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Abstract

This note is an announcement of a forthcoming paper [13] in collaboration with K. Pravda-Starov on global hypoelliptic estimates for Fokker-Planck and linear Landau-type operators. Linear Landau-type equations are a class of inhomogeneous kinetic equations with anisotropic diffusion whose study is motivated by the linearization of the Landau equation near the Maxwellian distribution. By introducing a microlocal method by multiplier which can be adapted to various hypoelliptic kinetic equations, we establish optimal global hypoelliptic estimates with loss of 4/3 derivatives in a Sobolev scale exactly related to the anisotropy of the diffusion.

1. Introduction

This paper is a short announcement of the article [13] and deals with regularization properties of some kinetic equations with intrinsic diffusion, such as Fokker-Planck, Landau or the Boltzmann equation without cut-off.

Concerning inhomogeneous kinetic equations, i.e. those describing the evolution of the system both in space and moment, one problem is that there is diffusion only in moment and not in space. In this sense they can be considered as degenerate. Anyway the regularization occurs in both variables thanks to a now well-understood mixing procedure called hypoellipticity (e.g. [15], [17], [24])

One step in studying regularization and hypoelliptic properties is to analyze the so-called subelliptic properties of the corresponding linearized operator. The aim of this article is to give optimal subelliptic estimates for Fokker-Planck and linear Landau type operators (without external potential) of the following form:

$$P = v.\partial_x - \Delta_v^2 + v^2, \quad \text{(Fokker-Planck operator)} \tag{1.1}$$

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and

$$P = v.\partial_x - \partial_v.\lambda(v)\partial_v - (v \wedge \partial_v).\mu(v)(v \wedge \partial_v) + F(v), \tag{1.2}$$

(linear Landau-type operators)

where $x, v \in \mathbb{R}^n$, $n \in \mathbb{N}$ in the Fokker-Planck case and $n = 3$ in the Landau case. Here ∂_x, ∂_v are the associated gradients and the diffusion is given by smooth positive functions λ, μ and F satisfying for all $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$,

$$|\partial_v^\alpha \lambda(v)| + |\partial_v^\alpha \mu(v)| \leq C_\alpha \langle v \rangle^{\gamma - |\alpha|}, \qquad |\partial_v^\alpha F(v)| \leq C_\alpha \langle v \rangle^{\gamma + 2 - |\alpha|}, \tag{1.3}$$

and there exists $C > 0$ such that for all $v \in \mathbb{R}^3$,

$$\lambda(v) \geq C \langle v \rangle^\gamma, \quad \mu(v) \geq C \langle v \rangle^\gamma, \quad F(v) \geq C \langle v \rangle^{\gamma+2}, \tag{1.4}$$

with $\gamma \in [-3, 1]$ and where $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$. In the following we shall use the notation $D_x = i^{-1}\partial_x$, $D_v = i^{-1}\partial_v$.

As a first (and essentially pedagogical) step we give a result concerning the Fokker-Planck operator:

Proposition 1.1. *Let P be the Fokker-Planck operator (1.1). Then, there exists a positive constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v})$,*

$$\|\langle D_x \rangle^{2/3} u\|_{L^2}^2 + \|\langle v \rangle^2 u\|_{L^2}^2 + \|\langle D_v \rangle^2 u\|_{L^2}^2 \leq C\Big(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2\Big), \tag{1.5}$$

where the notation $\| \cdot \|_{L^2}$ stands for the $L^2(\mathbb{R}^{2n}_{x,v})$-norm.

Some steps of the proof will be rapidly given in section 2 of this paper, and it is completely written in [13]. Note anyway that it was already essentially contained in [14] in the semi-classical framework (Sections 2, 8 and 9 there). We recall here some steps of the proof since the method used in the Landau case is essentially the same, although the proof is much harder.

Before giving the result in the Landau case we give some preliminary comments. Concerning the Fokker-Planck operator we notice that the exponent 2/3 appearing in (1.5) in the x-derivative has to be compared with the exponent 2 in the v-derivative. This is the typical exponent coming from the fact that *two* Poisson brackets between the real and the imaginary parts of the symbol are required to get a microlocally elliptic symbol (see chap 25 in [16]). Anyway the whole hypoelliptic theory will not be used here and we shall only use a pedestrian multiplier method, in the framework weyl-Hörmander symbolic calculus.

In the Landau case things are complicated by the fact that the real part of the symbol is not of order 2 anymore, and that there is an inhomogeneity in the v-derivatives because of the weights coming from the term $v \wedge \partial_v$. This prevents us from mimicking directly the proof provided in the Kramers-Fokker-Planck case. Nevertheless we are able to use again the same multiplier method, although in a gainless symbolic calculus. In order to handle this more complex situation, and get the right weights and the good exponents 2/3 and 2, we shall also use some elements of Wick calculus developed by N. Lerner in [18]. The main features and the definition of Wick calculus are recalled in a short self-contained exposition in an appendix of [13].

