

$$[w_k] < 0. \tag{19}$$

Lemma 5.7. — *Let the indices (i, j, k) be pairwise distinct. Then*

$$\mathrm{D}^2 w_k(r_i, r_j) = 0.$$

In particular, the signs of the the principal curvatures of the level set of w_k are equal to the signs of the numbers $\mathrm{D}^2 w_k(r_j, r_j)$ for $j \neq k$.

Proof. — Because of $\mathrm{D}^2 w_k((\mathrm{d}f - \lambda_k)h, k) = -\mathrm{d}w_k \cdot \mathrm{D}^2 f(h, k)$, the form

$$(h, k) \mapsto \mathrm{D}^2 w_k((\mathrm{d}f - \lambda_k)h, k)$$

is symmetric over $\ell_k^\perp$:

$$(\ell_k \cdot h = \ell_k \cdot k = 0) \Longrightarrow \mathrm{D}^2 w_k((\mathrm{d}f - \lambda_k)h, k) = \mathrm{D}^2 w_k((\mathrm{d}f - \lambda_k)k, h).$$

Taking $h = r_i$ and $k = r_j$, we deduce

$$(\lambda_i - \lambda_j)\mathrm{D}^2 w_k(r_i, r_j) = 0. \qquad \square$$

Instead of R_U, we may now consider the form

$$R'_U(h) := \mathrm{D}^2 w_k((\mathrm{d}f - \lambda_k)h, h), \qquad h, k \in \ell_k^\perp,$$

which has to be positive definite for local maximality. Because of Lemma 5.7, R'_U is diagonalized in the basis $(r_j)_{j \neq k}$, and we have

$$R'_U(r_j) = (\lambda_j - \lambda_k)\mathrm{D}^2 w_k(r_j, r_j).$$

Its signature is therefore related to the principal curvatures of the level set $\{u \mid w_k(u) = w_k(U)\}$:

PROPOSITION 5.8. — *If the critical point* U *is a local maximum of* D_{sm} *over* Π*, then the principal curvatures of the level set of* w_k *at* U *have the same signs as the differences* $\lambda_j - \lambda_k$ *when* j *runs over* $1, \dots, k-1, k+1, \dots, n$.

The situation is even better when (u_ℓ, u_r) is an *extreme shock*, that is when either $k = 1$ or $k = n$. For instance, if $k = 1$ then all the $\lambda_j - \lambda_1$ are positive, and the local maximality implies that the restriction of $\mathrm{D}^2 w_1$ to $\ker \mathrm{d}w_1$ is positive definite. This is exactly saying that the level set of w_1 is convex at U, with convexity turned toward u_r. When instead $k = n$, we find that the convexity turns toward u_ℓ. We point out that in this situation, the necessary condition is also sufficient, because the positive (or negative) definiteness of the tangential Hessian of w_k does imply $R'_U(r_j) > 0$, which means positive definiteness of R'_U. Let us summarize these results.

PROPOSITION 5.9 (Extreme shocks; rich case). — *Let* k *equal either* 1 *or* n*. We assume that the* k*th characteristic field is Genuinely non linear, Genuinely non Temple and is associated with a strong Riemann invariant* w_k*. The latter is oriented so that* $(\mathrm{d}\lambda_k \cdot r_k) \times (\mathrm{d}w_k \cdot r_k)$ *is positive. Let* (u_ℓ, u_r) *be a shock of small strength and* U *be the critical point of* D_{sm} *over* Π *mentioned in Proposition* 5.4*. Then the following statements are equivalent to each other:*

- *the restriction of* D_{sm} *to* Π *achieves a local maximum at* U*,*
- *the level set* $\{u \,|\, w_k(u) = w_k(U)\}$ *is convex at* U*, and its convexity is turned towards* u_r *(if* $k = 1$*) or* u_ℓ *(if* $k = n$*).*
- *the numbers* $\mathrm{D}^2 w_k(r_j, r_j)$ *for* $j \neq k$ *are positive if* $k = 1$ *(respectively negative for* $k = n$*).*

6. EXAMPLES AND COUNTER-EXAMPLES

6.1. THE EFFECT OF AN EULER–LAGRANGE TYPE TRANSFORMATION. — The phase space $\mathcal{U}$ in which $u(t,x)$ takes its values is usually a convex, strict subset of $\mathbb{R}^n$. It is therefore contained in a half-space. Because we are free to choose linear coordinates $(u_1, \dots, u_n)$, we may assume that $\mathcal{U}$ is contained in $\{u \,|\, u_1 > 0\}$.

