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Abstract

This is a report on recent progress concerning the global well-posedness problem for energy-critical nonlinear Schrödinger equations posed on specific Riemannian manifolds M with small initial data in $H^1(M)$. The results include small data GWP for the quintic NLS in the case of the $3d$ flat rational torus $M = \mathbb{T}^3$ and small data GWP for the corresponding cubic NLS in the cases $M = \mathbb{R}^2 \times \mathbb{T}^2$ and $M = \mathbb{R}^3 \times \mathbb{T}$. The main ingredients are bi-linear and tri-linear refinements of Strichartz estimates which obey the critical scaling, as well as critical function space theory. All results mentioned above have been obtained in collaboration with D. Tataru and N. Tzvetkov.

1. Introduction and main results

This is a report on recent progress [14, 15] concerning the small data global well-posedness problem for energy-critical nonlinear Schrödinger equations

$$(i\partial_t + \Delta)u = |u|^{\frac{4}{d-2}}u, \quad (d \geq 3) \tag{1.1}$$

posed on specific d-dimensional Riemannian manifolds M (without boundary) with initial data in $H^1(M)$. We are mostly interested in the case of compact manifolds or, more generally, in the case of manifolds with periodic geodesics. In the exposition we will restrict ourselves to defocusing nonlinearities, but all results remains true in the focusing case as we will only deal with small initial data.

For strong solutions u of (1.1) with initial data ϕ we have L^2-conservation

$$m(u(t)) = \frac{1}{2}\int_M |u(t,x)|^2 dx = m(\phi), \tag{1.2}$$

and energy conservation

$$e(u(t)) = \frac{1}{2}\int_M |\nabla u(t,x)|^2 dx + \frac{d-2}{2d}\int_M |u(t,x)|^{\frac{2d}{d-2}} dx = e(\phi). \tag{1.3}$$

Report on joint work with Daniel Tataru and Nikolay Tzvetkov.

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Thus, the natural energy space for this equation is $H^1(M)$, which is also the scale invariant space for (1.1) in the case of $M = \mathbb{R}^3$ in the following sense: if $u_\lambda(t,x) = \lambda^{1/2}u(\lambda^2 t, \lambda x)$ for some $\lambda > 0$ we have

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)} = \|u(\lambda^2 t,\cdot))\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Therefore, it is a natural problem to study the local well-posedness of (1.1) in $H^1(M)$, as it immediately implies small data global well-posedness.

First, let us review some selected general facts concerning space-time estimates for the linear Schrödinger equation without claiming to give a comprehensive list of references or a complete account on the history of the problem here. In the Euclidean case $M = \mathbb{R}^d$, the full range of sharp Strichartz estimates

$$\|e^{it\Delta}\phi\|_{L_t^p((0,T);L_x^q(M))} \lesssim \|\phi\|_{L^2(M)}, \qquad 2/p + d/q = d/2,\ p \geq 2, \tag{1.4}$$

is available, see e.g. [16], which imply small data global well-posedness [9]. The admissibility condition on the Strichartz pairs (p,q) comes from scaling considerations. Extensions to the case of asymptotically Euclidean and non-trapping metrics have been considered in [25, 24]. There are several examples where estimates of type (1.4) are known to fail, see [3, 26] for the failure of (1.4) on $M = \mathbb{T}^d$, see also [4, 5, 6, 7] for the case $M = \mathbb{S}^d$. It is a classical fact that non-degenerate and stable trapped geodesics allow spatial concentration of solutions and form a geometric obstructions to dispersion, see e.g. [1, 21, 22, 23] for precise statements. This may lead to instability results for nonlinear Schrödinger equations [27], and the failure of Strichartz estimates. Nevertheless, in many cases Strichartz estimates with a loss of derivatives or within a restricted range of admissible Strichartz pairs are known to hold true, see e.g. [3, 26, 2, 5, 10, 20] and references therein.