Now we give the main result in the Landau case:

Theorem 1.2. *Let P be a Landau-type operator (1.2) satisfying hypotheses (1.3-1.4) with $\gamma \in [-3, 1]$. Then there exists a constant $C \geq 0$ such that*

$$\left\|\langle v\rangle^{\gamma+2}u\right\|^2 + \left\|\,\langle v\rangle^{\gamma/3}\,|D_x|^{2/3}u\right\|^2 + \left\|\,\langle v\rangle^{\gamma/3}\,|v\wedge D_x|^{2/3}u\right\|^2 \\ + \left\|\,\langle v\rangle^{\gamma}\,|D_v|^2u\right\|^2 + \left\|\,\langle v\rangle^{\gamma}\,|v\wedge D_v|^2u\right\|^2 \leq C\left(\left\|Pu\right\|^2 + \left\|u\right\|^2\right)$$

for all $u \in \mathcal{S}(\mathbb{R}^6_{x,v})$.

Recently the problem of (global) regularity estimates for diffusive kinetic equation was studied with different angles of approach. Concerning the Fokker-Planck equation and similar models, we can mention the works [12], [11] and [8]. The local regularity of Landau-type equations is a direct consequence of their hypoelliptic structure and recently the (local) Gevrey regularity for diffusive models was studied in [5], [6], [7]. Concerning the Boltzmann equation without angular cutoff, existence and regularity results are given for example in [2] and references therein. The aim of this note is to propose and explain global and sharp estimates for the Landau and Fokker-Planck operators, whose proofs can be found in [13].

2. The Fokker-Planck operator

As mentioned in the introduction, we first consider the case of the Fokker-Planck operator without external potential,

$$P = iv.D_x + D_v^2 + v^2,\ \ x, v \in \mathbb{R}^n. \tag{2.1}$$

The aim of this section is to prove Proposition 1.1 and also illustrate in a simplified setting with good symbolic calculus the general method for proving optimal hypoelliptic estimates with loss of 4/3 derivatives. This microlocal method by multiplier can be adapted to various hypoelliptic kinetic equations; and as we shall see with linear Landau-type operators, it turns out to be sharp enough to handle anisotropic classes of symbols, even if in the latter case we shall have to deal with gainless symbolic calculus.

Coming back from now to the Fokker-Planck operator, we begin by performing a partial Fourier transform in the x variable and notice that one may reduce our study on the Fourier side to the analysis of the operator

$$P = iv.\xi + D_v^2 + v^2 = iv.\xi + \sum_{j=1}^{n} D_{v_j}^2 + \sum_{j=1}^{n} v_j^2, \qquad v, \xi \in \mathbb{R}^n,$$

depending on the parameter ξ. In this section, we shall therefore consider Weyl quantizations of symbols only in the velocity variable v and its dual variable η but not in the variable ξ, which will be considered here as a parameter

$$(a^w u)(v) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(v-\tilde{v}).\eta} a\Big(\frac{v+\tilde{v}}{2}, \eta\Big) u(\tilde{v})d\tilde{v}d\eta. \tag{2.2}$$

The Weyl symbol of the Fokker-Planck operator is then given by

$$p = iv.\xi + |\eta|^2 + |v|^2,$$

where $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^n$. Defining the symbol

$$\lambda = \Big(1 + |\eta|^2 + |v|^2 + |\xi|^2\Big)^{\frac{1}{2}}, \tag{2.3}$$

we shall see that Proposition 1.1 easily follows from the key hypoelliptic estimate

$$\|(\lambda^{2/3})^w u\|_{L^2}^2 \lesssim \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{2.4}$$

In order to explain how one can derive such an hypoelliptic estimate and justify the choice of multiplier introduced below, we first notice that the diffusive part of the Fokker-Planck operator gives a trivial control in the variables (v, η). Indeed, this control is just a consequence of the ellipticity of the real part of the symbol

$$\text{Re } p = |\eta|^2 + |v|^2,$$

in these variables. The main point in the estimate (2.4) is then to get a control of the term $|\xi|^{2/3}$. Notice that this control cannot be derived from the ellipticity of the symbol p and that we will need to consider the following iterated commutator

$$[(\text{Im } p)^w, [(\text{Re } p)^w, (\text{Im } p)^w]]$$

in order to get some ellipticity in the parameter ξ. Here Re p and Im p stand for the real and imaginary parts of the symbol p . Indeed, usual symbolic calculus (see Theorem 18.5.4 in [16]) or a direct computation shows that the Weyl symbol of this iterated commutator is exactly given by the iterated Poisson brackets

$$-\{\text{Im } p, \{\text{Re } p, \text{Im } p\}\} = \{\text{Im } p, \{\text{Im } p, \text{Re } p\}\} = 2|\xi|^2,$$

where we recall that the Poisson bracket of two symbols a and b is defined as

$$\{a, b\} = \frac{\partial a}{\partial \eta} \cdot \frac{\partial b}{\partial v} - \frac{\partial a}{\partial v} \cdot \frac{\partial b}{\partial \eta}.$$

