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László Székelyhidi Jr

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Abstract

These notes are based on a series of lectures given at the meeting Journées EDP in Roscoff in June 2015 on recent developments concerning weak solutions of the Euler equations and in particular recent progress concerning the construction of Hölder continuous weak solutions and Onsager's conjecture.

Contents

1. Introduction

The incompressible Euler equations describe the motion of a perfect incompressible fluid. Written down by L. Euler over 250 years ago, these are the continuum equations corresponding to the

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conservation of momentum and mass of arbitrary fluid regions. In Eulerian variables they can be written as

$$\begin{aligned} \partial_t v + (v \cdot \nabla) v + \nabla p = 0, \\ \operatorname{div} v = 0, \end{aligned} \tag{E}$$

where $v = v(x,t)$ is the velocity and $p = p(x,t)$ is the pressure. In this note we will focus on the 3-dimensional case with periodic boundary conditions. In other words we take the spatial domain to be the flat 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$.

A classical solution on a given time interval $[0,T]$ is defined to be a pair $(v,p) \in C^1(\mathbb{T}^3 \times [0,T])$. Despite the rich geometric structure underlying these equations (see e.g. [32] and references therein), little is known about smooth solutions except **(i)** local well-posedness (i.e. existence and uniqueness for short time) in Hölder spaces $C^{1,\alpha}$, $\alpha > 0$ [65] or Sobolev spaces H^s, $s > 5/2$ [49, 60] and **(ii)** the celebrated blow-up criterion of Beale-Kato-Majda [6] and its geometrically refined variants, see e.g. [33]. As a consequence of this deadlock and also motivated by physical applications, several weaker notions of solution have been proposed in the literature.

1.1. Notions of solutions

Although distributions were not yet developed in the 1920s, it was certainly recognized already at that time that one needs a notion of solution that allows discontinuities in the vorticity (vortex patches) and in the velocity (vortex sheets). Accordingly, weak solutions of (E) are defined in [65] as a pair $(v,p) \in C(\mathbb{T}^3 \times [0,T])$ such that, for any simply connected region $U \subset \mathbb{T}^3$ with C^1 boundary and any $t \in (0,T)$,

$$\begin{aligned} \int_U v(x,t)\,dx - \int_U v(x,0)\,dx + \int_0^t \int_{\partial U} v(v\cdot\vec{n}) + p\,dA\,ds = 0, \\ \int_{\partial U} v\cdot\vec{n}\,dA(x) = 0, \end{aligned} \tag{W}$$

where $\vec{n}$ is the unit outward normal to U. It is easy to see that if $(v,p) \in C^1$ is a solution of (W) then it is a classical solution of (E). Indeed, the derivation of (E) proceeds precisely this way: from the principles of continuum mechanics and the conservation laws of momentum and mass applied to arbitrary fluid regions U one obtains (W), and if in addition $(v,p) \in C^1$, the divergence theorem and a standard localization argument leads to (E).

This definition still includes the pressure. On the other hand it is well known (see e.g. [77]) that the pressure can be recovered (uniquely, upto an additive constant) from (E) via the equation

$$-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v).$$

Therefore one can eliminate the pressure from the equation by projecting the first equation of (E) onto divergence-free fields. In order to then define distributional solutions, one makes use of the following identity, which uses that $\operatorname{div} v = 0$:

$$(v\cdot\nabla v)_k = \left(\sum_i v_i \frac{\partial}{\partial x_i}\right) v_k = \sum_i \frac{\partial}{\partial x_i}(v_i v_k) = [\operatorname{div}(v\otimes v)]_k$$

for any $k = 1,2,3$. One then obtains from (E)

$$\int_0^T \int_{\mathbb{T}^3} \partial_t\varphi \cdot v + \nabla\varphi : v\otimes v\,dxdt + \int_{\mathbb{T}^3} \varphi(x,0)\cdot v_0(x)\,dx = 0 \tag{D}$$

for all $\varphi \in C^\infty(\mathbb{T}^3 \times [0,T);\mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$. Accordingly, the weakest possible notion of solution of (E) is given by a vectorfield $v \in L^2(\mathbb{T}^3 \times (0,T))$ with $\operatorname{div} v = 0$ in the sense of distributions such that (D) holds.

A stumbling block in obtaining a satisfactory existence theory of weak solutions is the lack of sufficiently strong *a priori* estimates. To overcome this difficulty, two "very weak" notions have been proposed in the literature, both based on considering weakly convergent sequences of Leray solutions of Navier-Stokes with vanishing viscosity: dissipative solutions of P. L. Lions [66] and measure-valued solutions of R. DiPerna and A. Majda [47]. The latter are based on the notion of

Young measure and can be described as follows: Given a sequence of velocity fields $v_k(x,t)$, it is known from classical Young measure theory (see e.g. [80, 2, 67]) that there exists a subsequence (not relabeled) and a parametrized probability measure $\nu_{x,t}$ on $\mathbb{R}^3$ such that for all *bounded* continuous functions f,

$$f(v_k(x,t)) \overset{*}{\rightharpoonup} \langle \nu_{x,t}, f\rangle \qquad \text{weakly* in } L^\infty(\mathbb{T}^3 \times (0,T)), \tag{1.1}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality bracket for $C_0^*(\mathbb{R}^3) = \mathcal{M}(\mathbb{R}^3)$. One can interpret the measure $\nu_{x,t}$ as the probability distribution of the velocity field at the point x at time t when the sequence (v_k) exhibits faster and faster oscillations as $k \to \infty$. Since the only known a priori estimate on solutions of the Euler equations is the energy bound, i.e. $v_k \in L^\infty(0,T;L^2(\mathbb{T}^3))$, concentrations could occur for unbounded f, in particular for the energy density $f(v) = \frac{1}{2}|v|^2$. DiPerna and Majda addressed this issue in [47], providing a framework in which both oscillations and concentrations can be described. Following [1] the generalized Young measure can be written as a triple $(\nu, \lambda, \nu^\infty)$, where $\nu = \nu_{x,t}$ is a parametrized probability measure on $\mathbb{R}^3$ as before (the oscillation measure), λ is a Radon measure on $\mathbb{T}^3 \times (0,T)$ (the concentration measure) and $\nu^\infty = \nu^\infty_{x,t}$ is a parametrized probability measure on S^2 defined λ-a.e. (the concentration-angle measure). Then (1.1) can be replaced by

$$f(v_k)dxdt \overset{*}{\rightharpoonup} \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda \tag{1.2}$$

in the sense of measures for every continuous $f : \mathbb{R}^3 \to \mathbb{R}$ that possesses an L^2-recession function f^∞ (i.e. such that $f^\infty(\theta) = \lim_{s\to\infty} s^{-2} f(s\theta)$ exists and is continuous). Note that for bounded f the formula in (1.2) reduces to (1.1) because $f^\infty = 0$ in this case.

In particular $(\nu, \lambda, \nu^\infty)$ is able to record oscillations and concentrations in the quadratic term $v \otimes v$ of the Euler equations (D). Denote by id the identity map $\xi \mapsto \xi$ and set $\sigma(\xi) = \xi \otimes \xi$, $\xi \in \mathbb{R}^3$. Noting that $\sigma^\infty = \sigma$, a measure-valued solution of the Euler equations is defined to be a generalized Young measure $(\nu, \lambda, \nu^\infty)$ such that $\operatorname{div} \langle \nu, id\rangle = 0$ in the sense of distributions and

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \phi \cdot \langle \nu, id\rangle + \nabla\phi : \langle \nu, \sigma\rangle \, dxdt + \iint_{\mathbb{T}^3\times(0,T)} \nabla\phi : \langle \nu^\infty, \sigma\rangle \, \lambda(dxdt)$$
$$= -\int_{\mathbb{T}^3} \phi(x,0) v_0(x)\, dx \tag{M}$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,T); \mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$.

Observe that (M) is simply a constraint on the first and second moments of the generalized Young measure, i.e. on

$$\overline{v} = \langle \nu_{x,t}, id\rangle, \qquad \overline{v\otimes v} = \langle \nu_{x,t}, \sigma\rangle + \langle \nu^\infty_{x,t}, \sigma\rangle \lambda(dxdt).$$

In particular a measure-valued solution merely gives information on *one-point statistics*, in the sense that there is no information about the correlation between the "statistics" of v_j at different points (x,t) and (x',t'). Moreover there are no *microscopic* constraints, that is, constraints on the distributions of the probability measures. This is very different from other contexts where Young measures have been used, such as conservation laws in one space dimension [46, 76], where the Young measures satisfy additional microscopic constraints in the form of commutativity relations (for instance as a consequence of the div-curl lemma applied to the generating sequence). Consequently, although the existence of measure-valued solutions for arbitrary initial data is guaranteed [47], there is a huge scope for unnatural non-uniqueness.

1.2. Non-uniqueness

In contrast with the local well-posedness for classical solutions of (E), solutions of (D) (or of (W), as we shall see) are in general quite "wild", and exhibit a behaviour which is very different from classical solutions. This behaviour is referred to as a form of h-principle.

Theorem 1.1. (i) [70, 71, 41] *There exist infinitely many non-trivial weak solutions $v \in L^\infty(\mathbb{T}^3 \times \mathbb{R})$ of* (D) *which have compact support in time.*

(ii) [78] *For any solenoidal $v_0 \in L^2(\mathbb{T}^3)$ there exist infinitely many global weak solutions $v \in L^\infty(0,\infty; L^2(\mathbb{T}^3))$ of* (D).

(iii) [75] *For any measure-valued solution of* (M) *there exists a sequence of weak solutions* $v_k \in L^2(\mathbb{T}^3 \times (0,T))$ *of* (D) *generating this measure-valued solution, in the sense that* (1.2) *holds.*

Part (i) was proved first by V. Scheffer [70] in two dimensions for $v \in L^2_{loc}(\mathbb{R}^2 \times \mathbb{R})$, A. Shnirelman [71] subsequently gave a different proof for $v \in L^2(\mathbb{T}^2 \times \mathbb{R})$. The statement for arbitrary dimension $d \geq 2$ for bounded velocities was obtained in [41]. Part (iii) shows that solutions of (D) and solutions of (M) are on the same level in terms of their "wild" behaviour.

1.3. Admissibility

It is a classical fact that C^1 solutions of (E) satisfy the following identity, which expresses the conservation of the kinetic energy in a local form:

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) = 0 . \tag{1.3}$$

Indeed, this follows from the following calculation:

$$v \cdot (v \cdot \nabla) v = \sum_{k,i} v_k v_i \frac{\partial}{\partial x_i} v_k = \sum_{k,i} v_i \frac{\partial}{\partial x_i} \frac{v_k^2}{2} = \operatorname{div}\left(v \frac{|v|^2}{2}\right) .$$

Integrating (1.3) in space we arrive at the conservation of the total kinetic energy

$$\frac{d}{dt} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = 0 . \tag{1.4}$$

In the previous section we have seen that solutions of (D) are in general highly non-unique and need not satisfy the energy conservation (1.4). It is therefore quite remarkable that, despite this high flexibility, the additional requirement that the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx$$

be non-increasing already suffices to single out the unique classical solution when it exists.

Theorem 1.2 (Weak-strong uniqueness)**.** *Let* $v \in L^\infty([0,T), L^2(\mathbb{T}^3))$ *be a solution of* (D) *with the additional property that* $\nabla v + \nabla v^T \in L^\infty$. *Assume that* $(\nu, \lambda, \nu^\infty)$ *is a solution of* (M) *satisfying the energy inequality*

$$\frac{1}{2} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\xi|^2 d\nu_{x,t}(\xi) \, dx + \frac{1}{2} \int_{\mathbb{T}^3} d\lambda_t(x) \ \leq \ \int_{\mathbb{T}^3} |v_0|^2(x) \, dx \qquad \text{for a.e. } t. \tag{1.5}$$

Then $(\nu, \lambda, \nu^\infty)$ *coincides with* v *as long as the latter exists, i.e.*

$$\nu_{x,t} = \delta_{v(x,t)} \ \text{ for a.a. } (x,t) \in \mathbb{T}^3 \times (0,T) \ \text{ and } \ \lambda \equiv 0 \ \text{ on } \mathbb{T}^3 \times (0,T).$$

This theorem was proved in [15], building upon ideas of [14, 16], where the authors dealt with the energy of measure-valued solutions to the Vlasov-Poisson system. More precisely, the proof of [15] yields the following information: if $\nu_{x,t}$ satisfies (1.5), then

$$\bar{v}(x,t) := \int_{\mathbb{R}^3} \xi \, d\nu_{x,t}(\xi) \quad (= \langle \xi, \nu_{x,t} \rangle)$$

is a *dissipative solution* of the Euler equations in the sense of P. L. Lions (see [66]). In fact, Lions introduced the latter notion to gain back the weak-strong uniqueness while retaining the weak compactness properties of the DiPerna-Majda solutions. Theorem 1.2 shows that this can be achieved in the framework of DiPerna and Majda by simply adding the natural energy constraint (1.5).