As we intend to prove estimates which allow us to solve H^1-critical equations we do not need to have the full range of sharp Strichartz estimates (1.4) at our disposal. However, we need estimates which respect the scaling of the problem and are sufficient for the given degree of the nonlinearity. Let us look at the example of the quintic nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u = |u|^4 u, \tag{1.5}$$

posed on the three dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. As a consequence of Littlewood-Paley theory and Bourgain's dyadic estimate [3, (3.117)] it holds

$$\|e^{it\Delta}\phi\|_{L^p(\mathbb{T}\times\mathbb{T}^3))} \lesssim \|\phi\|_{H^{s(p)}(\mathbb{T}^3)}, \qquad p > 4,\ s(p) = 3/2 - 5/p, \tag{1.6}$$

whereas these estimate are unknown for $p \leq 4$. Notice that this Strichartz type estimate obeys the scaling of the problem as it formally follows from the (unknown) Strichartz estimate (1.4) with $p > 4$ and the Sobolev embedding. However, if one tries to apply these estimates in order to control the quintic nonlinearity in $H^1(\mathbb{T}^3)$, one needs to combine them with the energy inequality

$$\|e^{it\Delta}\phi\|_{L_t^\infty H_x^1} \lesssim \|\phi\|_{H^1}$$

as this is the only known linear estimate without a loss of derivatives. Then, one is forced to put at least one factor in $L_t^p L_x^q$ for some $p \leq 4$, where no estimate without loss of derivatives is available. It is possible to rectify this strategy and to prove local well-posedness in $H^s(\mathbb{T}^3)$ for $s > 1$ with a modification of this argument, i.e. in the full sub-critical range, but not for initial data of critical regularity $s = 1$.

Our strategy will be to obtain a better share of derivatives by using multi-linear and scale invariant versions of Strichartz type estimates. This idea originates in [3] and has been successfully applied to many problems since then. The new difficulty which arises in this approach is that one needs to introduce appropriate function spaces in order to exploit the multi-linear structure of the nonlinearity. Bourgain's $X^{s,b}$ spaces are well-known examples, see [3, 12]. However, at the level of critical regularity refinements of $X^{s,\frac{1}{2}}$ are needed. We will employ a critical function space theory which originates in unpublished work on wave maps by D. Tataru and has been applied and developed e.g. in [17, 13]. In order to prove a tri-linear Strichartz estimate on the correct scales we need a refinement of these spaces which are sensible to finer than dyadic localizations. Our main result in the case $M = \mathbb{T}^3$ is

Theorem 1.1 (see [15]). *Let $d = 3$ and $M = \mathbb{T}^3$.*

(i) *The quintic nonlinear Schrödinger equation* (1.5) *is locally well-posed in $H^s(\mathbb{T}^3)$ for all $s \geq 1$.*

(ii) *The quintic nonlinear Schrödinger equation* (1.5) *is globally well-posed for small data in $H^s(\mathbb{T}^3)$ for all $s \geq 1$.*

In an ongoing work we also study the cubic NLS in dimension $d = 4$ with partially periodic data, i.e. (1.1) posed on $M = \mathbb{R}^2 \times \mathbb{T}^2$ and $M = \mathbb{R}^3 \times \mathbb{T}$. Variants of the above ideas apply, but with bi-linear instead of tri-linear refinements of Strichartz estimates, which involve somewhat more precise estimates but in a slightly easier analytic setup (due to the dispersive effect in the non-periodic directions). As a preliminary result we obtain

Theorem 1.2 (work in progress, see [14]). *Let $d = 4$, $M = \mathbb{R}^2 \times \mathbb{T}^2$ or $M = \mathbb{R}^3 \times \mathbb{T}$.*

(i) *The cubic nonlinear Schrödinger equation* (1.1) *is locally well-posed in $H^s(M)$ for all $s \geq 1$.*

(ii) *The cubic nonlinear Schrödinger equation* (1.1) *is globally well-posed for small data in $H^s(M)$ for all $s \geq 1$.*

In the next section we will elucidate the main ideas of the proof. We will restrict the presentation to the case $M = \mathbb{T}^3$, see [15] for further details.

