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Abstract

We discuss several model PDEs motivated by the incompressible Navier-Stokes equations. Some of the PDEs appear to be quite simpler, but basic questions about them are still open. In the last section we discuss uniqueness of weak solutions of the 3d incompressible Navier-Stokes in a natural energy class.

1. Introduction

A reasonable approach to dealing with the difficult problems in the PDE theory of the Navier-Stokes equations seems to be a study of simpler model problems, in the hope that we can learn enough from the models to make progress on the full equations. In these notes we will study several such models, and this will take up most of our time. In the last section we will discuss the uniqueness problem for the classical Cauchy problem for incompressible Navier-Stokes with initial datum $u_0 \in L^2$.

One of the main points we wish to make is that the PDE problems one faces in connection with the incompressible flows are not isolated examples in the PDE theory. There are quite a few model equations which appear to be easier that the Navier-Stokes or Euler equations, but for which basic PDE questions remain open.

2. 1d models

Although there are no non-trivial incompressible flows in 1d, there are interesting 1d models relevant to incompressible fluid dynamics. The reason is that the restriction of a incompressible flow to a lower-dimensional sub-manifold which is left invariant by the flow does not have to preserve the (lower-dimensional) volume on the sub-manifold.

Example 1 (after [12, 24])
Consider the 2d Boussinesq system in the upper half-plane $\Omega = \{(x,y) \in \mathbf{R}^2\,,\, y > 0\}$, in the vorticity form

$$\begin{aligned} \omega_t + u\omega_x + v\omega_y &= \theta_x\ , \\ \theta_t + u\theta_x + v\theta_y &= 0 \end{aligned} \tag{2.1}$$

complemented by the boundary condition $v|_{\partial\Omega} = 0$ and the usual Biot-Savart law expressing the div-free velocity field (u, v) in terms of ω. At the boundary the functions ω, θ, u are just functions of x (and time, of course), and we have

$$\begin{aligned} \omega_t + u\omega_x &= \theta_x\ , \\ \theta_t + u\theta_x &= 0\ . \end{aligned} \tag{2.2}$$

The 1d flow on $\partial\Omega$ given by $u(x,t)$ is typically not incompressible. System (2.2) is not closed, in the sense that at $\partial\Omega$ we cannot express u in terms of $\omega|_{\partial\Omega}$. However, one can introduce a new 1d

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Biot-Savart law, which to some degree approximates the real situation (see [12]):

$$u_x = H\omega\,, \qquad \lim_{x\to\infty} u(x,t) = \lim_{x\to-\infty} u(x,t) = 0 \tag{2.3}$$

where H is the Hilbert transform. The system (2.2) augmented by (2.3) is now closed. We will prove a finite-time blow-up result for it in subsection 2.2 .

Instead of working the half-space Ω, one can also work in a half-cylinder $\mathbf{S}^1 \times (0,\infty)$ and obtain the 1d system (2.2) on $\mathbf{S}^1$.

Example 2 (after [10, 11])
Consider the SQG equation in the plane

$$\theta_t + u\theta_x + v\theta_y = 0\,, \tag{2.4}$$

where the velocity field (u, v) is given by

$$u = -\psi_y\ ,\ v = \psi_x\ , \qquad \tilde{\Lambda}\psi = \theta\,, \tag{2.5}$$

where $\tilde{\Lambda} = -(-(\partial_x^2 + \partial_y^2))^{\frac{1}{2}}$. We will consider a special class of solutions, when $\psi = -yu(x,t)$. Using Fourier transform, it is not hard to see that we have $\tilde{\Lambda}(-yu(x,t)) = -y\Lambda u(x,t)$, where $\Lambda = -(-\partial_x^2)^{\frac{1}{2}}$, the operator on the real line with the Fourier multiplier $-|\xi|$. Taking the $y-$derivative of (2.4) at the line $y = 0$, we obtain a 1d equation (on the real line)

$$\omega_t + u\omega_x - u_x\omega = 0\,, \tag{2.6}$$

where $\omega = \omega(x,t) = -\Lambda u(x,t)$. This is the De Gregorio model, which was originally derived by De Gregorio [16] by different considerations, based on the Constantin-Lax-Majda model [13]. We will discuss the De Gregorio model in some detail in the next subsection.

Example 3 (after [5])
Our final example will not come from an incompressible flow, but from considerations generalizing a geometric interpretation of Euler's equations due to V. I. Arnold [1, 2]. From the point of view of the Lagrangian mechanics, one can think of the motion of an incompressible fluid in a domain $\Omega \subset \mathbf{R}^n$ as follows: the configuration space of the mechanical system given by the fluid is the group of volume-preserving diffeomorphisms[1] $G = \mathrm{SDiff}(\Omega)$ of Ω, and the action for a curve ϕ^t in G is given by $\int_{t_1}^{t_2}\int_\Omega \frac{1}{2}|\dot{\phi}^t(x)|^2\,dx\,dt$, the same as for the geodesics on G with respect to the natural L^2-metric.

One can now consider a 1d situation, take $\Omega = \mathbf{S}^1$ (the one dimensional circle), and $G = \mathrm{Diff}(\mathbf{S}^1)$ (the group of the diffeomorphisms of $\mathbf{S}^1$, or, if we like, its connected component containing the identity) and consider it with the metric induced from the tangent space $T_{\mathrm{id}}G$ by the $\dot{H}^s$-scalar product, and the requirement that the metric be right-invariant under the natural action of G on itself. The action then will be $\int_{t_1}^{t_2} ||\dot{\phi}^t \circ (\phi^t)^{-1}||^2_{\dot{H}^s}\,dt$. We note that $u(x,t) = \dot{\phi}^t((\phi^t)^{-1}(x))$ is the Eulerian velocity field. In the incompressible case it does not matter whether or not we compose $\dot{\phi}^t$ with $(\phi^t)^{-1}$ in the action integral, as ϕ^t is volume-preserving. For the group $\mathrm{Diff}(\mathbf{S}^1)$ this, of course, does matter.

An unexpected result of Michor and Mumford [39] is that for $s \le \frac{1}{2}$ the geodesic distance induced on the connected component of identity of $\mathrm{Diff}(\mathbf{S}^1)$ by the above action vanishes identically. The equations for geodesics still make sense, except that no non-trivial solution can be a minimizer. For $s = 0$ we obtain a variant of the Burgers equation

$$u_t + 3uu_x = 0\,, \tag{2.7}$$

which can, of course, develop singularities in finite time from smooth data. For $s = 1$ the resulting equation is the Hunter-Saxton equation, and is completely integrable, see [35]. It also can produce finite-time singularities from smooth data. In the context of other models we will be studying here, the most interesting case is perhaps $s = \frac{1}{2}$, which leads to

$$\omega_t + u\omega_x + 2u_x\omega = 0\,, \qquad u_x = -H\omega\,, \tag{2.8}$$

which is called the Wunch equation in [5]. (Note that if we change ω to $-\omega$ in the last equation, we can change the Biot-Savart law to $u_x = H\omega$.)

[1] We can also take its connected component containing identity.

2.1. The De Gregorio equation

We will now look in more detail at equation (2.6), which may be the most interesting one of the models above. Originally it appeared in a different context than in Example 2 above. Its history goes back to a model of Constantin-Lax-Majda from the 1980s [13], which we will refer to as the CLM model in what follows. It is the following equation on the unit circle $\mathbf{S}^1$

$$\omega_t = \omega\, H\omega\,, \tag{2.9}$$

where H is again the Hilbert transform. (The model can be defined of the real line, in the same way.) A remarkable feature of the model, discovered in [13], is that it can be solved explicitly. With a slight abuse of notation, let us denote use ω also for the harmonic extension of ω into the unit disc, and $H\omega$ for the harmonic extension of $H\omega$ into the unit disc. The function $f = \omega + iH\omega$ is then a holomorphic function in the unit disc, and one easily checks that the real part of the equation

$$f_t = -\frac{i}{2} f^2 \tag{2.10}$$

is exactly (2.9). The solution of (2.10) is

$$f(t) = \frac{f(0)}{1 + \frac{i}{2} f(0) t}\,, \tag{2.11}$$

and the explicit formula makes the analysis of singularity formation from smooth initial data easy. (We leave the details to the reader.)

The CLM model was motivated by a comparison with the vorticity form of the 3d Euler equation

$$\omega_t + u\nabla\omega = \omega\nabla u\,, \quad \operatorname{curl} u = \omega\,,\ \operatorname{div} u = 0\,. \tag{2.12}$$

We can write $\omega\nabla u$ as $\omega\, S(\omega)$, where S is a suitable singular integral operator (given by a $0-$homogeneous Fourier multiplier), and if we drop the term $u\nabla\omega$, we obtain a 3d version of (2.9). For a better comparison with the 3d Euler situation, we can write (2.9) as follows

$$\omega_t = \omega u_x\,, \qquad u_x = H\omega\,. \tag{2.13}$$

De Gregorio [16] suggested that changing (2.13) to

$$\omega_t + u\omega_x = \omega u_x\,, \qquad u_x = H\omega \tag{2.14}$$

might be a closer analogy to the 3d Euler than (2.13).

The condition $u_x = \omega$ only determines u up to a constant. Different choices of the constant (which can possibly depend on t) can be thought of as different "gauges" for the equation - the solutions corresponding to two different gauges can be transformed between themselves by a suitable choice of coordinates. A natural gauge on the circle is $\int_{\mathbf{S}^1} u(\theta,t)\, d\theta = 0$, but it is not always the most convenient one, see [29] for details.

We note that the De Gregorio equation (2.14) can also be written as

$$\omega_t + [u,\omega] = 0\,, \tag{2.15}$$

where we consider both u and ω as vector fields, and $[u,\omega]$ denotes their Lie bracket $u\omega_x - \omega u_x$. Geometrically, equation (2.15) describes the transport of the vector field ω by the flow generated by the velocity field u. If ϕ^t is the one-parameter family of diffeomorphisms generated by the flow $u(x,t)$, than $\omega(t) = \phi^t_{\#}\omega(0)$, where we use the usual notation $\phi_{\#}\xi$ for the push-forward of a vector field ξ by the diffeomorphism ϕ. For the 3d Euler equation (2.12) the situation is similar and the identity $\omega(t) = \phi^t_{\#}\omega(0)$ is just the Helmholtz law.

2.1.1. Local well-posedness

When considered on the real line, both the De Gregorio equation and the CLM model are invariant under the scaling

$$\omega(x,t) \to \omega(\lambda x, t)\,, \qquad \lambda > 0 \tag{2.16}$$

and hence the homogeneous part of the norm borderline spaces for the local well-posedness should be invariant under this scaling. For the Sobolev scale H^s (for ω_0), it is natural to expect the local-in-time well-posedness for compactly supported initial data when $s > \frac{1}{2}$, and this can indeed be proved using, for example, methods in [3]. On the other hand, we see from the explicit solutions of the CLM model, that (2.13) is not locally-in-time well-posed for (compactly supported) $\omega_0 \in \dot{H}^{\frac{1}{2}}$, and it is natural to expect the same for (2.15). For the same reason we do not expect local-in-time well-posedness for (compactly supported) $\omega_0 \in L^\infty$. On the other hand, it is likely that (2.15) is locally-in-time well-posed for compactly supported $\omega_0 \in B^{\frac{1}{2}}_{2,1}$ (where we use the usual notation for Besov spaces). The connection with the results in [3, 15] is probably best seen when one re-writes (2.15) as

$$u_t + uu_x = uu_x - \Lambda^{-1}\left(u\Lambda u_x - u_x \Lambda u\right) \stackrel{\text{def}}{=} B(u,u)\,. \tag{2.17}$$

2.1.2. The BKM criterion for regularity

A well-known result of Beal, Kato and Majda [6] says that for the incompressible Euler equation the regularity of solutions is controlled by the quantity $\int_0^T ||\omega(t)||_{L^\infty}$ (where ω is the vorticity) in the following sense: if T is a possible blow-up time for H^s solutions $s > \frac{n}{2}+1$, then $\int_0^T ||\omega(t)||_{L^\infty}\,dt = +\infty$. For the De Gregorio equation (2.15) one has the same type of results which can be proved by techniques similar to those used in the proof of the BKM criterion for incompressible Euler. For example, if $s > \frac{1}{2}$ and an H^s-solution blows up at time T, then $\int_0^T ||\omega(t)||_{L^\infty}\,dt = +\infty$. The proof for $s \geq 1$ can be found in [5]. The general case $s > \frac{1}{2}$ can be proved using the techniques in [3]. One can also replace the norm $||\omega(t)||_{L^\infty}$ by $||\omega(t)||_{H^{\frac{1}{2}}}$ in the statement.