The energy conservation for classical solutions expressed in (1.4) and the weak-strong uniqueness result Theorem 1.2 suggest that the notion of weak solution to (W) or (D) should be complemented with an additional *admissibility criterion*, which could be one of the conditions below:

(a) $\displaystyle\int |v(x,t)|^2 \, dx \leq \int |v_0(x)|^2 \, dx \qquad$ for a.e. t.

(b) $\displaystyle\int |v(x,t)|^2 \, dx \leq \int |v(x,s)|^2 \, dx \qquad$ for a.e. $t > s$.

(c) If in addition $v \in L^3_{loc}$, then

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \leq 0$$

in the sense of distributions (note that, since $-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v)$, the product pv is well-defined by the Calderon-Zygmund inequality).

Condition (c) has been proposed by Duchon and Robert in [48] and it resembles the admissibility criteria which are popular in the literature on hyperbolic conservation laws.

Next, denote by $L^2_w(\mathbb{T}^3)$ the space $L^2(\mathbb{T}^3)$ endowed with the weak topology. We recall that any weak solution of (D) in the *energy space*

$$L^\infty(0, T; L^2(\mathbb{T}^3))$$

can be modified on a set of measure zero so that $v \in C([0,T), L^2_w(\mathbb{R}^n)$ (this is a common feature of evolution equations in conservation form; see for instance Theorem 4.1.1 of [38]). Consequently v has a well-defined trace at every time and the requirements (a) and (b) can therefore be strengthened in the following sense:

(a') $\displaystyle\int |v(x,t)|^2\, dx \leq \int |v_0(x)|^2\, dx$ for *every* t.

(b') $\displaystyle\int |v(x,t)|^2\, dx \leq \int |v(x,s)|^2\, dx$ for *every* $t > s$.

Observe that the weak continuity in time and the energy inequality as above comes naturally when considering the inviscid limit. Indeed, it is not difficult to show that if $\{v_k\}_k$ is a sequence of Leray weak solutions of the Navier-Stokes equations on some time interval $[0,T]$ with viscosity $\nu_k \to 0$, and if $v_k \rightharpoonup v$ in $L^\infty(0,T; L^2(\mathbb{T}^3))$, then $v \in C([0,T), L^2_w(\mathbb{R}^n))$ and satisfies (b'). However, none of these criteria restore the uniqueness in general.

Theorem 1.3 (Non-uniqueness of admissible weak solutions)**.** *Let $n \geq 2$. There exist initial data $v_0 \in L^\infty \cap L^2$ for which there are infinitely many bounded solutions of* (D) *which are strongly L^2-continuous (i.e. $v \in C([0,\infty), L^2(\mathbb{R}^n))$) and satisfy (a'), (b') and (c).*

The conditions (a'), (b') and (c) hold with the equality sign for infinitely many of these solutions, whereas for infinitely many other they hold as strict *inequalities.*

This theorem is from [42]. The second statement generalizes the intricate construction of Shnirelman in [72], which produced the first example of a weak solution in $\mathbb{T}^3 \times [0,\infty[$ of (D) with strict inequalities in (a) and (b).

The initial data v_0 as in Theorem 1.3 are obviously not regular, since for regular initial data the local existence theorems and the weak-strong uniqueness (Theorem 1.2) ensure local uniqueness under the very mild condition (a). Nevertheless, the set of such “wild” initial data is dense in L^2:

Theorem 1.4 (Theorem 2 in [75])**.** *The set of initial data v_0 for which the conclusion of Theorem 1.3 holds is dense in the space of L^2 divergence-free vector fields.*

The non-uniqueness for admissible weak solutions seems to be closely related to strong instabilities in the Euler equations. In particular, consider the following solenoidal vector field in $\mathbb{T}^2$, related to the well-known Kelvin-Helmholtz instability:

$$v_0(x) = \begin{cases} (1,0) & \text{if } \theta_2 \in (-\pi, 0), \\ (-1,0) & \text{if } \theta_2 \in (0,\pi). \end{cases} \tag{1.6}$$

Theorem 1.5 ([73])**.** *For v_0 as in* (1.6) *there are infinitely many solutions of* (D) *on* $\mathbb{T}^2 \times [0,\infty)$ *which satisfy (b').*

See also [4] for another example of non-uniqueness which is also based on the instability of shear layers. We also refer to [43] for a discussion regarding possible selection criteria, a natural question in light of such examples of non-uniqueness.

1.4. Onsager's conjecture

Leaving the non-uniqueness aside, let us now turn to the question of energy conservation. As mentioned above in (1.3)-(1.4), for classical solutions (i.e. if $v \in C^1$) the energy is conserved in time, whereas part (i) of Theorem 1.1 shows that for weak solutions the energy need not be conserved. Nevertheless, it turns out that the question of energy conservation for weak solutions does have some physical relevance.

One of the cornerstones of three-dimensional turbulence is the so-called anomalous dissipation. This experimentally observed fact, namely that the rate of energy dissipation in the vanishing viscosity limit stays above a certain non-zero constant, is expected to arise from a mechanism of transporting energy from large to small scales (known as an energy cascade) via the nonlinear transport term in the Navier-Stokes equations, rather than the (dissipative) viscosity term. Motivated by this idea, L. Onsager stated in 1949 [69] the following:

Conjecture 1.6. *Consider solutions* (v, p) *of* (W) *satisfying the Hölder condition*

$$|v(x,t) - v(x',t)| \le C|x - x'|^\theta, \tag{1.7}$$

where the constant C *is independent of* $x, x' \in \mathbb{T}^3$ *and* t*. Then*

(a) *If* $\theta > \frac{1}{3}$*, any solution* (v, p) *of* (W) *satisfying* (1.7) *conserves the energy;*

(b) *For any* $\theta < \frac{1}{3}$ *there exist solutions* (v, p) *of* (W) *satisfying* (1.7) *which do not conserve the energy.*

This conjecture is also very closely related to Kolmogorov's famous K41 theory [62] for homogeneous isotropic turbulence in three dimensions. We refer the interested reader to [53, 52, 51].

Part (a) of the conjecture is by now fully resolved: it has first been considered by Eyink in [51] following Onsager's original calculations and proved by Constantin, E and Titi in [34]. Slightly weaker assumptions on v (in Besov spaces) were subsequently shown to be sufficient for energy conservation in [48, 25]. In the following, we recall the beautiful argument of [34].

We start with some estimates on convolutions. Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be a symmetric, non-negative mollifying kernel such that $\int \varphi = 1$, and set $\varphi_\ell(x) = \ell^{-3}\varphi(\frac{x}{\ell})$. Given $v \in C(\mathbb{T}^3)$ define

$$v_\ell(x) := v * \varphi_\ell(x) = \int_{\mathbb{R}^3} v(x-y)\varphi_\ell(y)\,dy.$$

Lemma 1.7. *Assume* $v \in C^\theta(\mathbb{T}^3)$*. Then we have*

$$\|v - v_\ell\|_0 \le C\ell^\theta [v]_\theta, \tag{1.8}$$

$$\|\nabla v_\ell\|_0 \le C\ell^{\theta-1}[v]_\theta, \tag{1.9}$$

$$\|(v \otimes v)_\ell - v_\ell \otimes v_\ell\|_0 \le C\ell^{2\theta}[v]_\theta^2. \tag{1.10}$$

Proof. For (1.8) observe that

$$\begin{aligned} |v_\ell(x) - v(x)| &= \left| \int \varphi_\ell(x-y)(v(y) - v(x))\,dy \right| \\ &\le \left| \int \varphi_\ell(x-y)|y-x|^\theta\,dy \right| [v]_\theta. \end{aligned}$$

For obtaining (1.9) we simply write

$$\nabla v_\ell(x) = \int \nabla\varphi_\ell(x-y)v(y)\,dy = \int \nabla\varphi_\ell(x-y)(v(y) - v(x))\,dy.$$

Finally, for (1.10) note that

$$\begin{aligned} (v \otimes v)_\ell(x) &= \int \varphi_\ell(x-y)v(y) \otimes v(y)\,dy \\ &= \int \varphi_\ell(x-y)(v(y) - v(x)) \otimes (v(y) - v(x))\,dy \\ &\qquad + v(x) \otimes v_\ell(x) + v_\ell(x) \otimes v(x) - v(x) \otimes v(x). \end{aligned}$$

Hence

$$(v\otimes v)_\ell(x) - v_\ell(x)\otimes v_\ell(x) = \int \varphi_\ell(x-y)(v(y)-v(x))\otimes(v(y)-v(x))\,dy \\ - (v(x)-v_\ell(x))\otimes(v(x)-v_\ell(x)).$$

Therefore (1.10) follows from (1.8). □

Next, let (v,p) be a Hölder-continuous solution of (W). Then $\operatorname{div} v_\ell = 0$ and

$$\partial_t v_\ell + v_\ell\cdot\nabla v_\ell + \nabla p_\ell = -\operatorname{div} R_\ell,$$

where

$$R_\ell = (v\otimes v)_\ell - v_\ell\otimes v_\ell.$$

Proceeding as in (1.3)-(1.4) we obtain the energy balance

$$\frac{dE_\ell}{dt} = \int \nabla v_\ell : R_\ell\,dx,$$

where $E_\ell = \frac{1}{2}\int |v_\ell(x)|^2\,dx$. From Lemma 1.7 it follows that, for any $T>0$,

$$|E_\ell(T) - E_\ell(0)| \le \int_0^T \int_{\mathbb{T}^3} \ell^{3\theta-1}[v(t)]_\theta^3\,dxdt.$$

Consequently, as $\ell\to 0$ we obtain $E(T)=E(0)$, provided

$$\int_0^T [v(t)]_\theta^3\,dx < \infty \quad \text{for some } \theta > 1/3. \tag{1.11}$$

This proves in particular part (a) of the conjecture. For sharper conditions, formulated in terms of the Littlewood-Paley decomposition of v, we refer to [25].

Concerning point (b) of Conjecture 1.6, the first mathematical statement in that direction is the theorem of V. Scheffer, formulated in part (i) of Theorem 1.1 concerning solutions of (D). In recent years a series of results concerning continuous solutions of (W) appeared, starting with [44]. Having fixed a certain specific space of (continuous) functions X, these results can be classified in the following two categories:

(A) There exists a nontrivial weak solution $v\in X$ of (E) with compact support in time.

(B) Given any smooth positive function $E=E(t)>0$, there exists a weak solution $v\in X$ of (E) with

$$\int |v(x,t)|^2\,dx = E(t) \quad \forall\, t.$$

Obviously both types lead to non-conservation of energy and would therefore conclude part (b) of Onsager's conjecture if proved for the space $X = L^\infty(0,T;C^{1/3}(\mathbb{T}^3))$. So far the best results are as follows.

Theorem 1.8.

- *Statement (A) is true for* $X = L^1(0,T;C^{1/3}(\mathbb{T}^3))$.
- *Statement (B) is true for* $X = L^\infty(0,T;C^{1/5-\epsilon}(\mathbb{T}^3))$.

Statement (B) has been shown for $X = L^\infty(0,T;C^{1/10-\epsilon})$ in [45], whereas P. Isett in [57] was the first to prove Statement (A) for $X = L^\infty(0,T;C^{1/5-\epsilon})$, thereby reaching the current best "uniform" Hölder exponent for Part (b) of Onsager's conjecture. Subsequently, T. Buckmaster, the two authors and P. Isett proved Statement (B) for $X = L^\infty(0,T;C^{1/5-\epsilon})$ in [20]. Finally, Statement (A) for $X = L^1(0,T;C^{1/3}(\mathbb{T}^3))$ has been proved very recently in [21].

The basic construction underlying all these results was first introduced in [44]. In these lectures this basic scheme will be presented in Section 3. The proof of Statement (B) will then be explained in Sections 4.1 and 4.2 and finally the key ideas towards Statement (A) will be outlined in Section 4.4.