2. Elements of Proof

Let $\mathcal{C}_N$ denote the collection of cubes $C \subset \mathbb{Z}^3$ of side-length $N = 2^n \geq 1$ with arbitrary center and orientation. For a given set C let P_C denote a Fourier multiplication operator which Fourier-localizes functions on $\mathbb{T}^3$ to C, and for a dyadic number $N \geq 1$ let $P_N = P_{\{\xi \in \mathbb{Z}^3 : |\xi| \sim N\}}$. As usual, the Sobolev space $H^s(\mathbb{T}^3)$ is defined as the space of all functions for which

$$\|f\|^2_{H^s(\mathbb{T}^3)} := \sum_{\xi \in \mathbb{Z}^3} \langle \xi \rangle^{2s} |f(\xi)|^2 < \infty.$$

Let us first recall Bourgain's L^p estimates of Strichartz type which already have been mentioned in Section 1.

Proposition 2.1 (Bourgain [3]). *Let $p > 4$. For all $N \geq 1$ we have*

$$\|P_N e^{it\Delta}\phi\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim N^{\frac{3}{2}-\frac{5}{p}}\|\Delta_N\phi\|_{L^2(\mathbb{T}^3)}. \tag{2.1}$$

More generally, for all $C \in \mathcal{C}_N$

$$\|P_C e^{it\Delta}\phi\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim N^{\frac{3}{2}-\frac{5}{p}}\|P_C\phi\|_{L^2(\mathbb{T}^3)}. \tag{2.2}$$

We would like to tri-linearize this estimate in function spaces which include linear solutions.

Additionally, we will be able to strengthen (2.2) in the case $p = 8$. Strictly speaking, this refinement is not necessary to derive small data GWP for the quintic NLS in the case $M = \mathbb{T}^3$. However, it is an interesting estimate in its own right and this circle of ideas will prove useful in the context of the $4d$ cubic NLS, so let us describe it here. Let $\mathcal{R}_M(N)$ be the collection of all rectangular sets in $\mathbb{Z}^3$ of arbitrary orientation and center, and side-lengths $N \times N \times M$.

Proposition 2.2. *For all $1 \leq M \leq N$ and $R \in \mathcal{R}_M(N)$ we have*

$$\|P_R e^{it\Delta}\phi\|_{L^8(\mathbb{T}\times\mathbb{T}^3)} \lesssim M^{\frac{1}{8}}N^{\frac{3}{4}}\|P_R\phi\|_{L^2(\mathbb{T}^3)}. \tag{2.3}$$

The proof of (2.3) is based on the fact $\|u\|_{L^8}^4 = \|uuuu\|_{L^2}$, Estimation of the Fourier transform by Cauchy-Schwarz and Landau's asymptotic [19] for the number of lattice points in $6d$ (rational) ellipsoids.

For the construction of the appropriate function spaces we follow the approach from [13, Section 2]. Let H be either $L^2(\mathbb{T}^3)$ or $\mathbb{C}$ and let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K \leq \infty$ of the real line. If $t_K = \infty$, we use the convention that $v(t_K) := 0$ for all functions $v : \mathbb{R} \to H$.

Definition 2.3. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset H$ with $\sum_{k=0}^{K-1}\|\phi_k\|_H^p = 1$ and we call the piecewise defined function $a : \mathbb{R} \to H$,

$$a = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}$$

a U^p-atom. We define the atomic space $U^p(\mathbb{R}, H)$ of all functions $u : \mathbb{R} \to H$ such that

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \ \text{ for } U^p\text{-atoms } a_j,\ \{\lambda_j\} \in \ell^1,$$

endowed with the norm

$$\|u\|_{U^p} := \inf\left\{\sum_{j=1}^{\infty}|\lambda_j| \ :\ u = \sum_{j=1}^{\infty}\lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j\ U^p\text{-atom}\right\}. \tag{2.4}$$

The normed spaces $U^p(\mathbb{R}, H)$ are complete and $U^p(\mathbb{R}, H) \hookrightarrow L^\infty(\mathbb{R}; H)$. Every $u \in U^p(\mathbb{R}, H)$ is right-continuous and u tends to 0 for $t \to -\infty$.

