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An overview on congestion phenomena in fluid equations

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Abstract

We review some recent analysis results and open perspectives around congestion phenomena in fluid equations. The PDE systems under study are based on Navier–Stokes equations in which congestion is encoded in a maximal density constraint. The paper is organized around three main topics: multi-scale issues, regularity issues and finally non-locality issues.

1. Handling congestion in fluid equations

If the Navier–Stokes equations present natural advantages compared to the microscopic point of view (number of parameters and unknowns, computational costs, etc.) and have proved to be relevant for simple fluids in standard flowing conditions, they may nevertheless fail to reproduce behaviors of heterogeneous flows. Typically, in highly dense regimes, dynamics of flows are altered, congested, and may exhibit some non-local effects due to the so-called *maximal packing constraint*, that is the non-overlap constraint on the microscopic components. Classical Navier–Stokes equations do not account for such microscopic constraints which are nevertheless crucial in the modeling of collective motions and dispersed mixtures like bubbly fluids or granular suspensions. From a general point of view, apparition of congestion can be seen as a phase transition phenomenon.

The purpose of this paper is to present and analyze some of the simplest models to handle congestion in fluid equations. We will assume that the microscopic packing constraint can be expressed at the continuous level through a maximal density constraint $\rho \leq \rho^*$ where the value ρ^* is supposed to be known, equal to 1 in the rest of this paper. A toy model for constrained flows based on the classical Navier–Stokes equations reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & \text{(1.1a)} \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div} \mathbb{S} = \rho f, & \text{(1.1b)} \\ 0 \leq \rho \leq 1, \quad \operatorname{spt} p \subset \{\rho = 1\}, \quad p \geq 0. & \text{(1.1c)} \end{cases}$$

The constrained variable $\rho \in [0,1]$ denotes the density. Depending on the application it can also represent the volume fraction of a component in a mixture or the height of the fluid in the case of liquid flows in closed pipes. The vector u is the velocity field, $\mathbb{S}$ is the viscous stress tensor and f an external force. Finally, the pressure p is the variable which encodes the two possible behaviors (i.e. dynamics) of the system. It is activated in the congested domain $\{\rho = 1\}$ and it ensures the maximal density constraint. More precisely, the fact that the medium cannot be more compressed in the congested regions can be translated into a constraint on the velocity field:

$$\operatorname{div} u \geq 0 \quad \text{in} \quad \{\rho = 1\}. \tag{1.2}$$

The pressure $p \geq 0$ can thus be seen as the Lagrange multiplier associated to the previous constraint on the velocity field in the congested domain.

System (1.1) is then a free boundary problem between a free phase where $\rho < 1$ and $p = 0$ (pressureless equations) and a congested phase where $\rho = 1$ and $p \geq 0$.

Keywords: Navier–Stokes equations, Euler equations, maximal packing constraint, phase transitions.

Remark 1.1. The condition $\operatorname{spt} p \subset \{\rho = 1\}$ is often replaced in the literature by the so-called *exclusion constraint* (or *unilateral constraint* in reference to obstacle problems, see for instance [3])

$$(1-\rho)p = 0, \tag{1.3}$$

which expresses as well the activation of p when $\rho = 1$. But the lack of regularity on p can be an obstacle to the rigorous justification of the exclusion constraint, we shall come back on this point and its consequences in Section 2.

In order to ensure the maximal density constraint another approach can be followed. Instead of switching the dynamics when the constraint is reached, one can introduce in the equations singular forces which play the role of barriers and guarantee that $\rho < 1$ almost everywhere. Classically, when the density approaches the maximal constraint the pressure is supposed to blow up in the following compressible Navier–Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.4a)\\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \operatorname{div} \mathbb{S} = \rho f, & (1.4b)\\ 0 \le \rho < 1, \quad P(\rho) \underset{\rho \to 1^-}{\longrightarrow} +\infty. & (1.4c) \end{cases}$$

The singular pressure P is interpreted as the macroscopic resultant of the repulsive forces between the microscopic particles (like social forces in collective motion). As we will see in Section 4, it can be relevant from the physical point of view to consider other singular forces like viscosities. Such models with singular forces are called *soft congestion models*, they always keep a compressible dynamics. By opposition, systems of type (1.1) are called *hard congestion models*. The link between the soft and the hard approaches is addressed in Section 2.

Creation of congested phases can be also at the origin of non-local effects. As said in [13]: *"A fundamental and vivid issue is the possible emergence of non-locality as the signature of a dynamical phase transition"*. We will discuss in details a particular type of time non-locality (*memory effects*) associated to congestion phenomena.

Remark 1.2. Of course many variations or extensions of the previous systems could be imagined and actually exist in the literature. A good introduction to the modeling of congestion phenomena both from microscopic and macroscopic viewpoints is proposed in [49]. Among the possible general extensions, let us mention the case of heterogeneous maximal density constraints $\rho \le \rho^*(t,x)$ (see for instance [24], [38], [60] or [7] for applications to traffic flows). Depending on the applications, other reference fluid systems can be preferred to the classical Navier–Stokes equations: non-linear shallow water or Boussinesq equations for wave-structure interactions (see [38], [9], [36]), gradient flow formulations in the modeling of crowds [50], porous media equations and Hele–Shaw free boundary problems for tissue growth modeling (see for instance [35], [61]), Bingham equations for complex geophysical flows [21], etc. For the clarity of the presentation, we shall stick in this paper to the two "toy" systems (1.1) and (1.4) which already raise important and difficult analysis problems.

The paper is organized around three main topics: first we address multi-scale issues where the rigorous link between Equations (1.4) and Equations (1.1) is investigated and we show how this link can be used to describe the formation and the dynamics of congested fronts. The second part is devoted regularity issues on the free boundary problem (1.1) and the final section is an opening towards more complex systems including non-local effects such as memory effects.

Remark 1.3. In this paper, we focus on theoretical analysis issues raised by the constrained systems (1.1) and (1.4): existence of solutions, regularity, stationary solutions, etc. Congestion phenomena also bring important numerical issues (e.g. design of stable and convergent numerical schemes, qualitative properties of the numerical solutions, etc.) that will be not discussed here. The interested reader is referred to [23], [64] and references therein for more information on the subject.

2. Multi-scale issues

We address in this section different issues in relation with the multi-scale nature of the congestion phenomena (discrete/"microscopic" approximations, asymptotic analysis, singular limits, etc.).

We begin this section with a first existence proof of weak solutions to (1.1) that is based on a discretization by congested blocks in the case where $\mathbb{S} = 0$. Then, we analyze soft congestion systems of type (1.4) and present a result on the singular limit towards system (1.1). We show that the approximation of hard congestion systems by their soft counterpart enables to get valuable insights on the formation and dynamics of congested domains.

2.1. Constrained Euler equations and approximation by discrete blocks

Originally Bouchut et al. in [11] introduced the following free-congested one-dimensional Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (2.1a) \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0, & (2.1b) \\ 0 \le \rho \le 1, \ \operatorname{spt} p \subset \{\rho = 1\}, \ p \ge 0, & (2.1c) \end{cases}$$

as an asymptotic model of biphasic liquid-gas equations. More precisely, as the ratio ρ_g/ρ_l between the reference gas and liquid densities tends to 0, the initial biphasic system with four equations (mass and momentum equations for each phase) is supposed to degenerate towards (2.1) where ρ is interpreted as the volume fraction of the liquid phase. The first two equations (2.1a)–(2.1b) are the limits of the mass and momentum equations on the liquid phase. In fact, a natural limit system should be the pressureless equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \end{cases}$$

but, as shown by Bouchut in [10], solutions of this system may develop concentration phenomena (i.e. apparition of Dirac masses) that are incompatible with the maximal volume fraction constraint $\rho \le 1$. To remedy the shortcomings of the pressureless gas system, Bouchut et al. propose in [11] to add in the momentum equation the ad hoc limit pressure p satisfying (2.1c). This pressure can be seen as an artificial residual effect of the gas phase in the saturated regions where $\rho = 1$.

In [11], discrete (weak) solutions of (2.1) are identified and used from a numerical point of view to approximate general weak solutions of (2.1). These special solutions generalize the *sticky particles* for the pressureless gas system (see [15]) and consist of discrete congested blocks (also called *sticky blocks*)

$$\rho(t,x) = \sum_{i=1}^{N} \mathbf{1}_{a_i(t)<x<b_i(t)}, \qquad \rho u(t,x) = \sum_{i=1}^{N} u_i(t) \mathbf{1}_{a_i(t)<x<b_i(t)}. \tag{2.2}$$

The number of blocks N depends on time, but is piecewise constant. These blocks are supposed to satisfy specific dynamics: as long as the blocks do not meet, they move at constant velocity u_i and as soon as a collision occurs between two (or more than two) blocks, the concerned blocks build a new block, whose size is the sum of the former sizes, and the momentum is the sum of former momenta. Considering general initial data $(\rho^0, (\rho u)^0)$ with

$$\rho^0 \in L^1(\mathbb{R}), \quad 0 \le \rho^0 \le 1 \text{ a.e.}, \quad u^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \tag{2.3}$$

which can be discretized into a finite number of blocks (2.2). Then one expects that the discrete dynamics of these blocks will lead by a stability argument to the existence of weak "continuous" solutions (ρ, u, p). This has been rigorously proved by Berthelin in [5].

Theorem 2.1 (Berthelin [5])**.** *Assume initially* (2.3)*, then there exists for any* $T > 0$*,* (ρ, u, p) *with regularities*

$$\begin{aligned} &\rho \in L^\infty(0,T; L^\infty \cap L^1(\mathbb{R})) \cap \mathcal{C}([0,T]; L^\infty_{w*}(\mathbb{R})), \\ &u \in L^\infty((0,T)\times\mathbb{R}), \\ &p \in \mathcal{M}_+((0,T)\times\mathbb{R}), \end{aligned}$$

satisfying Equations (2.1a)–(2.1b) *in the weak sense with* $\operatorname{spt} p \subset \{\rho = 1\}$ *and the constraint* $0 \le \rho \le 1$ *satisfied a.e.* $(t,x) \in (0,T)\times\mathbb{R}$.

The same discrete approximation have been extended next in [8] to a more sophisticated system, called constrained Aw–Rascle–Zhang system in traffic flow modeling, and also to the multi-dimensional setting in [6].

Remark 2.1. As said in the previous Remark 1.1, originally the condition $\operatorname{spt} p \subset \{\rho = 1\}$ was replaced by the exclusion constraint (1.3), namely $p = \rho p$. But, the fact that p belongs to $\mathcal{M}_+((0,T)\times\mathbb{R})$ prevents us to define the product ρp a.e.. An alternative (weak) formulation of the constraint (1.3) is proposed in [5] to bypass this difficulty.

Remark 2.2. Another existence result of weak solutions can be obtained in the class

$$\rho \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R})), \quad \rho_t \ll \mathcal{L}^1, \quad 0 \le \rho \le 1 \text{ a.e.},$$
$$u \in L^\infty(0,T;L^2(\mathbb{R},\rho_t(\mathrm{d}x))), \quad p \in \mathcal{M}_+((0,T)\times\mathbb{R}).$$

This result can be derived as a corollary of the existence result obtained in [58] for a slightly more complicated system (the model (4.1) is presented below in Section 4). It does not rely on compactness arguments like [5], but it is instead related to optimal transport tools through a Lagrangian reformulation of the dynamics in terms of the monotone rearrangement (see details in Section 4.1).

Remark 2.3. Note finally that without additional condition we do not have uniqueness of solutions of system (2.1) for a given initial datum $(\rho^0,(\rho u)^0)$. For instance, the initial data

$$\rho^0 = \mathbf{1}_{[-1,1]} \quad \text{with} \quad (\rho u)^0 = 0$$

admits the trivial solution $(\rho,u,p) = (\mathbf{1}_{[-1,1]},0,0)$ but also the solution

$$\rho(t,x) = \mathbf{1}_{-1-t<x<-t} + \mathbf{1}_{t<x<1+t}, \quad \rho u(t,x) = -\mathbf{1}_{-1-t<x<-t} + \mathbf{1}_{t<x<1+t},$$
$$p(t,x) = \delta_0(t)(1-|x|)\mathbf{1}_{[-1,1]}.$$

that describes the separation of two blocks. We have in fact an infinity of unphysical solutions corresponding to disaggregation of congested blocks. One can also exhibit different explicit solutions such that the kinetic energy decreases. As explained by Preux in [64] a possible way to recover the uniqueness would be to impose an additional inelastic collision law which reads

$$u(t^+,\cdot) = \mathrm{P}_{C_K(\rho)}(u(t^-,\cdot)) \tag{2.4}$$

where $C_K(\rho)$ is the set of admissible velocities, that is velocities which preserve the maximal density constraint. Formally, as we said in the introduction, the condition is $\operatorname{div} u \ge 0$ in $\{\rho = 1\}$. If $u(t,\cdot) \in L^2_x$ the set of admissible velocities can be expressed by duality (see also [51]) as follows

$$C_K(\rho) = \left\{ v \in L^2 \text{ s.t. } \int v\cdot\nabla q \le 0 \quad \forall\, q \in H^1,\ q \ge 0,\ q(1-\rho) = 0 \right\}.$$

2.2. Soft congestion systems

Let us now turn to the second type of congestion systems which take the form (1.4). This type of models involve through the pressure $P(\rho)$ short range repulsive forces which prevent the density to exceed 1. We show in this subsection how the maximal constraint can be recovered from the mathematical of view thanks to the energy estimate. We present then a global weak existence result as well as a weak-strong uniqueness result.

Existence of global weak solutions. Let $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3 and impose at the boundary Dirichlet condition on the velocity field u

$$u_{|\partial\Omega} = 0.$$

We assume that

$$P(\rho) = a\left(\frac{\rho}{1-\rho}\right)^\gamma, \quad \gamma > 1,\ a > 0. \tag{2.5}$$

Compared to the previous paragraph, we assume here that the system is dissipative and the viscous stress tensor is decomposed as follows

$$\mathbb{S} = 2\mu\mathrm{D}(u) + \lambda \operatorname{div} u\mathrm{I} \tag{2.6}$$

where $\mathrm{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ and I is the identity matrix. The coefficients μ and λ are respectively the shear viscosity and the bulk viscosity, they are supposed to be constant such that $\mu > 0$, $\lambda + 2\mu > 0$. Then the energy estimate writes

$$\sup_{t\in[0,T]} \int_\Omega \left[\frac{\rho|u|^2}{2} + H(\rho)\right] + \mu \int_0^T \int_\Omega |\nabla u|^2 + (\lambda + 2\mu) \int_0^T \int_\Omega |\mathrm{div}(u)|^2 \leq \int_\Omega \left[\frac{\rho^0|u^0|^2}{2} + H(\rho^0)\right] \quad (2.7)$$

where the internal energy H is such that

$$sH'(s) - H(s) = P(s). \quad (2.8)$$

Assuming that the pressure is sufficiently singular (namely $\gamma > 1$) close to 1, then the internal energy is also singular close to 1. Hence, if (ρ, u) satisfies the previous energy inequality (2.7) then the maximal density constraint holds almost everywhere

$$\mathrm{meas}\{(t,x),\ \rho(t,x) \geq 1\} = 0.$$

In the one-dimensional case treated in [18], refined bounds (from below and above) can be proved on the density and the existence and uniqueness of global strong solutions to the singular Navier–Stokes equations is ensured. Global strong solutions are not known to exist for the multi-dimensional compressible Navier–Stokes equations even for standard non-singular pressures (e.g. $P(\rho) = \rho^\gamma$). This is in particular due to the fact that we cannot control ρ far from vacuum in multi-d. Therefore, we shall look for global weak solutions which satisfy the equations in the sense of distributions and whose regularity and integrability is suggested by the previous energy inequality.