2.1.3. Conserved quantities

At the time of this writing, there are no known coercive conserved quantities for (2.15). One conserved quantity is

$$\int_{\mathbf{S}^1} \omega(x,t)\,dx\,. \tag{2.18}$$

We have

$$\frac{\partial}{\partial t}\int_{\mathbf{S}^1} \omega(x,t)\,dx = \int_{\mathbf{S}^1} -[u,\omega]\,dx = \int_{\mathbf{S}^1} 2(H\omega)\omega\,dx = 0\,. \tag{2.19}$$

In these lectures we will be dealing with the case when the initial condition ω_0 satisfies $\int_{\mathbf{S}^1}\omega_0(x)\,dx = 0$. The more general case with $\int_{\mathbf{S}^1}\omega_0(x)\,dx \neq 0$ may exhibit some additional interesting effects.

From the geometric interpretation of (2.15) as transport of the vector field $\omega(x)\frac{\partial}{\partial x}$, one sees that the invariants of the adjoint orbit $\mathcal{O}_{\omega_0} = \{\phi_\#\omega_0\,,\ \phi \in \mathrm{Diff}_0(\mathbf{S}^1)\}$ (where $\mathrm{Diff}_0(\mathbf{S}^1)$ is the connected component of the diffeomorphism group $\mathrm{Diff}(\mathbf{S}^1)$ and $\phi_\#$ is the push-forward of the vector field ω_0 by ϕ) are also invariant under the evolution. Under some non-degeneracy conditions, such invariants have been characterized by Hitchin in [27]. In the special case when $\omega_0'(x) \neq 0$ whenever $\omega_0(x) = 0$, the invariants are exactly the number of zeroes, and the derivatives of ω_0 at the zeroes (ordered up to a cyclic permutation). The conservation of $\omega_x(x(t),t)$ for $\omega(x(t),t) = 0$ can also be easily seen directly from the equation:

$$\frac{d}{dt}\omega(x(t),t) = \omega_x(x(t),t)\dot{x} + \omega_t(x(t),t) = u_x(x(t),t)\omega(x(t),t) = 0\,. \tag{2.20}$$

2.1.4. Numerical results concerning global well-posedness and long-time behavior

Numerical experiments performed on $\mathbf{S}^1$ for the case $\int_{\mathbf{S}^1}\omega(x,t)\,dx = 0$ suggest (see, for example, [41, 29]) that the equation (2.15) on the circle $\mathbf{S}^1$ is globally well-posed for smooth initial data. (We also expect this to be the case when the equation is considered on $\mathbf{R}$.) On the other hand, based on recent results in [18], one expects that the equation will not be globally well-posed in H^s for $s < \frac{3}{2}$, and one expects the same on $\mathbf{S}^1$ (although - as far as I know - the case of $\mathbf{S}^1$ has not been worked out in the literature).

The long-time behavior of the global solutions on $\mathbf{S}^1$ observed numerically is, generically, an approach to a manifold of stable equilibria. (These results are reported in [29]). One can easily check that the eigenfunctions of the operator Λ on $\mathbf{S}^1$ are equilibria: $\omega = \Lambda u = \lambda u$ clearly implies that $[u,\omega] = 0$. The equilibria corresponding to the lowest non-zero eigenvalue are of the form $\omega(x) = A\sin(x - x_0)$. These functions form a two-dimensional manifold $\mathcal{M}^2$ and numerical experiments suggest that generic solutions approach this manifold. The equilibria $A\sin m(x-x_0)$ with $m \geq 2$ appear to be unstable, with finite-dimensional unstable manifolds. We note that the equation is reversible, and hence the behavior for $t \to \infty$ is the same as the behavior for $t \to -\infty$.

The above long-time behavior observations concern the case $\int_{\mathbf{S}^1} \omega_0(x)\,dx = 0$. When the integral does not vanish, the behavior may be more complicated.

2.1.5. A rigorous stability result

The numerical behavior described above can be confirmed rigorously for solutions on $\mathbf{S}^1$ in the case when the initial condition ω_0 is sufficiently close to the manifold of stable equilibria $\mathcal{M}^2$. In [29] the following result is proved:

Theorem 2.1. *Consider the De Gregorio equation (2.15) on the circle $\mathbf{S}^1$. Assume that ω_0 is a C^2-function which is sufficiently close in the C^2-norm to an equilibrium of the form $A\sin(x-x_0)$. Then the De Gregorio equation (2.15) has a global C^2-solution $\omega(x,t)$. In addition, as $t \to \pm\infty$, for each $s < \frac{3}{2}$ the solution approaches in H^s equilibria $\Omega_{A^\pm, x_0^\pm} = A_\pm \sin(x - x_0^\pm)$. The constants $A_\pm$ can be determined explicitly from the conservation laws discussed in subsection 2.1.3.*

Remark: The convergence is typically not smooth, for $s > \frac{3}{2}$ the norm $||\omega(t) - \Omega_{A^\pm, x_0^\pm}||_{H^s}$ exponentially diverges to ∞ in the generic case.

A detailed proof can be found in [29]. We outline the main ideas. For the outline of the proof, let us switch notation, and write the point on $\mathbf{S}^1$ as

$$z = e^{i\theta}\,, \tag{2.21}$$

so that our variable x above will now be called θ (in the case when the equation is considered on $\mathbf{S}^1$). It is enough to prove the theorem for the equilibrium $\Omega = \Omega(\theta) = -\sin\theta$. The corresponding velocity field is then $U = U(\theta) = \sin\theta$. We linearize the equation at Ω, obtaining

$$\eta_t + [U,\eta] + [v,\Omega] = 0\,, \qquad v_\theta = H\eta\,, \tag{2.22}$$

which is the same as

$$\eta_t + [U,\eta + v] = 0\,, \qquad v_\theta = H\omega\,. \tag{2.23}$$

It turns out one has good estimates for this equation in two types of spaces:

(a) Energy estimate in a space closely related to $H^{\frac{3}{2}}(\mathbf{S}^1)$. We note that the exponent $s = \frac{3}{2}$ is optimal, as the $H^{\frac{3}{2}+\epsilon}$ norm of a typical solution is expected to blow-up for $t \to \infty$, for any $\varepsilon > 0$. The exponent $s = \frac{3}{2}$ can also be seen from the fact that near the equilibria of the field U, the evolution by the transport part of (2.23) is given essentially by scaling $\eta(\theta-\theta_0) \to \lambda\eta(\lambda^{-1}(\theta-\theta_0))$, with $\lambda \sim e^{-t}$ for the stable equilibrium and $\lambda = e^t$ for the unstable equilibrium.

(b) Exponential convergence to the kernel of the linear operator in a suitable L^2-based weighted space.

We note that, in close similarity with linearizations of incompressible Euler equation about non-trivial equilibria, the linearized equation (2.22) can be thought of in terms of a "leading part", which is the transport equation

$$\eta_t + [U,\eta] = 0 \tag{2.24}$$

and a non-local perturbation term $[v,\Omega]$. To avoid distractions by technicalities, we will explain the main ideas on a simpler model problem. The leading part (2.24) describes the transport of the vector field η by the flow generated by the vector field U, and can be solved more or less explicitly. The main difficulty is to control the influence of the non-local term $[v,\Omega]$, which is not as geometric, and its effects seem to be less transparent.

It is quite straightforward to establish good estimates for (2.24). The danger of the non-local term is that it can potentially change the spectrum of the linear operator in a way which would rule out the decay estimates we ultimately need for the non-linear perturbation theory, for example by creating non-trivial eigenfunctions with purely imaginary eigenvalues, which would be consistent with the energy estimate (a), but not with the exponential decay (b), as they would generate to periodic solutions of the linearized equation, ruling out (b).

The transport equation (2.24) can be "diagonalized" in many ways. For example, one can use a simple change of coordinates $dx = \frac{d\theta}{\sin\theta}$ to "straighten" the field $U(\theta)\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial\theta}$ in the interval $(0,\pi)$ to the field $\frac{\partial}{\partial x}$ on $\mathbf{R}$, and similarly on $(-\pi, 0)$. After this change of variables equation (2.24) becomes equivalent to two copies of $\xi_t + \xi_x = 0$ in $\mathbf{R}$, which is, of course, diagonalized by the Fourier transformation.

While this diagonalization works well for equation (2.24), it is not clear how to deal the "perturbative term" $[v,\Omega]$ in these coordinates. The term will remain non-local, with non-trivial interactions between the two copies of the diagonalized equation $\xi_t + \xi_x = 0$.

It turns out there is a different diagonalization of the transport operator in (2.24), which makes it much easier to handle the non-local term. Namely, one can split η into its holomorphic part η_+ and anti-holomorphic part η_-. On the subspace of functions with zero average, this splitting commutes with the evolution given by (2.24). Moreover, the nonlocal part $[v,\Omega]$ becomes much easier to handle, as the Biot-Savart law $v_\theta = H\eta$ restricted to holomorphic fields becomes

$$\eta(z) = -zv'(z)\,, \tag{2.25}$$

where $v' = \frac{dv}{dz}$.

A simplified model for equation (2.24).

Although equation (2.24) can be solved explicitly, it is instructive to consider a still simpler equation

$$f_t + \sin\theta\, f_\theta = 0\,. \tag{2.26}$$

for functions on the circle $\mathbf{S}^1$. In spite of its simplicity, the equation has some interesting features which illustrate well some of the issues coming up in the proof of Theorem 2.1.

The transport equation (2.24) can be brought to the form (2.26) by the change of variables

$$\eta = Uf\,. \tag{2.27}$$

Recalling that $U = \sin\theta$, it is easy to check that the evolution for f given by (2.24) and (2.27) is exactly (2.26).

We note that the vector field $\sin\theta\frac{\partial}{\partial\theta}$ considered on $\mathbf{S}^1$ has a holomorhphic extension to $\mathbf{C}$, given by

$$\frac{1}{2}(z^2-1)\frac{\partial}{\partial z}\,. \tag{2.28}$$

The flow map generated by the field, i. e. the solution of $\dot z = \frac{1}{2}(z^2-1)$, is given by conformal maps of the disc onto itself, of the form $(z-\alpha)/(1-\alpha z)$, with $\alpha = \tanh\frac{1}{2}t$. The evolution by (2.26) clearly preserves the class of functions which have a holomorphic extension to the unit disc $D = \{z\,,\,|z|<1\}$, and also the class of functions which have a bounded holomorphic extension to the complement of the unit disc. (The latter class coincides with the functions which have an anti-holomorphic extension to the disc.)