Definition 2.4. Let $1 \leq p < \infty$.

(i) $V^p(\mathbb{R}, H)$ is defined as the space of all functions $v : \mathbb{R} \to H$ such that

$$\|v\|_{V^p}^p := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_H^p < \infty. \tag{2.5}$$

(ii) $V^p_{rc}(\mathbb{R},H)$ denotes the closed subspace of all right-continuous functions $v:\mathbb{R}\to H$ such that $\lim_{t\to-\infty}v(t)=0$, endowed with the norm (2.5).

Notice that (2.5) defines a norm since for any $t\in\mathbb{R}$ the partition $\{t,\infty\}$ yields $\|v\|_{V^p}\geq|v(t)|$. The normed spaces $V^p(\mathbb{R},H)$, $V^p_{rc}(\mathbb{R},H)$ are complete and it holds $U^p(\mathbb{R},H)\hookrightarrow V^p_{rc}(\mathbb{R},H)\hookrightarrow L^\infty(\mathbb{R};H)$. For $1\leq p<q<\infty$ the embedding $V^p_{rc}(\mathbb{R},H)\hookrightarrow U^q(\mathbb{R},H)$ holds true. Corresponding to the linear Schrödinger flow we define

Definition 2.5. For $s\in\mathbb{R}$ we define $V^p_\Delta H^s$ and $U^p_\Delta H^s$ to be the spaces of all functions $u:\mathbb{R}\to H^s(\mathbb{T}^3)$ such that $t\mapsto e^{-it\Delta}u(t)\in U^p(\mathbb{R},H^s)$ and $t\mapsto e^{-it\Delta}u(t)\in V^p(\mathbb{R},H^s)$, endowed with the norms

$$\|u\|_{U^p_\Delta H^s}=\|e^{-it\Delta}u\|_{U^p(\mathbb{R},H^s)},\qquad \|u\|_{V^p_\Delta H^s}=\|e^{-it\Delta}u\|_{V^p(\mathbb{R},H^s)},\tag{2.6}$$

respectively.

Spaces of this type have been successfully used as replacements for $X^{s,\frac12}$ spaces which are still effective at critical scaling, see for instance [17, 18, 13]. However, it turns out that we need a refinement which is sensible to stronger localizations in Fourier space.

Definition 2.6. Let $s\in\mathbb{R}$.

(i) We define X^s as the space of all functions $u:\mathbb{R}\to H^s(\mathbb{T}^3)$ such that for every $\xi\in\mathbb{Z}^3$ the map $t\mapsto e^{it|\xi|^2}\widehat{u(t)}(\xi)$ is in $U^2(\mathbb{R},\mathbb{C})$, and for which

$$\|u\|^2_{X^s}:=\sum_{\xi\in\mathbb{Z}^3}\langle\xi\rangle^{2s}\|e^{it|\xi|^2}\widehat{u(t)}(\xi)\|^2_{U^2_t}<\infty.\tag{2.7}$$

(ii) We define Y^s as the space of all functions $u:\mathbb{R}\to H^s(\mathbb{T}^3)$ such that for every $\xi\in\mathbb{Z}^3$ the map $t\mapsto e^{it|\xi|^2}\widehat{u(t)}(\xi)$ is in $V^2_{rc}(\mathbb{R},\mathbb{C})$ such that

$$\|u\|^2_{Y^s}:=\sum_{\xi\in\mathbb{Z}^3}\langle\xi\rangle^{2s}\|e^{it|\xi|^2}\widehat{u(t)}(\xi)\|^2_{V^2_t}<\infty.\tag{2.8}$$

It is easy to relate the X^s and Y^s spaces with the previously defined $V^p_\Delta H^s$ and $U^p_\Delta H^s$. Indeed, the following embeddings hold:

$$U^2_\Delta H^s\hookrightarrow X^s\hookrightarrow Y^s\hookrightarrow V^2_\Delta H^s.$$