Theorem 2.2 (Perrin, Zatorska [60])**.** *Let $T > 0$, Ω be a bounded domain of $\mathbb{R}^3$. Assume that $\gamma > 3$ in* (2.5) *and assume initially $0 \leq \rho^0 < 1$ a.e. with moreover*

$$\int_\Omega \left[\frac{\rho^0|u^0|^2}{2} + H(\rho^0)\right] \leq E^0 < +\infty.$$

Then there exist global weak solutions (ρ, u) to the compressible system (1.4) *which satisfy the energy inequality* (2.7)*.*

The global weak solutions are obtained as limits as $\delta \to 0$ of weak solutions to the compressible Navier–Stokes equations with the truncated pressure

$$P_\delta(\rho) = \begin{cases} P(\rho) & \text{if} \quad \rho \leq 1 - \delta, \\ a\frac{\rho^\gamma}{\delta^\gamma} & \text{if} \quad \rho > 1 - \delta. \end{cases} \quad (2.9)$$

At $\delta > 0$ fixed, the existence of global weak solutions is ensured by the theory developed by Lions [43] and Feireisl [27]. The energy estimate (2.7) (satisfied uniformly wrt δ) and an additional equi-integrability property on the pressure P_δ (which requires in [60] $\gamma > 3$) provide then the necessary arguments for the weak convergence, as $\delta \to 0$, of the solutions towards a global weak solutions of system (1.4). We rigorously justify the constraint $\rho < 1$ a.e. thanks to the uniform control of the potential energy $H_\delta(\rho)$ which is singular close to 1:

$$\mathrm{meas}\{\rho_\delta \geq 1\} \leq C\delta^{\gamma-1}\|H_\delta(\rho_\delta)\|_{L^\infty L^1} \underset{\delta\to 0}{\longrightarrow} 0.$$

Remark 2.4. As shown by Feireisl et al. in [29], the condition $\gamma > 3$ which was used to prove the equi-integrability of the pressure, can actually be relaxed. For a slightly different system than (1.4), which also involves a singular pressure, the authors prove the equi-integrability of the pressure under the hypothesis $\gamma > 5/2$.

Weak-strong uniqueness. A legitimate question is to know if the weak solutions that we have constructed coincide with the unique local strong solution as long as this latter exists. A weak-strong uniqueness principle based on a relative entropy estimate for (1.4) was indeed proved by Feireisl et al. in [30]. For two couples starting from the same initial datum: (ρ, u) a weak solution to (1.4), and (r,U) the strong solution to (1.4) existing on the time interval $[0,T]$, the classical relative entropy functional reads

$$\mathcal{E}(\rho,u|r,U) = \int_\Omega \left[\frac{\rho|u-U|^2}{2} + \Big(H(\rho) - H(r) - H'(r)(\rho - r)\Big)\right]. \tag{2.10}$$

This functional satisfies for a.a. $t \in [0,T]$ the inequality

$$\mathcal{E}(\rho,u|r,U)(t) + \mu \int_0^t \int_\Omega |\nabla(u-U)|^2 + (2\mu+\lambda)\int_0^t\int_\Omega |\mathrm{div}(u-U)|^2 + \int_0^t\int_\Omega P(\rho)b(\rho) \leq \mathcal{E}(\rho,u|r,U)(0) + \mathcal{R} \tag{2.11}$$

where $b \in \mathcal{C}^1([0,1))$ is such that

$$|b'(s)|^{5/2} + |b(s)|^{5/2} \leq C\big(1+P(s)\big) \quad \text{for some} \quad C>0,\ \forall\, s\in[0,1)$$

and where the remainder term $\mathcal{R}$ is shown to be controlled under suitable assumptions on the pressure. In particular Feireisl et al. assume that $\gamma \geq 3$ which is more restrictive than the condition $\gamma > \frac{5}{2}$ that was supposed for the construction of weak solutions.

Remark 2.5. Compared to the classical entropy estimate for compressible Navier–Stokes equations (see for instance [28]), the inequality (2.11) includes in the left-hand side an original additional contribution:

$$\int_0^t\int_\Omega P(\rho)b(\rho).$$

This is basically due to the fact that the internal energy H is less singular than the pressure close to 1 (see (2.15) below) which means that it is dominated by the pressure for large densities. The additional integral involving the singular pressure is therefore necessary to control the most singular contributions appearing in the remainder $\mathcal{R}$ of the right-hand side.

Precisely, Feireisl et al. get the following result in the periodic setting.

Theorem 2.3 (Feireisl et al. [30])**.** *Assume that $\Omega = \mathbb{T}^d$, $d=2$ or 3, and that $\gamma \geq 3$. Let $T>0$, (ρ,u) be a global weak solution to (1.4), $(r,U) \in \mathcal{C}^1([0,T]\times\Omega)\times\Omega)\times\mathcal{C}^1([0,T];\mathcal{C}^2(\Omega))$ be a strong solution defined on $[0,T]$ such that $0<r<1$ in $[0,T]\times\Omega$. Assume moreover that initially the two couples (ρ,u), (r,U) coincide. Then for a.a. $t\in[0,T]$*

$$\mathcal{E}(\rho,u|r,U)(t) + \mu \int_0^t \int_\Omega |\nabla(u-U)|^2 + (2\mu+\lambda)\int_0^t\int_\Omega |\mathrm{div}(u-U)|^2 + \int_0^t\int_\Omega P(\rho)b(\rho) \leq \int_0^t \eta(s)\mathcal{E}(\rho,u|r,U)(s)\,\mathrm{d}s$$

for some $\eta \in L^1((0,T))$ and consequently, thanks to the Gronwall inequality,

$$(\rho,u) = (r,U) \quad \text{in } (0,T)\times\Omega.$$

Remark 2.6. The extension of the result to other boundary conditions is for the moment open. For instance, the case of Dirichlet boundary conditions (framework used in Theorem 2.2) would lead to a much more complicated remainder term $\mathcal{R}$ which cannot be easily tackled (see details in Remark 2.4 in [30]).

2.3. Singular limit towards hard congestion systems

The natural issue is now to link the soft approach (1.4) to the hard approach (1.1). Intuitively, setting $a=\varepsilon$ in the pressure (2.5) and letting ε tend to 0, then the pressure

$$p_\varepsilon(\rho) = P(\rho) = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\gamma \tag{2.12}$$

becomes more and more negligible for a fixed $\rho<1$ and it should converge to a limit pressure p which is activated only in the congested domain, i.e. such that $\mathrm{spt}\, p \subset \{\rho = 1\}$ (see Figure 2.1). This suggests that the limit $\varepsilon \to 0$ establishes a transition between the soft congestion model (1.4) with the singular pressure (2.12) and the hard congestion model (1.1).

This link is particularly interesting from the numerical point of view. If the limit hard congestion system may be difficult to handle due to the (unknown) sharp interface between the free and the congested domains, one can instead try to simulate numerically solutions of the approximate soft congestion system for which it is possible to adapt classical compressible numerical schemes. The interested reader is referred to the works of Degond and collaborators in [23, 25, 24] for more details on the subject.

From the theoretical point of view, we can justify rigorously the convergence of the weak solutions constructed in Theorem 2.2 towards global weak solutions of the hard congestion system (1.1). Consider the initial data $(\rho_{|t=0}, (\rho u)_{|t=0}) = (\rho^0, m^0)$ such that

$$0 \le \rho^0 \le 1 \text{ a.e.}, \quad \langle \rho^0 \rangle = \frac{1}{|\Omega|}\int_\Omega \rho^0(x)\,\mathrm{d}x < 1, \tag{2.13}$$

$$m^0 \in L^2(\Omega), \quad m^0 \mathbf{1}_{\{\rho^0=0\}} = 0 \text{ a.e.}, \quad \frac{|m^0|^2}{\rho^0}\mathbf{1}_{\{\rho^0>0\}} \in L^1(\Omega). \tag{2.14}$$

Theorem 2.4 (Perrin, Zatorska [60])**.** *Let $T > 0$, Ω be a bounded domain of $\mathbb{R}^3$ and assume initially* (2.13)*–*(2.14)*. Then, as $\varepsilon \to 0$, there exists a subsequence $(\rho_\varepsilon, u_\varepsilon)$ of global weak solutions constructed in Theorem 2.2 such that*

$$\begin{aligned} \rho_\varepsilon \to \rho \quad & \text{strongly in} \quad \mathcal{C}\big([0,T]; L^q(\Omega)\big) \quad \forall\, 1 \le q < +\infty \\ u_\varepsilon \to u \quad & \text{weakly in} \quad L^2\big(0,T; (H^1_0(\Omega))^d\big) \\ p_\varepsilon(\rho_\varepsilon) \to p \quad & \text{weakly in} \quad \mathcal{M}^+\big((0,T)\times\Omega\big) \end{aligned}$$

where (ρ, u, p) is a global weak solution of the hard congestion system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \nabla\left(\lambda \operatorname{div} u\right) - 2\operatorname{div}\big(\mu \mathrm{D}(u)\big) = 0, \\ 0 \le \rho \le 1, \ \operatorname{spt} p \subset \{\rho = 1\}, \ p \ge 0 \end{cases}$$

associated to the initial condition (ρ^0, m^0).

Idea of the proof. The cornerstone of the proof is the uniform control of the pressure in $L^1((0,T)\times\Omega)$. For classical Navier–Stokes equations, with for instance the pressure $P(\rho) = \rho^\gamma$, then the control of the energy (2.7) directly provides an estimate on $H(\rho) = \frac{\rho^\gamma}{\gamma-1}$ and thus on $P(\rho)$ in $L^\infty(0,T;L^1(\Omega))$. This is not the case here with the singular pressure $p_\varepsilon(\rho) = \varepsilon\big(\frac{\rho}{1-\rho}\big)^\gamma$, $\gamma > 5/2$, for which the free energy

$$H_\varepsilon(\rho) = \frac{\varepsilon}{\gamma-1}\frac{\rho^\gamma}{(1-\rho)^{\gamma-1}} \tag{2.15}$$

is less singular than p_ε close to the maximal density constraint. It means that additional work is required to control the pressure p_ε. The desired control is obtained by testing the momentum equation against a suitable test function ϕ. Consider $\phi = \mathcal{B}(\rho_\varepsilon - \langle\rho_\varepsilon\rangle)$ where $\mathcal{B}$ is the Bogovskii operator (an inverse of the divergence operator see Chapter 3.3 in [55] for definition and properties of this operator), the weak formulation of the momentum equation then rewrites

$$\int_0^T\!\!\int_\Omega p_\varepsilon(\rho)(\rho_\varepsilon - \langle\rho_\varepsilon\rangle) = -\int_0^T\!\!\int_\Omega \rho_\varepsilon u_\varepsilon \cdot \partial_t\phi - \int_0^T\!\!\int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla\phi + \int_0^T\!\!\int_\Omega \mathbb{S} : \nabla\phi$$

where all the integrals of the right-hand side are controlled thanks to the energy inequality. Split then the integral of p_ε into two parts, $\big\{\rho_\varepsilon \le \frac{1+\langle\rho_\varepsilon\rangle}{2}\big\}$ and $\big\{\rho_\varepsilon > \frac{1+\langle\rho_\varepsilon\rangle}{2}\big\}$, and observe that in the first domain the pressure $p_\varepsilon(\rho_\varepsilon)$ remains bounded since for ε small enough $\langle\rho_\varepsilon\rangle = \langle\rho^0_\varepsilon\rangle \le M^0 < 1$ (we recall hypothesis (2.13)). It means that the corresponding integral is bounded. Hence, we have the control

$$
\begin{aligned}
C &\geq \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)(\rho_\varepsilon - \langle\rho_\varepsilon\rangle)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+\langle\rho_\varepsilon\rangle}{2}\}} \\
&\geq \frac{1-\langle\rho_\varepsilon\rangle}{2}\int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+\langle\rho_\varepsilon\rangle}{2}\}} \\
&\geq \frac{1-M^0}{2}\int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)\mathbf{1}_{\{\rho_\varepsilon > \frac{1+\langle\rho_\varepsilon\rangle}{2}\}}.
\end{aligned}
$$

Using once again the assumption $M^0 < 1$, we finally deduce that

$$
\|p_\varepsilon(\rho_\varepsilon)\|_{L^1((0,T)\times\Omega)} \leq C. \tag{2.16}
$$

The rest of the proof is more or less standard and follows the compactness method developed by Lions [43]. In particular, the so-called *effective viscous flux*, defined as $(2\mu+\lambda)\operatorname{div} u_\varepsilon - p_\varepsilon(\rho_\varepsilon)$, is shown to satisfy a weak compactness property. This property is a key ingredient to demonstrate the strong convergence of the density ρ_ε. We refer to [60] for the complete proof. □

Remark 2.7. We are in fact able to show that the system follows the incompressible dynamics in the congested domain. This is given by the next lemma which relies on the theory of renormalized solutions.

Lemma 2.5 ([44, Lemma 2.1])**.** *Let* $(\rho, u) \in L^2\big((0,T)\times\Omega\big) \times L^2(0,T;H^1_0(\Omega))$ *satisfy the mass equation* (1.1a) *in the sense of distributions. Then the following two assertions are equivalent*

- $0 \leq \rho \leq 1$ *a.e.;*
- $\operatorname{div} u = 0$ *a.e. on* $\{\rho \geq 1\}$ *and* $0 \leq \rho_0 \leq 1$.

Hence, in the singular limit $\varepsilon \to 0$, we obtain the convergence of the compressible equations towards the incompressible equations locally where the density ρ_ε tends to 1 sufficiently fast with respect to ε. Let us emphasize the multi-scale nature of the problem which presents two competitive effects in the pressure $p_\varepsilon(\rho_\varepsilon) = \varepsilon\big(\frac{\rho}{1-\rho}\big)^\gamma$: $\varepsilon \to 0$ and ρ_ε that can tend to 1. The congested regions $p > 0$ are those where at the approximate level $1 - \rho_\varepsilon = \mathcal{O}(\varepsilon^{1/\gamma})$. To some extent, the congestion limit can be related to the well-known low Mach number limit which establishes a link between compressible fluid equations and incompressible ones (see for instance [31]). The analogy between the two singular limits has been successfully used for the design of numerical schemes in [23, 24].

Remark 2.8. Observe that the singular limit does not bring additional regularity on the limit pressure p compared to Theorem 2.1 obtained by Berthelin. Here, the low regularity of p is a consequence of the lack of equi-integrability of the sequence of approximate pressures $(p_\varepsilon(\rho_\varepsilon))_\varepsilon$. Nevertheless, we are able to characterize more precisely the regularity of p thanks to an analysis of the different terms of the momentum equation (1.1b). We have actually

$$
p \in W^{-1,\infty}(0,T;H^1(\Omega)) + L^{q_1}(0,T;L^{q_2}(\Omega)) \quad \text{for some} \quad q_1, q_2 > 1.
$$

Combining this result with the regularity of ρ, we are then able to give a sense the product ρp and we can finally justify the exclusion constraint (1.3) (see [60] for details).

Remark 2.9. Note that the previous relative entropy estimate (2.11) derived by Feireisl et al. in [30] does not hold uniformly with respect to ε if one sets $P(\rho) = p_\varepsilon(\rho)$ with (2.12). This prevents us to use the relative entropy method to justify the singular limit $\varepsilon \to 0$, which makes an important difference with the low Mach number limit (see [31]) previously mentioned.