Each function $f\colon \mathbf{S}^1 \to \mathbf{C}$ can be decomposed into a sum of such functions, and the decomposition is unique modulo constants (as the intersection of the two classes of functions consists of constant functions).

Energy estimate and a spectral decomposition

Let $\dot{\mathcal{H}}^{\frac12}(\mathbf{S}^1)$ be the functions in $\dot H^{\frac12}(\mathbf{S}^1)$ which have a holomorphic extension to the disc. We will slightly abuse notation and for $f \in \dot{\mathcal{H}}^{\frac12}(\mathbf{S}^1)$ we will also denote by f the holomorphic extension to the disc. For $f \in \dot{\mathcal{H}}^{\frac12}$ equation (2.26) is the same as

$$f_t + \frac{1}{2}(z^2-1)f_z = 0. \tag{2.29}$$

This represents the transport of f in the unit disc by the holomorphic vector field (2.28). As the integral $\int_D |f'(z)|^2\,\frac{i}{2}dz\wedge d\bar z$ is invariant under conformal transformations of D, and equal to the square of the $\dot H^{\frac12}-$norm of f, we see that the evolution given by (2.26) preserves the $\dot H^{\frac12}$ norm.

Let us now consider the conformal mapping of the unit disc D onto the strip $\mathcal{O} = \{w\,,\ -\pi/2 < \operatorname{Im} w < \pi/2\}$ given by

$$z \to w = \log \frac{1-z}{1+z}\,, \tag{2.30}$$

where we take $\log(re^{i\theta}) = \log r + i\theta$ for $r \in (0,\infty)$ and $\theta \in (-\pi,\pi)$. In the $w-$coordinate the equation becomes

$$f_t + f_w = 0\,. \tag{2.31}$$

The direct straightening of the field $\sin\theta \frac{\partial}{\partial\theta}$ to $\frac{\partial}{\partial x}$ through the change of coordinates $d\theta/\sin\theta = dx$ leads to

$$x = \log\tan\frac{\theta}{2} = \log i\frac{1-z}{1+z}\,, \qquad z = e^{i\theta}\,, \tag{2.32}$$

which only differs by shifting the strip $\mathcal{O}$ up by $\pi/2$. If we only work on the circle, with $\theta \in (-\pi,\pi)$, the change of coordinates splits the circle into two independent pieces (the two lines at the boundary of the shifted strip), which reflects that the spectrum of the operator $-\sin\theta\frac{\partial}{\partial\theta}$ has multiplicity two, and one might be inclined to decompose the space into the two parts corresponding to the two components. The decomposition into the holomorphic and anti-holomorphic parts gives another way to do the splitting, which works much better for our purposes here.

The equation (2.31) is of course diagonalized by the Fourier transformation

$$f(w) = \int_{-\infty}^{\infty} \phi(s) e^{isw}\, ds\,, \tag{2.33}$$

and a simple calculation shows that

$$\int_{\mathcal{O}} |f'(w)|^2\, \frac{i}{2}\, dw \wedge d\bar{w} = 2\pi \int_{-\infty}^{\infty} |\varphi(s)|^2 s \sinh \pi s\, ds\,. \tag{2.34}$$

Going back to the variable $z \in D$, we see that for holomorphic functions on D we can write

$$f(z) = \int_{-\infty}^{\infty} \varphi(s) \left(\frac{1-z}{1+z}\right)^{is} ds\,, \tag{2.35}$$

and

$$||f||^2_{\dot{\mathcal{H}}^{\frac12}(D)} \sim \int_{-\infty}^{\infty} |\varphi(s)|^2 s\, \sinh \pi s\ ds\,. \tag{2.36}$$

The functions

$$h(z,\lambda) = \left(\frac{1-z}{1+z}\right)^{\lambda} \tag{2.37}$$

satisfy

$$\frac12 (z^2-1)\frac{\partial}{\partial z}\, h(z,\lambda) = \lambda\, h(z,\lambda)$$

and can be thought of as generalized eigenfunctions of the operator $\frac12(z^2-1)\frac{\partial}{\partial z}$ in $\dot{\mathcal{H}}^{\frac12}$, with the decomposition (2.35) representing the spectral decomposition of the restriction of the operator $-\sin\theta\frac{\partial}{\partial\theta}$ to $\dot{\mathcal{H}}^{\frac12}$. The restricted operator is skew-adjoint with respect to the $\dot{\mathcal{H}}^{\frac12}$ scalar product, and its spectrum coincides with the imaginary axis. In the "coordinate" φ given by (2.35) the operator acts via $\varphi(s) \to is\varphi(s)$. Recalling that functions in $\dot{\mathcal{H}}^{\frac12}$ are considered only modulo constants, we see that the spectrum is absolutely continuous. It is worth noting that for $\lambda = is$ with s real the generalized eigenfunctions just barely miss $\dot{\mathcal{H}}^{\frac12}$, in the sense that they belong to $\dot{\mathcal{H}}^{\frac12-\varepsilon}$ for $\varepsilon > 0$. (The function $h(z,0)$ is constant and is of course in the kernel of the operator, but this function is equivalent to 0 in $\dot{\mathcal{H}}^{\frac12}$.)

In a slighly different language, the main point of the above consideration can be viewed as follows. Let us denote M the operator on $\dot{H}^{\frac12}(\mathbf{S}^1)$ given by $-\sin\theta\frac{\partial}{\partial\theta}$, and let H be the Hilbert transform. Then the commutator $[H,M]$ is given by

$$[H,M]f = \frac{i}{2}(f_1 + f_{-1})\cdot 1\,, \tag{2.38}$$

where f_1 and f_{-1} are the Fourier coefficients of f corresponding to $e^{i\theta}$ and $e^{-i\theta}$, respectively. In particular,

$$[H,M] = 0 \qquad \text{in } \dot{H}^{\frac12}(\mathbf{S}^1)\,. \tag{2.39}$$

This means the eigenspaces of H are invariant under M, which gives our splitting of the space $\dot{H}^{\frac{1}{2}}$ above.

From the spectral analysis above it can be easily seen that when the initial condition $f|_{t=0} = f_0$ belongs to $\dot{H}^{\frac{1}{2}}(\mathbf{S}^1)$, the solution $f(t)$ of (2.26) converges weakly to 0 in $\dot{H}^{\frac{1}{2}}(\mathbf{S}^1)$. This can, of course, also be seen from an explicit calculation. However, the explicit calculation is not available for (2.22), whereas the spectral analysis can be done along similar lines, although with more complicated arguments concerning the generalized eigenfunctions. (We recall that the functions in $\dot{H}^{\frac{1}{2}}$ are considered modulo constants. This is essential, as for a general $f_0 \in \dot{H}^{\frac{1}{2}}$ the constant part of the solution $f(t)$ cannot be controlled, and can exhibit arbitrarily large oscillations.)

The estimate of solution $f(t)$ of (2.26) in $\dot{H}^{\frac{1}{2}}(\mathbf{S}^1)$ discussed above illustrates in a simpler situation the energy estimate for (2.22) in a space equivalent to $\dot{H}^{\frac{3}{2}}$, or more precisely, its suitable factor space (which mods out the kernel of the linearized operator).

Estimate in a weighted L^2-space

There is another natural space in which (2.26) can be considered. To motivate its definition, we note that the value of the initial datum f_0 at $\theta = 0$ plays a decisive role for the evolution, as we have

$$f(z,t) = f_0\left(\frac{z + \tanh\frac{t}{2}}{1 + z\tanh\frac{t}{2}}\right), \qquad z = e^{i\theta}\,. \tag{2.40}$$

We now define an L^2-based space of functions $f\colon \mathbf{S}^1 \to \mathbf{C}$ which puts a special emphasis on the value of f at $\theta = 0$. Let us choose $\gamma \in (\frac{1}{2}, 1)$ and define

$$Y_0 = \{f \in L^2(\mathbf{S}^1)\,,\ \int_{-\pi}^{\pi} |f(\theta)|^2 |\sin(\frac{\theta}{2})|^{-2\gamma}\, d\theta\} \tag{2.41}$$

with the norm

$$||f||_{Y_0} = ||\, f\, |\sin\frac{\theta}{2}|^{-\gamma}||_{L^2(\mathbf{S}^1)}\,. \tag{2.42}$$

We also define

$$Y = Y_0 \oplus \mathbf{C}\cdot 1 \tag{2.43}$$

with the norm of $f = g + a \in Y\,,\ g \in Y_0\,, a \in \mathbf{C}$ given by

$$||f||_Y = ||g||_{Y_0} + |a|\,. \tag{2.44}$$

For $f \in Y$ as above we can define $f(0) = a$. With this definition, the mapping $f \to f(0)$ is clearly continuous on Y. It is easy to see that the Hölder space $C^\alpha(\mathbf{S}^1)$ is contained in Y when $\alpha > \gamma - \frac{1}{2}$. The suitability of the space Y for our problem is seen from the following: If $f_0 \in Y_0$ then

$$||f(t)||_{Y_0} \le Ce^{-\beta t}||f_0||_{Y_0}\,, \qquad \beta = \gamma - \frac{1}{2}\,. \tag{2.45}$$

It is not hard to obtain this estimate from (2.40) by a direct calculation, see Lemma 3.5 in [29] for a similar calculation done for (2.24).

The analogous exponential decay obtained for the linearized equation (2.22) is crucial for the proof of Theorem 2.1. While such as estimate can be obtained by a direct calculation in the case of the transport equation 2.24, its proof for the full linearized equation (2.22) relies on additional information one has to get about the spectrum and generalized eigenfunctions, which are more difficult to obtain.

An important point in the proof is (a variant of) the fact that the space Y is preserved under the Hilbert transform. In other words,

$$||Hf||_Y \le C||f||_Y\,, \qquad f \in Y\,. \tag{2.46}$$

See [29] for details.

For a use of un-isotropic Sobolev spaces for the analysis of the spectral properties of Morse-Smale flows and their action on differential forms (which is in some sense dual to the flow defined by (2.24)) we refer the reader to [14].

Generalized eigenfunctions of the full linearized equation

Here we outline the main points of the argument which is used in [29] to obtain information about the spectrum of the linearized operator in (2.23). We will denote by L the linearized operator in (2.22), i. e.

$$L\eta = -[U, \eta + v]\,, \qquad v_\theta = H\eta\,, \qquad \int_{\mathbf{S}^1} v\, d\theta = 0\,. \tag{2.47}$$

Lemma 2.1. *On the space of functions $\eta \in H_0^1(\mathbf{S}^1)$ with zero average, the operator L commutes with the Hilbert transform, i. e. $[H, L] = 0$.*

As in the simpler example discussed above, this means that we can restrict the study of L to the case when η has a holomorphic extension to the unit disc D. (The anti-holomorphic part is handled, *mutatis mutandis*, in the same way as the holomorphic one.)

Lemma 2.2. *L is skew-adjoint with respect to the Hermitian product corresponding to the semi-norm $||\eta||_*^2 = \sum_k c_k \eta_k \bar{\eta}_k$, where η_k are the Fourier coefficients of η and $c_k = (k^2 - 1)(|k| + 1)$ for $k \neq 0$, and $c_0 = 0$.*

This means that e^{tL} preserves $|| \cdot ||_*$, which is the energy estimate mentioned in point (a) following (2.23). The norm $|| \cdot ||_*$ is equivalent to the norm in the space $H_0^{\frac{3}{2}}/\mathrm{Ker}L$ (where the subindex 0 again means the zero average $\int_{\mathbf{S}^1} \eta\, d\theta$, and $\mathrm{Ker}L$ is the kernel of L in $H_0^{\frac{3}{2}}$, spanned by the functions $e^{i\theta}$ and $e^{-i\theta}$. This is the (complexified) tangent space at $\eta = -\sin\theta$ to the manifold of equilibria $\mathcal{M}^2$ discussed above.