Notice that we shall need the ellipticity of this iterated commutator only in the region of the phase space where $|\eta|^2 + |v|^2 \lesssim \lambda^{2/3}$, since one can directly rely on the real part of the symbol p in the region where $|\eta|^2 + |v|^2 \gtrsim \lambda^{2/3}$. This informal discussion accounts for the following choice of symbol multiplier. Let ψ be a $C_0^\infty(\mathbb{R}, [0, 1])$ function such that

$$\psi = 1 \text{ on } [-1, 1], \text{ and supp } \psi \subset [-2, 2]. \tag{2.5}$$

We define the real-valued symbol

$$g = -\frac{\xi . \eta}{\lambda^{4/3}} \psi\left(\frac{|\eta|^2 + |v|^2}{\lambda^{2/3}}\right), \tag{2.6}$$

where the function λ is defined in (2.3). The cutoff function ψ allows to localize the symbol multiplier in the region of the phase space where we need the ellipticity of the iterated commutator

$$[(\text{Im } p)^w, [(\text{Re } p)^w, (\text{Im } p)^w]],$$

whereas the factor $\lambda^{4/3}$ appearing in (2.6) will ensure that the symbol g defines a bounded operator on L^2. Following the usual notations introduced by L. Hörmander in [16, Chapter 18] (see also [19]) we consider the metric

$$\Gamma = \frac{dv^2 + d\eta^2}{M},$$

with

$$M = 1 + |v|^2 + |\eta|^2 + \lambda^{2/3}, \tag{2.7}$$

and the classes of symbols $S(m, \Gamma)$ associated to order functions m, that is, the class of all functions $a \in C^\infty(\mathbb{R}^{2n}_{v,\eta}, \mathbb{C})$ possibly depending on the parameter ξ and satisfying

$$\forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0, \forall (v, \eta, \xi) \in \mathbb{R}^{3n}, \quad |\partial^\alpha_{v,\eta} a(v, \eta, \xi)| \leq C_\alpha m(v, \eta, \xi) M(v, \eta, \xi)^{-|\alpha|/2}.$$

It is easy to check that this metric Γ is admissible (slowly varying, satisfying the uncertainty principle and temperate) with gain

$$\lambda_\Gamma(X) = \inf_{T \neq 0} \left(\frac{\Gamma^\sigma_X(T)}{\Gamma_X(T)} \right)^{1/2} = M(X), \ X = (v, \eta, \xi), \tag{2.8}$$

for symbolic calculus in the symbol classes $S(m, \Gamma)$. We refer to [16] or [19] for extensive presentations of symbolic calculus.

As a first step in the study, we are able to prove the following estimates: For any $m \in \mathbb{R}$, the following symbols belong to their respective symbol classes

$$i) \ \langle \xi \rangle^m \in S(\lambda^m, \Gamma); \quad ii) \ \lambda^m \in S(\lambda^m, \Gamma); \tag{2.9}$$

$$iii) \ g \in S(1, \Gamma); \quad iv) \ \mathrm{Re}\, p \in S(M, \Gamma); \tag{2.10}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. For the proof we refer to [13, Lemma 2.2]. Let us just mention that the metric is adapted both to the weight g and the (real part of the) symbol p. Recall also that in the "elliptic" regions, i.e. in the phase space where $\eta^2 + v^2$ is large, the weight g is zero, and it will be useful only in the regions where

$$|\eta|^2 + |v|^2 \lesssim \lambda^{2/3}.$$

and this explains the introduction of the cut off function, and it is also remarkable that it has the following symbolic property

$$\psi\left(\frac{|\eta|^2 + |v|^2}{\lambda^{2/3}} \right) \in S(1, \Gamma). \tag{2.11}$$

The next step is to show that up to controlled terms and a weight factor $\lambda^{4/3}$, the Poisson bracket

$$\{\mathrm{Im}\, p, g\},$$

makes appear the elliptic symbol of the iterated commutator

$$-\{\mathrm{Im}\ p, \{\mathrm{Re}\ p, \mathrm{Im}\ p\}\} = 2|\xi|^2,$$

in the region of the phase space where $|\eta|^2 + |v|^2 \lesssim \lambda^{2/3}$. This is the case and we are able to show that [13, Lemma2.3]

$$\{\mathrm{Im} p, g\} = \frac{|\xi|^2}{\lambda^{4/3}} \psi\left(\frac{|\eta|^2 + |v|^2}{\lambda^{2/3}} \right) + r, \tag{2.12}$$

with a remainder r belonging both to symbol classes $S\Big(|\eta|^2 + |v|^2, \Gamma\Big)$ and $S(M, \Gamma)$, uniformly with respect to the parameter $\xi \in \mathbb{R}^n$.