Remark 2.10. It could be possible to extend Theorem 2.4 to more general (possibly non-monotone) singular pressures p_ε than (2.12), provided that close to 1, for some $\gamma > 1$: $p_\varepsilon(\rho)(1-\rho)^\gamma \to +\infty$ as $\rho \to 1$. But, as explained in the previous subsection, the construction of global weak solutions at ε fixed is ensured only for pressures such that $p_\varepsilon(\rho)(1-\rho)^\gamma \to +\infty$ as $\rho \to 1$ for some $\gamma > 5/2$.

Another possible extension concerns the case of heterogeneous maximal constraints. Originally the result in [60] is stated for constraints $\rho \leq \rho^*(x)$ where ρ^* is $\mathcal{C}^1(\overline{\Omega})$ and bounded from below.

Moreover, the pressure term is assumed to be of the form $\rho^* \nabla p_\varepsilon(\rho/\rho^*)$ to ensure the compatibility with the energy estimate. In [25], Degond et al. develop the same analysis for a time and space dependent maximal constraint ρ^* which is transported by the flow. Finally, another extension which, to the knowledge of the author, has not been treated in the literature is the case of partially constrained flows where ρ^* would be infinite in a sub-domain of Ω. There are nevertheless some local well-posedness results in this direction for some models of waves-structures interactions (see Subsection 3.2 below).

Other approximations of the hard congestion system. Theorem 2.4 provides the existence of global weak solutions to the free/congested equations (1.1) via the approximation by the singular compressible Navier–Stokes equations (1.4), but this is not the only way to construct weak solutions to (1.1). Actually, the first existence result for the hard congestion system was obtained by Lions and Masmoudi [44] by a *penalty method*. Like for classical contact problems, the idea is to penalize the large density regions by considering a barotropic pressure of the form $p_n = a\rho^{\gamma_n}$ where γ_n is very large (see Figure 2.1). As $\gamma_n \to +\infty$ the limit pressure satisfies as well $\operatorname{spt} p \subset \{\rho = 1\}$.

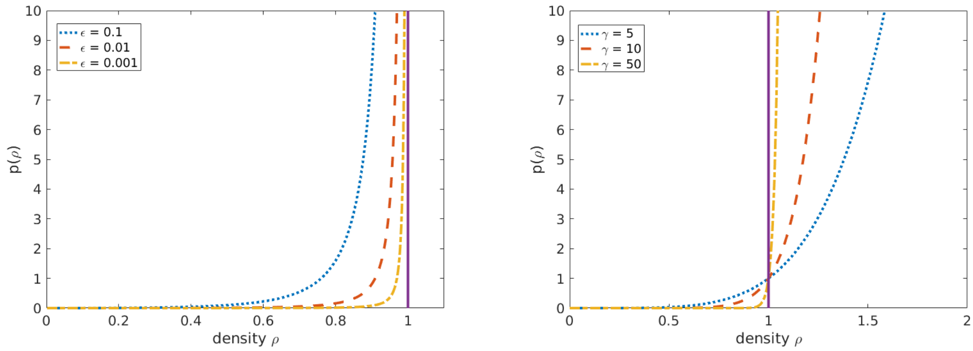

Figure 2.1: Different approximate pressures. On the left: singular pressures $p_\varepsilon = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\gamma$ $(\gamma = 2)$ corresponding to the soft congestion approach. On the right: power laws $p_n(\rho) = a\rho^{\gamma_n}$ $(a = 1)$ corresponding to the penalty method.

In [44], Lions and Masmoudi prove that global weak solutions to the compressible Navier–Stokes equations with the pressure $p_n = a\rho_n^\gamma$ converge weakly (in the same sense as in Theorem 2.4) towards global weak solutions to (1.1). This result has been then adapted by Labbe and Maitre in [37] to the case of density dependent viscosities $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ (with additional capillarity for a simplified model of droplet). This penalty approach is also intensively used in the tissue growth modeling like in [69] for the free/congested Navier–Stokes equations and in [61], [62] for the Hele–Shaw free boundary problem (in which the Navier–Stokes equations are replaced by the porous medium equation). In this context, the singular limit is sometimes called *mesa limit*.

In [34], [33], this is a *relaxation approach* which is adopted and used from the numerical point of view. It consists in considering at the approximate level a pressure $p_\lambda = \frac{(\rho-1)_+^2}{\lambda^2}$ and to let $\lambda \to 0$ like in usual relaxation methods. The parameter λ can be interpreted here as the Mach number which characterizes the compressibility of the flow.

In these alternative approaches, since the considered approximate pressure is not singular in the vicinity of $\rho = 1$, the maximal density constraint $\rho \leq 1$ will be not satisfied at the approximate level (n or λ fixed) but is only recovered at the limit ($n \to +\infty$ or $\lambda \to 0$). Let us emphasize that the choice of the approximate problem should be supported by the modeling arguments.

2.4. Congested fronts

We have presented in Theorem 2.1 and Theorem 2.4 two results showing the existence of global weak solutions for the free-congested fluid equations in the inviscid and viscous cases respectively. In the inviscid case the solutions are obtained as limits of discrete congested blocks while in the viscous case, there are constructed as limits of solutions of the soft congestion system. In this latter case, it seems that we do not get any information on the measure and the dynamics of the congested domain. We propose to investigate in this subsection the question of the transition at the frontier between the congested domain and the free domain. In the one-dimensional inviscid and viscous settings, we show that the singular congestion limit $\varepsilon \to 0$ provides valuable insights on the dynamics and the stability of congested zones. In the rest of this subsection we set $\Omega = \mathbb{R}$.

2.4.1. Partially congested solutions of the free-congested Euler equations

First of all, let us recall that the presence of constant positive viscosities was essential for the validity of the compactness arguments developed in [60] and presented in the previous section. Without the viscous dissipation, the singular limit $\varepsilon \to 0$ (i.e. Theorem 2.4) is still an open problem. Nevertheless, some studies have shown numerical evidences of the link between the soft and the hard approaches in the inviscid case, that is between equations

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, & \text{(2.17a)} \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x\left(\rho_\varepsilon u_\varepsilon^2 + p_\varepsilon(\rho_\varepsilon)\right) = 0. & \text{(2.17b)} \end{cases}$$

and the constrained Euler equations (already written in (2.1))

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{(2.18a)} \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0, & \text{(2.18b)} \\ 0 \le \rho \le 1, \ \operatorname{spt} p \subset \{\rho = 1\}, \ p \ge 0. & \text{(2.18c)} \end{cases}$$

The Riemann problem. In [23], Degond et al. compare the behavior of their numerical solutions, obtained by discretizing the approximate problem (2.17), with some exact solutions of the limit problem (2.18). These exact solutions are solutions of the Riemann problem (see (2.19) below). The idea is to describe the solutions of the Riemann problem on the soft compressible model and to study their asymptotic behavior as $\varepsilon \to 0$. This analysis is important as a first step in the understanding of the singular limit between the systems (2.17) and (2.18). Besides, the study of these solutions brings valuable insights on the different scales involved in passing to the limit $\varepsilon \to 0$. Riemann solutions are solutions to (2.17) associated to the initial condition

$$(\rho, u)(0, x) = \begin{cases} (\rho_l, u_l) & \text{if} \quad x < 0, \\ (\rho_r, u_r) & \text{if} \quad x \ge 0. \end{cases} \tag{2.19}$$

In [23], the pressure is assumed to be the same as the one considered in the previous section, i.e.

$$p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1-\rho}\right)^\gamma \quad \text{for some } \gamma > 1. \tag{2.20}$$

System (2.17), expressed in variables $(\rho_\varepsilon, q_\varepsilon = \rho_\varepsilon u_\varepsilon)$, admits the following characteristic speeds

$$\lambda_\varepsilon^1(\rho, q) = \frac{q}{\rho} - \sqrt{p_\varepsilon'(\rho)}, \quad \lambda_\varepsilon^2(\rho, q) = \frac{q}{\rho} + \sqrt{p_\varepsilon'(\rho)} \tag{2.21}$$

and characteristic fields

$$r_\varepsilon^1(\rho, q) = \begin{pmatrix} 1 \\ \frac{q}{\rho} - \sqrt{p_\varepsilon'(\rho)} \end{pmatrix}, \quad r_\varepsilon^2(\rho, q) = \begin{pmatrix} 1 \\ \frac{q}{\rho} + \sqrt{p_\varepsilon'(\rho)} \end{pmatrix}$$

which are both genuinely non-linear for positive ρ. Solutions of Problem (2.17)–(2.19) consist then of constant states connected by shocks (discontinuities) or rarefaction waves (continuous curves). Let (ρ_l, q_l) be our reference left state, we either have (see classical books on hyperbolic problems like [66])

- a *shock wave*: the set of right states that can be connected to (ρ_l, q_l) by a shock satisfying both the Rankine–Hugoniot and Lax entropy conditions satisfy

$$\begin{cases} \rho_r > \rho_l \\ q_r = S_1^\varepsilon(\rho_r) = \rho_r u_l - \sqrt{\frac{\rho_r}{\rho_l} \frac{p_\varepsilon(\rho_r) - p_\varepsilon(\rho_l)}{\rho_r - \rho_l}} (\rho_r - \rho_l) \end{cases} \tag{1-shock}$$

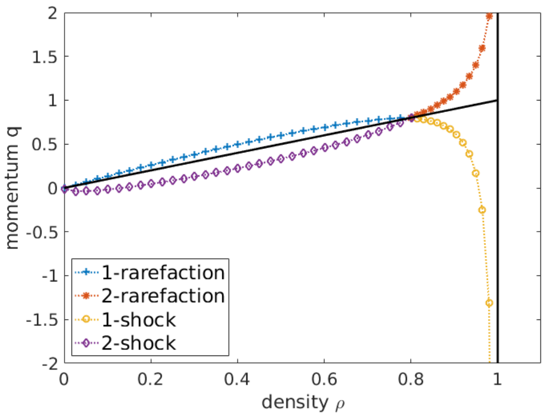

Figure 2.2: Rarefaction and shock curves for the reference left state $(\rho_l, q_l) = (0.8, 0.8)$ and parameters $\varepsilon = 0.01$, $\gamma = 2$ in the pressure law.

or

$$\begin{cases} \rho_r < \rho_l \\ q_r = S_2^\varepsilon(\rho_r) = \rho_r u_l + \sqrt{\frac{\rho_r}{\rho_l}\frac{p_\varepsilon(\rho_r)-p_\varepsilon(\rho_l)}{\rho_r-\rho_l}}(\rho_r - \rho_l) \end{cases} \qquad \text{(2-shock)}$$

and the respective speeds of these shocks, $\sigma_{1,2}^\varepsilon$, are

$$\sigma_{1,2}^\varepsilon = \frac{S_{1,2}^\varepsilon - q_l}{\rho_r - \rho_l} = u_l \mp \sqrt{\frac{\rho_r}{\rho_l}}\sqrt{\frac{p_\varepsilon(\rho_r) - p_\varepsilon(\rho_l)}{\rho_r - \rho_l}};$$

- or a *rarefaction wave*: the set of right states that can be connected to (ρ_l, q_l) by a rarefaction wave are

$$\begin{cases} \rho_r < \rho_l \\ q_r = R_1^\varepsilon(\rho_r) = \rho_r u_l - \rho_r \int_{\rho_l}^{\rho_r} \frac{\sqrt{p_\varepsilon'(s)}}{s}\,\mathrm{d}s \end{cases} \qquad \text{(1-rarefaction)}$$

or

$$\begin{cases} \rho_r > \rho_l \\ q_r = R_2^\varepsilon(\rho_r) = \rho_r u_l - \rho_r \int_{\rho_l}^{\rho_r} \frac{\sqrt{p_\varepsilon'(s)}}{s}\,\mathrm{d}s \end{cases} \qquad \text{(2-rarefaction)}$$

The shock $S_{1,2}^\varepsilon$ and rarefaction $R_{1,2}^\varepsilon$ curves are represented in the (ρ, q)-plane on Figure 2.2. Observe that in each case, if the density states stay far from 1, then all the pressure contributions vanish at the limit $\varepsilon \to 0$ and $q_r \to \rho_r u_l$. This means that the curves represented on Figure 2.2 tend to the union of the solid lines. From the theoretical point of view, like for the pressureless equations [10], we have the emergence on the limit Riemann solutions of contact waves and vacuum states.

Proposition 2.6 ([23, Propositions 1 and 2])**.** *Let (ρ_l, q_l) be a given left state such that $0 < \rho_l < 1$, the following facts hold in the limit $\varepsilon \to 0$.*

1. *The limit of the shock curves $S_{1,2}^\varepsilon$ as $\varepsilon \to 0$ is the union of the straight lines $\{q = \rho u_l\}$ and $\{\rho = 1\}$. This remains also true if $\rho_l = \rho_l^\varepsilon \to 1$.*

2. *The limit of the rarefaction curves $R_{1,2}^\varepsilon$ as $\varepsilon \to 0$ is the union of the straight lines $\{q = \rho u_l\}$ and $\{\rho = 1\}$.*

3. *Assume that $\rho_l = \rho_l^\varepsilon \to 1$ in such a way that $p_\varepsilon(\rho_l^\varepsilon) \to \bar{p} > 0$, then, for all $\rho_r < \rho_l^\varepsilon$, it holds*

$$|R_1(\rho_r) - u_l| \leq \mathcal{O}\left(\varepsilon^{1/(2\gamma)}\right). \qquad (2.22)$$

We are especially interested in the evolution of a congested front with a left almost congested state $(\rho_l^\varepsilon, q_l^\varepsilon)$ and a free right state (ρ_r, q_r). More precisely, we consider

$$\rho_r < 1, \quad \rho_l^\varepsilon = 1 - r_l \varepsilon^{1/\gamma} \text{ that is } p_l = r_l^{-\gamma} \tag{2.23}$$

and we assume that ε is small so that $\rho_r < \rho_l^\varepsilon$. According to the values of velocities u_l^ε and u_r, four cases arise

- $u_l^\varepsilon < u_r$: the two states are connected with two rarefaction waves
$$(\rho_l^\varepsilon, q_l^\varepsilon) \underset{\text{1-RW}}{\longrightarrow} (\hat\rho^\varepsilon, \hat q^\varepsilon) \underset{\text{2-RW}}{\longrightarrow} (\rho_r, q_r) \qquad \text{where} \quad \hat\rho < \rho_r;$$
- $u_l = u_r$: the two states are connected with one rarefaction wave and one shock wave
$$(\rho_l^\varepsilon, q_l^\varepsilon) \underset{\text{1-RW}}{\longrightarrow} (\hat\rho^\varepsilon, \hat q^\varepsilon) \underset{\text{2-SW}}{\longrightarrow} (\rho_r, q_r) \qquad \text{where} \quad \rho_r < \hat\rho < \rho_l^\varepsilon;$$
- $u_l^\varepsilon > u_r$: either $S_2^\varepsilon(\rho_l) < q_l$ and the solution consists of two shocks
$$(\rho_l^\varepsilon, q_l^\varepsilon) \underset{\text{1-SW}}{\longrightarrow} (\hat\rho^\varepsilon, \hat q^\varepsilon) \underset{\text{2-SW}}{\longrightarrow} (\rho_r, q_r) \qquad \text{where} \quad \rho_l^\varepsilon < \hat\rho,$$
or $S_2^\varepsilon(\rho_l) > q_l^\varepsilon$ and then we have a rarefaction wave followed by a shock
$$(\rho_l^\varepsilon, q_l^\varepsilon) \underset{\text{1-RW}}{\longrightarrow} (\hat\rho^\varepsilon, \hat q^\varepsilon) \underset{\text{2-SW}}{\longrightarrow} (\rho_r, q_r) \qquad \text{where} \quad \rho_r < \hat\rho < \rho_l^\varepsilon.$$

As $\varepsilon \to 0$, the characteristic speeds $\lambda_{1,2}^\varepsilon$ at $\rho_l^\varepsilon = 1 + r_l\varepsilon^{1/\gamma}$ tend to $\mp\infty$. On the contrary, for $(\rho^\varepsilon, q^\varepsilon)$ such that $p_\varepsilon(\rho^\varepsilon) \to 0$, we have $\lim_\varepsilon \lambda_1^\varepsilon(\rho^\varepsilon, q^\varepsilon) = \lim_\varepsilon \lambda_2^\varepsilon(\rho^\varepsilon, q^\varepsilon) = \frac{q}{\rho}$. As a consequence, a rarefaction wave linking two non-congested states (i.e. with zero limit pressure) degenerates into a contact discontinuity (CD) with speed $u = \frac{q}{\rho}$. In the congested domain, the rarefaction waves give rise to another type of discontinuities called *declustering waves* (DW) in [23]. They consist of a shock wave between two congested states with speed $-\infty$ and with a zero pressure for positive time.