Let us denote by X the space of holomophic functions in D which belong to $H_0^{\frac{3}{2}}/\mathrm{Ker}L$, considered with the norm $|| \cdot ||_*$.

Theorem 2.2. *The operator L is skew-adjoint on X and has an absolutely continuous spectrum, which coicides with the imaginary axis.*

We refer the reader to [29], but we outline the main idea behind the proof that L has no eigenvalues in X. The main point is that on both the holomorphic and anti-holomorphic section of the action of L the Biot-Savart law is given by a local operator. Let us take for example the holomorphic part, and assume that η, v are holomorphic functions on the unit disc D. Then the equation $v_\theta = H\eta\,, \int_{\mathbf{S}^1} v\, d\theta = 0$ becomes

$$\eta(z) = -zv'(z)\,, \qquad v' = \frac{dv}{dz}\,, \qquad v(0) = 0\,. \tag{2.48}$$

Assume

$$L\eta = \lambda\eta\,. \tag{2.49}$$

Setting $\eta = zf$ and $v = zF$, we obtain, after some calculation

$$F'' + \left[\frac{-1+\lambda}{z-1} + \frac{-1-\lambda}{z+1} + \frac{3}{z}\right] F' + \frac{2\lambda}{z(z^2-1)} F = 0\,. \tag{2.50}$$

The key point is that this is an ODE, rather than a non-local equation. This is the advantage which the splitting into the holomorphic and anti-holomorphic parts gives. Equation (2.50) is known: it is an ODE in the complex plane with four regular singular points (including the one at "infinity" - the north pole of the Riemann sphere). The solutions can be analyzed by standard tools of the theory of complex ODEs, and with the help of this analysis we can control the properties of the generalized eigenfunctions of L. A more precise analysis (carried out in [29]) shows that the potentially dangerous non-local term $[v, \Omega]$ in L will not create any new spectral effects, and the situation will be similar to the model (2.26), with relevant changes replacing the natural space $\dot{\mathcal{H}}^{\frac{1}{2}}$ for (2.26) by $\dot{\mathcal{H}}^{\frac{3}{2}}$ for the full linearized problem.

At some level the relation between the generalized eigenfunctions of L and the simplified model (2.26) can be seen from the substitution (2.27) taking (2.24) to (2.26). Locally, near the points $z = \pm 1$, the generalized eigenfunctions for (2.24) should be of the form Uh, where h is a generalized eigenfunction (2.37). This means that the leading order singular behavior $(1-z)^{is}$ of eigenfunctions $h(z, is)$ in for the model problem (2.26) should correspond to leading order singular behavior $(z-1)(1-z)^{is}$ for the generalized eigenfunctions associated with (2.24). The full

ODE (2.50) exhibits exactly the same behavior at the regular singular points ± 1. It is also worth noting that $(1-z)^{is}$ "just misses" (locally) the space $H^{\frac{1}{2}}$ near $z=1$, and $(z-1)(1-z)^{is}$ "just misses" (locally) the space $H^{\frac{3}{2}}$ near $z=1$.

The above discussion covers some of the main points of the linear theory, although there are still a few more steps in the full analysis, for which we refer the reader to [29].

The proof of Theorem 2.1 is based on the above linear theory together with some product estimates in the relevant function spaces, which are H^s and analogues of the spaces Y_0, Y above. One additional trick which has to be used is time renormalization, as the problem has certain quasi-linear features, see [29].

We conjecture that the behavior proved in Theorem (2.1) extends to generic initial data, not necessarily close to the manifold of the steady states given by $A\sin(\theta-\theta_0)$. The convergence of the solution to such steady states for $t\to\pm\infty$ will not hold for all initial data, as the steady states of the form $A\sin m(\theta-\theta_0)$ with $m=2,3,4,\dots$ have non-trivial stable manifold. We conjecture that any solution will converge to one of such steady states as $t\to\infty$ or $t\to-\infty$. It is quite conceivable that a proof of such result will be based on a completely different approach than the perturbation methods discussed above.

2.1.6. Equation $\omega_t + u\omega_x - au_x\omega = 0$

It is instructive to look at the more general evolution equation

$$\omega_t + u\omega_x - au_x\omega = 0\,, \tag{2.51}$$

where $a\in\mathbf{R}$ is a parameter, see, for example, [41, 18, 49, 45], The case $a=1$ corresponds to the De Gregorio equation, the case $a=-2$ corresponds to equation (2.8) from Example 3 in the introduction to Section 2. It can be shown that for $a<0$ many solutions starting from smooth data blow up in finite time, see [10, 45].

The Constantin-Lax-Majda model (2.9) corresponds, roughly speaking, to $a=+\infty$ (and also to $a=-\infty$), and, of course, exhibits finite time blow-up from smooth data. A recent paper [18] shows that this is also the case for large $a>0$ (in addition to the case $a<0$, mentioned above).

The case $a=0$ corresponds to the "1d model for 2d Euler"

$$\omega_t + u\omega_x = 0\,, \qquad u_x = H\omega\,. \tag{2.52}$$

for which one can show that there is no finite-time blopup from smooth data. A reasonable conjecture might be that for (2.51) blowup from smooth data (with suitable decay when the domain is $\mathbf{R}$) is possible if and only if $a\in\overline{\mathbf{R}}\setminus[0,1]$, where $\overline{\mathbf{R}}=\mathbf{R}\cup\infty$, with $a=\infty$ formally corresponding to the CLM model (2.9).

2.2. The 1d Boussinesq model

In this section we focus on the 1d model from Example 1, considered on the real line $\mathbf{R}$:

$$\begin{aligned} \omega_t + u\omega_x &= \theta_x\,, \\ \theta_t + u\theta_x &= 0\,, \end{aligned} \tag{2.53}$$

complemented by the Biot-Savart law

$$u_x = H\omega\,, \qquad u(x)\to 0 \text{ as } x\to\pm\infty\,. \tag{2.54}$$

Our goal is to outline a proof of a finite time blowup for a class of smooth, compactly supported initial data ω_0. We loosely follow [12], although some of the arguments are somewhat different in our exposition here.

The system (2.53), (2.54) can be considered as a model for the boundary behavior of the standard 2d Boussinesq system in the upper half-plane, which is in turn similar to certain regimes of 3D axi-symmetric flows with swirl. Both the 2d Boussinesq equations and the 3d axi-symmetric are conjectured to blow-up at the boundary in a way which is similar to the blow-up of the 1d model considered here, see [12] for a more detailed comments.

The Biot-Savart law (2.54) (considered here on the real line) can be described by the kernel

$$k(x) = \frac{1}{\pi}\log|x|\,, \tag{2.55}$$

which corresponds to taking $\omega = \delta$ (the Dirac mass) in (2.54). The velocity field u is obtained from ω by

$$u = k * \omega\,, \tag{2.56}$$

where $*$ denotes the convolution $\int_{\mathbf{R}} k(x-y)\omega(y)\,dt$.

We will consider a special class of solutions when ω is odd and compactly supported in $\mathbf{R}\setminus\{0\}$, and θ is even and compactlly supported in $\mathbf{R}$.

Under these symmetry assumptions we can restrict our attention to $x \in (0,\infty)$. Using that ω is odd and writing

$$u(x) = \int_{\mathbf{R}} k(x-y)\omega(y)\,dy = \int_0^\infty [k(x-y) - k(x+y)]\omega(y)\,dy\,, \tag{2.57}$$

we see that, with the above symmetries, we have

$$\frac{u(x)}{x} = -\frac{1}{\pi}\int_0^\infty \frac{y}{x}\log\left|\frac{x+y}{x-y}\right|\;\omega(y)\,\frac{dy}{y}\,. \tag{2.58}$$

This integral is of the form

$$\int_0^\infty M\left(\frac{x}{y}\right)\omega(y)\,\frac{dy}{y}\,, \tag{2.59}$$

which represents convolution in the multiplicative group $\mathbf{R}_+$ taken with respect to the natural invariant measure $\frac{dy}{y}$.

The kernel M is given by

$$M(s) = \frac{1}{s}\log\left|\frac{s+1}{s-1}\right|\;, \qquad s > 0\,. \tag{2.60}$$

We will use the decomposition of M into the symmetric and ati-symmetric part with respect to the inversion $s \to s^{-1}$.

$$M(s) = \frac{1}{2}\left(\frac{1}{s}+s\right)\log\left|\frac{s+1}{s-1}\right| + \frac{1}{2}\left(\frac{1}{s}-s\right)\log\left|\frac{s+1}{s-1}\right| = M_{\rm sym}(s) + M_a(s)\,. \tag{2.61}$$

We have

$$M_{\rm sym}\left(\frac{1}{s}\right) = M_{\rm sym}(s), \qquad M_a\left(\frac{1}{s}\right) = -M_a(s)\,. \tag{2.62}$$

We collect some simple properties of the function M in the following lemma.

Lemma 2.3. *The function M has the following properties:*

(i) *M is increasing on $(0,1)$ and decreasing on $(1,\infty)$.*

(ii) $\lim_{s\to 0_+} M(s) = 2\,,\quad \lim_{s\to 0_+} M'(s) = 0\,.$

(iii) *M_a is continuous and decreasing in $(0,\infty)$, with $\lim_{s\to 0_+} M_a(s) = 1$.*

(iv) $M(s) = \frac{2}{s^2} + O\left(\frac{1}{s^3}\right)\,,\quad s\to\infty.$

The proof is elementary.

In view of the above formulae, it seems natural to work with the variables ξ, $U(\xi)$, $\Omega(\xi)$, $\Theta(\xi)$ defined by

$$x = e^{-\xi}\,,\quad U(\xi) = -\frac{u(x)}{x}\,,\quad \Omega(\xi) = \omega(x)\,,\quad \Theta(\xi) = -\theta(x) + \theta(0)\,. \tag{2.63}$$

In these coordinates, the system (2.53) becomes

$$\begin{array}{rcl} \Omega_t + U\Omega_\xi &=& e^\xi\Theta_\xi\,,\\ \Theta_t + U\Theta_\xi &=& 0\,, \end{array} \tag{2.64}$$

with the Biot-Savart law

$$U = K * \Omega\,, \tag{2.65}$$

where

$$K(\xi) = \frac{1}{\pi}\, M(e^{-\xi}) \tag{2.66}$$

and $*$ denotes the standard convolution. The function K is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ with
$\lim_{\xi\to\infty} K(\xi) = \frac{2}{\pi}$, as one can easily see from Lemma 2.3.