When it is so, the first conservation law $\partial_t u_1 + \partial_x f_1 = 0$ is the compatibility condition for the existence of a function y such that

$$\partial_x y = u_1, \qquad \partial_t y = -f_1,$$

which may be use to design a change of variables $(x,t) \mapsto (y,t)$, because of $dy \wedge dt = u_1 dx \wedge dt \neq 0$. Remark that in gas dynamics, with $u_1 = \rho$, the density, then $(x,t) \mapsto (y,t)$ is the transformation from Eulerian coordinates into Lagrangian mass coordinates. It is shown in [27] (see also Wagner [32] for the system of gas dynamics) that there is a one-to-one correspondence between weak entropy solutions u of (1) and weak entropy solutions v of

$$\partial_t v + \partial_y g(v) = 0, \tag{20}$$

where $v_1 = 1/u_1$, $g_1 = -f_1/u_1$ and

$$v_j = \frac{u_j}{u_1}, \qquad g_j = f_j - \frac{u_j}{u_1} f_1$$

otherwise. We point out that the image $\mathcal{V}$ of $\mathcal{U}$ under $u \mapsto v$ is again convex, for if $a \cdot u \geqslant \alpha$ over $\mathcal{U}$, then

$$a_1 + a_2 v_2 + \cdots + a_n v_n \geqslant \alpha v_1,$$

and conversely.

If (1) admits an entropy-flux pair (η, q) in which $u \mapsto \eta$ is strongly convex, then (20) admits the entropy-flux pair $(\Phi := \eta/u_1, Q := q - f_1\eta/u_1)$, in which $v \mapsto \Phi$ is strongly convex. Our main observation is the following.

Proposition 6.1. — *Let $u, \overline{u} \in \mathcal{U}$ be given, and $v, \overline{v}$ be their images under the Euler–Lagrange type transformation. Then we have*

$$\eta(u|\overline{u}) = u_1 \Phi(v|\overline{v}).$$

Corollary 6.2. — *Under an Euler–Lagrange type transformation, we have*

$$\int_J \eta(u(x,t)|\overline{u})\, dx = \int_{J'} \Phi(v(y,t)|\overline{v})\, dy,$$

where J and J' are in correspondence through $x \mapsto y(x,t)$.

Proof. — Every function $h(u) = H(v)$ satisfies the following identities:

$$\frac{\partial h}{\partial u_1} = -v_1 \mathrm{d}H \cdot v, \qquad \frac{\partial h}{\partial u_j} = v_1 \frac{\partial H}{\partial v_j} \quad (j \geqslant 2).$$

From the above, we obtain $\mathrm{d}\eta \cdot u - \eta = -\partial\Phi/\partial v_1$, and conversely

$$\frac{\partial H}{\partial v_1} = -u_1 \mathrm{d}h \cdot u, \qquad \mathrm{d}\Phi \cdot v - \Phi = -\frac{\partial \eta}{\partial u_1}.$$

This yields the following calculations

$$\begin{aligned}
\eta(u|\overline{u}) &= \eta(u) - \mathrm{d}\eta(\overline{u}) \cdot u + \mathrm{d}\eta(\overline{u}) \cdot \overline{u} - \eta(\overline{u}) \\
&= u_1 \Phi(v) - u_1(\Phi(\overline{v}) - \mathrm{d}\Phi(\overline{v}) \cdot \overline{v}) - \sum_2^n u_j \overline{v}_1 \left. \frac{\partial(\Phi/v_1)}{\partial v_j} \right|_{\overline{v}} - \frac{\partial \Phi}{\partial v_1}(\overline{v}) \\
&= u_1 \left(\Phi(v) - \Phi(\overline{v}) - \mathrm{d}\Phi(\overline{v}) \cdot (v - \overline{v}) \right) \\
&= u_1 \Phi(v|\overline{v}).
\end{aligned}$$

□

These properties show that if some meaningful statement about the dissipation rate is true in the original formulation (system (1), variables t, x, u), then it is also true in the modified formulation (system (20), variables t, y, v). And conversely. This applies to various questions, like those about its sign or its critical points. We illustrate this principle with the constrained hyperplane defined by (5):

$$[\mathrm{d}\eta] \cdot u = [\mathrm{d}\eta \cdot u - \eta].$$

With the formulae above, the right-hand side is nothing but $-[\partial\Phi/\partial v_1]$, while the left-hand side rewrites as

$$u_1\left[\Phi - \mathrm{d}\Phi\cdot v\right] + \sum_2^n u_j\left[v_1\frac{\partial(u_1\Phi)}{\partial v_j}\right].$$