We now give the main step of the result concerning the Fokker-Planck operator.

Proposition 2.1. *There exists a positive constant $C > 0$ such that for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $u \in \mathcal{S}(\mathbb{R}^n_v)$,*

$$\||\xi|^{1/3}u\|^2_{L^2} + \|vu\|^2_{L^2} + \|D_v u\|^2_{L^2} \leq C\Big(\|\langle\xi\rangle^{-s}Pu\|_{L^2}\|\langle\xi\rangle^{s}u\|_{L^2} + \|u\|^2_{L^2}\Big),$$

where $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^n_v)$-norm. In particular we have

$$\||\xi|^{1/3}u\|^2_{L^2} + \|vu\|^2_{L^2} + \|D_v u\|^2_{L^2} \leq C\Big(\|Pu\|_{L^2}\|u\|_{L^2} + \|u\|^2_{L^2}\Big),$$

in the case when $s = 0$.

Proof. This is essentially [13, Propositions 2.4 and 2.5], but we give it here in a rather complete way since this is the core of the proof. We consider the multiplier $G = g^w$ defined by the Weyl quantization of the symbol g as in (2.2); and let ε be a positive parameter such that $0 < \varepsilon \leq 1$. For any $s \in \mathbb{R}$, we may write

$$\begin{aligned}\mathrm{Re}(\langle\xi\rangle^{-s}Pu, \langle\xi\rangle^{s}(1-\varepsilon G)u) = \|D_v u\|^2_{L^2} + \|vu\|^2_{L^2}\\ - \varepsilon\mathrm{Re}(iv.\xi u, Gu) - \varepsilon\mathrm{Re}(|D_v|^2 u, Gu) - \varepsilon\mathrm{Re}(|v|^2 u, Gu).\end{aligned} \tag{2.13}$$

We need to estimate the terms appearing on the second line of (2.13). We begin by noticing from (2.9) and the Calderón-Vaillancourt Theorem that the operator G is bounded on L^2. This implies that

$$\begin{aligned}|\mathrm{Re}(|D_v|^2 u, Gu)| = |\mathrm{Re}(D_v u, D_v G u)|\\ \leq |\mathrm{Re}(D_v u, [D_v, G]u)| + |\mathrm{Re}(D_v u, G D_v u)| \lesssim \|D_v u\|^2_{L^2} + \|[D_v, G]u\|^2_{L^2},\end{aligned} \tag{2.14}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. Symbolic calculus shows that the symbol of the commutator $[D_v, G]$ is exactly given by $i^{-1}\partial_v g$. In view of (2.9), this symbol belongs to the symbol class $S(1, \Gamma)$. We therefore deduce from the Calderón-Vaillancourt Theorem that

$$|\mathrm{Re}(|D_v|^2 u, Gu)| \lesssim \|D_v u\|^2_{L^2} + \|u\|^2_{L^2}, \tag{2.15}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. A similar reasoning gives the estimate

$$|\mathrm{Re}(|v|^2 u, Gu)| \lesssim \|vu\|^2_{L^2} + \|u\|^2_{L^2}, \tag{2.16}$$

uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. Regarding the last term, we may write

$$-\mathrm{Re}(iv.\xi u, Gu) = \frac{1}{2}\mathrm{Re}([iv.\xi, G]u, u),$$

since the operators G and $iv.\xi$ are respectively formally selfadjoint and skew-selfadjoint. Symbolic calculus then shows that the symbol of the commutator

$$\frac{1}{2}[iv.\xi, G],$$

is exactly given by

$$\frac{1}{2}\{v.\xi, g\} = \frac{1}{2}\frac{|\xi|^2}{\lambda^{4/3}}\psi\left(\frac{|\eta|^2 + |v|^2}{\lambda^{2/3}}\right) + \frac{r}{2},$$

where we know from (2.12) that r belongs both to symbol classes $S\Big(|\eta|^2 + |v|^2, \Gamma\Big)$ and $S(M, \Gamma)$, uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. Notice from (2.9) and (2.8) that $|\eta|^2+|v|^2$ and r are both first order symbols belonging to the class $S(M,\Gamma)$. Using that the estimate

$$|r| \lesssim |\eta|^2+|v|^2,$$

holds uniformly with respect to the parameter $\xi \in \mathbb{R}^n$, since $r \in S\big(|\eta|^2+|v|^2,\Gamma\big)$, we deduce from the Gårding inequality (Theorem 2.5.4 in [19]) that

$$|(r^w u,u)| \lesssim \|D_v u\|_{L^2}^2+\|vu\|_{L^2}^2+\|u\|_{L^2}^2.$$

Setting

$$\Psi=\frac{|\xi|^2}{2\lambda^{4/3}}\psi\left(\frac{|\eta|^2+|v|^2}{\lambda^{2/3}}\right), \tag{2.17}$$

we can therefore find a positive constant $C>0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^n)$ and $\xi \in \mathbb{R}^n$,

$$-\operatorname{Re}(iv.\xi u,Gu) \geq (\Psi^w u,u)-C\|D_v u\|_{L^2}^2-C\|vu\|_{L^2}^2-C\|u\|_{L^2}^2. \tag{2.18}$$