2. The h-principle

The homotopy principle was introduced by M. Gromov [54] as a general principle encompassing a wide range of existence problems in differential geometry. Roughly speaking, the *h*-principle applies to situations where the problem of existence of a certain object in differential geometry can be reduced to a purely topological question and thus treated with homotopic-theoretic methods. We quote Gromov [55]:

The infinitesimal structure of a medium, abiding by this principle does not effect the global geometry but only the topological behaviour of the medium.

In a sense the *h*-principle is the opposite of the classical local-to-global principle, where global behaviour is directly affected by infinitesimal laws. A paradigm example where the interaction of both principles can be seen is that of isometric embeddings. To fix ideas, let us consider embeddings of the standard 2-sphere S^2 into $\mathbb{R}^3$, i.e. maps

$$u : S^2 \hookrightarrow \mathbb{R}^3.$$

A continuous map u is said to be isometric if it preserves the length of curves:

$$\ell(u \circ \gamma) = \ell(\gamma) \qquad \text{for all rectifiable curves } \gamma \subset S^2. \tag{2.1}$$

If u is continuously differentiable, i.e. $u \in C^1(S^2; \mathbb{R}^3)$, this condition is equivalent to preserving the metric, which in local coordinates amounts to the system of partial differential equations

$$\partial_i u \cdot \partial_j u = g_{ij} \qquad i, j = 1, 2, \tag{2.2}$$

with g_{ij} being the metric on S^2. The equivalence of (2.1) and (2.2) is a first simple instance of the local-to-global principle: the length of a curve (a global quantity) can be obtained from the metric (an infinitesimal quantity) by integrating.

It is easy to construct Lipschitz isometric embeddings of S^2 which are not equivalent to the standard embedding: consider reflecting a spherical cap cut out by a plane slicing the standard sphere. More generally, one can imagine a sphere made out of paper, and crumpling it. This process will necessarily create creases, meaning that the associated embedding is only Lipschitz but not C^1. Nevertheless, such maps will still easily satisfy both (2.1) and also (2.2) almost everywhere. Indeed, if u is merely Lipschitz, the system (2.2) still makes sense *almost everywhere*, since by Rademacher's theorem u is differentiable almost everywhere. However, in this case (2.2) a.e. is not equivalent to (2.1) – see (iii) in the theorem below.

As the preceeding discussion indicates, the class of isometric embeddings very much depends on the regularity assumption on u:

Theorem 2.1. (i) **Rigidity.** [56] [30] *If $u \in C^2$ is isometric, then u is equal to the standard embedding of $S^2 \subset \mathbb{R}^3$, modulo rigid motion.*

(ii) ***h*-principle.** [68], [63] *Any short embedding can be uniformly approximated by isometric embeddings of class C^1.*

(iii) **Lipschitz maps.** [54, p218] *There exist $u \in Lip$ such that* (2.2) *is satisfied almost everywhere, but* (2.1) *fails: certain curves on S^2 get mapped to a single point.*

A short embedding is simply one that shrinks the length of curves, i.e. $\ell(u \circ \gamma) \leq \ell(\gamma)$ for all rectifiable curves $\gamma \subset S^2$.

The rigidity statement (i) is a prominent example of the local-to-global principle in geometry: a local, differential condition leads to a strong restriction of the global behaviour. The theorem of Nash-Kuiper in (ii) signifies the *failure* of this local-to-global principle if u is not sufficiently differentiable, whereas (iii) shows that for Lipschitz maps satisfying (2.2) almost everywhere even the simple local-to-global principle on the length of curves fails.

2.1. Relaxation and residuality

The h-principle amounts to the vague statement that local constraints do not influence global behaviour. In differential geometry this leads to the fact that certain problems can be solved by purely topological or homotopic-theoretic methods, once the "softness" of the local (differential) constraints has been shown. In turn, this softness of the local constraints can be seen as a kind of relaxation property.

In order to gain some intuition let us again look at the system of partial differential equations (2.2) with some fixed smooth g, and consider a sequence of (smooth) solutions $\{u^k\}_k$, $u^k : S^2 \to \mathbb{R}^3$. Then the sequence of derivatives $|\partial_i u^k|^2 = g_{ii}$ is uniformly bounded, hence by the Arzelà-Ascoli theorem there exists a subsequence $u_{k'}$ converging uniformly to some limit map u. The limit u must be Lipschitz and an interesting question is whether u is still a solution (i.e. isometric). This would follow from some better convergence, for instance in the C^1 category. If the metric g has positive curvature and the maps u^k are sufficiently smooth, their images will be convex surfaces: this, loosely speaking, amounts to some useful information about second derivatives which will improve the convergence of u^k and result in a limit u with convex image.

If instead we only assume that the sequence u^k consists of approximate solutions, for instance in the sense that

$$\partial_i u^k \cdot \partial_j u^k - g_{ij} \to 0 \text{ uniformly},$$

then even if g has positive curvature and the u^k are smooth, their images will not necessarily be convex. Let us nonetheless see what we can infer about the limit u. Consider a smooth curve $\gamma \subset S^2$. Then $u^k \circ \gamma$ is a C^1 Euclidean curve and our assumption implies

$$\ell(u^k \circ \gamma) \to \ell(\gamma). \tag{2.3}$$

On the other hand the curves $u^k \circ \gamma$ converge uniformly to the (Lipschitz) curve $u \circ \gamma$ and it is well-known that under such type of convergence the length might shrink but cannot increase. We conclude that

$$\ell(u \circ \gamma) \leq \ell(\gamma)\,, \tag{2.4}$$

in other words the map u is *short*. Recall that, by Rademacher's theorem, u is differentiable almost everywhere: it is a simple exercise to see that, when (2.4) holds for every (Lipschitz) curve γ, then

$$\partial_i u \cdot \partial_j u \leq g_{ij} \qquad \text{a.e.}, \tag{2.5}$$

in the sense of quadratic forms. Thus, loosely speaking, one possible interpretation of Theorem 2.1 (ii) is that the system of partial differential inequalities (2.5) is the “relaxation” of (2.2) with respect to the C^0 topology.

2.2. Differential inclusions

In order to explain this better, let us simplify the situation further, and consider the case $\Omega \subset \mathbb{R}^2$ with the flat metric $g_{ij} = \delta_{ij}$, to be embedded isometrically into $\mathbb{R}^3$. Then the system (2.2) is equivalent to the condition that the full matrix derivative $Du(x)$ is a linear isometry at every point x, i.e. that

$$Du(x) \in O(2,3) \tag{2.6}$$

for every x. Note also that the inequality (2.5) is similarly equivalent to

$$Du(x) \in \text{co } O(2,3), \tag{2.7}$$

where, for a compact set K we denote by co K its convex hull.

Let

$$X = \left\{u \in Lip(\Omega; \mathbb{R}^3) : \, Du(x) \in \text{co } O(2,3) \ \text{ a.e. } x\right\}.$$

The discussion in the previous paragraph amounts to the statement that, equipped with the topology of uniform convergence X is a compact metric space. The local aspect of the h-principle expressed in Theorem 2.1 (ii) can then be stated as follows: The set

$$\left\{u \in C^1(\Omega; \mathbb{R}^3) : \, Du(x) \in O(2,3) \ \text{ for all } x\right\}$$

is *dense* in X.

The functional analytic background behind this kind of statement can be viewed as a version of the Krein-Milman theorem. Indeed, consider the following one-dimensional version, the inclusion problem

$$u'(x) \in \{-1, 1\} \quad \text{a.e. in } (0,1).$$

Of course C^1 solutions need to have constant derivative ± 1, but Lipschitz solutions may be rather wild. In fact, it is not difficult to show that the closure in C^0 of the set $\mathcal{S} := \{u \in \mathrm{Lip}(0,1) : |u'| = 1 \text{ a.e.}\}$ coincides with the convex hull $\mathcal{R} := \{u \in \mathrm{Lip}(0,1) : |u'| \leq 1 \text{ a.e.}\}$. Since the topology of uniform convergence in this setting (uniform Lipschitz bound) is equivalent to weak* convergence of the derivative in L^∞, the latter statement can be interpreted as a form of the Krein-Milman theorem. Moreover, it was observed in [22] that $\mathcal{R} \setminus \mathcal{S}$ is a meager set in the Baire Category sense, cf. also [17].

More generally, as an illustration of the methods and ideas involved, let us treat the same problem in general dimensions $m \geq n$. Thus, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\Gamma \subset \overline{\Omega}$ a closed set of zero Lebesgue measure, and define, for

$$X_0 = \left\{ u \in C^1(\Omega; \mathbb{R}^m) : Du^T Du(x) < \mathrm{Id} \text{ for all } x \in \Omega \text{ and } u|_\Gamma = 0 \right\}$$

and let X be the closure of X_0 with respect to the topology of uniform convergence. Note that the inequality in the expression for X_0 is again interpreted in the sense of quadratic forms, and amounts to the geometric statement that u should be *strictly short*.

Theorem 2.2. *The set*

$$S = \left\{ u \in X : Du(x) \in O(n,m) \text{ a.e. } x \in \Omega \text{ and } u|_\Gamma = 0 \right\}$$

is residual in X in the sense of Baire category.

Observe that Theorem 2.2 essentially provides a proof of part (iii) of Theorem 2.1.

Before giving the proof, we start with a few preliminary remarks. As in the previous example, X is a (non-empty) compact metric space. Since elements of X are differentiable almost everywhere, we can consider the gradient operator as a map

$$\nabla : X \to L^1(\Omega).$$

Lemma 2.3. *The map $\nabla : X \to L^1(\Omega)$ is of class Baire-1, i.e. the pointwise limit of continuous mappings.*

Proof. Consider $F_\delta(u) := \nabla(u * \varphi_\delta) = u * \nabla\varphi_\delta$, where φ_δ is a standard mollifying kernel and the convolution is defined by extending u outside Ω by zero. Obviously $F_\delta : X \to L^1(\Omega)$ is continuous. Furthermore, for any $u \in X$ we have that $F_\delta(u) \to \nabla u$ in $L^1(\Omega)$ as $\delta \to 0$. Therefore ∇ is Baire-1. □

Although a Baire-1 mapping need not be continuous, it is continous in some sense at "most" points of X. More precisely, the set of continuity points is a residual set in X (i.e. the complement of a meager set, hence in particular dense). On the other hand, intuitively we would not expect $\nabla : X \to L^1(\Omega)$ to be continuous anywhere, since on X we put the uniform topology. A typical example of a sequence of functions $u_k : (0,1) \to \mathbb{R}$ converging uniformly to zero, but whose derivatives ∇u_k do not converge to zero, is

$$u_k(x) = \frac{1}{k} \sin(kx).$$

It is not difficult to construct similar examples for mappings $\Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. The apparent contradiction between the intuition coming from such examples and the statement of Lemma 2.3 is that elements $u \in X$ satisfy a uniform bound for the gradient: $Du^T Du \leq \mathrm{Id}$. For elements, where this bound is saturated everywhere (e.g. convex extreme points of X), the above simple construction fails. This is quantified in the following lemma:

Lemma 2.4. *For all $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that, for all $\delta > 0$ and all matrices $A \in \mathbb{R}^{m \times n}$ with $A^T A < \mathrm{Id}$ and $\lambda_{max}(\mathrm{Id} - A^T A) \geq \alpha$ there exists $w \in C^1_c(B_1(0); \mathbb{R}^m)$ such that*

(i) $(A + Dw(x))^T (A + Dw(x)) < \mathrm{Id}$ *for all x;*

(ii) $\sup_x |w(x)| \leq \delta$;

(iii) $\int |Dw|\,dx \geq \varepsilon$.