Gathering these expression in (5) and dividing by u_1, we end up with

$$[\mathrm{d}\Phi]\cdot v = [\mathrm{d}\Phi\cdot v - \Phi].$$

Finally, we obtain that the Euler-Lagrange type transformation sends the u-hyperplane of constraints onto the v-hyperplane of constraints.

6.2. The p-system. — Let us consider the well-known 2×2 system

$$\partial_t u_1 + \partial_x u_2 = 0, \qquad \partial_t u_2 + \partial_x p(u_1) = 0. \tag{21}$$

It is strictly hyperbolic if $p' > 0$. We assume genuine nonlinearity, namely that p'' does not vanish. This is a situation where (GNL) implies (GNT).

The wave velocities are $\lambda_\pm = \pm\sqrt{p'(u_1)}$. The eigenvectors, eigenforms and Riemann invariants are

$$r_\pm = p''\begin{pmatrix}\pm 1\\ \sqrt{p'}\end{pmatrix}, \qquad \ell_\pm = p''\left(\pm\sqrt{p'}\ 1\right), \qquad w_\pm = (\mathrm{sgn}p'')\left(u_2 \pm \int^{u_1}\sqrt{p'(s)}\,ds\right),$$

where the factor p'' or its sign have been chosen in order that $\mathrm{d}\lambda_k\cdot r_k$, $\ell_k\cdot r_k$ and $\mathrm{d}w_k\cdot r_k$ are all positive.

With $\mathrm{D}^2 w_\pm = \pm\frac{|p''|}{\sqrt{p'}}(du_1)^2$, we have

$$\mathrm{D}^2 w_\pm(r_\mp, r_\mp) = \pm\frac{|p''|^3}{\sqrt{p'}}.$$

With the notations of the previous paragraph, this gives

$$\mathrm{D}^2 w_1(r_2, r_2) < 0, \qquad \mathrm{D}^2 w_2(r_1, r_1) > 0,$$

which are the exact opposite of the third statement in Proposition 5.9. We deduce that in the p-system, the critical point of D_{sm} furnished by Proposition 5.4 is actually a local minimum instead of a maximum.

This analysis tells us that the supremum of D_{sm} is obtained when letting the point u tend to one of the extremities of the line of constraints Π. As a matter of fact, Π has an equation of the form

$$\sigma u_1 + u_2 = \mathrm{cst},$$

where σ is the shock velocity. This implies that along Π,

$$D_{sm} = \mathrm{cst} + [u_2](\sigma^2 u_1 - p(u_1)).$$

It is well-known that when the system is GNL, that is when p'' keeps a constant sign, then $[u_2]p''$ is negative. We infer that D_{sm} is a strongly convex function of u_1 along Π. If (m, M) denotes the domain of definition of p, we have $\sup_\Pi D_{sm} = +\infty$, unless $\tau\mapsto p(\tau)-\sigma^2\tau$ is uniformly bounded. The latter instance is unlikely; for instance, if (m, M) is unbounded, this and the convexity/concavity of p would imply that p is affine, contradicting the genuine non linearity.

Comments. — System (21) models either one-D elasticity or isentropic gas dynamics in Lagrangian coordinates, when u_1 is the specific length and u_2 the material velocity. Thanks to the previous paragraph, we know that this counter-example translates into another one about the Eulerian form of isentropic gas dynamics. – It is a bit astonishing that for *every* genuinely nonlinear equation of state, the L^2-stability of shock waves in the p-system is not handable by Leger's technique. The reason for this is subtle: the same differential quantity p'' determines simultaneously whether the fields are linear or not, and whether they are Temple or not. This is no longer true for general systems, as the Temple property and the Genuine nonlinearity are distinct properties. Thus we should not take this example as an argument to reject Leger's technique.