We then deduce from (2.13), (2.15), (2.16) and (2.18) that there exists a constant

$$0<\varepsilon_0 \leq 1,$$

and a new positive constant $C>0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^n)$ and $\xi \in \mathbb{R}^n$,

$$\operatorname{Re}(\langle\xi\rangle^{-s}Pu,\langle\xi\rangle^{s}(1-\varepsilon G)u) \geq \frac{1}{2}(\|D_v u\|_{L^2}^2+\|vu\|_{L^2}^2)+\varepsilon_0(\Psi^w u,u)-C\|u\|_{L^2}^2. \tag{2.19}$$

By considering separately the two regions of the phase space,

$$|\eta|^2+|v|^2 \lesssim \lambda^{2/3} \quad \text{and} \quad |\eta|^2+|v|^2 \gtrsim \lambda^{2/3}$$

and according to the support of the function

$$\psi\left(\frac{|\eta|^2+|v|^2}{\lambda^{2/3}}\right),$$

we notice that one can find a positive constant $\varepsilon_1>0$ such that for all $(v,\eta,\xi) \in \mathbb{R}^{3n}$,

$$\begin{aligned}\varepsilon_0\frac{|\xi|^2}{2\lambda^{4/3}}\psi\left(\frac{|\eta|^2+|v|^2}{\lambda^{2/3}}\right)+\frac{1}{2}(|v|^2+|\eta|^2) &\geq \varepsilon_1\lambda^{2/3}+\frac{1}{4}(|v|^2+|\eta|^2)\\ &\geq \varepsilon_1(|\xi|^{2/3}+|v|^2+|\eta|^2).\end{aligned} \tag{2.20}$$

This estimate is the crucial step where we combine the ellipticity in the variables (v,η) of the real part of the symbol p together with the ellipticity in the variable ξ of the iterated commutator

$$[(\operatorname{Im} p)^w,[(\operatorname{Re} p)^w,(\operatorname{Im} p)^w]]=2|\xi|^2,$$

in order to derive the optimal hypoelliptic estimate with loss of 4/3 derivatives. Notice from (2.9), (2.11) and (2.7) that

$$\varepsilon_0\frac{|\xi|^2}{2\lambda^{4/3}}\psi\left(\frac{|\eta|^2+|v|^2}{\lambda^{2/3}}\right)+\frac{1}{2}(|v|^2+|\eta|^2)$$

and

$$\varepsilon_1(|\xi|^{2/3}+|v|^2+|\eta|^2),$$

are both first order symbols belonging to the class $S(M,\Gamma)$. Recalling (2.17) and (2.19), we can then deduce from (2.20) and another use of the Gårding inequality that there exists a new positive constant $C > 0$ such that for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $u \in \mathcal{S}(\mathbb{R}^n_v)$,

$$\mathrm{Re}\big(\langle\xi\rangle^{-s}Pu, \langle\xi\rangle^{s}(1-\varepsilon G)u\big) \geq \varepsilon_1(\|D_v u\|^2_{L^2} + \|vu\|^2_{L^2} + \||\xi|^{1/3}u\|^2_{L^2}) - C\|u\|^2_{L^2}.$$

Notice that

$$\langle\xi\rangle^{s}(1-\varepsilon G) = (1-\varepsilon G)\langle\xi\rangle^{s}.$$

Recalling that the multiplier G defines a bounded operator on L^2, Proposition 2.1 then follows from the Cauchy-Schwarz inequality. □

Of course the estimates in Proposition 2.1 are not optimal, but with some additional work and substituting $\langle\xi\rangle^{1/3}u$ to u we get

Proposition 2.2. *There exists a positive constant $C > 0$ such that for all $\xi \in \mathbb{R}^n$ and $u \in \mathcal{S}(\mathbb{R}^n_v)$,*

$$\|\langle\xi\rangle^{2/3}u\|^2_{L^2} + \|\langle v\rangle^2 u\|^2_{L^2} + \|\langle D_v\rangle^2 u\|^2_{L^2} \leq C(\|Pu\|^2_{L^2} + \|u\|^2_{L^2}), \tag{2.21}$$

where $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^n_v)$-norm.