The limit behaviors are described in the next proposition.

Proposition 2.7 ([23, Proposition 5])**.** *Assume that the left state $(\rho_l^\varepsilon, u_l^\varepsilon, p_\varepsilon(\rho_l^\varepsilon))$ tends to (ρ_l, u_l, p_l) with $\rho_l = 1$, $p_l > 0$, as $\varepsilon \to 0$. Depending on the sign of $u_l - u_r$ we have the three following cases:*

1. *$u_l < u_r$: then the two states are connected by the succession of one declustering wave and two contact discontinuities*
$$(1, q_l, p_l) \underset{DW}{\longrightarrow} (1, q_l, 0) \underset{CD}{\longrightarrow} (0, \hat q, 0) \underset{vacuum}{\longrightarrow} (0, q_r, 0) \underset{CD}{\longrightarrow} (\rho_r, q_r, 0);$$
2. *$u_l = u_r$: then the two states are connected by the succession of one declustering wave and one contact discontinuity*
$$(1, q_l, p_l) \underset{DW}{\longrightarrow} (1, \hat q, 0) \underset{CD}{\longrightarrow} (\rho_r, q_r, 0);$$
3. *$u_l > u_r$: then the two states are connected by the succession of shock waves*
$$(1, q_l, p_l) \underset{1\text{-}SW}{\longrightarrow} (1, q_l, \hat p) \underset{2\text{-}SW}{\longrightarrow} (\rho_r, q_r, 0)$$
where the intermediate pressure is
$$\hat p = \frac{\rho_r}{1-\rho_r}(u_l - u_r)^2 \tag{2.24}$$
and the speed shocks are
$$\sigma_1 = -\infty, \quad \sigma_2 = u_r + \sqrt{\frac{1}{\rho_r}}\sqrt{\frac{\hat p}{1-\rho_r}}. \tag{2.25}$$

Remark 2.11. An interesting feature of the limit solutions is that they do not depend on the soft approximation, namely on the choice of pressure we make at $\varepsilon > 0$. The limit pressure state $p_l = r_l^{-\gamma}$ (cf (2.23)) which depends on the parameter γ, is instantaneously changed into 0 or $\hat p$ (independent of γ) since the declustering wave (Cases 1 and 2) and the 1-shock (Case 3) have infinite speed. Note however, that this is true only for a right state ρ_r which stays far away from the singularity. In the case where $\rho_r = \rho_r^\varepsilon \to 1$, the limit solution will instead depends explicitly on the parameter γ, i.e. on the choice of the approximate pressure law (cf. [23, Proposition 6]).

Creation of congestion and shocks. We have exhibited in the previous proposition some stationary partially congested solutions to the constrained Euler equations (2.18). From the dynamical point of view, another interesting question is the one of the formation of a congested domain. This is precisely the purpose of the study of Bresch and Renardy in [19]. More precisely, introducing the Lagrangian coordinates (t, x)

$$\mathrm{x} = \int_{-\infty}^{x} \rho(t, y)\, \mathrm{d}y, \tag{2.26}$$

system (2.18) rewrites

$$\begin{cases} \partial_t \left(\frac{1}{\rho}\right) - \partial_{\mathrm{x}} u = 0 & (2.27a) \\ \partial_t u + \partial_{\mathrm{x}} p = 0 & (2.27b) \\ \frac{1}{\rho} \geq 1, \quad \operatorname{spt} p \subset \{\rho = 1\}, \quad p \geq 0. & (2.27c) \end{cases}$$

If initially $\rho^0(\mathrm{x}) < 1$ for all $\mathrm{x} \in \mathbb{R}$, then we have the free evolution

$$u(t, \mathrm{x}) = u^0(\mathrm{x}), \quad \frac{1}{\rho(t, \mathrm{x})} = \frac{1}{\rho^0(\mathrm{x})} + (u^0)'(\mathrm{x}) t$$

at least during a short time, as long as $\rho < 1$. If we assume that $(u^0)' < 0$ and $u^0(\mathrm{x}^*) = 0$ then the density ρ reaches 1 at the position x^* and time $t^* = \frac{\rho^0(\mathrm{x}^*)-1}{\rho^0(\mathrm{x}^*)(u^0)'(\mathrm{x}^*)} > 0$. After the first contact time t^*, we expect that the congested domain becomes an interval $[\mathrm{x}_-(t), \mathrm{x}_+(t)]$ which continues to grow due to the initial concentrating velocity u^0. In this congested (incompressible) domain we have

$$\rho(t, \mathrm{x}) = 1, \quad u(t, \mathrm{x}) = \bar{u}(t), \quad p(t, \mathrm{x}) = -\bar{u}(t)\mathrm{x} + \bar{p}(t).$$

To determine $\bar{u}$, $\bar{p}$ and the evolution of the congested domain, that is the dynamics of $\mathrm{x}_\pm(t)$, Bresch and Renardy make in [19] two assumptions. On one hand they assume that at (t^*, x^*)

$$\partial_{\mathrm{x}} \rho(t^*, \mathrm{x}^*) = 0, \quad \partial^2_{\mathrm{xx}} \rho(t^*, \mathrm{x}^*) = -2q_1 < 0, \quad (u^0)'(\mathrm{x}^*) = -q_2 < 0.$$

On the other hand they assume that two shocks directly occur after t^* at the transitions with the free domain, i.e. at $\mathrm{x}_\pm(t)$. To simplify, let us set $(t^*, \mathrm{x}^*) = (0, 0)$. At leading order we have

$$\frac{1}{\rho(t, \mathrm{x})} - 1 = q_1 \mathrm{x}^2 - q_2 t, \quad u^0(\mathrm{x}) = -q_2 \mathrm{x}, \tag{2.28}$$

which we plug into the systems expressing the Rankine–Hugoniot conditions at the transition points $\mathrm{x}_\pm$. An explicit solution of this system is

$$\mathrm{x}_+(t) = -\mathrm{x}_-(t) = \sqrt{3\frac{q_2}{q_1} t}, \quad \bar{u}(t) = 0, \quad \bar{p}(t) = \frac{3}{2}\frac{q_2^2}{q_1}. \tag{2.29}$$

and a perturbation argument shows in [19] that this solution is unique.

Link with the study of the Riemann problem. In the light of the above Proposition 2.7 (from Degond et al. [23]), the assumption that shocks occur as soon as the maximal value is attained, is somehow natural since we have seen that only shocks may connect a congested state to a free state for the Riemann problem ($u_r < u_l$ Case 3 of Proposition 2.7). Observe also that (2.29) can be recovered from (2.24). Indeed we have here at x_+ $u_l = 0$, $\rho_l = 1$ and

$$u_r = -q_2 \mathrm{x}_+(t) = -\sqrt{3\frac{q_2^3}{q_1} t}, \quad \frac{1}{\rho_r} = 1 + q_1 \mathrm{x}_+^2(t) - q_2 t = 1 + 2q_2 t.$$

Therefore by (2.24), we recover

$$\bar{p} = \frac{\rho_r}{1-\rho_r}(u_l - u_r)^2 = \frac{3q_2^2}{2q_1}.$$

Despite this formal link with the explicit solutions of the Riemann problem, the rigorous justification of apparition of shocks at the congestion fronts is, to the knowledge of the author, still an open issue.

Soft approximation and numerical evidences for the creation of shocks. Bresch and Renardy also deduce from (2.28) a natural parabolic scaling between the time and space variables $t = \mathrm{x}^2$, which turns out to be crucial to observe the creation of shocks on numerical simulations. Let us come back to the approximate soft congestion model

$$\begin{cases} \partial_t\left(\frac{1}{\rho_\varepsilon}\right) - \partial_{\mathrm{x}} u_\varepsilon = 0 \\ \partial_t u_\varepsilon + \partial_{\mathrm{x}} p_\varepsilon(\rho_\varepsilon) = 0 \end{cases}$$

with the pressure

$$p_\varepsilon(\rho) = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\gamma, \quad \gamma > 1.$$

Given a pressure p, the density is then given by

$$\frac{1}{\rho_\varepsilon} - 1 = \frac{\varepsilon^{1/\gamma}}{p^{1/\gamma}}.$$

In the congested domain we have

$$p = \bar{p} > 0,$$

which leads us to set

$$\frac{1}{\rho_\varepsilon} - 1 = \varepsilon^{1/\gamma} r.$$

and thus by (2.28)

$$t = \varepsilon^{1/\gamma}\tilde{t}, \quad \mathrm{x} = \varepsilon^{1/(2\gamma)}\tilde{\mathrm{x}}, \quad \text{and} \quad u = \varepsilon^{1/(2\gamma)}\tilde{u}.$$

The rescaled variables satisfy now the system

$$\begin{cases} \partial_{\tilde{t}} r - \partial_{\tilde{\mathrm{x}}}\tilde{u} = 0, \\ \partial_{\tilde{t}}\tilde{u} + \partial_{\tilde{\mathrm{x}}}\left(\frac{1}{r^\gamma}\right) = 0, \end{cases}$$

which describes in a refined way the transition between the congested domain and the free domain. Numerical simulations of solutions of the rescaled system are provided in [19] and they clearly show two shocks connecting the congested and the free phases.

As we said before, the creation of the shocks at the transition has not been demonstrated yet and seems to be a challenging issue. Another interesting question is the question of the stability of the congested shocks explicitly described in Proposition 2.7. Besides, one may ask if it is possible to find steady partially congested solutions in the viscous framework, and if yes, what happens at the transition between the free and the congested phases. These issues are addressed in the next paragraph.

2.4.2. Partially congested profiles in Navier–Stokes equations

The one-dimensional free-congested Navier–Stokes equations written in Lagrangian coordinates (t, x) read

$$\begin{cases} \partial_t v - \partial_{\mathrm{x}} u = 0, & (2.30a) \\ \partial_t u + \partial_{\mathrm{x}} p - \partial_{\mathrm{x}}\left(\frac{\partial_{\mathrm{x}} u}{v}\right) = 0, & (2.30b) \\ v \geq 1, \ \operatorname{spt} p \subset \{v = 1\}, \ p \geq 0. & (2.30c) \end{cases}$$

where $v = 1/\rho$ denotes the specific volume. The Navier–Stokes system is supplemented with the far field condition

$$(v, u, p)(t, \mathrm{x}) \underset{\mathrm{x}\to\pm\infty}{\longrightarrow} (v_\pm, u_\pm, p_\pm) \quad \forall\, t \geq 0. \tag{2.31}$$

Description of partially congested fronts. As in the previous paragraph, we are interested in finding explicit free-congested solutions of (2.30). Assume that $v_- = 1$, $v_+ > 1$, $p_+ = 0$, namely that we have a congested left state and a free right state, and suppose moreover $u_- > u_+$. The calculations performed in [22] show that we can found a traveling front solution, that is a solution under the form $(v, u, p)(t, \mathrm{x}) = (\mathfrak{v}, \mathfrak{u}, \mathfrak{p})(\mathrm{x} - st)$, provided that the left pressure state satisfies

$$p_- = \frac{(u_- - u_+)^2}{v_+ - 1}.$$

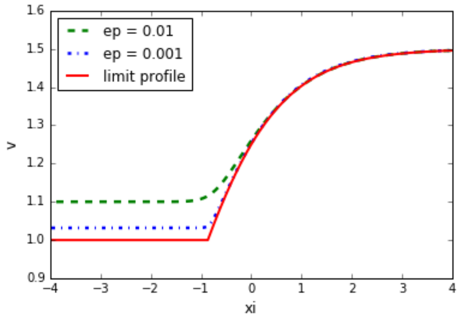

Figure 2.3: Asymptotic behavior of the profiles $\mathfrak{v}_\varepsilon$ joining the end-states $v_-^\varepsilon = 1 + \varepsilon^{1/\gamma}$ and $v_+ = 1.5$.

The shock speed is then given by the Rankine–Hugoniot relation

$$s = \frac{u_- - u_+}{v_+ - 1} > 0.$$

Let us set $\xi = \mathrm{x} - st$. If the transition between the two dynamics is located at $\xi^* \in \mathbb{R}$, we have the congested dynamics

$$\mathfrak{v}(\xi) = 1, \quad \mathfrak{u}(\xi) = u_-, \quad \mathfrak{p}(\xi) = p_- \quad \forall\, \xi < \xi^*,$$

and the free dynamics in $(\xi^*, +\infty)$ characterized by

$$\begin{cases} \mathfrak{v}'(\xi) = \frac{s}{\mu}(v_+ - \mathfrak{v}(\xi))\mathfrak{v}(\xi) \\ \mathfrak{u}'(\xi) = -s\mathfrak{v}'(\xi) \\ \mathfrak{p}(\xi) = 0 \end{cases} \quad \forall\, \xi > \xi^* \quad \text{and} \quad \begin{cases} \mathfrak{v}(\xi^*) = 1 \\ \mathfrak{u}(\xi^*) = u_- \end{cases}$$

$\mathfrak{v}$ and $\mathfrak{u}$ being continuous at ξ^*.

The solutions $(\mathfrak{v}, \mathfrak{u}, \mathfrak{p})$ are approximated in [22] by viscous traveling waves $(v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) = (\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)(t - s_\varepsilon \mathrm{x})$ solutions of the following *soft congestion* system

$$\begin{cases} \partial_t v_\varepsilon - \partial_\mathrm{x} u_\varepsilon = 0, & (2.32a) \\ \partial_t u_\varepsilon + \partial_\mathrm{x} p_\varepsilon(v_\varepsilon) - \mu \partial_\mathrm{x}\left(\frac{\partial_\mathrm{x} u_\varepsilon}{v_\varepsilon}\right) = 0, & (2.32b) \\ v_\varepsilon > 1,\ p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma},\ \gamma > 1. & (2.32c) \end{cases}$$

provided some appropriate far field conditions

$$(v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) \underset{\mathrm{x}\to-\infty}{\longrightarrow} (v_-^\varepsilon, u_-^\varepsilon), \quad (v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) \underset{\mathrm{x}\to+\infty}{\longrightarrow} (v_+, u_+) \quad \forall\, t \geq 0. \tag{2.33}$$

We illustrate the asymptotic behavior of the viscous shock profiles on Figure 2.3.