6.3. Non-isentropic gas dynamics. — One-D full gas dynamics obeys to a 3×3 system, whose extreme (acoustic) fields are not integrable (i.e., they are not rich). Thanks to Paragraph 6.1, we are free to choose between the Eulerian and the Lagrangian formulations to carry out the calculations. The latter looks easier to deal with. We thus consider the system

$$\begin{aligned} \partial_t \tau - \partial_x v &= 0, \\ \partial_t v + \partial_x q(\tau, e) &= 0, \\ \partial_t \Big(\frac{1}{2} v^2 + e\Big) + \partial_x (qv) &= 0, \end{aligned}$$

where τ is the specific length and q the pressure. We have

$$u = \begin{pmatrix} \tau \\ v \\ \frac{1}{2} v^2 + e \end{pmatrix}, \qquad f(u) = \begin{pmatrix} -v \\ q \\ qv \end{pmatrix}.$$

The wave velocities are $0, \pm c$ with $c = \sqrt{q q_e - q_\tau}$. The corresponding eigenfields are

$$r_0 = \begin{pmatrix} q_e \\ 0 \\ -q_\tau \end{pmatrix}, \qquad r_\pm = \begin{pmatrix} -1 \\ \pm c \\ q \pm vc \end{pmatrix}$$

and

$$\ell_0 = \begin{pmatrix} q & -v & 1 \end{pmatrix}, \qquad \ell_\pm = \begin{pmatrix} q_\tau & \pm c - q_e v & q_e \end{pmatrix}.$$

In other words, we have $\ell_\pm = \mathrm{d}q \pm c \mathrm{d}v$, and

$$\mathrm{d}\tau \cdot r_0 = q_e, \quad \mathrm{d}v \cdot r_0 = 0, \quad \mathrm{d}e \cdot r_0 = -q_\tau, \quad \mathrm{d}\tau \cdot r_\pm = -1, \quad \mathrm{d}v \cdot r_\pm = \pm c, \quad \mathrm{d}e \cdot r_\pm = q.$$

Remark also

$$\mathrm{d}q \cdot r_0 = 0, \qquad \mathrm{d}q \cdot r_\pm = c^2.$$

Now, we have

$$\ell_\pm \mathrm{D}^2 f = (\pm c - q_e v) \mathrm{D}^2 q + q_e \mathrm{D}^2 (qv) = \pm c \mathrm{D}^2 q + 2 q_e \mathrm{d}v \, \mathrm{d}q.$$

We deduce the following formulae

$$\begin{aligned}\ell_\pm \mathrm{D}^2 f r_0 \otimes r_0 &= \pm c(q_e^2 q_{\tau\tau} - 2q_e q_\tau q_{e\tau} + q_\tau^2 q_{ee})\\ \ell_\pm \mathrm{D}^2 f r_0 \otimes r_\mp &= \pm c(-q_e q_{\tau\tau} + (q_\tau + q q_e) q_{e\tau} - q q_\tau q_{ee})\\ \ell_\pm \mathrm{D}^2 f r_\mp \otimes r_\mp &= \pm c(q_{\tau\tau} - 2q q_{e\tau} + q^2 q_{ee} - 2c^2 q_e).\end{aligned}$$

We point out that for an ideal gas, meaning that $q = (\gamma-1)e/\tau$ with $\gamma > 1$ the *adiabatic constant*, one has $\ell_\pm \mathrm{D}^2 f r_0 \otimes r_0 \equiv 0$. If the restriction of $\ell_\pm \mathrm{D}^2 f$ over $\ker \ell_\pm$ is going to be semi-definite, then we need also that

$$\ell_\pm \mathrm{D}^2 f r_0 \otimes r_\mp = \mp\gamma(\gamma-1)^2 \frac{ce}{\tau^4}$$

vanish, which is impossible. This shows that the ideal gas cannot be treated by our method of relative entropy.

Appendix. Proof of Lemma 2.2

Since $u \in BV_{\mathrm{loc}}$, for every Lipschitz function $t \to h(t)$, we define for almost every $t > 0$:

$$\begin{aligned}U_{\max}(t) &= \max\{V_\epsilon(u(t, h(t)-), V_\epsilon(u(t, h(t)+))\},\\ U_{\min}(t) &= \min\{V_\epsilon(u(t, h(t)-), V_\epsilon(u(t, h(t)+))\}.\end{aligned}$$