Proof. [13, Proposition 2.6]. □

When coming back to the direct Fourier side and integrating with respect to the x variable, Proposition 1.1 directly follows from Proposition 2.2. This proves the optimal hypoelliptic estimate fulfilled by the Fokker-Planck operator without external potential. □

3. Linear Landau-type operators

Before giving some elements on the proof of Theorem 1.2, we recall some details about the linearized Landau operator. Details about the full Landau equation may be found for example in the works by Y. Guo [10], C. Mouhot and L. Neumann [20], or C. Villani [25], and we may only recall here that the Landau equation reads as the evolution equation of the density of particles

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = Q_L(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{3.1}$$

where Q_L is the so-called Landau collision operator

$$Q_L(f,f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} \mathbf{A}(v-v_*)\Big(f(v_*)(\nabla_v f)(v) - f(v)(\nabla_v f)(v_*)\Big)dv_*\Big). \tag{3.2}$$

Here, $\mathbf{A}(z)$ is a symmetric nonnegative matrix depending on a parameter $z \in \mathbb{R}^3$,

$$\mathbf{A}(z) = |z|^2\Phi(|z|)\mathbf{P}(z),$$

with $\Phi(|z|) = |z|^\gamma$, and where $\mathbf{P}$ the orthogonal projection onto $z^\perp$,

$$\mathbf{P}(z) = \mathrm{Id} - \frac{1}{|z|^2}z.z^\perp,$$

matrix whose entries are

$$\Big(\mathbf{P}(z)\Big)_{i,j} = \delta_{i,j} - \frac{z_i z_j}{|z|^2},\ 1 \le i,j \le 3.$$

The original Landau collision operator describing collisions among charged particles interacting with Coulombic force and introduced by Landau in 1936, corresponds to the case $\gamma = -3$. As in the Boltzmann equation, it is well-known that Maxwellians are steady states to the Landau equation

$$\mathcal{M}(x,v) = (2\pi)^{-3/2} e^{-|v|^2/2}. \tag{3.3}$$

Following the standard procedure described in [10] or [20], we linearize the Landau equation around $\mathcal{M}$ by posing

$$f = \mathcal{M} + \sqrt{\mathcal{M}} u,$$

and one can check that *after linearization* the Landau equation for the perturbation $u(t,x,v)$ now reads as

$$\partial_t u + iv.D_x u - Lu = 0, \tag{3.4}$$

with $D_x = i^{-1}\partial_x$. The transport part of the equation $iv.D_x$ is unchanged, whereas one can prove that the operator L may write as

$$L = L_* - D_v A(v) D_v - F(v), \tag{3.5}$$

with F a positive smooth function satisfying the estimates (1.3) and (1.4). Here, the operator L_* is a convolution-type term bounded on L^2, which only has a (big) influence on the lower part of the spectrum of the operator $iv.D_x - L$, whereas the other term

$$A(v) = (\mathbf{A} * \mathcal{M})(v), \tag{3.6}$$

inherits the properties of the projection $\mathbf{P}$. More specifically, for each vector $v \in \mathbb{R}^3$, the matrix $A(v)$ is symmetric with a simple eigenvalue $\lambda(v)$ associated to the eigenvector v; and a double eigenvalue $\lambda_\perp(v)$ associated to the eigenspace $v^\perp$; which satisfy the estimates

$$\forall \alpha \in \mathbb{N}^3, \exists C_\alpha > 0, \forall v \in \mathbb{R}^3, \quad |\partial_v^\alpha \lambda(v)| \le C_\alpha \langle v\rangle^{\gamma-|\alpha|}, \quad |\partial_v^\alpha \lambda_\perp(v)| \le C_\alpha \langle v\rangle^{\gamma+2-|\alpha|},$$

giving rise to the anisotropy of the diffusion. Up to a bounded operator, this explains why the linearization of the Landau equation essentially reduces to the study of a linear Landau-type operator

$$P = iv.D_x + D_v.\lambda(v) D_v + (v \wedge D_v).\mu(v)(v \wedge D_v) + F(v),$$

with $\mu(v) \sim \frac{\lambda_\perp(v)}{\langle v\rangle^2}$ and a perhaps slightly modified function $\lambda(v)$ so that the estimates (1.4) hold. This motivates the present work on the hypoellipticity of these operators.

Now we give some elements about the proof of the optimal anisotropic hypoelliptic estimate with loss of 4/3 derivatives given in Theorem 1.2. This is done in section 3 of [13] for the following generalized linear Landau-type operators

$$P = iv.D_x + \sum_{j,k=1}^{n} D_{v_j} A_{j,k}(v) D_{v_k} + F(v); \tag{3.7}$$

where $x, v \in \mathbb{R}^n$. Here $A(v) = (A_{j,k}(v))_{1 \le j,k \le n}$ stands for a positive definite symmetric matrix with real-valued smooth entries verifying

$$|\partial_v^\alpha A_{j,k}(v)| \lesssim \langle v\rangle^{\gamma+2-|\alpha|},\ \alpha \in \mathbb{N}^n,\ 1 \le j,k \le n, \tag{3.8}$$

and F is a smooth positive function verifying (1.3). We assume that we may write

$$A(v) = B(v)^T B(v), \tag{3.9}$$

where $B(v)$ is a matrix with real-valued smooth entries verifying

$$|\partial_v^\alpha B_{j,k}(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|},\ \alpha \in \mathbb{N}^n,\ 1 \leq j,k \leq n; \tag{3.10}$$

and $B(v)^T$ is its adjoint. Moreover, we assume that there exists a constant $c > 0$ such that for all $v, \eta \in \mathbb{R}^n$,

$$A(v)\eta.\eta = |B(v)\eta|^2 \geq c\langle v \rangle^\gamma |\eta|^2. \tag{3.11}$$