Proposition 2.8 (Dalibard, Perrin [22]). *Consider the approximate left state $(v_-^\varepsilon, u_-^\varepsilon)$ such that $v_-^\varepsilon = 1 + \frac{\varepsilon^{1/\gamma}}{p_-^{1/\gamma}} < v_+$, $u_-^\varepsilon > u_+$ with $(u_+ - u_-^\varepsilon)^2 = -(v_+ - v_-^\varepsilon)(p_\varepsilon(v_+) - p_-)$. Then there exists a unique (up to a shift) traveling front $(v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) = (\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)(\mathrm{x} - s_\varepsilon t)$ solution of (2.32)–(2.33) with the shock speed*

$$s_\varepsilon = \frac{u_-^\varepsilon - u_+}{v_+ - v_-^\varepsilon} = \sqrt{-\frac{p_\varepsilon(v_+) - p_-}{v_+ - v_-^\varepsilon}}.$$

In addition

$$\lim_{\varepsilon\to 0} \sup_{\xi\in\mathbb{R}} \inf_{C\in\mathbb{R}} \left|\mathfrak{v}_\varepsilon(\xi + C) - \mathfrak{v}(\xi)\right| = 0.$$

Remark 2.12. Observe that the pressure profile $\mathfrak{p}$ is discontinuous across the interface (although $\mathfrak{v}$ and $\mathfrak{u}$ are continuous) and does not depend on the viscosity μ in the congested domain. Actually, we recover again the same value as the one derived by Degond, Hua and Navoret in [23] for the free-congested Euler system (see Proposition 2.7 Case 3).

A refined description of the behavior of the solutions in the vicinity of ξ^* is also proposed in [22]. It enables the derivation of quantitative error estimates in the zone of the transition between the two dynamics.

Asymptotic nonlinear stability of the approximate profiles $(\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)$. Thanks to a suitable weighted energy method, the approximate profiles $(\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)$ are proved to be asymptotically stable uniformly with respect to the parameter ε.

Theorem 2.9 (Dalibard, Perrin [22]). *Assume that initially the couple $(v_\varepsilon^0, u_\varepsilon^0)$ satisfies*

$$v_\varepsilon^0 - \mathfrak{v}_\varepsilon \in H^2(\mathbb{R}) \cap L_0^1(\mathbb{R}), \quad u_\varepsilon^0 - \mathfrak{u}_\varepsilon \in H^1(\mathbb{R}) \cap L_0^1(\mathbb{R}), \tag{2.34}$$

where $L_0^1(\mathbb{R})$ denotes the space of integrable function with zero integral. If the initial perturbation $(v_\varepsilon^0 - \mathfrak{v}_\varepsilon, u_\varepsilon^0 - \mathfrak{u}_\varepsilon)$ is sufficiently small in some appropriate weighted energy spaces, then there exists a unique global strong solution $(v_\varepsilon, u_\varepsilon)$ of (2.32)–(2.33) with the initial data $(v_\varepsilon^0, u_\varepsilon^0)$, which is such that

$$v_\varepsilon(t, \mathrm{x}) > 1 \quad a.e.$$

and

$$\sup_{\mathrm{x}\in\mathbb{R}} \big| \big((v_\varepsilon, u_\varepsilon)(t, \mathrm{x}) - (\mathfrak{v}_\varepsilon, \mathfrak{u}_\varepsilon)(t, \mathrm{x}) \big) \big| \underset{t\to+\infty}{\longrightarrow} 0.$$

As said before, this result relies on an energy method. More precisely, the key points of the proof are the passage to the variables (v, w) where w is the so-called *effective velocity* $w = u - \mu\partial_\mathrm{x} \ln v$ (see also Section 4.2.1), and the derivation of the following weighted energy estimate on the integrated quantities (V, W), such that $\partial_\mathrm{x} V = v_\varepsilon - \mathfrak{v}_\varepsilon$, $\partial_\mathrm{x} W = w_\varepsilon - \mathfrak{w}_\varepsilon$,

$$\sum_{k=0}^{2} \varepsilon^{\frac{2k}{\gamma}} \left[\sup_{t\geq 0} \int_\mathbb{R} \left(\frac{|\partial_\mathrm{x}^k W|^2}{-p_\varepsilon'(\mathfrak{v}_\varepsilon)} + |\partial_\mathrm{x}^k V|^2 \right) + \int_0^t \int_\mathbb{R} \left(\partial_\mathrm{x}\mathfrak{v}_\varepsilon |\partial_\mathrm{x}^k W|^2 + |\partial_\mathrm{x}^{k+1} V|^2 \right) \right] \leq \delta \varepsilon^{\frac{5}{\gamma}},$$

for some small $\delta > 0$. Note that these two points both strongly rely on the presence of the viscosity $\mu > 0$, such a stability result in the inviscid case is still an open question.

3. Regularity issues

Theorem 2.4 gives the existence of global weak solutions to the free-congested Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \nabla(\lambda \operatorname{div} u) - 2\operatorname{div}(\mu \mathrm{D}(u)) = \rho f, \\ 0 \leq \rho \leq 1, \quad \operatorname{spt} p \subset \{\rho = 1\}, \quad p \geq 0, \end{cases} \tag{3.1}$$

but the existence of local strong solutions is still an open question. The first part of the section presents the originality and the difficulties raised by (3.1) in terms of regularity while the second part is devoted to the study of the *floating body problem* which turns out to be closely related to (3.1) and for which the local well-posedness has been proved recently by Iguchi and Lannes in [36].

3.1. General difficulties raised by the hard free boundary problem

Regularity of the pressure. Theorem 2.4 states that global weak solutions of the soft congestion system (1.4) tend to global weak solutions of the hard congestion system (1.1) as the parameter ε in the singular pressure $p_\varepsilon = \varepsilon\big(\frac{\rho}{1-\rho}\big)^\gamma$ tends to 0. In particular, the sequence of approximate pressures $(p_\varepsilon)_\varepsilon$ converges only in the sense of measures and the limit p is shown to satisfy

$$p \in W^{-1,\infty}(0, T; H^1(\Omega)) + L^{q_1}(0, T; L^{q_2}(\Omega)) \quad \text{for some} \quad q_1, q_2 > 1 \tag{3.2}$$

as said in Remark 2.8. The low regularity in time of the limit pressure is inherent to our approximation by the soft congestion system (1.4). At the approximate level $\varepsilon > 0$, we are not able to guarantee the equi-integrability of the sequence $(p_\varepsilon)_\varepsilon$. The natural question is to know if better regularity could be obtained by a direct study of Equations (3.1) or by another approximation procedure.

Regularity of the interface. We have highlighted in Remark 2.7 the hybrid nature of System (3.1) which couples compressible (pressureless) dynamics in the free domain and incompressible dynamics in the congested domain. Hence, as for classical incompressible fluids, one expects to recover the pressure p by solving a Poisson problem $\Delta p = F(f, u, \nabla u)$. Nevertheless, the frontier of the congested domain on which this elliptic equation is posed, is completely unknown and its regularity is not guaranteed. It seems that even changes of topology of the congested domain cannot a priori be excluded. This question of the preservation of geometric structures and of the propagation of regularity is to some extend in connection with the *density patch problem* introduced by Lions in [42] which has been intensively studied in the context of inhomogeneous incompressible fluids (viscous or not). On the other hand, several local well-posedness results have been recently obtained for free boundary problems between two compressible and incompressible fluids (see the review paper [26] and the references given therein). In all cases, the interface between the two phases is supposed to be closed (impermeable), which means that the following kinematic condition holds on the boundary Γ_t

$$\partial_t F + u \cdot \nabla F = 0$$

if Γ_t is defined by $F(t,x) = 0$ locally, or written differently, $\Gamma_t = \{x = X(t,y) \,|\, y \in \Gamma_0\}$ with

$$\frac{\mathrm{d}X}{\mathrm{d}t} = u(t,X). \tag{3.3}$$

As suggested by the previous formulation, the analysis of the free boundary problem then relies on the passage to Lagrangian coordinates $(t, \varphi(t,y))$,

$$\varphi(t,y) = y + \int_0^t u(\tau, y)\,\mathrm{d}\tau, \tag{3.4}$$

which fixes the interface. The free-congested Navier–Stokes equations (3.1) are really different in the sense that the interface between the compressible and the incompressible domains is not closed since there are mass exchanges between the free and the congested phases. It means that the previous kinematic condition (3.3) is not relevant for the congestion problem and the tools developed for instance in [26] (or in [67]) become useless. Recent results have nevertheless been obtained outside the framework of a closed interface, they are presented in the next section.

3.2. Free boundary problems for wave-structure interactions

This section concerns local well-posedness issues for the so-called *floating body problem* which describes the interaction of surface water waves with a floating structure. The problem consists basically of two free boundary problems: the first one is the free interface liquid-air in the exterior domain (i.e. the classical problem of water waves), while the second one is the evolution of the *contact line*, that is the interface solid-air-liquid at the boundary with the structure. More generally this type of problems arises for partially free surface flows like flows in closed pipes (see for instance [34], [12]). We present here the recent results obtained by Iguchi and Lannes in [36] for a system which is closely related to the free-congested Euler equations (2.1).

Formulation of the 1d floating body problem. We consider the one-dimensional situation illustrated on Figure 3.1. The domain is decomposed into three parts a constrained/interior part denoted $\mathcal{I}(t)$ which is assumed to be an interval $\mathcal{I}(t) = (x_-(t), x_+(t))$, and two free infinite domains $\mathcal{E}_\pm(t)$ on both sides of $\mathcal{I}(t)$. The union of the two exterior free domains is denoted $\mathcal{E}(t)$. The problem of the contact line consists in describing the evolution of the interface $\Gamma_t = \{x_\pm(t)\}$. In the following f_e and f_i will denote the restrictions of f to the external (free) and to internal (constrained) domains respectively. In the shallow water regime, we are interested in the evolution of $(Z_{e,i}, Q_{e,i})$, $Z_{e,i}$ being the surface elevation of the fluid such that $H_{e,i} = h_0 + Z_{e,i}$ (h_0 the mean depth which is supposed to be constant here), and $Q_{e,i}$ being the horizontal flux such that

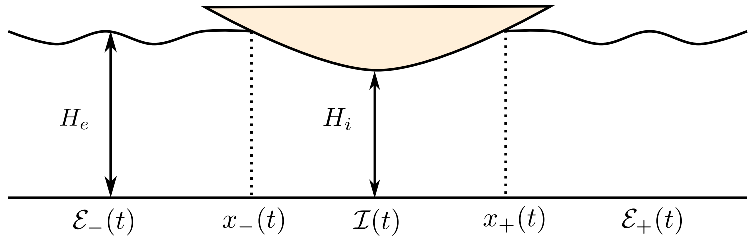

Figure 3.1: Floating structure.

$Q_{e,i} = H_{e,i}\bar{V}$ ($\bar{V}$ the vertically averaged horizontal velocity). The equations write

$$\begin{cases} \partial_t Z_e + \partial_x Q_e = 0 \\ \partial_t Q_e + \partial_x \left(\frac{Q_e^2}{H_e} + \frac{1}{2}gH_e^2\right) = 0 \end{cases} \quad \text{in } \mathcal{E}(t), \tag{3.5}$$

$$\begin{cases} \partial_t Z_i + \partial_x Q_i = 0 \\ \partial_t Q_i + \partial_x \left(\frac{Q_i^2}{H_i} + \frac{1}{2}gH_i^2\right) = -\frac{1}{\rho}H_i\partial_x P_i \end{cases} \quad \text{in } \mathcal{I}(t). \tag{3.6}$$

The system is then supplemented with transmission conditions at the interface Γ_t which express the continuity of the different variables (the structure has no vertical wall):

$$Z_e = Z_i, \quad Q_e = Q_i, \quad P_i = P_{\text{atm}} \quad \text{at} \quad x_\pm(t), \tag{3.7}$$

where P_{atm} the pressure at the surface in $\mathcal{E}(t)$ is supposed to be constant. Then, these equations are coupled with the dynamics of the structure which can be fixed, $H_{\text{sol}}(t,x) = \text{cst}$, or can follow a forced motion (with a prescribed $H_{\text{sol}}(t,x)$), or finally can evolve freely in interaction with the fluid. In this latter case, Equations (3.5)–(3.7) are coupled with Newton's laws for conservation of linear and angular momentum of the structure (see the full system in [36]).

Associated free-congested problem. As explained by Lannes in [38], the previous system can be expressed as a hard congestion system similar to (1.1). Let us assume that the vertical position of the lower boundary of the solid structure is parametrized by $H_{\text{sol}} = h_0 + Z_{\text{sol}}$ then Equations (3.5)–(3.6) can be rewritten as

$$\begin{cases} \partial_t H + \partial_x Q = 0, & (3.8a) \\ \partial_t Q + \partial_x \left(\frac{Q^2}{H} + \frac{1}{2}gH^2\right) = -\frac{1}{\rho}H\partial_x P, & (3.8b) \\ H \le H_{\text{sol}}, \quad (H_{\text{sol}} - H)(P - P_{\text{atm}}) = 0. & (3.8c) \end{cases}$$

Compared to (1.1), we have exactly the same structure (with the additional contribution of the hydrostatic pressure gH^2 in the momentum equation) where we have substituted H to ρ and Q to ρu. Note that the floating body problem prescribes explicitly through (3.7) the continuity of H and Q at the interface between the two domains which was not the case in the initial formulation of the free-congested Navier–Stokes equations (3.1).

Transmission conditions and evolution of the interface. For hyperbolic boundary problems with a fixed boundary only one boundary condition is required at each interface x_+, x_- (see for instance [4]). Here, we have two conditions, one on Z and the other one on Q. But this overdetermination is in fact necessary to derive an equation for the evolution the interface. Compared to the compressible-incompressible free boundary problems where the dynamics of the interface is given by the kinematic condition (3.3), here the information on the interface is implicitly encoded in the transmission condition

$$Z_e(t, x_\pm(t)) = Z_i(t, x_\pm(t)).$$

The dynamics of the interface $\Gamma_t = \{x_\pm(t)\}$ can be recovered by differentiation of this equality with respect to time

$$\partial_t Z_e + \dot{x}_\pm \partial_x Z_e = \partial_t Z_i + \dot{x}_\pm \partial_x Z_i.$$

Since the continuity equation holds on each domain, we can next replace $\partial_t Z_{e,i}$ by $-\partial_x Q_{e,i}$ to get finally

$$\dot{x}_\pm \big(\partial_x Z_e(t, x_\pm(t)) - \partial_x Z_i(t, x_\pm(t))\big) = \partial_x Q_e(t, x_\pm(t)) - \partial_x Q_i(t, x_\pm(t)). \tag{3.9}$$

The presence of $\partial_x Z_{e,i}$, $\partial_x Q_{e,i}$ (which are discontinuous at the contact points) makes the problem more singular than the compressible-incompressible free boundary problem treated in [26] for which the kinematic condition (3.3) holds. In this latter case, $\dot{x}$ ($\partial_t X$ in (3.3)) has the same regularity as the trace of the solution whereas there is a loss of one space derivative in (3.9). For a kinematic condition, the Lagrangian transformation (3.4) was applied to the system to fix the moving domain $\mathcal{E}(t)$. This is no longer the case here and another kind of diffeomorphisms φ has to be introduced to compensate the loss of regularity induced by (3.9). For the floating structure problem, the following mapping is considered by Iguchi and Lannes

$$\varphi(t, \cdot\,) : \underline{\mathcal{E}} = (-\infty, x_-(0)) \cup (x_+(0), +\infty) \;\to\; \mathcal{E}(t)$$

$$\varphi(t, x) = \begin{cases} x + \psi\Big(\frac{x - x_-(0)}{\eta}\Big)(x_-(t) - x_-(0)) & \text{for} \quad x \in \underline{\mathcal{E}}_- \\ x + \psi\Big(\frac{x - x_+(0)}{\eta}\Big)(x_+(t) - x_+(0)) & \text{for} \quad x \in \underline{\mathcal{E}}_+ \end{cases}$$

for a fixed $\eta > 0$ and a cut-off function $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\psi(x) = 1$ for $|x| \leq 1$. In the new coordinates, denoting $\zeta_{e,i} = Z_{e,i} \circ \varphi$, $h_{e,i} = H_{e,i} \circ \varphi$, $q_{e,i} = Q_{e,i} \circ \varphi$, the problem rewrites in the exterior domain

$$\begin{cases} \partial_t^\varphi \zeta_e + \partial_x^\varphi q_e = 0 \quad \text{in} \quad \underline{\mathcal{E}} \\ \partial_t^\varphi q_e + 2\frac{q_e}{h_e} \partial_x^\varphi q_e + \Big(g h_e - \frac{q_e^2}{h_e^2}\Big) \partial_x^\varphi \zeta_e = 0 \quad \text{in} \quad \underline{\mathcal{E}} \\ \zeta_e = \zeta_i, \quad q_e = q_i \quad \text{on} \quad \partial\underline{\mathcal{E}} \end{cases} \tag{3.10}$$

where $\partial_t^\varphi = \partial_t - (\partial_t\varphi)(\partial_x\varphi)^{-1}\partial_x$, $\partial_x^\varphi = (\partial_x\varphi)^{-1}\partial_x$ and $\partial\underline{\mathcal{E}} = x_\pm(0)$. The equations satisfied in the interior domain $\underline{\mathcal{I}} = (x_-(0), x_+(0))$ then depend on the dynamics of the structure (fixed, prescribed motion or free motion). In the case of a fixed structure, one can show that the dynamics in the interior domain reduces to the following ODE (written in the original Eulerian coordinates) on $Q_i(t, x) = Q_i(t)$

$$\dot{Q}_i = -\frac{1}{\int_{\mathcal{I}(t)} H_i} \left[\!\!\left[\frac{Q_i^2}{2H_i^2} + g H_i \right]\!\!\right] \tag{3.11}$$

where $[\![F]\!] := F(t, x_+(t)) - F(t, x_-(t))$ and $H_i = H_i(x) = h_0 + Z_{\rm sol}(x)$. The interior pressure can next be determined explicitly in terms of Q_i, H_i.