Proof. Let A as in the Lemma so that, by assumption, there exists a unit vector $\xi \in \mathbb{R}^n$, $|\xi| = 1$ such that

$$\mathrm{Id} - A^T A \geq \alpha \xi \otimes \xi. \tag{2.8}$$

Let $\eta \in C_c^\infty(B_1(0))$ be a cut-off function, i.e. such that $0 \leq \eta \leq 1$, and let $\zeta \in \mathbb{R}^n$ a unit vector such that $A^T\zeta = \mu\xi$ for some $\mu \in \mathbb{R}$. Such ζ always exists, since either $\ker A^T \neq \{0\}$ (in which case we set $\mu = 0$) or A^T is invertible (and $m = n$). Moreover, $|\mu| = |\xi \cdot A^T\zeta| = |A\xi \cdot \zeta| \leq |A\xi| \leq 1$. Then, define $w : \mathbb{R}^n \to \mathbb{R}^m$ for some $\lambda \gg 1$ by

$$w(x) = \frac{\varepsilon}{\lambda}\eta(x)\sin(\lambda x \cdot \xi)\zeta.$$

Since

$$\begin{aligned}(A + Dw(x))^T&(A + Dw(x)) \\ &= A^T A + 2\varepsilon\mu\eta(x)\cos(\lambda x \cdot \xi)\xi \otimes \xi + \varepsilon^2\eta(x)^2\cos(\lambda x \cdot \xi)^2 \xi \otimes \xi + O\left(\frac{1}{\lambda}\right),\end{aligned}$$

it follows from (2.8) that condition (i) is satisfied for λ sufficiently large, provided

$$\varepsilon(2\mu\cos(s) + \varepsilon\cos^2(s)) \leq \alpha/2 \quad \text{for all } s.$$

Condition (ii) is easily satisfied by choosing λ sufficiently large, and condition (iii) by choosing η appropriately and observing that $\int |Dw|\,dx$ is bounded below independently of λ. □

Proof of Theorem 2.2. Being a Baire-1 mapping, we know that the set of continuity points of ∇ is a residual set in X. Therefore, in order to prove the theorem it suffices to show that continuity points of $\nabla : X \to L^1(\Omega)$ are contained in S. We argue by contradiction and assume that $u \in X$ is a point of continuity for ∇ such that the set

$$\left\{x : \Omega : \ Du^T Du \neq \mathrm{Id}\right\}$$

has positive Lebesgue measure. Then there exists $\alpha > 0$ and $\beta > 0$ such that

$$\mathcal{L}^n\left(\{x \in \Omega : \ \lambda_{max}(\mathrm{Id} - Du^T Du) \geq 2\alpha\}\right) \geq \beta.$$

Let $\varepsilon = \varepsilon(\alpha)$ be the constant from Lemma 2.4. Since ∇ is assumed to be continuous at $u \in X$, there exists $\delta > 0$ such that

$$\|Du - D\tilde{u}\|_{L^1} < \frac{1}{2}\varepsilon\beta \quad \text{whenever} \quad \|u - \tilde{u}\|_{C^0} < \delta. \tag{2.9}$$

Next, using the density of $X_0 \subset X$ with respect to uniform convergence, combined with Egorov's theorem and the fact that u is a point of continuity for $\nabla : X \to L^1$, we can find $v \in X_0$ such that

$$U := \{x \in \Omega : \ \lambda_{max}(\mathrm{Id} - Dv^T Dv) > \alpha \text{ and } v|_\Gamma \neq 0\}$$

satisfies

$$\mathcal{L}^n(U) \geq \beta/2$$

and moreover

$$\|u - v\|_{C^0} < \delta/2, \quad \|Du - Dv\|_{L^1} \leq \varepsilon\beta/2.$$

Now we can apply Lemma 2.4 and a simple covering argument in the open set U (i.e. filling up U with rescaled and translated copies of the perturbation w from the lemma) to obtain a mapping $w \in C_c^1(\Omega \setminus \Gamma)$ such that $\tilde{u} := v + w \in X_0$ but

$$\|w\|_{C^0} < \delta/2 \quad \text{and} \quad \|Dw\|_{L^1} \geq 2\varepsilon\beta.$$

This contradicts (2.9), thereby concluding the proof. □

Baire category arguments for differential inclusions have a long history, see [39, 22] for ordinary differential inclusions and [24, 36, 61] for partial differential inclusions. We also refer to the survey [23].

Note that the Lipschitz solutions produced by such methods are in general highly non-smooth, e.g. nowhere C^1, c.f. [61, Proposition 3.35]. For the weak isometric map problem corresponding to Theorem 2.2, solutions can also be constructed by folding [37], but such maps have an altogether different structure both from the Nash-Kuiper C^1 solution and from typical solutions produced by the Baire category method. In this example the mere existence of many Lipschitz solutions is not surprising. Next, we discuss the Euler equations, where already a weak form of the h-principle is rather striking.

2.3. Euler Subsolutions

We start by recalling the concept of *Reynolds stress tensor*. It is generally accepted that the appearance of high-frequency oscillations in the velocity field is the main reason responsible for turbulent phenomena in incompressible flows. One related major problem is therefore to understand the dynamics of the coarse-grained, in other words macroscopically averaged, velocity field. If $\overline{v}$ denotes the macroscopically averaged velocity field, then it satisfies

$$\begin{aligned} \partial_t \overline{v} + \operatorname{div}(\overline{v} \otimes \overline{v} + R) + \nabla \overline{p} = 0 \\ \operatorname{div} \overline{v} = 0, \end{aligned} \tag{2.10}$$

where

$$R = \overline{v \otimes v} - \overline{v} \otimes \overline{v} = \overline{w \otimes w} \tag{2.11}$$

and

$$w = v - \overline{v} \tag{2.12}$$

is the “fluctuation”. The symmetric 2-tensor R is called Reynolds stress and arises because the averaging does not commute with the nonlinearity $v \otimes v$. On this formal level the precise definition of averaging plays no role, be it long-time averages, ensemble-averages or local space-time averages. The latter can be interpreted as taking weak limits. Indeed, weak limits of Leray solutions of the Navier-Stokes equations with vanishing viscosity have been proposed in the literature as a deterministic approach to turbulence (see [3, 5, 29, 64]).

A slightly more general version of this type of averaging follows the framework introduced by L. Tartar [76] and R. DiPerna [46] in the context of conservation laws. We start by separating the linear equations from the nonlinear constitutive relations. Accordingly, we write (2.10) as

$$\begin{aligned} \partial_t \overline{v} + \operatorname{div} \overline{u} + \nabla \overline{q} = 0 \\ \operatorname{div} \overline{v} = 0, \end{aligned}$$

where $\overline{u}$ is the traceless part of $\overline{v} \otimes \overline{v} + R$. Since $R = \overline{w \otimes w}$ can be written as an average of positive semidefinite terms, it is clear that $R \geq 0$, i.e. R is a symmetric positive semidefinite matrix. In terms of the coarse-grained variables $(\overline{v}, \overline{u})$ this inequality can be written as

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \tfrac{2}{3} \overline{e} \, \mathrm{Id},$$

where Id is the 3×3 identity matrix and

$$\overline{e} = \overline{\tfrac{1}{2}|v|^2}$$

is the macroscopic kinetic energy density. Motivated by these calculations, we define subsolutions as follows. Since they will appear often, we introduce the notation $\mathcal{S}_0^{3\times 3}$ for the vector space of symmetric traceless 3×3 matrices.

Definition 2.5 (Subsolutions). Let $\overline{e} \in L^1(\mathbb{T}^3 \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple

$$(v, u, q) : \mathbb{T}^3 \times (0,T) \to \mathbb{R}^3 \times \mathcal{S}_0^{3\times 3} \times \mathbb{R}$$

such that

$$\left\{ \begin{array}{l} \partial_t v + \operatorname{div} u + \nabla q = 0 \\ \operatorname{div} v = 0, \end{array} \right. \qquad \text{in the sense of distributions;} \tag{2.13}$$

and

$$v \otimes v - u \leq \tfrac{2}{3}\overline{e}\,\mathrm{Id} \quad \text{ a.e. .} \tag{2.14}$$

Observe that subsolutions automatically satisfy $\frac{1}{2}|v|^2 \leq \overline{e}$ a.e. (the inequality follows from taking the trace in (2.14)). If in addition we have the equality sign $\frac{1}{2}|v|^2 = \overline{e}$ a.e., then the v component of the subsolution is in fact a weak solution of the Euler equations. As mentioned above, in passing to weak limits (or when considering any other averaging process), the high-frequency oscillations in the velocity are responsible for the appearance of a non-trivial Reynolds stress. Equivalently stated, this phenomenon is responsible for the inequality sign in (2.14).

In terms of the relaxation as in the previous section, we can view (2.13)-(2.14) as the analogue of short maps, i.e. the relaxation of the Euler equations (E). Indeed, the analogy can be made even more direct by noting that (2.2) can be written for the gradient mapping $A := Du$ as

$$A^T A = g \quad \text{ and } \quad \mathrm{curl}\, A = 0,$$

whereas (E) with the extra condition $\frac{1}{2}|v|^2 = \overline{e}$ can be written for the variables (v, u, q) as

$$v \otimes v - u = \tfrac{2}{3}\overline{e}\,\mathrm{Id} \quad \text{ and } \quad (2.13).$$

Observe also that (2.13) can be written as

$$\mathrm{Div}_{(x,t)} \begin{pmatrix} v \otimes v + q\mathrm{Id} & v \\ v^T & 0 \end{pmatrix} = 0,$$

where $\mathrm{Div}_{(x,t)}$ means applying the divergence on each row of the matrix and treating t as the 4th variable.

The corresponding "weak h-principle statement", i.e. the analogue of Theorem 2.2 is the following (see [41], and [42] for more refined versions):

Theorem 2.6. *Let $\overline{e} \in C^\infty \cap L^1(\mathbb{T}^3 \times (0,T))$ and $(\overline{v}, \overline{u}, \overline{q})$ be a subsolution with kinetic energy $\overline{e}$. Then there exists a sequence of bounded weak solutions (v^k, p^k) of* (D) *on $\mathbb{T}^3 \times (0,T)$ such that*

$$\tfrac{1}{2}|v^k|^2 = \overline{e} \qquad \text{for almost every } (x,t) \tag{2.15}$$

and $v^k \rightharpoonup v$ weakly in $L^2(\mathbb{T}^3 \times (0,T))$.

In analogy with the proof of Theorem 2.2, proving Theorem 2.6 involves defining the space of smooth, strict subsolutions X_0, i.e.

$$X_0 = \Big\{ (v,u) \in C^\infty : \text{ (2.13) holds for some } q;\ v \otimes v - u < \tfrac{2}{3}\overline{e}\mathrm{Id} \text{ in } \mathbb{T}^3 \times (0,T) \Big\},$$

equipped with the topology of weak convergence in $L^2(\mathbb{T}^3 \times (0,T))$, and define X to be the closure of X_0. Since the inequality (2.14) implies a uniform bound on v and u, the set X is bounded on L^2 and hence the weak topology is metrizable, with metric d_{weak}. The analogue of Lemma 2.4 is the following "perturbation property", which we state for simplicity for the case $\overline{e} \equiv \frac{3}{2}$:

Lemma 2.7. *For all $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that, for all $\delta > 0$ and all constant $(\overline{v}, \overline{u}) \in \mathbb{R}^3 \times \mathcal{S}_0^{3\times 3}$ with $\overline{v} \otimes \overline{v} - \overline{u} < \mathrm{Id}$ and $\lambda_{max}(\mathrm{Id} - (\overline{v} \otimes \overline{v} - \overline{u})) \geq \alpha$ there exists $(v, u, q) \in C^1_c(B_1(0); \mathbb{R}^3 \times \mathcal{S}_0^{3\times 3} \times \mathbb{R})$ such that*

(i) $(\overline{v} + v) \otimes (\overline{v} + v) - (\overline{u} + u) < \mathrm{Id}$ *for all (x,t);*

(ii) $d_{weak}(v, 0) \leq \delta$*;*

(iii) $\int |v|^2\, dx\, dt \geq \varepsilon$.

For the proof and for the general formulation encompassing both Theorem 2.2 and Theorem 2.6 we refer to the lecture notes [74] and the survey [43].

2.4. The Nash-Kuiper construction

In this section we provide a sketch proof of the Nash-Kuiper theorem, already alluded to in Theorem 2.1 (ii). For convenience of the reader we restate it in the following general form:

Theorem 2.8 (Nash-Kuiper)**.** *Let (M^n, g) be a smooth compact manifold, $m \geq n+1$ and $u : M^n \to \mathbb{R}^m$ a short immersion. Then u can be uniformly approximated by C^1 isometric immersions. If in addition u is an embedding, then the approximation also holds with embeddings.*

Recall that a C^1 map u is an immersion if the total derivative $Du(x)$ has full rank at every x. The Nash-Kuiper theorem seems not to be accessible by Baire category arguments. Although the mappings obtained are still highly irregular, a constructive scheme with estimates on the C^0 and C^1 norms is necessary. For a comprehensive proof of Theorem 2.8 we refer to [50] and [74]. Here we merely explain the main analytic ideas involved.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary, which we can think of as a coordinate patch on M^n, let g be a smooth metric on Ω and consider immersions $u : \Omega \to \mathbb{R}^{n+2}$ - codimension 2 and higher was the case dealt with by Nash in [68], the case of embeddings $u : M \hookrightarrow \mathbb{R}^{n+1}$ requires a modification [63].