2.2.1. Monotonicity

We set

$$I(\Omega, \xi) = \int_{-\infty}^{\xi} U_\xi(\eta)\Omega(\eta)\, d\eta\,. \tag{2.67}$$

Lemma 2.4. *For any smooth compactly supported $\Omega \geq 0$ we have $I(\Omega, \xi) \geq 0$ for all ξ.*

Proof:
Let us write $\Omega = \Omega_l + \Omega_r$, where

$$\Omega_l = \Omega\, \chi_{(-\infty,\xi]}\,, \qquad \Omega_r = \Omega\, \chi_{(\xi,\infty)}\,. \tag{2.68}$$

We have

$$U = U_l + U_r, \quad U_l = K * \Omega_l, \quad U_r = K * \Omega_r\,, \tag{2.69}$$

and

$$I = I(\Omega, \xi) = \int_{\mathbf{R}} U_\xi(\eta)\Omega_l(\eta)\, d\eta\, = \int_R U_{l\xi}\Omega_l\, d\eta + \int_R U_{r\xi}\Omega_l\, d\eta\,. \tag{2.70}$$

We claim that the in the last expression both integrals are non-negative. For the first integral we have

$$\int_{\mathbf{R}} U_{l\xi}\Omega_l\, d\eta = \int_R \int_{\mathbf{R}} K'(\eta-\zeta)\Omega_l(\zeta)\Omega_l(\eta)\, d\eta\, d\zeta = \int_R \int_{\mathbf{R}} K_a'(\eta-\zeta)\Omega_l(\zeta)\Omega_l(\eta)\, d\eta\, d\zeta \geq 0\,, \tag{2.71}$$

as K_a is increasing. The second integral is equal to

$$\int_{\mathbf{R}} \int_{\mathbf{R}} K'(\eta-\zeta)\Omega_r(\zeta)\Omega_l(\eta)\, d\,\zeta\, d\eta, \tag{2.72}$$

and we note that the integration can be restricted to the domain $\{\eta < \zeta\}$, as the integrand vanishes elsewhere. As $K'(\xi) > 0$ for $\xi < 0$, the result follows.

Remark: The positivity of $\int_{\mathbf{R}} U_\xi \Omega\, d\xi$ can be also seen in the original coordinates, see also [24]. It amounts to the positivity of $\int_0^\infty -\frac{u\omega_x}{x}\, dx$. Using Fourier transform, one can check that

$$\int_{\mathbf{R}} -\frac{u\omega_x}{x} \sim \left|\left|\frac{u(x)}{x}\right|\right|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R})}\,. \tag{2.73}$$

The equation (2.53) then gives

$$\frac{\partial}{\partial t}\left(\frac{u(x,t)}{x}\right)_{x=0} \sim \left|\left|\frac{u(x)}{x}\right|\right|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R})}\,. \tag{2.74}$$

Note that the $\dot{H}^{\frac{1}{2}}$ norm just narrowly fails to control the sup-norm. If it controled the sup-norm, (2.74) would immediately imply a finite time blow-up for the equation (2.53). In the ξ coordinate we have the identity

$$\frac{d}{dt}\int_{\mathbf{R}} \Omega(x,t)\, d\xi = \int_{\mathbf{R}}\int_{\mathbf{R}} K_a'(\xi-\eta)\Omega(\xi,t)\Omega(\eta,t)\, d\eta d\xi\,. \tag{2.75}$$

This equation shows that unless the support of Ω “spreads”, the solution will blow-up.

2.2.2. Finite time blowup

Let us consider an initial condition $\theta(x,0) = \theta_0(x)$ which is compactly supported in $(-1,1)$ even, non-decreasing on $[0,\infty]$, with $\theta_0(0) = -1$. Then $\rho(\xi) = \Theta_\xi(\xi)$ can be considered as a density of total mass 1 supported in $(0,\infty)$. It satisfies the continuity equation (obtained by differentiating the second equation in (2.64))

$$\rho_t + (U\rho)_\xi = 0\,. \tag{2.76}$$

Let us now look at the quantity

$$\bar\xi = \bar\xi(t) = \int \xi\rho\, d\xi\,, \tag{2.77}$$

which can be thought of as an average coordinate ξ with respect to the probability density ρ. From the continuity equation we have

$$\frac{d}{dt}\bar\xi = \int U\rho\, d\xi\,. \tag{2.78}$$

For simplicity we will assume that $\Omega(\xi,0) \equiv 0$. Then, letting $\gamma = \frac{2}{\pi}$, we have

$$U(\xi,t) \geq \gamma \int_0^\xi \Omega(\eta,t)\, d\eta = \gamma \int_0^t ds \frac{d}{ds}\int_0^{\zeta(s)} d\eta\; \Omega(\eta,s)\,, \tag{2.79}$$

where $\zeta(s)$ "moves with the flow", i. e.

$$\frac{d}{ds}\zeta(s) = U(\zeta(s),s)\,, \tag{2.80}$$

and $\zeta(t) = \xi$. If we wish to indicate explicitly the dependence of ζ on ξ we will write $\zeta(s,\xi)$ rather than $\zeta(s)$. From the first equation of (2.64) and Lemma 2.4 we have

$$\frac{d}{ds}\int_0^{\zeta(s)} d\eta\; \Omega(\eta,s) = \int_0^{\zeta(s)} \left[e^\eta \rho(\eta,s) + U_\xi \Omega\right]\, d\eta \;\geq\; \int_0^{\zeta(s)} e^\eta\rho(\eta,s)\, d\eta\,. \tag{2.81}$$

To estimate (2.78), we need to integrate the last expression with respect to the measure $\rho(\xi,t)\,d\xi$. Let us define $\tilde\zeta(s,\eta)$ as the solution ξ of the equation $\zeta(s,\xi) = \eta$. We have

$$\int_0^\infty d\xi\rho(\xi,t)\int_0^{\zeta(s,\xi)} d\eta\, e^\eta\rho(\eta,s) = \int_0^\infty d\eta\, e^\eta\rho(\eta,s)\int_{\tilde\zeta(s,\eta)}^\infty d\xi\,\rho(\xi,t)\,. \tag{2.82}$$

Recalling that $\Theta(\xi,t)$ is assumed to be increasing, approaching 0 as $\xi\to\infty$, we have

$$\int_{\tilde\zeta(s,\eta)}^\infty d\xi\,\rho(\xi,t) = -\Theta(\tilde\zeta(s,\eta),t) = -\Theta(\eta,s)\,, \tag{2.83}$$

where the last equality follows from the fact that the function Θ "moves with the flow" and that $\tilde\zeta(s,\eta) = \xi$ is by definition the same as $\zeta(s,\xi) = \eta$ and it means that the "fluid particle" with coordinate ξ at time t had coordinate η at time s. Using (2.83) together with $\rho(\eta,s) = \Theta_\eta(\eta,s)$, wee see that the double integrals in (2.82) are estimated from above by

$$\int_0^\infty d\eta\; e^\eta\,\Theta_\eta(-\Theta)\, d\eta \;= \frac{1}{2}\int_0^\infty e^\eta\Theta^2\, d\eta + \frac{1}{2}\,. \tag{2.84}$$

where the functions are taken at time s. We have

$$\bar\xi(s) = \int_0^\infty \xi\rho\, d\xi = \int_0^\infty -\Theta\, d\xi = \int_0^\infty -\Theta e^{\frac{\xi}{2}} e^{\frac{-\xi}{2}}\, d\xi \leq \left(\int_0^\infty e^\xi\Theta^2\, d\xi\right)^{\frac{1}{2}}\,. \tag{2.85}$$

We see that the integral on the right-hand side of (2.84) is bounded from below by $\bar\xi^{\,2}(s)$. We conclude that

$$\frac{d}{dt}\bar\xi(t) \geq \frac{\gamma}{2}\int_0^t \left[\bar\xi^{\,2}(s) + 1\right]\, ds\,, \tag{2.86}$$

which implies that the solution must blow up in finite time. We note that in the estimate (2.85) we have not used that $|\Theta| \leq 1$. If we use this additional information, we can get a "faster" blow-up rate.[2]

We see that we have proved

[2] The problem of finding an optimal estimate replacing the elementary estimate (2.85) seems to be interesting.

Theorem 2.3. *Under the assumptions on the initial date introduced above, the solutions of the 1d Boussinesq model (2.53), (2.54) develop a singularity in finite time.*

2.3. Other 1d models

2.3.1. Scalar Burgers equations with fractional dissipartion

One of the oldest models in 1d is the Burgers equation

$$u_t + uu_x = \nu u_{xx}\,, \tag{2.87}$$

where $\nu \geq 0$ represents viscosity. It has been studied in great detail, with well-known classical results due to Hopf [28], Oleinik [42] and many others. More recently, the model with fractional (and non-local) dissipation

$$u_t + uu_x = -(-\partial_x^2)^\alpha u \tag{2.88}$$

has been studied, see, for example [32]. (We could also put a viscosity coefficient ν in front of the practional viscosity term, but, just as in the classical case $\alpha = 1$, when $\nu > 0$ we can always change variables to get the situation with $\nu = 1$.) A natural energy estimate for this equation is

$$u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^\alpha \tag{2.89}$$

The scaling symmetry of the equation (when considered for $x \in (-\infty, \infty)$ is

$$u(x,t) \to \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t)\,. \tag{2.90}$$

It is easy to check that the norm in the energy space (2.89) is invariant under the scaling iff $\alpha = \frac{3}{4}$. Based on this one expects that the critical exponent α for both local and global well-posedness of the equation for the Cauchy problem with the initial data $u_0 \in L^2$ is $\alpha = \frac{3}{4}$. In this case the situation is similar to the 2d Navier-Stokes equations. This may perhaps be optimal if we wish to deal with $u_0 \in L^2$, especially as far as uniqueness of the solution in the energy class is concerned.

When the initial datum is more regular (say, smooth, compactly supported), an interesting question is for which α singularities can develop. If we had no other estimate than the energy estimate, one might still expect $\alpha = \frac{3}{4}$ to be critical, but the inviscid part $u_t + uu_x = 0$ satisfies many other estimates, beyond the energy estimate. Based on this one can prove it is in fact sufficient to have $\alpha \geq \frac{1}{2}$ to prevent singularity formation from smooth compactly supported initial datum u_0, see [32]. On the other hand, for $\alpha < \frac{1}{2}$ finite time singularities can develop from smooth compactly supported data, see [32].

What happens if we do not have all the special properties of the inviscid scalar equation $u_t + uu_x = 0$? One interesting class of equations where this question seems to be interesting is the vector-valued analogues of (2.88), which we will not describe.

2.3.2. Vector Burgers equations with fractional dissipation

We consider vector-values function $u(x,t) = (u_1(x,t), \ldots, u_r(x,t))$, where $(x,t) \in \mathbf{R} \times [0,\infty)$. A natural generalization of the scalar equation (2.88) seems to be

$$u_t + b(u, u_x) = -(-\partial_x^2)^\alpha u\,, \tag{2.91}$$

where $b(u, u_x)$ bi-linear in the sense that for some constants a_{klm}

$$[b(u,u_x)]_k = a_{klm} u_l u_{xm}\,, \tag{2.92}$$

with summation over repeated indices is understood. To we will also require that $b(u, u_x)$ satisfies

$$\int_{\mathbf{R}} b(v, v_x) v\, dx = 0 \quad \text{for each smooth } v\colon \mathbf{R} \to \mathbf{R}^r \text{ with compact support}\,. \tag{2.93}$$

Letting $I(v) = \int_{\mathbf{R}} b(v, v_x) v\, dx$ and using that the variational derivatives $\delta I(v)/\delta v^i$ have to vanish for each v when (2.93) is satisfied, it is not not hard to see that (2.93) is equivalent to

$$a_{klm} + a_{lkm} - a_{lmk} - a_{mlk} = 0\,, \qquad k, l, m = 1, \ldots, r\,. \tag{2.94}$$

Let us call tensors a_{klm} satisfying $a_{klm} = a_{lkm}$ partially symmetric and the tensor satisfying $a_{klm} = -a_{lkm}$ to be partially antisymmetric. Each tensor a_{klm} can be uniquely decomposed into a sum of a partial symmetric and a partially anti-symmetric tensor. The forms b corresponding to the partially anti-symmetric tensors are easy to understand:

$$b(u,u_x) = A(u_x)u\,, \tag{2.95}$$

where $A(u_x)$ is an anti-symmetric $r\times r$ matrix depending linearly on the vector u_x.