Local well-posedness. Under some compatibility conditions on the initial data (that come from the general theory of initial boundary value problems [4] and from the well-posedness theory of shallow water equations), Iguchi and Lannes prove the local well-posedness of problem in the three configurations: fixed structure, prescribed motion and free motion. In the case of a fixed structure the result is the following.

Theorem 3.1 (Iguchi, Lannes [36])**.** *Let $m \geq 2$ be an integer and I_f an open interval. Assume that $Z_{\rm sol} \in W^{m,\infty}(I_f)$ and initially $x_\pm^0 \in I_f$, $\zeta_e^0, q_e^0 \in H^m(\underline{\mathcal{E}})$, $Q_i^0 \in \mathbb{R}$. Provided some compatibility conditions satisfied by the initial datum $(\zeta_e^0, q_e^0, x_\pm^0, Q_i^0)$, there exist $T > 0$ and a unique solution $(\zeta_e, q_e, x_\pm, Q_i)$ to* (3.10)–(3.11) *such that*

$$\zeta_e, q_e \in \bigcap_{j=0}^{m-1} \mathcal{C}^j([0,T]; H^{m-j}(\underline{\mathcal{E}})),$$

$$x_\pm \in H^m(0,T), \quad Q_i \in H^{m+1}(0,T).$$

We refer to [36, Theorems 13-14] for the cases of forced and free motions.

Remark 3.1. Beyond the local well-posedness of the floating body problem, the study of Iguchi and Lannes [36] also brings new results in the general theory of one-dimensional initial boundary value problems. In particular the first part the paper is devoted to the derivation new sharp estimates (in terms of regularity requirements on the initial data and boundary condition) on 2×2 quasi-linear hyperbolic systems.

Back to the hard congestion problem. In the situation described above (see Figure 3.1), it was natural to assume the continuity of the different quantities across the interface Γ_t but other transmission conditions could a priori be considered within the general theory developed by Iguchi and Lannes. In view of the numerical results obtained by Bresch and Renardy, and discussed in Section 2.4, the treatment of discontinuous conditions would be crucial for the free-congested Euler equations. Discontinuous transmission conditions were recently studied by Shibata [67] for the compressible-incompressible free boundary problem but there, the interface was supposed to be impermeable (i.e. the kinematic condition holds at the interface). For the floating structures, the presence of vertical walls (see [9], [38]) is a case where a jump on H at the interface is considered. This leads to two corrections terms in the pressure $(P_i)_{|\Gamma_t}$, namely

$$P_i = P_{\text{atm}} + \rho g(H_e - H_i) + P_{cor} \quad \text{on} \quad \Gamma_t$$

where $\rho g(H_e - H_i)$ is the hydrostatic contribution while P_{cor} is a correction which ensures conservation of the energy. In [9] Bocchi proves the local well-posedness of the problem in the axisymmetric case, that is the case where only vertical displacements of the structure are permitted. In this framework, the interface is therefore automatically fixed which eliminates the compressible-incompressible free boundary problem.

Transmission conditions are not prescribed in the initial formulation of the free-congested Euler equations but the soft approximation presented in the previous section seems, at least for the Riemann problem solutions described in Proposition 2.7, to select discontinuities independent of the approximation process. It would be interesting to investigate the stability of such shocks by means of the tools developed by Iguchi and Lannes.

To conclude, let us mention that the question of the extension to viscous systems of the previous local well-posedness results would be also very interesting for the Navier–Stokes system (3.1) and also in view of results recently obtained in [22]. A further step would consist next in studying the vanishing viscosity limit for which we expect interesting behaviors in the vicinity of the interface Γ_t.

4. Non-local issues, memory effects in congested media

The previous sections were concerned with the two toy models (1.1) and (1.4) which are to some extent the "simplest" fluid systems to handle congestion phenomena. The purpose of this section is to present and analyze more complex models including non-local effects. If non-locality is more or less natural in the modeling of collective motion where active particles interact through social forces (alignment, attraction, repulsion), non-local effects can be also important for non-active materials like granular flows. A first evidence of non-locality in dense granular flows is the existence of large spatial correlations: when approaching the quasi-static regime, correlations in the force network and in the velocity fluctuations can be observed (see [63]). Non-local models are also presumed to remedy to the shortcomings of the popular (local) $\mu(I)$-rheology, shortcomings which mainly concern transitions between "liquid-like" phases and "solid-like" phases. For instance the stopping and starting properties of avalanches are not correctly described within the $\mu(I)$ framework. . . . As we recalled in the introduction of this paper, non-locality is actually intimately linked with congestion and more generally with phase transitions. Since the last few years, theses issues have been the subject of numerous physical studies reviewed in particular by Bouzid et al. in [13]. Among them, several develop a phase field approach which consists in describing the transition between the liquid and solid behaviors through an additional state parameter, usually called *fluidity* but which can refer to different physical quantities (inverse of viscosity, distance to the maximal volume fraction, etc.). The non-locality is then encoded in the equation satisfied by the fluidity parameter. For the moment, it seems that there is no consensus around the definition and the dynamics satisfied by the fluidity parameter. The mathematical analysis could perhaps bring new comparison elements between the different models proposed in the literature.

The goal of this section is to investigate a particular type of non-local (in time) models. These models have been introduced in relation with congestion phenomena by Lefebvre–Lepot and Maury in [41]. In the one-dimensional setting, the equations write

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (4.1a) \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f, & (4.1b) \\ \partial_t \gamma + u \partial_x \gamma = -p, & (4.1c) \\ 0 \leq \rho \leq 1, \ \operatorname{spt} \gamma \subset \{\rho = 1\}, \ \gamma \leq 0. & (4.1d) \end{cases}$$

This system shares with (2.1) the same structure based on compressible, incompressible Euler equations where p plays again the role of the Lagrange multiplier in the congested domain. The difference is that p (which has no sign a priori) is now linked to an additional variable γ, called *adhesion potential*, which captures the amount of compression that the fluid is exposed to, but cannot accommodate due to the maximal constraint $\rho \leq 1$. More precisely, in the congested phases the pressure p balances the external force f and relates therefore f to the adhesion potential γ through equation (4.1c). Hence, the adhesion potential keeps track of the history of the constraint exerted on the system over the course of time (see also (4.13) for a microscopic case). It reveals non-local memory effects and can be interpreted as a state or fluidity parameter which is activated only in the congested domain.

We detail hereafter recent analysis results obtained in relation with memory effects. First, we present an existence result for (4.1). Then, we show that the model can be approximated by a *soft* congestion system with singular bulk viscosities. As an interesting corollary of this study, we explain in the final part how memory effects bring a new point of view on the analysis of fluids with pressure dependent viscosity.

4.1. Existence of global weak solutions to the system with memory effects

Let us first observe that Equations (4.1) share the same structure as the well-known pressureless gas equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{cases} \tag{4.2}$$

Indeed, differentiation of Equation (4.1c) yields

$$\partial_t \partial_x \gamma + \partial_x(u \partial_x \gamma) = -\partial_x p \tag{4.3}$$

that we can subtract from (4.1b) to obtain the modified pressureless system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (4.4a) \\ \partial_t\big(\rho u - \partial_x \gamma\big) + \partial_x\big((\rho u - \partial_x \gamma) u\big) = \rho f, & (4.4b) \\ 0 \leq \rho \leq 1, \ \operatorname{spt} \gamma \subset \{\rho = 1\}, \ \gamma \leq 0, & (4.4c) \end{cases}$$

where $\partial_x \gamma$ can be seen as an additional momentum that activates in the congested regions.

Among the large literature existing on the pressureless gas equations (4.2), one approach introduced by Natile and Savaré [53] (then also used in [14], [20]) turns out to be very interesting for the construction of global weak solutions to the system (4.4). The approach relies on a Lagrangian characterization of the dynamics. More precisely to every probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ (i.e. with finite quadratic moment), we associate a unique transport map $X \in K$, K being the closed convex cone of non-decreasing maps in $L^2(0,1)$, such that $\rho = X_{\#} \mathcal{L}^1_{|(0,1)}$ with $\mathcal{L}^1_{|(0,1)}$ the Lebesgue measure restricted to $[0,1]$. The Lagrangian point of view consists in describing the dynamics of the transport $X_t = X(t,\cdot\,)$ and of the Lagrangian velocity $U_t = \frac{\mathrm{d}}{\mathrm{d}t} X_t$. This latter is linked to the Eulerian velocity by $u_t(X_t(w)) = U_t(w)$ for a.a. $w \in (0,1)$. Using an approximation by discrete *sticky particles*, Natile and Savaré construct Eulerian weak solutions (ρ, u) of (4.2) for which the associated transport X is such that

$$X_t = \mathrm{P}_K(\bar{X} + t\bar{U}) \quad \forall\, t \geq 0, \tag{4.5}$$

where P_K is the $L^2(0,1)$ projection onto the closed convex set K and $\bar{X}, \bar{U}$ are respectively the transport and the Lagrangian velocity associated to the initial data $(\bar{\rho}, \bar{u})$. The map $\bar{X} + t\bar{U}$

represents the free motion path, which is at the discrete level the transport corresponding to the case where the particles do not interact with each other.

Conversely, Cavalletti et al. showed in [20] that, starting from (4.5), one can define for almost all time t a Lagrangian velocity $U_t := \frac{\mathrm{d}}{\mathrm{d}t} X_t$ which is essentially constant where X_t is constant and from which one can recover an Eulerian velocity $u_t \in L^2(\mathbb{R}, \rho_t)$. The Eulerian couple (ρ, u) is then a global weak solution of (4.2).

Coming back to the constrained system (4.4) one can adapt this Lagrangian point of view in order to take into account the maximal density constraint and the external force f. First, observe that from the push-forward formula (Lemma 5.5.3 in [1]), the following equality holds for ρ_t-a.e. $x \in \mathbb{R}$

$$\rho_t(x) = \frac{1}{\partial_w X_t\big(X_t^{-1}(x)\big)}$$

(with a small abuse of notations we identify the absolute continuous measure $\rho_t(\mathrm{d}x)$ with its Lebesgue density), which means that the maximal density constraint will be ensured provided that $\partial_w X_t \geq 1$, that is

$$X_t \in \widetilde{K} = K + \mathrm{Id}\,.$$

Next, we need to extend the notion of free transport $\bar{X} + t\bar{U}$ to the case where the external force f is applied on the system. We are naturally led to define a *free velocity*

$$U_t^{\mathrm{free}} := \bar{U} + \int_0^t f(s, X_s)\,\mathrm{d}s, \tag{4.6}$$

integrating the external force f along the trajectories $t \mapsto X_t(y)$ starting at y. The *free trajectory* would be then be given by the formula

$$X_t^{\mathrm{free}} := \bar{X} + \int_0^t U_s^{\mathrm{free}}\,\mathrm{d}s,$$

and in analogy with (4.5), we consider

$$X_t := \mathrm{P}_{\widetilde{K}}(X_t^{\mathrm{free}}) = \mathrm{P}_{\widetilde{K}}\left(\bar{X} + \int_0^t U_s^{\mathrm{free}}\,\mathrm{d}s\right). \tag{4.7}$$

Note that (4.6)–(4.7) form a coupled system for which existence and uniqueness requires some Lipschitz regularity on the external force f. The existence of global weak Eulerian solutions is stated in the next theorem.

Theorem 4.1 (Perrin, Westdickenberg [59]). *Let $T > 0$ and f be a given external force such that*

$$f \in L^\infty\big(0, T; \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R})\big).$$

Suppose that the initial Eulerian datum $(\bar{\rho}, \bar{u})$ is such that

$$\bar{\rho} \in \mathcal{P}_2(\mathbb{R}) \text{ with } \bar{\rho} \ll \mathcal{L}^1 \text{ and } 0 \leq \bar{\rho} \leq 1 \text{ a.e.}$$

and that $\bar{u} \in L^2(\mathbb{R}, \bar{\rho})$. Define the Lagrangian quantities $\bar{X}$ and $\bar{U}$ associated to $(\bar{\rho}, \bar{u})$, and set $X_0 = \bar{X}$, $U_0^{\mathrm{free}} = \bar{U}$. There exists a curve $[0, T] \ni t \mapsto X_t \in \widetilde{K}$ that is differentiable for a.e. $t \in (0, T)$ and solves the coupled system of equations (4.6)–(4.7). The following quantities are well-defined:

$$U_t(w) := \dot{X}_t(w),\ \Gamma_t(w) := \int_0^w \Big(U_t(w) - U_t^{\mathrm{free}}(w)\Big)\,\mathrm{d}w \tag{4.8}$$

for $w \in (0,1)$ and a.e. $t \in (0,T)$. There exist $(u_t, \gamma_t) \in L^2(\mathbb{R}, \rho_t) \times W^{1,1}(\mathbb{R})$, such that

$$U_t = u_t \circ X_t,\ \Gamma_t = \gamma_t \circ X_t \quad \text{where} \quad \rho_t := (X_t)_\# \mathcal{L}^1_{|[0,1]}.$$

The triple (ρ, u, γ) is a global weak solution of system (4.4).

One key step of the proof is to show that the Lagrangian velocity U_t is the orthogonal projection of the free velocity U_t^{free} onto the subset of $L^2(0,1)$ of functions that are essentially constant on the congested domain. Note that this property corresponds also to a Lagrangian equivalent of the collision law imposed by Preux in [64] and discussed in Remark 2.3. Finally, this property ensures then that the Lagrangian adhesion potential defined in (4.8) is non-positive and activates only in the congested domain (cf. [59, Proposition 3.12]).