Notice that linear Landau-type operators are particular generalized linear Landau-type operators when taking

$$B(v) = \begin{pmatrix} \sqrt{\lambda(v)} & -v_3\sqrt{\mu(v)} & v_2\sqrt{\mu(v)} \\ v_3\sqrt{\mu(v)} & \sqrt{\lambda(v)} & -v_1\sqrt{\mu(v)} \\ -v_2\sqrt{\mu(v)} & v_1\sqrt{\mu(v)} & \sqrt{\lambda(v)} \end{pmatrix}, \tag{3.12}$$

with λ and μ being the functions defined in (1.3) and (1.4). Indeed, we have for any $\eta \in \mathbb{R}^3$,

$$|B(v)\eta|^2 = |\sqrt{\lambda(v)}\eta + \sqrt{\mu(v)}v \wedge \eta|^2 = |\sqrt{\lambda(v)}\eta|^2 + |\sqrt{\mu(v)}v \wedge \eta|^2 \geq c\langle v \rangle^\gamma |\eta|^2. \tag{3.13}$$

In order to prove Theorem 1.2 in [13], we use a multiplier method inspired from the one presented in the previous section for the Fokker-Planck operator without external potential. Recalling (3.9), the Weyl symbol of a generalized linear Landau-type operator (3.7) may write as

$$iv.\xi + |B(v)\eta|^2 + F(v) + \textit{Lower order terms}.$$

By denoting

$$\tilde{p} = iv.\xi + |B(v)\eta|^2 + F(v),$$

we shall take advantage of the ellipticity in the variables (v, η) of the real part of the symbol $\tilde{p}$,

$$\text{Re } \tilde{p} = |B(v)\eta|^2 + F(v).$$

As in the case of the Fokker-Planck operator, the main point in proving Theorem 1.2 is then to get a control of the ξ variable. Notice again that this control cannot be derived from the ellipticity of the symbol $\tilde{p}$ and that we will need to consider the following iterated commutator

$$[(\text{Im } \tilde{p})^w, [(\text{Re } \tilde{p})^w, (\text{Im } \tilde{p})^w]]$$

in order to get some ellipticity in the ξ variable. Indeed, usual symbolic calculus (see Theorem 18.5.4 in [16]) or a direct computation shows that the Weyl symbol of this iterated commutator is exactly given by the iterated Poisson brackets

$$-\{\text{Im } \tilde{p}, \{\text{Re } \tilde{p}, \text{Im } \tilde{p}\}\} = \{\text{Im } \tilde{p}, \{\text{Im } \tilde{p}, \text{Re } \tilde{p}\}\} = 2|B(v)\xi|^2.$$

The structure of this iterated poisson bracket suggests to introduce the following anisotropic symbol

$$\lambda = \Big(1 + |B(v)\xi|^2 + |B(v)\eta|^2 + F(v)\Big)^{1/2}, \tag{3.14}$$

which defines an anisotropic Sobolev scale which is exactly related to the anisotropy of the diffusion. As in the case of the Fokker-Planck operator, we aim at establishing an optimal hypoelliptic estimate with loss of 4/3 derivatives in this anisotropic Sobolev scale

$$\|(\lambda^{2/3})^w u\|_{L^2}^2 \lesssim \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2.$$

By noticing that for a generalized linear Landau-type operator

$$\{\mathrm{Re}\ \tilde{p}, \mathrm{Im}\ \tilde{p}\} = 2B(v)\xi.B(v)\eta,$$

it is natural to consider the following multiplier: Let Ψ be a $C_0^\infty(\mathbb{R},[0,1])$ function such that

$$\psi = 1 \text{ on } [-1,1], \text{ and supp } \psi \subset [-2,2]. \tag{3.15}$$

Define the real-valued symbol

$$g = -\frac{B(v)\xi.B(v)\eta}{\lambda^{4/3}}\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right), \tag{3.16}$$

where λ is the symbol defined in (3.14). The main difference with the Fokker-Planck case is that this multiplier does not belong anymore to a symbol class with good symbolic calculus. Indeed, because of the anisotropy of the symbol $\tilde{p}$, we will have to deal with gainless symbolic calculus. As a consequence, the implementation of the method developed for the Fokker-Planck operator is more complex and requires more advanced microlocal analysis. In order to handle this setting with gainless symbolic calculus, we use in [13] some elements of Wick calculus developed by N. Lerner in [18] (a short self-contained presentation is given in the appendix of [13]). For the complete proofs, we refer to the original article [13].