To understand the forms b corresponding to the partially symmetric tensors, we use the following simple lemma.

Lemma 2.5. *A partially symmetric tensor a_{klm} satisfying (2.94) is symmetric.*

Here we use the usual terminology that a tensor a_{klm} is symmetric if the value a_{klm} does not change if we permute the indices k,l,m. The symmetric tensor a_{klm} are of course in one-to-one correspondence with the cubic homogeneous polynomials

$$P(u) = a_{klm}u_k u_l u_m\,, \qquad \text{(summation over repeated indices understood).} \tag{2.96}$$

Hence the forms $b(u,u_x)$ corresponding to partially symmetric tensors a_{klm} are of the form

$$b(u,u_x)_k = \frac{\partial}{\partial x}\frac{\partial P}{\partial u_k}(u)\,, \tag{2.97}$$

where $P(u)$ is a cubic homogeneous polynomial in u.

In the symmetric case the inviscid part of (2.91) can be written as a system of conservation laws

$$u_t + Q_x = 0\,, \qquad Q_k(u) = \frac{\partial P}{\partial u_k}(u)\,. \tag{2.98}$$

We can call the above class of equations *vector-valued Burgers equations with fractional dissipation.*

It is easy to see from the point of view of well-posedness and regularity, the exponent $\alpha = \frac{3}{4}$ is a critical exponent for this class of equations. In this case the situation is similar to the 2d Navier-Stokes, and the regularity problem, as well as the well-posedness problem for $u_0 \in L^2$ are "critical", with perturbation techniques being sufficient to handle the problems.

In the inviscid case the solution will typically develop discontinuities (shocks), and one expects that this will still be the case for $\alpha < \frac{1}{2}$, when the dissupation is too weak to prevent this.

The interesting case is $\frac{1}{2} \le \alpha < \frac{3}{4}$. Can one have singularities then? I do not know any examples, and - as far as I can tell - the problem is open.

3. Models in higher dimensions

It is easy to generalize the class of equations considered in subsection 2.3.2 to higher dimension and incompressible flows. Let us mention one simple ad-hoc modification of the Navier-Stokes equation without any particular physical meaning. In $\mathbf{R}^3 \times (0,\infty)$ let us consider

$$\begin{array}{rcl} u_{1t} + u_2 u_{1,2} + u_3 u_{1,3} + p_{,1} - \Delta u_1 & = & 0\,, \\ u_{2t} + u_1 u_{2,1} + u_3 u_{2,3} + p_{,2} - \Delta u_2 & = & 0\,, \\ u_{3t} + u_1 u_{3,1} + u_2 u_{3,2} + p_{,3} - \Delta u_3 & = & 0\,, \\ \operatorname{div} u & = & 0\,. \end{array} \tag{3.1}$$

One can expect that a large part of existing Navier-Stokes regularity theory will work also for this equation, in spite of its artificiality. The regularity problem for this equation in $\mathbf{R}^3$ seems to be open.

3.1. Local models

Here we will mention one local model, which a special case of the equations of the form

$$u_t + b(u,\nabla u) - \nabla u = 0\,, \qquad (x,t)\in \mathbf{R}^n \times [0,\infty) \tag{3.2}$$

with bi-linear b and energy conservation. The classification of bilinear forms $b(u,\nabla u)$ with energy conservation in higher dimension is a simple generalization of the 1d calculations done in subsection 2.3.2. Writing

$$[b(u,\nabla u)]_k = a^j_{klm} u_l u_{m,j} \qquad \text{(the summation convention understood)}$$

it is easy to see that energy conservation (in the sense of subsection 2.3.2) is equivalent to the condition

$$a^j_{klm} + a^j_{lkm} - a^j_{lmk} - a^j_{mlk} = 0\,, \qquad j = 1, \dots, n\,, \quad k, l, m = 1, \dots, r\,. \tag{3.3}$$

One reason often given for the difficulty of the Navier-Stokes regularity problem is the non-locality of the incompressible Navier-Stokes equations. Equations of the form (3.2) are local, but there regularity in space dimension three and four seems to be open. As we will see, in dimension five and higher, finite time blowup from smooth compactly supported initial conditions can happen.

We will restrict our attention to an example of (2.3.2) with $r = n$, in divergence form (i. e. , of the form $\operatorname{div} \tilde{b}(u, u)$), and with the maximal symmetry. The example is

$$u_t + \operatorname{div}\,(u \otimes u + \frac{1}{2}|u|^2 I) = \Delta u\,, \qquad (x, t) \in \mathbf{R}^n \times (0, \infty)\,, \tag{3.4}$$

where $u = (u_1, \dots, u_n)$ and I is the identity matrix. In coordinates,

$$u_{it} + \partial_j (u_i u_j + \frac{1}{2}|u|^2 \delta_{ij}) = \Delta u_i \tag{3.5}$$

This equation has appeared independently in several works (see, e. g. , [25, 43]), and the term *template matching equation* has been used for it in [25]. The inviscid part of the equation is nothing but the higher-dimensional analogue of Example 3 from the introduction. It arises from geodesics in the diffeomorphism group $G = \mathrm{Diff}(\mathbf{R}^n)$ equipped with the right-invariant metric, given the natural L^2-scalar product on tangent space to G at identity. For $n = 1$ this gives the form 2.7 of the Burgers equation.

Much of regularity theory for Navier-Stokes equation can be applied to (3.4). For example, one can show that for $n = 2$ the Cauchy problem is globally well-posed for the initial data $u_o \in L^2(\mathbf{R}^2)$. For $n = 3$ one can construct weak solution, prove their partial regularity, etc. For $n = 3, 4$ it is not known whether the equation can develop singularities from smooth initial data with fact decay. It is expected that for $n \geq 5$ or for $n = 3$ and slowly decaying initial data singularities can develop, see for example [43].

Insights into (3.4) can be obtained from looking at solutions of the form

$$u(x) = -x\, v(r)\,, \qquad r = |x|\,. \tag{3.6}$$

This gives

$$v_t = v_{rr} + \frac{n+1}{r} v_r + 3rvv_r + (n+2)v^2\,. \tag{3.7}$$

A version of this equation also appears in connection with a chemotaxis model, see [23]. With a suitable rescaling, the equation from the chemotaxis model is

$$q_t = q_{rr} + \frac{n+1}{r} q_r + 3rqq_r + 3nq^2\,. \tag{3.8}$$

For $n = 1$ the two equations coincide. In dimensions $n = 2, 3, 4$ the behavior of the two equations is different: for compactly supported smooth data, equation (3.7) admits global smooth solutions, whereas (3.8) may blow-up in finite time (if the initial datum is sufficiently large). For $n \geq 5$ both equation can produce singularities from smooth, compactly supported data.

Here we will outline the argument for regularity in the case of smooth, rapidly decaying data for $n = 3, 4$.

3.1.1. Regularity and blowup for solutions of equation 3.7

Equation (3.7) is a scalar equation to which standard comparison principles can be applied. It turns out that in dimensions $n \leq 4$ steady states provide enough information to rule out singularity formation from smooth, compactly supported data. The steady states have been studied in [43], and we briefly recall the main points here. We note that the equation

$$v_{rr} + \frac{n+1}{r} v_r + 3rvv_r + (n+2)v^2 = 0 \tag{3.9}$$

has a scaling symmetry

$$v(r) \to \lambda^2 v(\lambda r)\,. \tag{3.10}$$

Therefore it is natural to change variables as follows

$$v = \frac{w}{r^2}\,, \qquad r = e^s\,. \tag{3.11}$$

This gives

$$w'' + (n-4)w' + 2(2-n)w + 3ww' + (n-4)w^2 = 0\,, \qquad w' = \frac{dw}{ds}\,. \tag{3.12}$$

There is a restriction on the rate of decay of $u(x)$ from the energy conservation, which is related to Liouville theorem for the steady solutions of the (incompressible) Navier-Stokes, see [21]. Assuming $\partial_j(u_iu_j + \frac{1}{2}|u|^2\delta_{ij}) = \Delta u_i$, taking a scalar product with $u_i\varphi$ with a compactly supported smooth φ, integrating by parts, and optimizing over φ, it is not hard to see that, for $n = 3$, any solution $u \in L^{\frac{9}{2}}(\mathbf{R}^3)$ has to vanish. If we think in terms of the fastest possible decay of $|u(x)|$ for non-zero solutions, we wee that, for the scale $|x|^{-\varkappa}$ (for large $|x|$) the decay of $|u(x)|$ cannot be faster than $O(|x|^{-\frac{2}{3}})$. (A similar argument in dimension $n = 4$ shows that in that case any solution $u \in L^4(\mathbf{R}^4)$ has to vanish and the decay of a non-trivial solution cannot be faster than $O(|x|^{-1})$.) Focusing on the case $1 < n < 4$, we see $w \to \infty$ for $s \to \infty$, and hence we expect that the leading order term behavior of the relevant solutions of (3.12) for large s will be determined by the non-linear term: $w(3w' + (n-4)w) \sim 0$. This means $w \sim e^{\frac{4-n}{3}}s$. Going back to the variable u, we obtain

$$|u| \sim r^{-\frac{n-1}{3}}\,, \quad r \to \infty\,, \qquad 1 < n < 4\,. \tag{3.13}$$

In the case $n = 4$ the behavior of the solution at ∞ is $\sim \frac{4}{3}r^{-1}\log r$, whereas for $n > 4$ is is simply $\sim r^{-1}$.

All this can be proved rigorously in several ways. For example, one can use the change of variables

$$w = \tan\phi\,, \quad w' = \frac{\tan\psi}{\cos^2\phi} \tag{3.14}$$

which maps the phase-space of (3.12) with coordinates (w, w') into a torus with coordinates (ϕ, ψ), in which the image of the vector field in w, w' generating the flow is given by a smooth field on the torus multiplied the function $[\cos\phi\cos\psi]^{-1}$ (which adjusts time along trajectories). See [43] for details. In our context here, the most important conclusion from our analysis of the steady states is the following:

Theorem 3.1. *Let $V(r)$ be the steady-state solution of (3.7) satisfying $V(0) = 1$ and $V'(0) = 0$, defined at first locally in $[0, r_0)$ for some $r_0 > 0$, by ODE methods. Then V is global, and has the following decay as $r \to \infty$:*

(i) $V(r) \sim r^{-\frac{n+2}{3}}$ *for* $1 < n < 4\,,$

(ii) $V(r) \sim r^{-2}(\frac{4}{3}\log r + C)$ *for* $n = 4\,,$

(iii) $V(r) \sim r^{-2}$ *for* $n > 4\,.$

For the proof we refer the reader to [43] .

Corollary 3.1. *Assume that $v_0\colon [0,\infty) \to \mathbf{R}$ is a bounded function.*

(i) *If $1 < n < 4$ and $\limsup_{r\to\infty} r^{\frac{n+2}{3}} v_0(r) < +\infty$, then the Cauchy problem for (3.7) with the initial datum $v(r, 0) = v_0(r)$ has a unique global bounded solution.*

(ii) *If $n = 4$ and $\limsup_{r\to\infty} \frac{r^2}{\frac{4}{3}\log r} v_0(r) < 1$, then the Cauchy problem for (3.7) with the initial datum $v(r, 0) = v_0(r)$ has a unique global bounded solution.*

Proof: Under the assumptions of the Corollary, it is easy to see that one can find $C > 0$ and $\lambda > 0$ such that

$$-C \le v_0(r) \le \lambda^2 V(\lambda r)\,. \tag{3.15}$$

As $\lambda^2 V(\lambda r)$ is a steady-state of (3.7) and the solution of (3.7) with $v(r, 0) = -C$ (given by the ODE $\dot{v} = (n+2)v^2, v(0) = -C$) is bounded, the result easily follows from standard parabolic theory and the comparison principle.