Remark 4.1. The formula (4.7) turns out to be also interesting from the numerical point of view since the problem can then be reduced to optimization under constraint: at each time step one has to minimize the objective function

$$\phi_{t_h}(X) = \left\| \bar{X} + \int_0^{t_h} U_s^{\text{free}}\, \mathrm{d}s - X \right\|_{L^2}^2$$

under the constraint $X \in \widetilde{K}$. Numerical simulations are provided in [59] to illustrate the memory effects.

4.2. Effects of compression in the bulk viscosity and memory effects

Singular bulk viscosity in suspension flows. Originally, Lefebvre–Lepot and Maury have introduced Equations (4.1) in [40] from a one-dimensional system for suspension flows which reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(4.9a)} \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x\left(\frac{\nu}{1-\rho}\partial_x u\right) = \rho f & \text{(4.9b)} \end{cases}$$

It describes at the continuous level a system of aligned rigid spheres that are immersed in a viscous liquid. This system is rigorously derived in the stationary case from the discrete level via a kind of homogenization process. The singular dissipation term is the macroscopic counterpart of the lubrication force that is exerted on the particles by the interstitial fluid. At the microscopic level, if one considers a system formed by two particles, denoted 1 and 2, the lubrication force on particle 1 reads at leading order

$$F_{1\to 2} = -\kappa \frac{u_2 - u_1}{d} \tag{4.10}$$

where d is the distance between the two particles and κ is a constant depending on the viscosity of the interstitial fluid (and on the size of the particles). Observe that this force is singular as d tends to 0 which prevents the contact in finite time between the particles. At the continuous level, the distance d is replaced by the quantity $1-\rho$ and the lubrications forces are now modeled by the bulk viscosity term $\partial_x\left(\lambda(\rho)\partial_x u\right)$ with $\lambda(\rho) = \frac{\nu}{1-\rho}$. Similarly to the microscopic case, the singularity prevents the formation of congested areas characterized by $\rho = 1$.

Expected asymptotics, activation of memory effects. As the viscosity coefficient $\nu = \varepsilon$ tends to 0, Lefebvre–Lepot and Maury [40] expect that (4.9) degenerates towards the system (4.1) where the memory effects are seen as residual effects of the lubrication forces. To understand this, let us come back to a discrete system studied by Maury in [48]. This system is formed by one single particle, immersed in a viscous liquid, that moves along an axis under an external force f (e.g. gravity). On its trajectory is placed a solid wall that the particle cannot cross. The dynamics is described by the differential equation

$$\begin{cases} \ddot{q}(t) + \varepsilon \frac{\dot{q}}{q}(t) = f(t) \\ q(t) > 0 \end{cases} \tag{4.11}$$

where q is the distance between the particle and the wall, $-\varepsilon\frac{\dot{q}}{q}$ is the lubrication force on the particle (cf (4.10)).

Theorem 4.2 (Maury [48])**.** *Let $T > 0$, $f \in L^1(0,T)$, $q_0 > 0$ be given.*

- *let $\varepsilon > 0$, then (4.11) admits a unique global solution $q_\varepsilon \in W^{1,\infty}(0,T)$;*
- *as $\varepsilon \to 0$, there exists a subsequence $(q_\varepsilon)_\varepsilon$ such that*

$$q_\varepsilon \longrightarrow q \text{ uniformly}$$
$$\gamma_\varepsilon = \varepsilon \ln q_\varepsilon \longrightarrow \gamma \text{ weakly-* in } L^\infty(0,T)$$

where the couple $(q,\gamma) \in W^{1,\infty}(0,T) \times L^\infty(0,T)$ is a solution of

$$\begin{cases} \dot{q} + \gamma = \bar{u} + \int_0^t f(s)\,\mathrm{d}s, \\ q \geq 0, \quad \gamma \leq 0, \quad q\,\gamma = 0. \end{cases} \tag{4.12}$$

System (4.12) describes the two possible states of the system: *free* when $q > 0$ (that is, the particle evolves freely under the external force f), and *stuck* whenever $q = 0$. In this latter case, the adhesion potential is activated and

$$\gamma(t) = u^{\text{free}}(t) = \bar{u} + \int_0^t f(s)\,\mathrm{d}s, \tag{4.13}$$

which is the velocity the particle would have if there was no wall on its trajectory. It ensures that the particle cannot cross the wall and that it sticks to the wall as long as $\gamma < 0$. Note the analogy with the definition (4.8) of the Lagrangian potential in the previous section. System (4.12) is in fact equivalent to the following second order system (see [39])

$$\begin{cases} \ddot{q} = f + \lambda & (4.14\text{a}) \\ \dot{q}(t^+) = \mathrm{P}_{C_{q,\gamma}(t)}\dot{q}(t^-) & (4.14\text{b}) \\ \mathrm{spt}(\lambda) \subset \{t \colon q(t) = 0\} & (4.14\text{c}) \\ \dot{\gamma} = -\lambda & (4.14\text{d}) \\ q \geq 0, \quad \gamma \leq 0, & (4.14\text{e}) \end{cases}$$

where $C_{q,\gamma}(t)$ denotes the set of admissible velocities

$$C_{q,\gamma}(t) = \begin{cases} \{0\} & \text{if} \quad \gamma(t^-) < 0 \\ \mathbb{R}^+ & \text{if} \quad \gamma(t^-) = 0,\ q(t) = 0 \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

System (4.1) is then the macroscopic extension of (4.14): the equation on the macroscopic potential

$$\partial_t \gamma + u \partial_x \gamma = -p$$

which relates the time derivative in Lagrangian coordinates of γ to the pressure p, is the natural counterpart of (4.14d).

Inspired by this discrete case, we formulate the following conjecture.

Conjecture 4.3. *As ε tends to 0, there exists a subsequence of solutions $(\rho_\varepsilon, u_\varepsilon)$ of the suspension system*

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x\left(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\right) = \rho_\varepsilon f_\varepsilon \\ \lambda_\varepsilon = \frac{\varepsilon}{1-\rho_\varepsilon} \end{cases}$$

which converges to (ρ, u, γ) a weak solution of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ 0 \leq \rho \leq 1,\ \mathrm{spt}\,\gamma \subset \{\rho = 1\},\ \gamma \leq 0. \end{cases}$$

This limit is completely analogous to the singular limit treated in Section 2.3 between soft and hard congestion systems, except that this is now the bulk viscosity λ_ε instead of the pressure which encodes the effects of congestion. In particular, as $\varepsilon \to 0$ the bulk viscosity $-\partial_x(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon)$ would have to degenerate towards a limit pressure p. In the regions where ρ_ε tends to 1, $\lambda_\varepsilon(\rho_\varepsilon)$ tends to $+\infty$ but it is compensated by $\partial_x u_\varepsilon$ which is expected to tend to 0 (the incompressibility condition $\partial_x u = 0$ will be satisfied in the congested domain). Up to now, the full justification of this singular limit is still open. The main difficulty relies in the lack of control on $\partial_x u_\varepsilon$ or $\partial_x \rho_\varepsilon$ which prevents to pass to the limit in the nonlinear convective terms (this is discussed in [58]).

4.2.1. Justification of the asymptotics in the viscous case

One possible framework within which it is possible to justify the singular limit passage is the case treated in [57] where additional dissipation is considered. More precisely, it consists in adding a supplementary density-dependent shear viscosity $\mu_\varepsilon(\rho)$ in system (4.9) which has to be such that

$$\mu_\varepsilon(\rho) = \rho + \mu_\varepsilon^1(\rho) \quad \text{and} \quad \lambda_\varepsilon(\rho) = 2\big((\mu_\varepsilon^1)'(\rho)\rho - \mu_\varepsilon^1(\rho)\big). \tag{4.15}$$

Note that if λ_ε is sufficiently singular close to 1, then μ_ε also blows up at $\rho = 1$. We also assume that μ_ε^1 degenerates close to vacuum:

$$\mu_\varepsilon^1(\rho) \underset{0}{\sim} C\rho^\gamma, \quad \gamma \geq 1$$

to be consistent with the classical expression of effective viscosity of dilute suspensions (first derived by Einstein, see [54]). In multi-d the modified suspension system reads

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, & (4.16a) \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \nabla\left(\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon\right) - 2\operatorname{div}\left(\mu_\varepsilon(\rho_\varepsilon)\mathrm{D}(u_\varepsilon)\right) = 0, & (4.16b) \\ 0 \leq \rho < 1. & (4.16c) \end{cases}$$

Despite its restrictiveness, relation (4.15) is compatible with the physical literature in which many singular constitutive laws may be found (like the empirical law of Krieger–Dougherty, cf. [2]). The main advantages of (4.15) is, first, that it guarantees a nice structure of the equations and, second, that it enables to give a simple interpretation of the adhesion potential in terms of the shear viscosity. We detail these two points in the next paragraphs.

Density-dependent viscosities and entropy estimate. Formally Equation (4.15) can be associated to the mass equation (4.16a) to get

$$\partial_t \mu_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\frac{1}{2}\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon. \tag{4.17}$$

This is the well-known relation introduced by Bresch and Desjardins [16] to derive an additional "entropy estimate" for compressible Navier–Stokes equations with degenerate viscosities. Let us detail this point. Differentiation of (4.17) yields

$$\partial_t \nabla\mu_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\nabla\mu_\varepsilon(\rho_\varepsilon) \otimes u_\varepsilon) + \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon)\mathrm{D}(u_\varepsilon)) \\ - \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon)\mathrm{A}(u_\varepsilon)) + \nabla((\mu_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon - \mu_\varepsilon(\rho_\varepsilon)) \operatorname{div} u_\varepsilon) = 0 \tag{4.18}$$

where

$$\mathrm{D}(u) = \frac{\nabla u + \nabla^t u}{2}, \quad \mathrm{A(u)} = \frac{\nabla \mathrm{u} - \nabla^t \mathrm{u}}{2}.$$

We can now combine (4.18) with (4.16b), under the condition (4.15), which leads to

$$\partial_t\big(\rho_\varepsilon u_\varepsilon + 2\nabla\mu_\varepsilon(\rho_\varepsilon)\big) + \operatorname{div}\left((\rho_\varepsilon u_\varepsilon + 2\nabla\mu_\varepsilon(\rho_\varepsilon)) \otimes u_\varepsilon\right) - 2\operatorname{div}(\mu_\varepsilon(\rho_\varepsilon)\mathrm{A}(u_\varepsilon)) = 0. \tag{4.19}$$

Now, as observed by Bresch and Desjardins [16], we can define the *effective velocity*

$$v_{eff} = u_\varepsilon + 2\frac{\nabla\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}$$

and test (4.19) against v_{eff} to get the so-called *BD entropy*

$$\sup_{t\in[0,T]} \int_\Omega \frac{\rho_\varepsilon |v_{eff}|^2}{2} + 2\int_0^T \int_\Omega \mu_\varepsilon(\rho_\varepsilon)|\mathrm{A}(u_\varepsilon)|^2 \leq \int_\Omega \frac{\rho_\varepsilon^0 |v_{eff}^0|^2}{2}. \tag{4.20}$$

From the classical energy estimate

$$\sup_{t\in[0,T]} \int_\Omega \frac{\rho_\varepsilon |u_\varepsilon|^2}{2} + 2\int_0^T \int_\Omega \mu_\varepsilon(\rho_\varepsilon)|\mathrm{D}(u_\varepsilon)|^2 + \int_0^T \int_\Omega \lambda_\varepsilon(\rho_\varepsilon)(\operatorname{div}(u_\varepsilon))^2 \leq \int_\Omega \frac{\rho_\varepsilon^0 |u_\varepsilon^0|^2}{2}$$

we do not guarantee any control on ∇u_ε since μ_ε degenerates close to vacuum, nevertheless the two previous estimates lead to a control of $\frac{\nabla\mu_\varepsilon(\rho_\varepsilon)}{\sqrt{\rho_\varepsilon}}$ in $L_t^\infty L_x^2$. As detailed in [57], this control allows on one hand to ensure the maximal density constraint $\rho_\varepsilon < 1$ a.e. and on the other hand, to show the strong convergence of the density (see also [16]). Finally, in order to pass to the limit in the non-linear convective term $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon$ appearing in (4.16b), additional control of $\sqrt{\rho_\varepsilon}|u_\varepsilon|$ is required (the energy only provides a bound in $L_t^\infty L_x^2$). In the one-dimensional case [58], the so-called *Mellet–Vasseur estimate* (initially derived in [52]) is proved to be satisfied uniformly with respect to ε. Unfortunately, as explained in [58], the extension of this estimate to the multi-dimensional case remains an open question in the case of singular viscosities $\lambda_\varepsilon, \mu_\varepsilon$. This difficulty is bypassed in [57] by considering an additional turbulent drag term $r\rho_\varepsilon|u_\varepsilon|u_\varepsilon$ in the momentum equation (4.16b).

Density-dependent viscosities and memory effects. Observe that in the previous paragraph, the steps (4.17)–(4.18) we perform to arrive at (4.19), are really similar to those made in Section 4.1 (cf. (4.1), (4.3), (4.4)). In fact, if we want to justify the singular limit from the suspension model (4.9) to the hybrid model (4.1), we are naturally led to define an approximate adhesion potential γ_ε satisfying

$$\partial_t\gamma_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\gamma_\varepsilon(\rho_\varepsilon)u_\varepsilon) = \lambda_\varepsilon(\rho_\varepsilon)\operatorname{div} u_\varepsilon,$$

equation that will degenerate to (4.1c), namely

$$\partial_t\gamma + u\cdot\nabla\gamma = -p$$

as $\varepsilon \to 0$ (we recall that $\operatorname{div} u = 0$ where $\rho = 1$). Actually, by imposing (4.15) we set: $\gamma = -\frac{\mu}{2}$. Hence, the addition of the singular shear viscosity is not only relevant from the physical point of view, but is also a way to understand the activation of memory effects in free/congested granular flows (see also the next Section 4.3).

Existence of solutions to the suspension model and limit $\varepsilon \to 0$. We state here a result in the two-dimensional periodic setting proved in [57] (see [58] for a one-dimensional result).

Theorem 4.4 (Perrin [57])**.** *Let $\Omega = \mathbb{T}^2$, $\mu_\varepsilon(\rho) = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\gamma + \rho$ with $\gamma > 1$. Assume that initially $0 \le \rho_\varepsilon^0 < 1$ a.e. and*

$$\int_\Omega \rho_\varepsilon^0\frac{|u_\varepsilon^0|^2}{2} + \int_\Omega\left[\mu_\varepsilon(\rho_\varepsilon^0) + \frac{|\nabla\mu_\varepsilon(\rho_\varepsilon^0)|^2}{\rho_\varepsilon^0}\right] \le E^0 \tag{4.21}$$

for some $E_0 > 0$ independent of ε. Then

- *for r small enough, there exists a global weak solution to* (4.16) *and moreover the following equation holds in the sense of distributions*
$$\partial_t\mu_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon)u_\varepsilon) = -\frac{1}{2}\lambda_\varepsilon(\rho_\varepsilon)\operatorname{div} u_\varepsilon;$$
- *as $\varepsilon \to 0$, there exists a subsequence $(\rho_\varepsilon, u_\varepsilon, -\lambda_\varepsilon(\rho_\varepsilon)\operatorname{div} u_\varepsilon, \mu_\varepsilon^1(\rho_\varepsilon))$ converging weakly to (ρ, u, Π, μ) a global weak solution of the following free-congested granular system*
$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla\Pi - 2\operatorname{div}\big((\mu+\rho)\mathrm{D}(u)\big) + r\rho|u|u = 0\\ \partial_t\mu + u\cdot\nabla\mu = \Pi/2\\ 0\le\rho\le 1,\ \operatorname{spt}\mu\subset\{\rho=1\},\ \mu\ge 0\\ \operatorname{div} u = 0 \quad \textit{on } \{\rho = 1\}\end{cases} \tag{4.22}$$

Remark 4.2. Originally, the result given in [57] involves an additional singular pressure $p_\varepsilon(\rho)$ which has the same behavior as the shear viscosity close to the maximal density constraint $p_\varepsilon(\rho) = \mu_\varepsilon^1(\rho)$. As a consequence of the previous theorem, we can consider a sequence of initial approximate densities $(\rho_\varepsilon^0)_\varepsilon$ which tends to 1 a.e. on Ω and get at the limit weak solutions to a model of fully incompressible fluids with pressure dependent viscosities $\mu(p)$. This result and its consequences are presented in Section 4.4.