Consider the classic mollifier function defined on $\mathbb{R}$ for any positive integer m

$$\delta_m(x) = m\delta_1(mx),$$

where δ_1 is a smooth non-negative function, compactly supported in $(0,1)$, with integral equal to 1. We define on $\mathbb{R}^+ \times \mathbb{R}$

$$U_m(t,x) = \int_{\mathbb{R}} \delta_m(y) V_\epsilon(u(t, x-y))\, dy.$$

The function U_m is Lipschitz in x. We consider h_m the (unique) solution to the ODE:

$$\begin{aligned}\dot{h_m} &= U_m(t, h_m),\\ h_m(0) &= 0.\end{aligned}$$

The function h_m is uniformly Lipschitz in time with respect to m. Hence there exists a Lipschitz function $t \to h_\epsilon(t)$ such that (up to a subsequence) h_m converges to h_ϵ when m goes to infinity, in $C^0(0,T)$, for every $T > 0$. Note that $\dot{h_m}$ converges weakly-* in L^∞ to $\dot{h}_\epsilon$. We consider $U_{\max}$ and $U_{\min}$ as above for this particular fixed function h_ϵ. We show that for almost every $t > 0$

$$\begin{aligned}\lim_{m\to\infty} [\dot{h_m}(t) - U_{\max}(t)]_+ &= 0,\\ \lim_{m\to\infty} [U_{\min}(t) - \dot{h_m}(t)]_+ &= 0.\end{aligned}$$

Both limits are proved the same way. Let us focus on the first one. We have

$$\begin{aligned}
[\dot{h_m}(t) - U_{\max}(t)]_+ &= \left[\int_{\mathbb{R}} V_\epsilon(u(t, h_m(t) - y))\delta_m(y)\,dy - U_{\max}(t)\right]_+ \\
&= \left[\int_{\mathbb{R}} (V_\epsilon(u(t, h_m(t) - y)) - U_{\max}(t))\delta_m(y)\,dy\right]_+ \\
&\leqslant \int_{\mathbb{R}} [V_\epsilon(u(t, h_m(t) - y)) - U_{\max}(t)]_+\,\delta_m(y)\,dy \\
&\leqslant \operatorname*{ess\,sup}_{y\in(0,1/m)} [(V_\epsilon(u(t, h_m(t) - y)) - U_{\max}(t))]_+\,, \\
&\leqslant \operatorname*{ess\,sup}_{z\in(-\epsilon_m,\epsilon_m)} [V_\epsilon(u(t, h_\epsilon(t) - z)) - U_{\max}(t)]_+\,,
\end{aligned}$$

where, for a given $t > 0$, $\epsilon_m \to \infty$ is chosen such that $h_m(t)-h_\epsilon(t) \in (-\epsilon_m, \epsilon_m-1/m)$. Since $u \in BV_{\mathrm{loc}}$, for almost every $t > 0$, the last term above converges to 0 when $m \to \infty$. This proves that for almost every $t > 0$, $\dot{h}_\epsilon \in I(V_\epsilon(u_-), V_\epsilon(u_+))$.

To show that $(u_-, u_+, \dot{h})$ is an admissible entropic discontinuity (or else $u_- = u_+$), we consider

$$\psi_m(x) = \int_x^\infty (\delta_m(y) - \delta_m(-y))\,dy.$$

Note that ψ_m is a nonnegative compactly supported function which converges to 0 in $L^1(\mathbb{R})$ when m tends to infinity. Since $u \in L^\infty \cap BV_{\mathrm{loc}}$ and δ_m is compactly supported in $(0, 1)$, for every continuous function g, and for almost every t

$$\int_{\mathbb{R}} \psi_m'(x - h_\epsilon(t))g(u(t, x))\,dx$$

converges, when m goes to infinity to $g(u_-) - g(u_+)$. For any nonnegative compactly supported smooth function ϕ we use

$$(t, x) \longrightarrow \phi(t)\psi_m(x - h_\epsilon(t))$$

as a test function for the equation (1). We get

$$-\int \phi'(t)\psi_m(x-h_\epsilon(t))u(t, x)\,dx\,dt+\int \phi(t)\psi_m'(x-h_\epsilon(t))(\dot{h}_\epsilon(t)u(t, x)-f(u))\,dx\,dt = 0.$$

The first integral converges to 0 when m goes to infinity. the second one converges to

$$\int_0^T [f(u_+) - f(u_-) - \dot{h}(u_+ - u_-)]\phi\,dt = 0.$$

A similar treatment of equation (2) gives

$$\int_0^T [q(u_+) - q(u_-) - \dot{h}(\eta(u_+) - \eta(u_-))]\phi\,dt \leqslant 0.$$

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