Given a strictly short map u_q, $q \in \mathbb{N}$, a better approximation will be obtained with the perturbation

$$\tilde{u}_q(x) = u_q(x) + \frac{a_q(x)}{\lambda_{q+1}}\Big(\sin(\lambda_{q+1}x \cdot \nu_q)\eta_q(x) + \cos(\lambda_{q+1}x \cdot \nu_q)\zeta_q(x)\Big), \tag{2.16}$$

where a_q is an amplitude, λ_{q+1} a (large) frequency, ν_q is a unit coordinate direction and η_q, ζ_q are normal vector fields to the image $u_q(\Omega) \subset \mathbb{R}^{n+2}$. A short calculation gives

$$\partial_i \tilde{u}_q \cdot \partial_j \tilde{u}_q = \partial_i u_q \cdot \partial_j u_q + a_q^2 \nu_q^i \nu_q^j + O(\lambda_{q+1}^{-1}), \tag{2.17}$$

so that, choosing the frequency λ_{q+1} sufficiently large, one can achieve a correction to the metric by $a_q^2(x)\nu_q^i\nu_q^j$ plus a small error. On the other hand a decomposition of the metric error as

$$(g - Du_q^T Du_q)(x) = \sum_k a_{q,k}(x)\nu_k \otimes \nu_k \tag{2.18}$$

allows one to choose ν_q and a_q at each step $q \in \mathbb{N}$ suitably to achieve an iterative correction of the error. Such a decomposition (where the unit vectors ν_k do not depend on x) can be obtained from the following lemma from [68], giving a kind of partition of unity on $\mathcal{P}$, the space of positive definite matrices.

Lemma 2.9 (Decomposing the metric error)**.** *There exists a sequence $\{\xi^k\}$ of unit vectors in $\mathbb{R}^n$ and a sequence $\gamma_k \in C_c^\infty(\mathcal{P}; [0, \infty))$ such that*

$$A = \sum_k \gamma_k^2(A)\xi^k \otimes \xi^k \qquad \forall A \in \mathcal{P},$$

and there exists a number $n_* \in \mathbb{N}$ depending only on n such that, for all $A \in \mathcal{P}$ at most n_* of the $\gamma_k(A)$ are nonzero.

With this lemma at hand, we can define

$$a_{q,k}(x) = \gamma_k\bigg(g(x) - Du_q(x)^T Du_q(x)\bigg),$$

so that (2.18) holds. Observe that the sum in (2.18) is finite consisting of, say, N terms, and for any x there are at most n_* nonzero terms. Then, define inductively

$$u_{q,k+1}(x) = u_{q,k}(x) + \frac{a_{q,k}(x)}{\lambda_{q,k}}\Big(\sin(\lambda_{q,k}x \cdot \nu_k)\eta_{q,k}(x) + \cos(\lambda_{q,k}x \cdot \nu_k)\zeta_{q,k}(x)\Big),$$

where $u_{q,0} = u_q$, $\eta_{q,k}, \zeta_{q,k}$ are the unit normal vector fields to $u_{q,k}(\Omega)$ and $\lambda_{q,k}$ is chosen inductively so that the error terms in (2.17) remain small. After finite number of steps we arrive at u_{q+1}, which satisfies

$$Du_{q+1}^T Du_{q+1} = g + \sum_{k=1}^N O(\lambda_{q,k}^{-1}).$$

By iterating the previous construction, we can successively remove the error and arrive at an isometric map u.

The final map will have the form

$$u(x) = \sum_{q=0}^{\infty}\sum_{k=1}^{N}\frac{1}{\lambda_{q,k}}w_{q,k}(x,\lambda_{q,k}x),$$

where each $w_{q,k}$ is one such spiral. Ensuring that the final map is C^1 then just requires controlling the amplitudes $\delta_q^{1/2} := \sup_{x,k}|w_{q,k}|$ so that $\sum_q \delta_q^{1/2} < \infty$. Such control is possible since the amplitude $\sup_x |w_{q,k}| \sim \sup_x |a_{q,k}|$ only depends on the metric error at step q through the decomposition (2.18), but not on the frequency $\lambda_{q,k}$. We refer to the lecture notes [74] for a detailed expository proof.

Recently the construction of Nash (more precisely the construction of Kuiper, where the spiral from (2.16) needs to be replaced by a corrugation) has been visualized for the flat 2-torus in [13], where beautiful pictures showing the fractal nature of the construction have been presented.

2.5. $C^{1,\theta}$ isometric embeddings

In light of part (i) and (ii) of Theorem 2.1 an interesting question, that has been raised in several places ([54], [79]) is what happens with isometric immersions of S^2 of class $u \in C^{1,\theta}$. Yu. F. Borisov investigated isometric embeddings of class $u \in C^{1,\theta}$. He showed in [9, 10] the validity of the rigidity statement (i) in Theorem 2.1 for $u \in C^{1,\theta}$ with $\theta > 2/3$. The Nash-Kuiper construction has been revisited in [11, 12, 35], where sharper estimates on the approximating sequence have been obtained. In particular, it can be shown that one can additionally ensure that (i) $N = n_*$ and (ii) the estimate

$$\delta_q^{1/2} \lesssim \lambda_{q,k}^{-\frac{1}{1+2n_*}} \tag{2.19}$$

holds for all k. Such an estimate immediately leads to an improved regularity:

Theorem 2.10 ([12, 35])**.** *For any positive definite $g_0 \in \mathbb{R}^{n\times n}$ there exists $r > 0$ such that the following holds: For any smooth bounded $\Omega \subset \mathbb{R}^n$ equipped with a smooth Riemannian metric g such that $\|g - g_0\|_{C^0} \le r$, there exists a constant $\delta_0 > 0$ such that, if $u \in C^2(\overline{\Omega};\mathbb{R}^{n+1})$ is such that*

$$\|Du^T Du - g\|_0 \le \delta_0$$

then for any $\theta < \frac{1}{1+2n_*}$ there exists $v \in C^{1,\theta}(\overline{\Omega};\mathbb{R}^{n+1})$ with

$$Dv^T Dv \ = \ g$$

and moreover

$$\|v - u\|_{C^1} \ \le \ C\,\|Du^T Du - g\|_{C^0}^{1/2}.$$

The condition (i) above is achieved by ensuring that the metric error $g - Du_q^T Du_q$ is contained in a single patch of the decomposition in Lemma 2.9 (namely the patch containing g_0 in the theorem), and (ii) requires estimating the $O(\lambda_{q,k}^{-1})$ terms in (2.17). Even if condition (i) is not satisfied, one can adapt the construction above so that the number of terms in the sum (2.18) is bounded by a fixed number depending only on the dimension n. In this way one is lead to the following global version of Theorem 2.10.

Theorem 2.11 ([35])**.** *The Nash-Kuiper theorem remains valid for isometric embeddings of class $C^{1,\theta}$ with $\theta < \frac{1}{1+2(n+1)n_*}$.*

Observe that for $n = 2$ we have $n_* = 3$, so that Theorem 2.10 guarantees the existence of local isometric maps (i.e. for instance non-trivial bendings of convex surfaces) of class $C^{1,\theta}$ with $\theta < 1/7$. It turns out that, by utilizing conformal coordinates the exponent in this case can be improved to $\theta < 1/5$, thereby confirming a conjecture of Borisov from [11]:

Theorem 2.12 ([40])**.** *Let g be a C^2 metric on $\overline{D}_1$, the unit disc in $\mathbb{R}^2$ and $u \in C^1(\overline{D}_1,\mathbb{R}^3)$ a short embedding. For every $\theta < 1/5$ and $\delta > 0$ there exists an isometric embedding $v \in C^{1,\theta}$ of $(\overline{D}_1, g)$ into $\mathbb{R}^3$ such that $\|u - v\|_{C^0} < \delta$.*

In the case of isometric embeddings there does not seem to be a universally accepted sharp exponent θ_0 separating cases (i) and (ii) of Theorem 2.1 (see Problem 27 in [79]), even though 1/2 and 1/3 both seem relevant (compare with the discussion in [12]). For instance, consider an isometric map $u \in C^{1,\theta}$ with $\theta > 1/2$ and fix a symmetric mollifying kernel φ as in Lemma 1.7. Since $Du^T Du = g$, analogously to estimate (1.10) one obtains

$$\|Du_\ell^T Du_\ell - g_\ell\|_1 \le C\ell^{2\theta-1}[Du]_\theta^2,$$

where $u_\ell = \varphi_\ell * u$. By considering the expression for the Christoffel symbols of a Riemannian manifold in terms of the metric, we then deduce that

$$(\Gamma_\ell)^i_{jk} \to \Gamma^i_{jk} \quad \text{uniformly},$$

where Γ_ℓ denotes the Christoffel tensor for the induced surface by u_ℓ and Γ denotes the Christoffel tensor corresponding to the metric g. In turn, this implies that (extrinsic) parallel transport on the embedded $C^{1,\theta}$ surface can be defined via the (intrinsic) metric g (corresponding to the results of Borisov in [7, 8]) and hints at the absence of h–principle for $C^{1,1/2+\varepsilon}$ immersions. One might further notice that the regularity $C^{1,1/3+\varepsilon}$ is still enough to guarantee a very weak notion of convergence of the Christoffel symbols.

3. The Euler-Reynolds system

In the remaining sections we show the key ideas leading to the proofs of Theorem 1.8. Although the basic scheme follows the one introduced in [44], the presentation here uses crucial ideas that were introduced subsequently in the PhD Theses of T. Buckmaster [18] and of P. Isett [57].

The construction of continuous and Hölder-continuous solutions of (W) follows the basic strategy of Nash in the sense that at each step of the iteration, a highly oscillatory correction as the spiral in (2.16) is added. Note that both (E) and the equation of isometries (2.2) is quadratic – the oscillatory perturbation is chosen in such a way as to minimize the linearization, making the quadratic part of leading order. In turn, a finite-dimensional decomposition of the error (c.f. (2.18)) is used to control the quadratic part. There are, however, two important differences:

- The linearization of (2.2) is controlled easily by using the extra codimension(s) in the Nash proof. For Euler, the linearization of (E) leads to a transport equation, which is very difficult to control over long times and leads to a kind of CFL condition, c.f. Lemma 4.1 below. This issue is still the main stumbling block in the full resolution of Onsager's conjecture and is the subject of Section 4 below.

- The exponent 1/3 of Onsager's conjecture requires a sufficiently good correction of the error at each single step, whereas in the Nash iteration several steps (n_* steps) are required – this leads to the exponent $(1 + 2n_*)^{-1}$ in Theorem 2.10. Consequently one-dimensional oscillations, as used in the Nash-Kuiper scheme and, more generally, in convex integration, cannot be used[1] for part (b) of Conjecture 1.6. Thus, instead of convex integration, we use Beltrami flows, a special family of periodic stationary flows, as the replacement of (2.16) (compare (2.18) with (3.26)).

3.1. Inductive estimates

In analogy with the Nash-Kuiper construction explained in Section 2.4, we construct a sequence of triples $(v_q, p_q, \mathring{R}_q)$, $q \in \mathbb{N}$, solving the Euler-Reynolds system (see [44, Definition 2.1]):

$$\begin{aligned} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q &= -\operatorname{div} \mathring{R}_q, \\ \operatorname{div} v_q &= 0\,, \end{aligned} \tag{3.1}$$

where (v_q, p_q) is an approximate solution and $\mathring{R}_q$ is a traceless symmetric 3×3 tensor, i.e. $\mathring{R}_q(x,t) \in \mathcal{S}_0^{3\times 3}$. Here (v_q, p_q) is thought of as the approximation (corresponding to Du_q in Section 2.4) and $\mathring{R}_q$ is the analogue of the metric error $g - Du_q^T Du_q$.

[1]However, see also [59], where one-dimensional oscillations more closely following the Nash iteration are used for a general class of active scalar equations, albeit leading to suboptimal Hölder exponents.