Remarks.

1. The solutions of (3.7) with $v|_{t=0} = c > 0$ are given by the ODE $\dot{v} = (n+2)v^2$ with the initial datum $v(0) = c$, and obviously blow up in finite time. We conjecture that blow-up solutions with initial datum $rv_0 \in L^p$ with $p > \frac{3n}{n-1}$ can be constructed, which would mean that equation (3.4) is not globally well-posed in L^p for $p > \frac{3n}{n-1}$.

2. In dimensions $n > 4$, singularity formation from suitable smooth, compactly supported data is possible. Some strong evidence in that direction (which can be made rigorous) is in [43].

3. There appears to be some similarity in the regularity/blowup mechanisms of the radial solution of (3.4) (described by the Ansatz (3.6)) and the dyadic models studied in [4]. For example, the dimensions in which one expects singularity formation from localized smooth data come out similar. A dyadic model which exhibits blowup in dimension three is constructed in [48]. It would be interesting to know if one can have such an example in dimension three for equation 3.2.

3.2. Complexified equations

We recall the point of view of V. I. Arnold ([1, 2], in which the incompressible Euler equation is derived from the structure of the Lie algebra of the div-free vector field (and the L^2-scalar product) as follows.

Let L be a real Lie algebra equipped with a scalar product. At this point the two structures can be thought of being quite independent of each other (perhaps modulo some continuity requirements), there are no "compatibility conditions". We will use the following notation

$$\begin{array}{lll} [a,b] & \dots\dots\dots\dots\dots & \text{Lie bracket of } a \text{ and } b\,, \\ (a,b) & \dots\dots\dots\dots\dots & \text{scalar product of } a \text{ and } b\,. \end{array} \tag{3.16}$$

The two structures can be used to define the *Arnold form* $(c,a) \to B(c,a)$ on $L \times L$ by

$$([a,b],c) = (B(c,a),b)\,. \tag{3.17}$$

The *Euler-Arnold equation* for a trajectory $a = a(t)$ in the Lie algebra L is then given by

$$\dot{a} = B(a,a)\,. \tag{3.18}$$

Note that the equation automatically conserves the "energy" $\frac{1}{2}(a,a)$:

$$\frac{d}{dt}\frac{1}{2}(a,a) = (\dot{a},a) = (B(a,a),a) = 0\,, \tag{3.19}$$

because $[a,a] = 0$ for each $a \in L$. In fact, we see that

$$(B(c,a),a) = 0\,, \qquad a,c \in L\,. \tag{3.20}$$

We can now complexify the Lie algebra L, let us denote the complexification by $L_{\mathbf{C}}$. The Lie bracket of $a, b \in L_{\mathbf{C}}$ will still be denoted by $[a,b]$. Writing the elements of $L_{\mathbf{C}}$ as $a = a_1 + ia_2$, with $a_1, a_2 \in L$, we can define a Hermitian product in $L_{\mathbf{C}}$, still denoted (a,b), by

$$(a,b) = (a_1 + ia_2, b_1 - ib_2) = (a_1,b_1) + (a_2,b_2) + i[(a_2,b_1) - (a_1,b_2)]\,. \tag{3.21}$$

Its real part

$$\langle a,b\rangle = \operatorname{Re}(a,b) = (a_1,b_1) + (a_2,b_2) \tag{3.22}$$

is then a natural real scalar product on $L_{\mathbf{C}}$.

Therefore the Euler-Arnold equation on L generates also an Euler-Arnold equation on $L_{\mathbf{C}}$. In the coordinates $a = a_1 + ia_2$ the equation is given by

$$\begin{array}{lll} \dot{a}_1 & = & B(a_1,a_1) + B(a_2,a_2)\,, \\ \dot{a}_2 & = & -B(a_1,a_2) + B(a_2,a_1)\,. \end{array} \tag{3.23}$$

One can verify directly that the "energy"

$$\frac{1}{2}\langle a,a\rangle = \frac{1}{2}(a_1,a_1) + \frac{1}{2}(a_2,a_2) \tag{3.24}$$

is conserved by (3.23).

3.2.1. Extension of the Navier-Stokes equation to complex-valued vector fields

Applying the above setting to the Lie algebra of the div-free vector fields in $\mathbf{R}^n$, we obtain what might be called the "complex Euler equation" for complex-valued div-free fields. The equation reads as follows:

$$u_{kt} + \bar{u}_l u_{k,l} + u_l \bar{u}_{l,k} + \pi_{,k} = 0\,, \qquad \operatorname{div} u = 0\,. \tag{3.25}$$

Here we use the usual notation $\bar{u}$ for complex conjugation, i. e. $\overline{(u_1 + iu_2)} = u_1 - iu_2$.

The corresponding Navier-Stokes equation then is

$$u_{kt} + \bar{u}_l u_{k,l} + u_l \bar{u}_{l,k} + \pi_{,k} - \Delta u_k = 0\,, \qquad \operatorname{div} u_k = 0\,, \tag{3.26}$$

where the "pressure" π may not be the physical pressure even in the case when u is real.

One can also consider the usual Navier-Stokes equations and allow the velocity and pressure fields to be complex, see [8]. However, in contract with (3.26), this direct complexification will not have the energy inequality (or, in the inviscid case, energy conservation). This can be seen already in 1d models related to the group $\mathrm{Diff}(\mathbf{S}^1)$ with the L^2-induced right-invariant metric. In the real-valued case we get a variant of the viscous Burgers equation, the inviscid case of which already appeared in (2.7),

$$u_t + 3uu_x = u_{xx}\,. \tag{3.27}$$

This can be considered for complex-valued function u, and in that case it is not hard to show that singularities can develop from compactly supported smooth initial condition, see [44].

On the other hand, the geometric complexification above leads to

$$u_t + \bar{u}u_x + 2u\bar{u}_x = u_{xx}\,. \tag{3.28}$$

There is no problem to prove full regularity for this equation, as it has an energy estimate. One can also look at a modification of this model in the spirit of subsection 2.3 and consider

$$u_t + \bar{u}u_x + 2u\bar{u}_x = -(-\partial_x^2)^\alpha u\,. \tag{3.29}$$

One can again easily show regularity for $\alpha \geq \frac{3}{4}$, and one expects blow-up for $\alpha < \frac{1}{2}$. The situation for $\alpha \in [\frac{1}{2}, \frac{3}{4})$ is less clear. The question is again closely related to L^∞ estimates for the inviscid part of the system, similarly to the situation in subsection 2.3

Equation (3.26) seems to be the natural extension of the Navier-Stokes equation to complex-valued vector fields. I am not sure if the equation has any good physical interpretation, but mathematically it looks interesting. In dimension $n = 2$ the standard Navier-Stokes theory will presumably work without much problems also for this equation and one should be able to proof the standard 2d results. This is no longer the case for the 2d complex Euler equation, where the standard proofs of existence depend on more detailed properties of vorticity, which may not be shared by the complex equation.

Letting $\omega = \operatorname{curl} u$ as usual, one can write

$$\bar{u}_l u_{k,l} = \bar{u}_l(u_{k,l} - u_{l,k}) + \bar{u}_l u_{l,k} = (\omega \wedge \bar{u})_k + \bar{u}_l u_{l,k} \tag{3.30}$$

and we see that an equivalent form of (3.25) is

$$u_t + \omega \wedge \bar{u} + \nabla(|u|^2 + \pi) = 0\,. \tag{3.31}$$

Taking curl, we obtain

$$\omega_t + [\bar{u}, \omega] = 0\,, \tag{3.32}$$

where the Lie bracket $[\cdot,\cdot]$ is defined in the usual way:

$$[a,b]_k = a_l b_{k,l} - b_l a_{k,l}\,. \tag{3.33}$$

In dimension $n = 2$ the vorticity equation is

$$\omega_t + \bar{u}\nabla\omega = 0\,, \tag{3.34}$$

with u obtained from ω by the usual Biot-Savart law (extended to complex-valued fields by linearity over $\mathbf{C}$). As already discussed, regularity of solutions of this equation (for $n = 2$) is not clear, except in the case when ω is real, when classical results apply.

We note that the procedure of complexification can be repeated: starting with L we construct $L_{\mathbf{C}}$, consider it as the real algebra, complexify again, etc. This way we can obtain from (3.25) a larger set of equations, which we will not write down here.

4. Leray-Hopf solutions and uniqueness

In this section we will discuss the classical incompressible Navier-Stokes equations in the whole space $\mathbf{R}^3$. We consider the classical Cauchy problem

$$\left.\begin{array}{rcl} u_t + u\nabla u + \nabla p - \Delta u & = & 0 \\ \operatorname{div} u & = & 0 \end{array}\right\} \qquad \text{in } \mathbf{R}^3 \times (0,\infty)\,, \tag{4.1}$$

$$u|_{t=0} = u_0 \qquad \text{in } \mathbf{R}^3\,, \tag{4.2}$$

We recall the energy inequality

$$\int_{\mathbf{R}^3} \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_{\mathbf{R}^3} |\nabla u(x,t)|^2\,dx\,dt = \int_{\mathbf{R}^3} \frac{1}{2}|u(x,t_1)|^2\,dx\,, \tag{4.3}$$

for $0 \le t_1 \le t_2$, and the scaling symmetry

$$\begin{array}{ccccc} u(x,t) & & u_\lambda(x,t) & & \lambda u(\lambda x, \lambda^2 t)\,, \\ p(x,t) & \longrightarrow & p_\lambda(x,t) & = & \lambda^2 p(\lambda x, \lambda^2 t)\,, \\ u_0(x) & & u_{0\lambda}(x) & & \lambda u_0(\lambda x)\,, \end{array} \tag{4.4}$$

where λ is any number in $(0,\infty)$. These transformations are easily seen to preserve the solutions of the equations. The fields u, p, u_0 and other quantities which are invariant under this symmetry will be called *scale-invariant*. We have

$$\int_{\mathbf{R}^3} \frac{1}{2}|u_{0\lambda}(x)|^2\,dx = \lambda^{-1}\int_{\mathbf{R}^3} \frac{1}{2}|u_0|^2\,dx\,. \tag{4.5}$$

This means that the value of the kinetic energy is by itself not an important quantity for the regularity theory of the equation - it can be scaled to any value by a symmetry. (Note that this is completely different in dimension $n = 2$, where the energy is invariant under the scaling symmetry.)

In dimension $n = 3$, the scale-invariant L^p-norm of u_0 is $||u_0||_{L^3}$.

4.1. Weak solutions and the problem of their local-in time uniqueness

J. Leray [36] built a theory of weak solution of the Cauchy problem (4.1), (4.2) based on the energy inequality (4.3). The natural function space for these solutions is $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$, and the natural class of initial data is $u_0 \in L^2$. The main problem in the theory of weak solutions is that of uniqueness. From the Newtonian point of view, this is a fundamental issue. In Newtonian mechanics the evolution of the system should be uniquely determined by its current state. The full information about the state of the fluid is given by the positions and velocities of the “fluid particles”, which in our continuum model are represented by points, and changes in their positions are represented by volume-preserving diffeomorphisms. In an incompressible fluid of constant density the fluid particles are indistinguishable, and hence the state of the system is determined by u_0, modulo relabeling of the particles.[3] Therefore the uniqueness of the solutions of the Cauchy problem seems to be one of the most important questions about the model. The issue was already investigated in Leray’s paper [36], where the first *weak-strong uniqueness theorem* was proved. The theorem says that for initial conditions u_0 for which one can construct solutions which are sufficiently regular, the weak solution will conicide with the regular solution. Leray’s uniqueness result has since been generalized by many authors, see for example [46, 22, 33]. The technical details of the statements are not important for our purposes here, except for the following:

(i) $u_0 \in L^3$ is sufficient for ensuring the existence of a sufficiently regular solution u on some time interval $(0,T)$, with $T = T(u_0)$, and for $u_0 \in L^3 \cap L^2$ the weak solution coincides with the regular solution.