The motivation for the introduction of the X^s and Y^s spaces lies in the fact that for any partition $\mathbb{Z}^3=\cup C_k$ it holds

$$\sum_k\|P_{C_k}u\|^2_{V^2_\Delta H^s}\lesssim\|u\|^2_{Y^s}.$$

For a time interval $I\subset\mathbb{R}$ we also consider the restriction spaces $X^s(I)$, etc. Due to the atomic structure of U^2 the linear solution $u(t):=e^{it\Delta}\phi$ satisfies $u\in X^s([0,T))$ for any $\phi\in H^s(\mathbb{T}^3)$. By duality, see [13, Theorem 2.8 and Proposition 2.10], one also obtains (for sufficiently nice f) the estimate

$$\Big\|\int_0^t e^{i(t-s)\Delta}f(s)\Big\|_{X^s([0,T))}\leq\sup\int_0^T\int_{\mathbb{T}^3}f(t,x)\overline{v(t,x)}dxdt,\tag{2.9}$$

where the supremum is taken over all $v\in Y^{-s}([0,T))$ satisfying $\|v\|_{Y^{-s}}=1$.

The Y^s-norms are strong enough to prove the following:

Proposition 2.7. *There exists $\delta > 0$, such that for all dyadic numbers $N_1 \geq N_2 \geq N_3 \geq 1$ and intervals $I \subset [0, 2\pi]$ it holds*

$$\Big\| \prod_{j=1}^{3} P_{N_j} u_j \Big\|_{L^2(I\times\mathbb{T}^3)} \lesssim N_2 N_3 \max\Big\{ \frac{N_3}{N_1}, \frac{1}{N_2} \Big\}^{\delta} \prod_{j=1}^{3} \|P_{N_j} u_j\|_{Y^0}. \tag{2.10}$$

Proof. First of all, it suffices to prove (2.10) in the case $I = [0, 2\pi]$ (in the sequel we will write $L^2 = L^2([0, 2\pi] \times \mathbb{T}^3)$ for brevity), and when the first factor is further restricted to a cube $C \in \mathcal{C}_{N_2}$,

$$\|P_{N_1} P_C u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim N_2 N_3 \Big(\frac{N_3}{N_1} + \frac{1}{N_2} \Big)^{\delta} \prod_{j=1}^{3} \|P_{N_j} u_j\|_{Y^0}. \tag{2.11}$$

This is because given a partition $\mathbb{Z}^3 = \cup C_j$ into cubes $C_j \in \mathcal{C}_{N_2}$, the outputs $P_{N_1} P_{C_j} u_1 P_{N_2} u_2 P_{N_3} u_3$ are almost orthogonal, while

$$\|u\|_{Y_0}^2 = \sum_j \|P_{C_j} u\|_{Y_0}^2.$$

In (2.11) we can use $Y^0 \hookrightarrow V_\Delta^2 L^2$ to replace Y^0 by $V_\Delta^2 L^2$. Then (2.11) follows by interpolation (a multi-linear version of [13, Proposition 2.20]) from the following two tri-linear estimates:

$$\begin{aligned} &\|P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \\ &\lesssim N_2^{3-\frac{10}{p}} N_3^{\frac{3}{2}-\frac{5}{q}} \|P_{N_1} u_1\|_{U_\Delta^p L^2} \|P_{N_2} u_2\|_{U_\Delta^p L^2} \|P_{N_3} u_3\|_{U_\Delta^q L^2} \end{aligned} \tag{2.12}$$

for $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$ and $4 < p < 5$, respectively

$$\|P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim N_2 N_3 \Big(\frac{N_3}{N_1} + \frac{1}{N_2} \Big)^{\delta} \prod_{j=1}^{3} \|P_{N_j} u_j\|_{U_\Delta^2 L^2}. \tag{2.13}$$