Observe that, in terms of approximations, we have written the error in the right hand side as the divergence of a traceless symmetric tensor. That this involves no loss of generality is the consequence of the following lemma:

Lemma 3.1 (The operator div^{-1}). *There exists a homogeneous Fourier-multiplier operator of order -1, denoted*

$$\mathrm{div}^{-1} : C^\infty(\mathbb{T}^3;\mathbb{R}^3) \to C^\infty(\mathbb{T}^3;\mathcal{S}_0^{3\times 3})$$

such that, for any $f \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ with average $\fint_{\mathbb{T}^3} f = 0$ we have

(a) $\mathrm{div}^{-1} f(x)$ *is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;*

(b) $\mathrm{div}\,\mathrm{div}^{-1} f = f$.

Proof. The proof follows from direct calculation by defining div^{-1} as

$$\mathrm{div}^{-1} f := \frac{1}{4}\left(\nabla\mathcal{P}g + (\nabla\mathcal{P}g)^T\right) + \frac{3}{4}\left(\nabla g + (\nabla g)^T\right) - \frac{1}{2}(\mathrm{div}\, g)\mathrm{Id},$$

where $g \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ is the solution of $\Delta g = f - \fint_{\mathbb{T}^3} f$ in $\mathbb{T}^3$ and $\mathcal{P}$ is the Leray projector onto divergence-free fields with zero average. □

The *size* of the perturbation

$$w_q := v_q - v_{q-1}$$

will be measured by two parameters:

$$\textit{amplitude: } \delta_q^{1/2}, \qquad \textit{frequency: } \lambda_q,$$

where, along the iteration, we will have $\delta_q \to 0$ and $\lambda_q \to \infty$ at a rate that is (at least) exponential. For the sake of definiteness and for comparison with the Littlewood-Paley approach to turbulence (see [31, 26]) we may think

$$\lambda_q \sim a^q \quad \text{for some } a > 1,$$

(although in the actual proofs a slightly super-exponential growth is required). Here and in what follows, $A \lesssim B$ means that $A \le cB$ for some universal constant c, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Then, up to controllable errors, w_q will be a function with Fourier-support localized at frequencies comparable to λ_q (in other words a single Littlewood-Paley piece). The more precise formulation is that, denoting the sup-norm by $\|\cdot\|_0$,

$$\|w_q\|_0 \lesssim \delta_q^{1/2}\,, \tag{3.2}$$

$$\|\nabla w_q\|_0 \lesssim \delta_q^{1/2}\lambda_q\,, \tag{3.3}$$

and similarly,

$$\|p_q - p_{q-1}\|_0 \lesssim \delta_q\,, \tag{3.4}$$

$$\|\nabla(p_q - p_{q-1})\|_0 \lesssim \delta_q\lambda_q\,. \tag{3.5}$$

In constructing the iteration, the new perturbation w_{q+1} will be chosen so as to balance the previous Reynolds error $\mathring{R}_q$, in the sense that (cf. equation (2.11)) we have $\|w_{q+1}\otimes w_{q+1}\|_0 \sim \|\mathring{R}_q\|_0$. In terms of estimates this is formalized as

$$\|\mathring{R}_q\|_0 \lesssim \delta_{q+1}\,, \tag{3.6}$$

$$\|\nabla\mathring{R}_q\|_0 \lesssim \delta_{q+1}\lambda_q\,, \tag{3.7}$$

We might think of w_q as a mathematical realization of the concept of *eddy* in phenomenological descriptions of turbulence, c.f. [53, Ch 7]. Then, corresponding to eddies at "scale q" we have the following characteristic scales:

- Eddy length scale: $L_q \sim \frac{1}{\lambda_q}$;
- Eddy velocity scale: $U_q \sim \delta_q^{1/2}$;
- Eddy time scale: $T_q = \frac{L_q}{U_q} \sim \frac{1}{\delta_q^{1/2}\lambda_q}$.

To see that this is consistent with our estimates above, observe that from (3.1) we obtain $(\partial_t + v_q \cdot \nabla)v_q = \operatorname{div} \mathring{R}_q - \nabla p_q$. Hence, using (3.5) and (3.7),

$$\|(\partial_t + v_q \cdot \nabla)v_q\|_0 \lesssim (\delta_{q+1} + \delta_q)\lambda_q \lesssim \delta_q\lambda_q,$$

which agrees with $\frac{U_q}{T_q}$. Similarly, we will also impose the estimate

$$\|(\partial_t + v_q \cdot \nabla)\mathring{R}_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\,. \tag{3.8}$$

The idea to control the transport derivative $(\partial_t + v_q \cdot \nabla)$ instead of the pure time derivative ∂_t of (v_q, p_q, R_q) was introduced to the scheme by P. Isett in [57].

On the one hand (3.2), (3.4) and (3.6) will imply the convergence of the sequence v_q to a continuous weak solution of the Euler equations. On the other hand the precise dependence of λ_q on δ_q will determine the critical Hölder regularity, similarly to the Nash-Kuiper scheme and Theorem 2.11 above. Finally, control on the energy will be ensured by

$$\left| e(t)(1-\delta_{q+1}) - \int |v_q|^2(x,t)\,dx \right| \le \frac{1}{4}\delta_{q+1}e(t)\,. \tag{3.9}$$

3.2. Conditions on the fluctuation

We define

$$\rho_q(t) = \frac{1}{3(2\pi)^3}\left(e(t)(1-\delta_{q+1}) - \int |v_q|^2(x,t)\,dx \right) \tag{3.10}$$

and

$$R_q(x,t) = \rho_q(t)\mathrm{Id} + \mathring{R}_q(x,t). \tag{3.11}$$

It is not difficult to check that (3.9) ensures

$$\rho_q(t) \sim \delta_{q+1}e(t), \tag{3.12}$$

so that $\|R_q\|_0 \sim \delta_{q+1}$ (c.f. with (3.6)). Moreover, since ρ_q is a function of time only, we can write the Euler-Reynolds system (3.1) as

$$\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = -\operatorname{div} R_q,$$
$$\operatorname{div} v_q = 0$$

in analogy with the Reynolds-averaged system (2.10). Our aim is to choose the next perturbation w_{q+1} in such a way as to model the fluctuation w in (2.12) leading to the Reynolds stress. Following the idea of Nash and the spiral from (2.16) we make the *ansatz*

$$w_{q+1}(x,t) = W\Big(v_q(x,t), R_q(x,t), \lambda_{q+1}x, \lambda_{q+1}t\Big) + w_{\mathrm{corrector}}(x,t). \tag{3.13}$$

The corrector $w_{\mathrm{corrector}}$ is added to ensure that $\operatorname{div} w_{q+1} = 0$, but for the sake of not overburdening this exposition with technicalities, we will assume it to be negligible subsequently. The key point is how to choose the function $W = W(v, R, \xi, \tau)$.

We make the following assumptions on W:

- $\xi \mapsto W(v, R, \xi, \tau)$ is 2π-periodic with vanishing average, i.e.

$$\langle W \rangle := \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} W(v, R, \xi, \tau)\,d\xi = 0; \tag{H1}$$

- The average stress is given by R, i.e.

$$\langle W \otimes W \rangle = R \tag{H2}$$

  for all R in a suitable cone containing the identity matrix;

- The "cell problem" is satisfied:

$$\begin{aligned} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi(W \otimes W) + \nabla_\xi P &= 0, \\ \operatorname{div}_\xi W &= 0, \end{aligned} \tag{H3}$$

  where $P = P(v, R, \xi, \tau)$ is a suitable pressure;

- W is smooth in all its variables and satisfies the estimates
$$|W| \lesssim |R|^{1/2},\ |\partial_v W| \lesssim |R|^{1/2},\ |\partial_R W| \lesssim |R|^{-1/2}. \tag{H4}$$

Observe that (H1)-(H2) correspond to (2.11), (H3) arises from plugging the ansatz (3.13) into Euler, and (H4) are estimates consistent with (H2).

As a consequence of (H1)-(H2) we obtain
$$\int_{\mathbb{T}^3} |v_{q+1}|^2\, dx \sim \int_{\mathbb{T}^3} |v_q|^2\, dx + \int_{\mathbb{T}^3} \langle |W|^2 \rangle\, dx = \int_{\mathbb{T}^3} |v_q|^2\, dx + 3(2\pi)^3 \rho_q(t),$$
so that (3.9) can be ensured inductively. The main issue is therefore to show that indeed, $\delta_q \to 0$ with $q \to \infty$ (so that the scheme converges) and to obtain a relationship between δ_q and λ_q. More precisely, we are aiming at a relationship of the form
$$\delta_q^{1/2} \sim \lambda_q^{-\theta_0} \qquad \text{for all } q \in \mathbb{N}$$
since, comparing with (3.2)-(3.3), this implies that the limit function $v = \lim_{q\to\infty} v_q$ will be Hölder continuous with any exponent $\theta < \theta_0$ (c.f. (2.19) and Theorem 2.11 above). In order to obtain such a relationship, we need to estimate the new "Reynolds stress" $\mathring{R}_{q+1}$.

3.3. Estimating the new Reynolds stress

Assuming the existence of a function W satisfying (H1)-(H4) above, we can use the ansatz from (3.13) to obtain an estimate on the new "Reynolds stress" $\mathring{R}_{q+1}$. Indeed, since $v_{q+1} = v_q + w_{q+1}$, we have
$$\begin{aligned}
\mathring{R}_{q+1} &= \operatorname{div}^{-1}\Big[\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1}\Big] \\
&= \operatorname{div}^{-1}\Big[\partial_t w_{q+1} + v_q \cdot \nabla w_{q+1}\Big] && (3.14)\\
&\quad + \operatorname{div}^{-1}\Big[\operatorname{div}(w_{q+1} \otimes w_{q+1} - R_q) + \nabla(p_{q+1} - p_q)\Big] && (3.15)\\
&\quad + \operatorname{div}^{-1}\Big[w_{q+1} \cdot \nabla v_q\Big] && (3.16)\\
&= \mathring{R}^{(1)}_{q+1} + \mathring{R}^{(2)}_{q+1} + \mathring{R}^{(3)}_{q+1}\,, && (3.17)
\end{aligned}$$
where div^{-1} is the operator of order -1 from Lemma 3.1.

Consider first the term (3.16) (and remember that we ignore the corrector $w_{\text{corrector}}$). Recalling condition (H1) on W, we can expand $\xi \mapsto W(v, R, \xi, \tau)$ in a Fourier-series and write
$$\mathring{R}^{(3)}_{q+1} = \operatorname{div}^{-1}\Big[W \cdot \nabla v_q\Big] = \operatorname{div}^{-1} \sum_{k\in\mathbb{Z}^3, k\neq 0} a_k(x,t) e^{i\lambda_{q+1} k\cdot x}, \tag{3.18}$$
where, using (H2)
$$\|a_k\|_0 \lesssim \|W\|_0 \|\nabla v_q\|_0 \lesssim \|R_q\|_0^{1/2} \|\nabla v_q\|_0.$$
Since a_k depends on v_q and R_q, which, owing to our inductive estimates are localized in frequency space to frequencies $\sim \lambda_q$, and since we assume $\lambda_{q+1} \gg \lambda_q$, one may hope for an estimate from (3.18) of the type
$$\left\|\operatorname{div}^{-1}\left[a_k(x,t) e^{i\lambda_{q+1}k\cdot x}\right]\right\|_0 \lesssim \frac{1}{\lambda_{q+1}} \|a_k\|_0, \tag{3.19}$$
provided $k \neq 0$ (since in that case $|k| \geq 1$, there is no issue about small divisors). This estimate can be made rigorous in Hölder spaces using stationary phase arguments, essentially using integration by parts and Schauder estimates (see [44]). For the sake of simplicity in the presentation, let us assume that (3.19) is correct. Using our inductive estimates we then obtain
$$\|\mathring{R}^{(3)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}}.$$

Next, consider (3.14)-(3.15). Here one needs to differentiate W in x and t, and one needs to differentiate between "slow" and "fast" derivatives – where we refer to "fast derivatives" if the term involves a factor of λ_{q+1}. For instance

$$\partial_t W = \underbrace{\partial_v W \partial_t v_q + \partial_R W \partial_t R_q}_{\text{slow}} + \underbrace{\lambda_{q+1}\partial_\tau W}_{\text{fast}}.$$

However, owing to condition (H3) (the "cell problem") the fast derivatives in $\mathring{R}^{(1)}_{q+1} + \mathring{R}^{(2)}_{q+1}$ vanish identically. Hence, by some abuse of notation, we may replace (3.14) and (3.15) by

$$\mathring{R}^{(1)}_{q+1} = \operatorname{div}^{-1}\Big[(\partial_t + v_q\cdot\nabla)^{\text{slow}} W\Big], \tag{3.20}$$

$$\mathring{R}^{(2)}_{q+1} = \operatorname{div}^{-1}\Big[\operatorname{div}^{\text{slow}}(W\otimes W - R_q)\Big]. \tag{3.21}$$