4. Short conclusion

As a conclusion we just quote some developments of this work currently in preparation. The first direction deals with spectral and pseudospectral estimates for The Landau operator, which are naturally associated to the (sharp) subelliptic estimates as in the Fokker-Planck case ([12], [8], [14]..). The second direction deals with the Boltzmann equation without cut-off, for which a lot of attention has been given recently (see e.g. [1], [2]...) and for which the same multiplier method may work as well.

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References

[1] R. Alexandre, Y. Morimoto, S. Ukai, C-J. Xu, T. Yang, *Uncertainty principle and kinetic equations*, J. Funct. Anal. **255**, no. 8, 2013-2066 (2008).

[2] R. Alexandre, Y. Morimoto, S. Ukai, C-J. Xu, T. Yang, *The Boltzmann equation without angular cutoff. Global existence and full regularity of the Boltzmann equation without angular cutoff. Part I : Maxwellian case and small singularity*, preprint (2009), http://arxiv.org/abs/0912.1426

[3] P. Bolley, J. Camus, J. Nourrigat, *La condition de Hörmander-Kohn pour les opérateurs pseudo-différentiels*, Comm. Partial Differential Equations, **7**, no. 2, 197-221 (1982).

[4] F. Bouchut, *Hypoelliptic regularity in kinetic equations*, J. Math. Pures Appl. (9) **81**, no. 11, 1135-1159 (2002).

[5] H. Chen, W-X. Li, C-J. Xu, *Propagation of Gevrey regularity for solutions of Landau equations*, Kinet. Relat. Models, **1**, no.3, 355-368 (2008).

[6] H. Chen, W-X. Li, C-J. Xu, *Gevrey regularity for solution of the spatially homogeneous Landau equation*, Acta Math. Sci. Ser. B Engl. Ed. **29**, no. 3, 673-686 (2009).

[7] H. Chen, W-X. Li, C-J. Xu, *Gevrey hypoellipticity for linear and non-linear Fokker-Planck equations*, J. Differential Equations, **246**, no. 1, 320-339 (2009).

[8] J-P. Eckmann, M. Hairer, *Spectral properties of hypoelliptic operators*, Comm. Math. Phys. **235**, no. 2, 233-253 (2003).

[9] C. Fefferman, D.H. Phong, *The uncertainty principle and sharp Gårding inequalities*, Comm. Pure Appl. Math. **34**, no. 3, 285-331 (1981).

[10] Y. Guo, *The Landau equation in a periodic box*, Comm. Math. Phys. **231**, no. 3, 391-434 (2002).

[11] B. Helffer, F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, **1862**, Springer-Verlag, Berlin (2005).

[12] F. Hérau, F. Nier, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, Arch. Ration. Mech. Anal. **171**, no. 2, 151-218 (2004).

[13] F. Hérau, K. Pravda-Starov, *Anisotropic hypoelliptic estimates for Landau-type operators*, submitted (2010).

[14] F. Hérau, J. Sjöstrand, C. Stolk, *Semiclassical analysis for the Kramers-Fokker-Planck equation*, Comm. Partial Differential Equations, **30**, no. 4-6, 689-760 (2005).

[15] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119**, 147-171 (1967).

[16] L. Hörmander, *The analysis of linear partial differential operators*, vol. I-IV, Springer-Verlag (1985).

[17] J.J. Kohn, *Pseudodifferential operators and hypoellipticity*, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), pp. 61-69, Amer. Math. Soc., Providence, R.I. (1973).

[18] N. Lerner, *The Wick calculus of pseudo-differential operators and some of its applications*, Cubo Mat. Educ. **5**, no. 1, 213-236 (2003).

[19] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators, Theory and Applications, Vol. 3, Birkhäuser (2010).

[20] C. Mouhot, L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus*, Nonlinearity, **19**, no. 4, 969-998 (2006).

[21] Y. Morimoto, C-J. Xu, *Hypoellipticity for a class of kinetic equations*, J. Math. Kyoto Univ. **47**, no. 1, 129-152 (2007).

[22] Y. Morimoto, C-J. Xu, *Ultra-analytic effect of Cauchy problem for a class of kinetic equations*, J. Differential Equations, **247**, no. 2, 596-617 (2009).

[23] K. Pravda-Starov, *Subelliptic estimates for quadratic differential operators*, to appear in American Journal of Mathematics (2010), http://arxiv.org/abs/0809.0186

[24] L.P. Rothschild, E.M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137**, no. 3-4, 247-320 (1976).

[25] C. Villani, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam (2002).

[26] C-J. Xu, *Fourier analysis of non-cutoff Boltzmann equations*, Lectures on the Analysis of Nonlinear Partial Differential Equations, Vol. 1, Morningside Lectures in Mathematics, Higher Education Press and International Press Beijing-Boston, 585-197 (2009).

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