[3]The relabeling of the particles by volume-preserving diffeomorphism can be thought of as a symmetry of the system. In spite of being seemingly trivial, in the inviscid case this symmetry generates via Noether’s theorem the Helholtz law that the vorticity moves with the fluid, one of the most important insight into the fluid flows.

(ii) $u_0 \in L^{3,\infty}$ (the weak L^3 space) with a sufficiently small norm $||u_0||_{L^{3,\infty}}$ is sufficient for global existence of a sufficiently regular solution. In addition, when $u_0 \in L^2 \cap L^{3,\infty}$ and the norm $||u||_{L^{3,\infty}}$ is sufficiently small, any weak solution coincides with the regular solution. (The space $L^{3,\infty}$ can be replaced by various other spaces in this statement.)

(iii) All existing proofs of (ii) do need a smallness condition.

The weak solutions discussed above are all in $L^\infty_t L^2_x$, and hence their total energy is finite. An important generalization to the case of locally finite energy has been established (under some natural assumptions) by Lemarié-Rieusset, see [33, 34], where one can also find an excellent presentation of many other topics relevant to the themes discussed here.

One can also consider weak solutions in less regular functions spaces. The minimal regularity needed for formulating the equations in the sense of distributions is $L^2_t L^2_x$.

Very recently, non-uniqueness for weak solutions in the class $C^0_t L^2_c \cap L^2_t H^s_x$ for small s has been established by Buckmaster and Vicol [7] via an adaptation of convex integration techniques.

4.2. Scale-invariant solutions

Recently a possible non-uniqueness scenario for the weak solutions of Leray appeared in connection with the Cauchy problem for the scale-invariant solutions. Based on that scenario it appears that the uniqueness results in (i)-(ii) above are close to optimal. In particular, it appears that in the situation (ii) above the smallness of the norm is genuinely needed, and there are examples $u_0 \in L^2 \cap L^{3,\infty}$ for which the uniqueness of Leray's week solution fails.

Perturbation theory arguments need some small quantity. For example, even when we have a function $u_0 \in L^3$ which is not small, there is a "hidden" smallness quantity around: the integrals

$$\int_{B_r} |u|^3 \, dx \tag{4.6}$$

(where B_r represents balls of radius r) are small when r is small. By rescaling u to $ru_0(rx)$ we will have the same statement with $r = 1$ for the re-scaled function. This is essentially what makes it possible to prove the short-time existence for large L^3 data.

By contrast, the space $(\mathrm{BMO})^{-1}$, the Morrey space $M^{2,1}$ with the norm given by

$$\sup_{x,r} \frac{1}{r} \int_{B_{x,r}} |u_0(y)|^2 \, dy \,, \tag{4.7}$$

or the Lorentz space $L^{3,\infty}$ (the weak L^3-space), are examples of spaces where functions do not behave in this way. The spaces $(\mathrm{BMO})^{-1}$, $M^{1,2}$ and $L^{3,\infty}$ contain $(-1)-$homogeneous functions u_0 smooth away from the origin. The scaling $u_0 \to u_{0\lambda}$ leaves such functions invariant. We of course have $|u(x)| \le a|x|^{-1}$, but we do not seem to have any "hidden" smallness condition which would be useful for the Navier-Stokes perturbation theory, unless the coefficient a is already small. In that case one can indeed quite easily establish existence and uniqueness via the Picard iteration (in suitable spaces).

We see that there are essentially two types of critical spaces. One type is represented by $\dot H^{\frac12}, L^3$ or certain Besov spaces. With any function in these spaces one can associate a small quantity (related to a "uniform continuity condition") useful for the Navier-Stokes theory, one can prove local-in-time well-posedness results for any function in the space.

The other type are the space which contain $(-1)-$homogeneous functions, where the perturbation method works only for functions with a small norm.

Very likely, this is not just a technical point, but it reflects the behavior of the actual solutions of the Navier-Stokes equations.

4.3. Scale-invariant solutions for large data

Let us take a $(-1)-$homogeneous div-free vector field $w_0(x)$ which is smooth away from the origin, and let us look at the Cauchy problem (4.1), (4.2) with

$$u_0 = u_0^{(\kappa)} = \kappa w_0(x) \,. \tag{4.8}$$

One expects that the solutions u of the Cauchy problem should also be scale-invariant (in $\mathbf{R}^3 \times (0,\infty)$ This means that it is of the form

$$u(x,t) = \frac{1}{\sqrt{t}}\, U\left(\frac{x}{\sqrt{t}}\right)\,. \tag{4.9}$$

The profile function U satisfies the following elliptic equations

$$\begin{array}{rcl} -\Delta U - \frac{1}{2}x\nabla U - \frac{1}{2}U + U\nabla U + \nabla P & = & 0\,, \\ \operatorname{div} U & = & 0 \end{array} \tag{4.10}$$

in the space $\mathbf{R}^3$, and the “boundary condition”

$$U(x) = u_0(x) + o(|x|^{-1})\,, \quad |x| \to \infty\,. \tag{4.11}$$

For small κ one can show existence and uniqueness of the solution of the Cauchy problem (in space-time) by simple versions of perturbation theory. For example, standard perturbation theory arguments can be appliead in the space X of div-free vector fields $u(x,t)$ in $\mathbf{R}^3 \times (0,\infty)$ with the norm

$$||u||_X = \sup_{(x,t)\in\mathbf{R}^3\times(0,\infty)} |u(x,t)|\sqrt{|x|^2+t}\,.$$

For a proof using Besov spaces see [9].

For large κ the perturbation theory no longer works, of course. It turns out one can establish the existence of the solutions of elliptic problem (4.10) with the boundary conditions (4.11) using a topological argument, see [30]. However, unlike with the perturbative approach, the argument no longer gives uniqueness.

A possible non-uniqueness of the solution of the elliptic problem would give non-uniqueness for solutions of the Cauchy problems for scale-invariant initial data. The onset of the non-uniqueness might related to a bifurcation in the solutions of the elliptic problem.

Let us look at this scenario in more detail. Let $U^{(\kappa)}\,, \kappa \in [0,\kappa_1)$ be a curve of solutions of (4.10), (4.11), with $u_0 = u_0^{(\kappa)}$. Let $L^{(\kappa)}$ be the linearized operator at $U^{(\kappa)}$, i. e.

$$L^{(\kappa)}v = -\Delta v - \frac{1}{2}x\nabla v - \frac{1}{2}v + U^{(\kappa)}\nabla v + v\nabla U^{(\kappa)} + \nabla q \tag{4.12}$$

which is considered on a suitable space Y of div-free vector fields v. A bifurcation in the curve of solutions $U^{(\kappa)}$ would correspond to a non-trivial solution of the problem

$$\begin{array}{rcl} L^{(\kappa)}v & = & 0 \\ \operatorname{div} v & = & 0 \\ v(x) & = & o(|x|)\,, \quad x \to \infty\,. \end{array} \tag{4.13}$$

If one allows bifurcations to time-dependent solution, one can consider a broader class of solutions, namely eigenfunctions of $L^{(\kappa)}$ with a purely imaginary eigenvalue:

$$\begin{array}{rcl} L^{(\kappa)}v & = & i\beta \\ \operatorname{div} v & = & 0 \\ v(x) & = & o(|x|)\,, \quad x \to \infty\,, \end{array} \tag{4.14}$$

where $\beta \in \mathbf{R}$. One can think about the situation in the following way: For $\kappa = 0$ the operator $L^{(\kappa)}$ is well-known and can be identified with the Stokes operator in self-similarity coordinates. Its spectrum in a suitable space Y is in a half-plane $\{z\,, \operatorname{Re} z \le -\alpha\}$ for some $\alpha > 0$ (which may depend of the choice of Y). The operator $L^{(0)}$ may have a continuous spectrum, with relativelly slowly decaying (generalized) eigenfunctions, together with an imbedded point spectrum, with faster decaying eigenfunctions.

As we increase κ the spectrum can change, and some of the discrete eigenvalues may move to the right, eventually crossing the imaginary axis. A transversal crossing of the axis by an eigenvalue (or a pair of eigenvalues) would indicate a bifurcation, and this bifurcation would indicate non-uniqueness of the solutions of the Cauchy problem in the for the scale-invariant datum given by $u_0^{(\kappa)}$ (for the value of κ at which the bifurcation occurs).

The scenario is simple enough at a conceptual level, but at a technical level one has to deal with several issues. Some of the difficulties are posed by the infinite domain $\mathbf{R}^3$ of the operators $L^{(\kappa)}$ together with the term $x\nabla v$. These features lead to a large continuous spectrum of $L^{(\kappa)}$ in spaces Y

which we found suitable for our purposes. However, it is possible to find a good functional-analytic setup, where everything works, see [31].

4.4. Possible non-uniqueness of Leray-Hopf solutions

Once the bifurcation for $L^{(\kappa)}$ discussed above is found, it is possible to show (under some assumptions) non-uniqueness of solutions of the Cauchy problem for certain scale-invariant initial data. All non-trivial initial data which are scale-invariant must necessarily have infinite energy, as the only scale-invariant function in $L^2(\mathbf{R}^3)$ is the identical zero.

Can the situation with a non-unique solution of the Cauchy problem be used to produce non-uniqueness for finite-energy solutions? This is not completely implausible, as from the point of view of energy considerations, the singularity at $x = 0$ of the scale-invariant datum u_0 which is smooth away from the origin is quite mild. It turns out that one can indeed localize the situation, and preserve the effect for initial data with compact support and the same singular behavior near $x = 0$. The relevant initial data u_0 will be compactly supported, smooth away from the origin, and -1-homogeneous in a neighborhood of the origin. From the point of view of L^2, these are not very wild functions.

The following result can be established rigorously:

Theorem 4.1. *Let $Y = L^2 \cap L^4$. Assume that some points of the point spectrum of $L^{(\kappa)}$ (in the space Y) cross the imaginary axis as κ increases (with some transversality assumptions, see* [31] *or* [26] *for details). Then the Leray solutions of the Cauchy problem are not unique, with examples of the initial data for which non-uniqueness can occur having compact support and being smooth away from the origin while being locally -1-homogeneous near the origin.*

Such examples would be essentially optimal, at the borderline of the known weak-strong uniqueness theorems.

4.5. Numerical evidence for the spectral condition

In a recent work [26], numerical investigation of the spectral condition in Theorem 4.1 have been carried our in the axi-symmetric case. The results of the simulations identify a -1-homogeneous initial condition u_0 for which the numerics gives a convincing confiramtion that the spectral condition in Theorem (4.1) is satisfied. In cylindtrical coordinates (r, θ, z) the field u_0 is as follows:

$$u_0(r, \theta, z) = \frac{e^{-4\frac{z^2}{r^2}}}{\sqrt{r^2+z^2}}\ \frac{\partial}{r\partial\theta}\,. \tag{4.15}$$

The field is invariant under the reflection $(r, \theta, z) \to (r, \theta, -z)$. The bifurcation in the solutions of the problem (4.10), (4.11) is a classical pitchfork bifurcation which breaks this $\mathbf{Z}_2$-symmetry. A physical interpretation of the bifurcation can be found in the introduction of [26].

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