Observe that the expression in (3.20) is linear in W, hence the same stationary phase argument as above applies. We calculate:

$$(\partial_t + v_q\cdot\nabla)^{\text{slow}} W = \partial_v W\,(\partial_t + v_q\cdot\nabla)v_q + \partial_R W\,(\partial_t + v_q\cdot\nabla)R_q$$

so that, writing as before,

$$\mathring{R}^{(1)}_{q+1} = \operatorname{div}^{-1}\sum_{k\in\mathbb{Z}^3, k\neq 0} b_k(x,t)e^{i\lambda_{q+1}k\cdot x}$$

for some b_k. Using (H4), we have

$$\|b_k\|_0 \lesssim \|R_q\|_0^{1/2}\|(\partial_t + v_q\cdot\nabla)v_q\|_0 + \|R_q\|_0^{-1/2}\|(\partial_t + v_q\cdot\nabla)R_q\|_0.$$

From the inductive estimates on v_q and R_q in Section 3.1 we then deduce

$$\begin{aligned}\|\mathring{R}^{(1)}_{q+1}\|_0 &\lesssim \frac{1}{\lambda_{q+1}}\left(\delta_{q+1}^{1/2}\delta_q\lambda_q + \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\right)\\ &\lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.\end{aligned}$$

Finally, observe that in (3.21) we have $\langle W\otimes W\rangle = R_q$ because of condition (H2), so that once more, in the expansion of W as a Fourier-series in ξ there is no term $k=0$. Hence the same stationary phase estimate can be applied once more. Writing

$$\mathring{R}^{(2)}_{q+1} = \operatorname{div}^{-1}\sum_{k\in\mathbb{Z}^3, k\neq 0} c_k(x,t)e^{i\lambda_{q+1}k\cdot x}$$

and using (H4) we have the estimate

$$\begin{aligned}\|c_k\|_0 &\lesssim \|W\|_0\|\partial_v W\|_0\|Dv_q\|_0 + \|W\|_0\|\partial_R W\|_0\|DR_q\|_0\\ &\lesssim \|R_q\|_0\|Dv_q\|_0 + \|DR_q\|_0,\end{aligned}$$

so that

$$\begin{aligned}\|\mathring{R}^{(2)}_{q+1}\|_0 &\lesssim \frac{1}{\lambda_{q+1}}\left(\delta_{q+1}\delta_q^{1/2}\lambda_q + \delta_{q+1}\lambda_q\right)\\ &\lesssim \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1}}.\end{aligned} \tag{3.22}$$

Summarizing, we obtain

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}. \tag{3.23}$$

Of course, this is just one of the inductive estimates in Section 3.1, similar estimates should be obtained for all the other quantities (3.2)-(3.8). However, this estimate already implies a relationship between δ_q and λ_q. Indeed, comparing (3.6) and (3.23), the inductive step requires

$$\delta_{q+2} \sim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.$$

Assuming $\lambda_q \sim \lambda^q$ for some fixed $\lambda \gg 1$, this would lead to

$$\delta_q^{1/2} \sim \lambda^{-q/3} \sim \lambda_q^{-1/3}, \tag{3.24}$$

which, comparing with (3.2)-(3.3), *precisely gives exponent* $1/3$ *as the critical Hölder regularity.* Unfortunately, there are several assumptions made in the derivation above. Most importantly, we have assumed the existence of W with properties (H1)-(H4).

3.4. Beltrami flows

The estimates in the preceeding section were based on the existence of a "fluctuation profile" $W = W(v, R, \xi, \tau)$ satisfying the conditions (H1)-(H4). In order to construct such a fluctuation profile, our starting point is the class of Beltrami flows. These are a special class of stationary flows in $\mathbb{T}^3$ based on the identity

$$\operatorname{div}(W \otimes W) = W \times \operatorname{curl} W - \tfrac{1}{2}\nabla|W|^2.$$

In particular any eigenspace of the curl operator, i.e. solution space of the system

$$\operatorname{curl} W = \lambda_0 W$$
$$\operatorname{div} W = 0$$

leads to a *linear* space of stationary flows. These can be written as

$$\sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi} \tag{3.25}$$

for normalized complex vectors $B_k \in \mathbb{C}^3$ satisfying

$$|B_k| = 1, \quad k \cdot B_k = 0 \quad \text{and} \quad ik \times B_k = \lambda_0 B_k,$$

and arbitrary coefficients $a_k \in \mathbb{C}$. Choosing $B_{-k} = -\overline{B_k}$ and $a_{-k} = \overline{a_k}$ ensures that W is real-valued. A calculation then shows

$$\langle W \otimes W \rangle = \frac{1}{2} \sum_{|k|=\lambda_0} |a_k|^2 \Big(\mathrm{Id} - \frac{k \otimes k}{|k|^2}\Big). \tag{3.26}$$

This identity leads to the following version of Lemma 2.9:

Lemma 3.2. *For every* $N \in \mathbb{N}$ *we can choose* $0 < r_0 < 1$ *and* $\bar{\lambda} > 1$ *with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \bar{\lambda}\} \qquad j \in \{1, \dots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id})) \qquad j \in \{1, \dots, N\}, k \in \Lambda_j$$

such that

(a) $k \in \Lambda_j$ *implies* $-k \in \Lambda_j$ *and* $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$*;*

(b) *For each* $R \in B_{r_0}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2} \sum_{k\in\Lambda_j} \Big(\gamma_k^{(j)}(R)\Big)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \qquad \forall R \in B_{r_0}(\mathrm{Id}). \tag{3.27}$$

There are two differences to Lemma 2.9. First of all, Lemma 3.2 provides several "independent" decompositions of R, corresponding to the N families $(\gamma_k^j)_k$, $j = 1 \dots N$. We will return to the significance of this in the next section. Secondly and more importantly, the decomposition in (3.27) is valid only in the r_0-neighbourhood of the identity matrix. This corresponds to a single patch of the decomposition in Lemma 2.9 – compare with the discussion in Section 2.4 concerning the estimate (2.19).

This lemma, taken from [44] (see also [57] for a geometric proof) allows us to choose the amplitudes as

$$a_k = \rho^{1/2} \gamma_k \left(\frac{R}{\rho}\right),$$

where we write (c.f. (3.11)) $R = \rho\mathrm{Id} + \mathring{R}$ (i.e. $\rho = \frac{1}{3}\mathrm{tr}\, R$). Observe that the restriction to lie in $B_{r_0}(\mathrm{Id})$ then translates into

$$|\mathring{R}| \le r_0 \rho.$$

In light of the assumption (3.6) and the choice of ρ_q (see (3.12)) this requirement can be satisfied by an appropriate choice of constants.

With this choice of $a_k = a_k(R)$, $W = W(R, \xi)$ defined by the Beltrami-flow (3.25) satisfies (H1), (H2), (H4). Moreover, (H3) is also valid *for* $v = 0$, but the transport part of the cell problem (i.e. the term $\partial_\tau + v \cdot \nabla_\xi$) poses problems if $v \neq 0$.

4. The role of time

In Section 3.4 we showed that an appropriate family (given by the decomposition of Lemma 3.2) of stationary Beltrami flows can be used to define a fluctuation profile $W = W(R, \xi)$ as in (3.25), for which (H1), (H2) and (H4) are satisfied. The key point remaining is to modify W so that also (H3) holds. Unfortunately it turns out that this is not possible without modifications and additional error terms. Before discussing these modifications, let us take a second look at (H3). The difficulties are caused by the linear part of the equation, namely the term $(\partial_t + v \cdot \nabla)W$, representing the linearization of the Euler equations. As pointed out at the beginning of Section 3, the convex integration method relies on oscillatory perturbations, where the quadratic term will be of leading order and the linear term will be small - at variance to the Newton scheme. This can be seen very clearly in the proof of the Nash-Kuiper theorem, compare with (2.16) and (2.17). However, a key point is to use an ansatz for the perturbation which also guarantees smallness of *derivatives* of the perturbation appearing linearly in the equation. In the isometric embedding problem this is achieved by the choice of normal vectorfields in (2.16). For the Euler equations the corresponding term $(\partial_t + v \cdot \nabla)W$ is more difficult to control, and this seems to be the central obstruction in the construction of 1/3-Hölder continuous weak solutions. Nevertheless, the time-dependence helps to at least in some sense distribute the error – indeed, the methods below do not work for the stationary Euler equations, whereas there are analogous “weak h-principle” results to Theorem 2.6 for the stationary case, see [28].

4.1. Approximate Galilean transformations

In [44, 45] a “phase function” $\phi_k(v, \tau)$ was introduced to deal with the transport in the cell problem. By considering W of the form

$$\sum_{|k|=\lambda_0} a_k(R)\phi_k(v,\tau) B_k e^{ik\cdot\xi} \tag{4.1}$$

the cell problem in (H3) leads to the equation

$$\partial_\tau \phi_k + i(v \cdot k)\phi_k = 0.$$

However, the exact solution

$$\phi_k(v, \tau) = e^{-i(v\cdot k)\tau}, \tag{4.2}$$

which would correspond to a Galilean transformation, is incompatible with the requirement (H4) because $|\partial_v \phi_k| \sim |\tau|$ is unbounded. Instead, an approximation is used[2] such that

$$\partial_\tau \phi_k + i(v \cdot k)\phi_k = O\left(\mu_q^{-1}\right), \qquad |\partial_v \phi_k| \lesssim \mu_q$$

for some parameter μ_q. This leads to the following corrections to (H3) and (H4): (H3) is only satisfied approximately:

$$\partial_\tau W + v \cdot \nabla_\xi W + \mathrm{div}_\xi (W \otimes W) + \nabla_\xi P = O(\mu_q^{-1})$$

and in (H4) the second inequality is replaced by

$$|\partial_v W| \lesssim \mu_q |R|^{1/2}.$$

[2]To be precise, the approximation involves a partition of unity over the space of velocities with 8 families.

Carrying out the calculations in Section 3.3 with these corrections leads to[3]

$$\|\mathring{R}^{(1)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\mu_q} + \frac{\delta_{q+1}^{1/2}\mu_q\delta_q\lambda_q}{\lambda_{q+1}}$$

whereas

$$\|\mathring{R}^{(2)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}\lambda_q + \delta_{q+1}\mu_q\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.$$

The term (3.16) remains as before

$$\|\mathring{R}^{(3)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}.$$

By assuming $\delta_{q+1} \leq \delta_q$, we can easily identify the worst error terms (both coming from (3.14)) as

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\mu_q} + \frac{\delta_{q+1}^{1/2}\mu_q\delta_q\lambda_q}{\lambda_{q+1}}$$

instead of (3.23). We can optimize in μ_q by setting

$$\mu_q = \delta_q^{-1/2}\lambda_q^{-1/2}\lambda_{q+1}^{1/2},$$

since this choice minimizes the right hand side. Comparing then with the inductive estimate as in Section 3.3 one arrives at

$$\delta_q^{1/2} \sim \lambda_q^{-1/6}, \tag{4.3}$$

corresponding to Hölder exponents $\theta < 1/6$ (in fact $\theta < 1/10$ in [45] because of the worse estimate for $\|\mathring{R}^{(1)}_{q+1}\|_0$).