The first bound (2.12) follows from the atomic structure of $U_\Delta^p L^2$ and (2.2) by Hölder's inequality. For the second bound (2.13) we can use the atomic structure of $U_\Delta^2 L^2$ to reduce the problem to the similar estimate for the product of three solutions to the linear Schrödinger equation $u_j = e^{it\Delta}\phi_j$:

$$\|P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2} \lesssim N_2 N_3 \Big(\frac{N_3}{N_1} + \frac{1}{N_2} \Big)^{\delta} \prod_{j=1}^{3} \|P_{N_j} \phi_j\|_{L^2}. \tag{2.14}$$

Let ξ_0 be the center of C. We partition $C = \cup R_k$ into almost disjoint strips of width $M = \max\{N_2^2/N_1, 1\}$ which are orthogonal to ξ_0,

$$R_k = \Big\{ \xi \in C : \xi \cdot \xi_0 \in [|\xi_0| M k, |\xi_0| M (k+1)) \Big\}, \qquad |k| \approx N_1/M$$

It is $R_k \in \mathcal{R}_M(N_2)$ and we decompose

$$P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3 = \sum_k P_{R_k} P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3$$

and observe that the summands are almost orthogonal in $L^2(\mathbb{T} \times \mathbb{T}^3)$. This orthogonality no longer comes from the spatial frequencies, but from the time frequency. Hence,

$$\|P_C P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2}^2 \lesssim \sum_k \|P_{R_k} P_{N_1} u_1 P_{N_2} u_2 P_{N_3} u_3\|_{L^2}^2$$

On the other hand the estimates (2.3) and (2.2) yield

$$\|P_{R_k}P_{N_1}u_1P_{N_2}u_2P_{N_3}u_3\|_{L^2} \lesssim N_2^{3-\frac{10}{p}}N_3^{\frac{3}{2}-\frac{5}{q}}\left(\frac{M}{N_2}\right)^{\delta} \|P_{R_k}P_{N_1}\phi_1\|_{L^2}\|P_{N_2}\phi_2\|_{L^2}\|P_{N_3}\phi_3\|_{L^2},$$

for $\frac{2}{p}+\frac{1}{q}=\frac{1}{2}$ and $4<p<5$. Then (2.14) follows by summing up the squares with respect to k. □

Proposition 2.7 and (2.9) and dyadic summation yield

Proposition 2.8. *For all $0<T\leq 2\pi$ and $u,v\in X^1([0,T))$ the estimate*

$$\Big\|\int_0^t e^{i(t-s)\Delta}\Big(|u|^4u(s)-|v|^4v(s)\Big)ds\Big\|_{X^1([0,T))} \lesssim (\|u\|^4_{X^1([0,T))}+\|v\|^4_{X^1([0,T))})\|u-v\|_{X^1([0,T))}.$$

holds true.

As already mentioned above, in the function spaces $X^1([0,T))$ this estimate can be derived from Bourgain's results in Proposition 2.1 and additional orthogonality arguments without resorting to the refinement (2.3). A similar estimate holds true for higher Sobolev regularity.

The proof of Theorem 1.1 can be completed by performing the standard Picard iteration procedure for the local well-posedness, and additionally using the conservation laws (1.2) and (1.3) for the global results for small initial data.

3. Conclusion and Outlook

We have been able to prove small data global well-posedness for the energy critical NLS on specific manifolds with periodic geodesics, where the standard arguments of the Euclidean theory fail. Of course, a global well-posedness theory for large data remains a difficult open problem in the non-Euclidean setting, cf. [11] for the quintic NLS on $M=\mathbb{R}^3$.

A further interesting problem is the quintic NLS on the 3-sphere $\mathbb{S}^3$. The example of zonal eigenfunctions with large eigenvalues shows that the Strichartz estimates fail in this setting, see [5]. However, for linear solutions there is a tri-linear replacement which can be used to control the second Picard-iteration for this problem, see [8]. However, the argument in [8] uses strongly separated interactions of the spatial and the time frequencies, which is a serious obstruction for our arguments. As a result, we are not able to extend this estimate to our function spaces at this moment and it remains a challenging project.

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