4.3. Compression effects, pressure versus bulk viscosity

We have previously seen two ways of describing resistance to compression effects, i.e. two ways of handing a maximal congestion constraint in compressible fluids: either with a singular pressure $p_\varepsilon(\rho)$ or with a singular bulk viscosity $\lambda_\varepsilon(\rho)$. In the first case (Theorem 2.4), the congestion limit $\varepsilon \to 0$ consists of a simple compressible-incompressible two-phase system where the limit pressure p is the Lagrange multiplier associated to the incompressibility constraint in the congested domain. In the second case (Theorem 4.4), this is the whole bulk viscosity $-\lambda_\varepsilon \operatorname{div} u_\varepsilon$ which converges to a limit pressure Π and activates additional memory effects in the congested domain.

A natural question is then to know how the choice of the constitutive laws, p_ε, λ_ε, at the approximate level impacts the dynamics of congested zones as $\varepsilon \to 0$, that is to determine pressures and bulk viscosities which activate the memory effects at the limit. This is the purpose of the study [17] which focuses on the singular compressible Brinkman equations (set on $\mathbb{T}^3$)

$$\left\{\begin{aligned} &\partial_t\rho + \operatorname{div}(\rho u) = 0 && (4.23a)\\ &\nabla p_\varepsilon(\rho) - \nabla(\lambda_\varepsilon(\rho)\operatorname{div} u) - 2\operatorname{div}(\mu \mathrm{D}(u)) + ru = f && (4.23b)\end{aligned}\right.$$

where

$$p_\varepsilon(\rho),\ \lambda_\varepsilon(\rho) \underset{\rho\to 1}{\longrightarrow} +\infty, \qquad \mu = \mathrm{cst} > 0$$

and ru, $r > 0$, is a drag term. As in the previous section, let us introduce an "approximate" adhesion potential, here denoted $\Lambda_\varepsilon(\rho)$, such that

$$\partial_t\Lambda_\varepsilon(\rho) + \operatorname{div}\big(\Lambda_\varepsilon(\rho)u\big) = -\lambda_\varepsilon(\rho)\operatorname{div} u_\varepsilon.$$

We expect the following limit system as $\varepsilon \to 0$

$$\left\{\begin{aligned} &\partial_t\rho + \operatorname{div}(\rho u) = 0 && (4.24a)\\ &\nabla p + \nabla\Pi - 2\operatorname{div}(\mu \mathrm{D}(u)) + ru = f && (4.24b)\\ &\partial_t\Lambda + \operatorname{div}(\Lambda u) = \Pi && (4.24c)\\ &0 \le \rho \le 1,\ \operatorname{spt} p \subset \{\rho = 1\},\ \operatorname{spt}\Lambda \subset \{\rho = 1\},\ p \ge 0,\ \Lambda \ge 0 && (4.24d)\end{aligned}\right.$$

where an additional equation has to be derived to close the system. To simplify, let us consider the following constitutive laws

$$p_\varepsilon(\rho) = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\gamma \qquad \lambda_\varepsilon(\rho) = \varepsilon\left(\frac{\rho}{1-\rho}\right)^\beta, \qquad \gamma,\beta > 1,$$

then we have

$$\Lambda_\varepsilon(\rho) = \frac{\rho}{\beta - 1}\varepsilon^{\frac{1+\gamma-\beta}{\gamma}}\big(p_\varepsilon(\rho)\big)^{\frac{\beta-1}{\gamma}}. \tag{4.25}$$

This means that, depending on the sign of $1 + \gamma - \beta$ appearing in the exponent of ε, we activate either the pressure p or the memory effects via Λ:

- $\gamma > \beta - 1$ pressure but no memory effects
$$\Lambda = 0; \tag{4.26}$$
- $\gamma < \beta - 1$ memory effects but no pressure
$$p = 0; \tag{4.27}$$
- $\gamma = \beta - 1$ memory effects and pressure satisfying the relation
$$p = \gamma\Lambda. \tag{4.28}$$

Theorem 4.5 (Bresch, Necasova, Perrin [17])**.** *Let $f \in L^2(0,T;(L^q(\mathbb{T}^3))^3)$ for some $q > 3$. Assume that initially*

$$0 \le \rho_\varepsilon^0 \le R_\varepsilon < 1, \quad \langle\rho_\varepsilon^0\rangle \le M^0 < 1,$$

and

- *if $1 < \beta < \gamma$: $\int_{\mathbb{T}^3} \frac{\varepsilon}{(1-\rho_\varepsilon^0)^{\gamma-1}} \le E^0 < +\infty$,*
- *if $1 < \gamma \le \beta$: $\int_{\mathbb{T}^3} \big(\Lambda_\varepsilon(\rho_\varepsilon^0)\big)^2 \le \Lambda^0 < +\infty$,*

for some $E^0, \Lambda^0 > 0$ independent of ε. Then

- *there exists a global weak solution $(\rho_\varepsilon, u_\varepsilon)$ to Equations (4.23);*
- *as $\varepsilon \to 0$, there exists a subsequence converging weakly to (ρ, u, p, Λ) a global weak solution to (4.24) complemented by (4.26), or (4.27), or (4.28) depending on the respective values of γ and β.*

Remark 4.3. This result can of course be extended to more general pressures and bulk viscosities. Anyone who needs convincing of this can come back to (4.25) where we compare through the values of γ, β the respective influences of Λ_ε and p_ε close to 1.

Remark 4.4. The previous result concerns the Brinkman equations where, compared to the Navier–Stokes equations, the acceleration $\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)$ is supposed to be negligible. This assumption is particularly relevant for fluids in porous media (with possible applications in biology, cf. [61]) but is also necessary from the mathematical point of view. Actually, the existence of global weak solutions for compressible flows with a density-dependent bulk viscosity $\lambda(\rho) = \rho^\beta$ and a constant shear viscosity, is still a challenging issue in three-dimensional domains. We refer the interested reader to [68], [56] for results in dimension two.

4.4. Incompressible fluids with pressure dependent viscosities

If the issue of pressure dependent viscosities for compressible barotropic flows is now well understood since the works of Bresch, Desjardins [16], because it boils down to the study of density dependent viscosities, the subject remains largely open in the case of incompressible flows. The incompressible models with pressure dependent viscosities are nevertheless relevant in many applications like in elastohydrodynamics and several experiments have shown the importance of such dependency (see references given in [46]). After recalling the main theoretical difficulties raised by such models, we show how the introduction of memory effects, viewed as a particular type of pressure dependent viscosities, brings an original new approach to the problem.

Difficulties, state of the art. The incompressible Navier–Stokes system with a pressure dependent viscosity reads

$$\begin{cases} \operatorname{div} u = 0 & (4.29a) \\ \partial_t u + (u \cdot \nabla) u + \nabla p - 2 \operatorname{div}(\mu(p) \mathrm{D}(u)) = f & (4.29b) \end{cases}$$

where the momentum equation (4.29b) can be rewritten as

$$\partial_t u + (u \cdot \nabla) u + \nabla p - \mu(p) \Delta u + (\mathrm{I} - 2\mu'(p) \mathrm{D}(u)).\nabla p = f.$$

First, note that unlike classical Navier–Stokes system where only the gradient of the pressure is involved in the equations, it is necessary here to prescribe the average pressure (or the value on the boundary) in order to recover uniqueness. The second difference with the constant viscosity case is the possible loss of ellipticity of the pressure equation. In the classical case, the Hodge projection enables to eliminate the pressure from the equations. Moreover by taking the divergence of the momentum equation we arrive at the Poisson problem

$$\Delta p = F(f, u, \nabla u)$$

which gives the existence and uniqueness (up to a constant) of the pressure. In the present case, the Laplacian operator Δp is replaced by the nonlinear operator

$$\operatorname{div}\left((\mathrm{I} - 2\mu'(p) \mathrm{D}(u)).\nabla p\right) \tag{4.30}$$

which degenerates as $\mathrm{D}(u) = \frac{1}{2\mu'(p)} \mathrm{I}$. System (4.29) differs thus significantly from the classical incompressible Navier–Stokes equations and no global well-posedness theory can be expected.

The first mathematical study of system (4.29) seems to be the work of Renardy [65] which concerns the local well-posedness of strong solutions. The idea is to guarantee the ellipticity of the pressure equation by ensuring that the eigenvalues of $\mathrm{D}(u)$ remain below $1/2\mu'(p)$. For that purpose, Renardy assumes that the viscosity law is such that the growth of μ with respect to p is less than linear which allows to bound $1/\mu'(p)$ from below by $(\lim_{p \to +\infty} \mu'(p))^{-1}$.

Another point of view is adopted by Màlek, Necas and Rajagopal in [45]. There, an additional dependency of the viscosity μ with respect to the shear is assumed, i.e.

$$\mu = \mu(p, |\mathrm{D}(u)|^2)$$

and the operator defined in (4.30) becomes

$$\operatorname{div}\left(\left(\mathrm{I} - 2\frac{\partial \mu}{\partial p}(p, |\mathrm{D}(u)|^2) \mathrm{D}(u)\right).\nabla p\right). \tag{4.31}$$

Considering viscosities such as

$$\mu(p,|\mathrm{D}(u)|^2) = (1+\gamma(p)+|\mathrm{D}(u)|^2)^{\frac{r-2}{2}}, \quad r \in (1,2]$$

with for instance

$$\gamma(p) = \frac{1}{\sqrt{1+\alpha^2 p^2}},$$

one ensures that

$$\max \left| 2\frac{\partial \mu}{\partial p}(p,|\mathrm{D}(u)|^2)\mathrm{D}(u) \right| < 1$$

for all velocity fields u, and thus the non-degeneracy of the second order operator defined in (4.31). More generally, appropriate conditions on the form of the viscosity have to be assumed to guarantee the existence of global weak solutions [45]. The interested reader is referred to the paper [46] that reviews the existing results on the subject in various frameworks (steady or unsteady flows, with various boundary conditions, etc.).

A new approach through memory effects. We consider the following incompressible Navier–Stokes system with memory effects

$$\begin{cases} \operatorname{div} u = 0 & (4.32a) \\ \partial_t u + (u\cdot\nabla)u + \nabla p + \nabla\Pi - 2\operatorname{div}\big((\mu_0+\bar\mu p)\mathrm{D}(u)\big) + r|u|u = 0 & (4.32b) \\ \partial_t p + u\cdot\nabla p = \frac{1}{2\bar\mu}\Pi & (4.32c) \end{cases}$$

with $\mu_0, \bar\mu, r$ three positive constants. These equations can be seen as a full congested version of Equations (4.22) where an additional pressure p such that $\mu = \mu_0 + \bar\mu p$ has been added to the momentum equation. This system is thus an incompressible model with pressure dependent viscosity where a second pressure, Π is also present and is linked to the first pressure p via (4.32c). It also corresponds to the critical case (4.28) considered in Section 4.3 with additional inertial term and pressure dependent shear viscosity $\mu = \mu_0 + \bar\mu p$ in the momentum equation. Theorem 4.4 makes us hope that the existence of global weak solutions to this system could be obtained by means of an approximation by a singular compressible system of type (4.16) with additional singular pressure p_ε. The existence of global weak solutions is actually a corollary of Theorem 4.4, it suffices to show that from the initial incompressible datum (u^0, p^0) one can define a suitable approximate (compressible) initial datum $(\rho^0_\varepsilon, u^0_\varepsilon)$ which satisfies the condition (4.21). This result is proved in [57].

Theorem 4.6 (Perrin [57])**.** *Let $\Omega = \mathbb{T}^2$, $T > 0$. Assume that initially*

$$(u^0,p^0) \in (L^2(\Omega))^2 \times (L^\infty(\Omega)\cap W^{1,2}(\Omega)) \quad \textit{with} \quad \operatorname{div} u^0 = 0, \quad p^0 > 0.$$

Then, provided that r is small enough, there exists a global weak solution (u,p,Π) to (4.32) with the following properties

$$u \in L^\infty(0,T;(L^2(\mathbb{T}^2))^2) \cap L^2(0,T;(W^{1,2}(\mathbb{T}^2))^2)$$
$$p \geq 0, \quad p \in L^\infty(0,T;W^{1,2}(\mathbb{T}^2))$$
$$\Pi \in W^{-1,\infty}(0,T;L^{q_1}(\mathbb{T}^2)) + L^\infty(0,T;W^{-1,q_2}(\mathbb{T}^2)), \quad q_1 \in [1,+\infty),\ q_2 \in [1,2).$$

Theorem 4.6 shows that we can ensure the global existence of weak solutions without imposing an additional dependency of the shear viscosity with respect to $|\mathrm{D}(u)|$, provided that memory effects are taken into account in the constitutive law. Compared to the initial incompressible system (4.29), we no longer have the problem of the possible degeneracy of the operator (4.30). Indeed, the pressure Π, which is the Lagrange multiplier associated to the incompressibility constraint, is not involved in the viscosity and satisfies thus a standard Poisson equation for given u and p

$$\Delta\Pi = F(u,\nabla u,p).$$

A more interesting feature of system (4.32) compared (4.29) is an additional entropy estimate inherited from the *BD entropy* (4.20) that ensures the uniform bounds on the approximate quantities $\rho_\varepsilon, u_\varepsilon, p_\varepsilon(\rho_\varepsilon), \lambda_\varepsilon(\rho_\varepsilon)\operatorname{div} u_\varepsilon$, etc. For simplicity, let us drop the drag term $r|u|u$ from the equations and introduce the *effective velocity* $v = u + 2\nabla(\mu_0 + \bar\mu p)$ which satisfies

$$\partial_t v + u\cdot\nabla v + \nabla\Pi - 2\operatorname{div}\big((\mu_0+\bar\mu p)\mathrm{A}(u)\big) = 0.$$

This equation tested against v, combined with the classical energy estimate, then leads to the following entropy estimate

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^2}\left[\frac{|u|^2}{2}+\frac{|\nabla p|^2}{2}\right]+\int_0^T\int_{\mathbb{T}^2}(\mu_0+\bar{\mu}p)\big(|\mathrm{D}(u)|^2+|\mathrm{A}(u)|^2\big)+\int_0^T\int_{\mathbb{T}^2}|\nabla p|^2\leq C.$$

The existence of more regular solutions to the incompressible system (4.32) is an interesting open question and the previous estimate could play an important role in such study.

Solid zones. To conclude this section, observe that from (4.32b) we could expect that $|\mathrm{D}(u)|\to 0$ in the regions where $p\to+\infty$. Namely, only rigid motions (translations and rotations) would be allowed and the media would behave like a solid. In the modeling of visco-plastic flows such regions are called *plug zones* and the determination of their geometry has attracted a lot of interest since the 60's both from the physical and the mathematical points of view. Some theoretical results based on variational methods (see [32]) are known in specific configurations (anti-plane, Poiseuille) and precise numerical studies have been performed recently (see [47] and references therein). A refined analysis of the regularity of the interface of the plug zones seems to be an interesting and challenging issue in relation with the questions addressed in Section 3 for the congestion problem.

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