4.2. Transporting microstructure

One can obtain an improvement on the previous estimate (4.3) by realizing that it is better to make an error in the quadratic term of (H3) rather than the linear transport term. This was one key new idea in [57] and, following [20] leads to the modified *ansatz* (c.f. (3.13))

$$w_{q+1}(x,t) = W\Big(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)\Big) + w_{\text{corrector}}(x,t), \tag{4.4}$$

where $\Phi_q(x,t)$ is the solution of

$$\partial_t\Phi_q + v_q \cdot \nabla\Phi_q = 0, \tag{4.5}$$

with $\Phi_q(x,0) = x$. Recall that $\Phi_q(\cdot,t)$ above is the inverse flow of v_q, i.e. $\Phi_q(\cdot,t) = X_q^{-1}(\cdot,t)$, where X_q is the flow generated by v_q, i.e.

$$\frac{d}{dt}X_q = v_q(X_q,t)$$

with $X_q(x,0) = x$. Applying formula (4.4) to the Beltrami flows leads to a fluctuation of the form

$$\sum_k a_k\big(R_q(x,t)\big)B_k e^{i\lambda_{q+1}k\cdot\Phi_q(x,t)} = \sum_k a_k\big(R_q(x,t)\big)\phi_k(x,t)B_k e^{i\lambda_{q+1}k\cdot x},$$

where the new phase function is given by

$$\phi_k(x,t) = e^{i\lambda_{q+1}k\cdot(\Phi_q(x,t)-x)}.$$

Observe that

$$\Phi(x,t) - x = -v_q(x,0)t + O(t^2) \quad \text{as } t \to 0$$

so that this ansatz is another way of approximating (4.2). However, now we have abandoned the idea that the phase ϕ should depend explicitly in v_q and t, and abandoned the original ansatz (3.13), where W was a function of v_q, R_q, ξ, τ. Now the dependence on v_q is through the equation (4.5), i.e. via Lagrangian rather than Eulerian coordinates.

By defining w_{q+1} via the formula (4.4), we automatically take care of the transport part of the cell problem (i.e. the "fast derivatives"). On the other hand there will be a new error to the

[3] In [45] it was not used that estimating $(\partial_t + v_q\cdot\nabla)v_q$ (by using the equation) is better than estimating separately $\partial_t v_q$ and $v_q \cdot \nabla v_q$. Therefore the estimate in [45] is $\mathring{R}^{(1)}_{q+1} \lesssim \frac{\delta_{q+1}^{1/2}}{\mu_q} + \frac{\delta_{q+1}^{1/2}\mu_q\delta_q^{1/2}\lambda_q}{\lambda_{q+1}}$.

quadratic part of the cell problem due to the deformation matrix $D\Phi_q$. Furthermore, as we expect v_q to converge to a Hölder continuous flow with $\|Dv_q\|_0 \to \infty$ as $q \to \infty$, one only has control over $D\Phi_q$ for very short times. Indeed, we have the following well-known lemma, whose proof is a simple application of Gronwall's inequality.

Lemma 4.1. *Let $v \in C^\infty(\mathbb{T}^3 \times \mathbb{R}; \mathbb{R}^3)$ be a smooth vectorfield and $\Phi(t, \cdot)$ the solution of*

$$\partial_t \Phi + v \cdot \nabla \Phi = 0,$$
$$\Phi(x, t_0) = x$$

for some t_0. Then

$$\|D\Phi(t) - \mathrm{Id}\|_0 \le e^{(t-t_0)\|Dv\|_0} - 1.$$

To handle this problem we consider a partition of unity $(\chi_j)_j$ on the time interval $[0, T]$ such that the support of each χ_j is an interval I_j of size $\frac{1}{\mu_q}$ for some $\mu_q \gg 1$. In each time interval I_j we set $\Phi_{q,j}$ to be the solution of the transport equation (4.5) which satisfies

$$\Phi_{q,j}(x, t_j) = x,$$

where t_j is the center of the interval I_j. Recalling that $\|Dv_q\|_0 \lesssim \delta_q^{1/2}\lambda_q$, Lemma 4.1 leads to

$$\|D\Phi_{q,j}\|_0 = O(1) \quad \text{and} \quad \|D\Phi_{q,j} - \mathrm{Id}\|_0 \lesssim \frac{\delta_q^{1/2}\lambda_q}{\mu_q} \tag{4.6}$$

provided

$$\mu_q \ge \delta_q^{1/2}\lambda_q, \tag{4.7}$$

an estimate we will henceforth assume. Observe also that $|\partial_t \chi_j| \lesssim \mu_q$.

The new fluctuation will take the form

$$\begin{aligned} w_{q+1}(x,t) &= \sum_j \chi_j(t) \sum_{|k|=\lambda_0} a_{kj}(R_q) B_k e^{i\lambda_{q+1} k \cdot \Phi_{q,j}(x,t)} + w_{\text{corrector}}(x,t) \\ &= \sum_j \chi_j(t) \sum_{|k|=\lambda_0} a_{kj}(R_q) \phi_{kj}(x,t) B_k e^{i\lambda_{q+1} k \cdot x} + w_{\text{corrector}}(x,t). \end{aligned}$$

Actually, in order to make sure that (3.26) holds in overlapping temporal regions of the form $I_j \cap I_{j+1}$, we make use of two families of Beltrami flows Λ_{even} and Λ_{odd} (c.f. Lemma 3.2) and set

$$a_{kj}(R_q) = \rho_q^{1/2} \gamma_k^{(j)} \left(\frac{R_q}{\rho_q} \right)$$

if $k \in \Lambda_{even}$ for j even, or if $k \in \Lambda_{odd}$ for j odd, and $a_{kj}(R_q) = 0$ otherwise. Recall also that R_q and ρ_q are as in (3.10) and (3.11). In this way we can ensure that, modulo terms involving the corrector (which we again choose to ignore),

$$\begin{aligned} w_{q+1} \otimes w_{q+1} &\sim \frac{1}{2} \sum_{j,k} \chi_j^2 |a_{kj}|^2 \left(\mathrm{Id} - \frac{k \otimes k}{|k|^2} \right) + \\ &\quad + \sum_{j,j',k+k' \neq 0} \chi_j \chi_{j'} a_{kj} a_{k'j'} \phi_{kj} \phi_{k'j'} B_k \otimes B_{k'} e^{i\lambda_{q+1}(k+k')\cdot x} \\ &= R_q + \sum_{k'' \neq 0} c_{k''}(x,t) e^{i\lambda_{q+1} k'' \cdot x}. \end{aligned} \tag{4.8}$$

This way we recover condition (H2), which was the crucial point in the calculations for $\mathring{R}_{q+1}^{(2)}$ in Section 3.3 leading up to estimate (3.22).

We are now in a position to estimate $\mathring{R}_{q+1}$ given in (3.14)-(3.17). Note that, due to (3.9), (3.12) and (3.6) we have

$$\|a_{kj}\|_0 \lesssim \delta_{q+1}^{1/2},$$

in agreement with (H4). Therefore it is easy to see that the term (3.16) will be exactly as in Section 3.3, leading to

$$\|\mathring{R}_{q+1}^{(3)}\|_0 \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}}. \tag{4.9}$$

Next, consider the linear transport term (3.14): we obtain

$$(\partial_t + v_q \cdot \nabla) w_{q+1} = \sum_{j,k} \chi_j' a_{kj} \phi_{kj} B_k e^{i\lambda_{q+1} x\cdot k} + \sum_{j,k} \chi_j \nabla a_{kj} (\partial_t + v_q \cdot \nabla) R_q \phi_{kj} B_k e^{i\lambda_{q+1} x\cdot k},$$

owing to (4.5). Using the stationary phase estimate (3.19) leads to

$$\begin{aligned} \|\mathring{R}^{(1)}_{q+1}\|_0 &\lesssim \frac{\delta_{q+1}^{1/2}\mu_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} \\ &\lesssim \frac{\delta_{q+1}^{1/2}\mu_q}{\lambda_{q+1}}, \end{aligned} \tag{4.10}$$

where the second line follows by using (4.7).

Finally, for the quadratic term (3.15) we obtain, using (3.21) and (4.8)

$$\operatorname{div}^{\text{slow}}(w_{q+1} \otimes w_{q+1} - R_q) = \sum_{k''\neq 0} \operatorname{div} c_{k''}(x,t) e^{i\lambda_{q+1} x\cdot k''},$$

where

$$c_{k''} = \sum_{j,j'} \sum_{k+k'=k''} \chi_j \chi_{j'} a_{kj}(R_q) a_{k'j'}(R_q) \phi_{kj}\phi_{k'j'} B_k \otimes B_{k'}.$$

Using (3.7) and (4.6) we obtain from the stationary phase estimate (3.19)

$$\|\mathring{R}^{(2)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1}\delta_q^{1/2}\lambda_q}{\mu_q}. \tag{4.11}$$

Thus, overall we obtain

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{1/2}\mu_q}{\lambda_{q+1}} + \frac{\delta_{q+1}\delta_q^{1/2}\lambda_q}{\mu_q} \tag{4.12}$$

instead of (3.23). Optimizing in μ_q leads to the choice

$$\mu_q = \delta_q^{1/4}\lambda_q^{1/2}\delta_{q+1}^{1/4}\lambda_{q+1}^{1/2},$$

which is precisely the geometric mean between the "C^1-norms" $\delta_q^{1/2}\lambda_q$ and $\delta_{q+1}^{1/2}\lambda_{q+1}$, hence satisfies

$$\delta_q^{1/2}\lambda_q \ll \mu_q \ll \delta_{q+1}^{1/2}\lambda_{q+1}.$$

This is consistent with (4.7). Comparing with the inductive estimate as in Section 3.3 one arrives at

$$\delta_q^{1/2} \sim \lambda_q^{-1/5} \tag{4.13}$$

and the Hölder exponent $\theta < 1/5$.

4.3. Scholia

Up to now we have been deliberately been sloppy and ignored certain details. First of all, we have ignored the corrector term $w_{\text{corrector}}$. Secondly, in the previous sections we have shown how to obtain estimates for $\|\mathring{R}_{q+1}\|_0$ from the inductive estimates (3.2)–(3.8) and deduced a relationship linking δ_q and λ_q, leading in turn to a certain Hölder exponent. However, to close the induction we need to in addition verify all of the inductive estimates (3.2)–(3.8) for $q+1$. But note that, for instance $\|\mathring{R}_{q+1}\|_0$ already required knowledge of $\|\mathring{R}_q\|_1$, and similarly, estimating $\|\mathring{R}_{q+1}\|_1$ requires knowledge of $\|\mathring{R}_q\|_2$ and so on. In order to avoid this "loss of derivative", we need to mollify $(v_q, p_q, \mathring{R}_q)$ at each step.

In order to rigorously carry out the construction with all estimates, we need to make certain assumptions on the parameters (δ_q, λ_q). Given $b > 1$ and $\theta < 1/5$ we will call a sequence (δ_q, λ_q), $q \in \mathbb{N}$, (b,θ)-*admissible* if the inequalities

$$\delta_{q+1} \le \tfrac{1}{2}\delta_q, \quad \lambda_q^{\frac{b+1}{2}} \le \lambda_{q+1}, \quad 1 \le \delta_q^{1/2}\lambda_q^{\theta} \le 2 \tag{4.14}$$

are satisfied for any $q \in \mathbb{N}$. It is easy to see (c.f. [20, Section 6]) that if

$$\delta_q = a^{-b^q}, \quad a^{cb^{q+1}} \le \lambda_q \le 2a^{cb^{q+1}}, \tag{4.15}$$

for some $c > 5/2$ and $a, b > 1$, then (δ_q, λ_q) is (b, θ)-admissible for

$$\theta = \frac{1}{2bc} < \frac{1}{5},$$

provided $a \gg 1$ is sufficiently large (depending only on $b > 1$). Note that, at variance with the calculations in previous sections (see in particular the end of Section 3.3), here the sequence of frequencies λ_q is required to grow slightly super-exponentially (only slightly, because $b \to 1$ and $c \to 5/2$ as $\theta \to 1/5$). The main reason is technical, because the estimate (3.19) is not valid as stated. However, it is valid with an ε-loss in the exponent due to the use of Schauder estimates[4] . More precisely, we have

Lemma 4.2. *Let $k \in \mathbb{Z}^3/\{0\}$ be fixed. For a smooth vector field $a \in C^\infty(\mathbb{T};\mathbb{R}^3)$, let $F(x) := a(x)e^{i\lambda k\cdot x}$. Then, for any $\varepsilon > 0$ we have*

$$\|\mathrm{div}^{-1}(F)\|_\varepsilon \leq \frac{C}{\lambda^{1-\varepsilon}}\|a\|_0 + \frac{C}{\lambda^{m-\varepsilon}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\varepsilon},$$

where $C = C(\varepsilon, m)$.

Next, we introduce a strengthening of the notion of subsolution:

Definition 4.3 (Strong subsolutions)**.** A *strong subsolution* is a triple

$$(v, p, R) : \mathbb{T}^3 \times (0, T) \to \mathbb{R}^3 \times \mathbb{R} \times S^{3\times 3}$$

such that

$$\begin{cases} \partial_t v + \mathrm{div}\, v \otimes v + \nabla p = -\mathrm{div}\, R \\ \mathrm{div}\, v = 0, \end{cases} \tag{4.16}$$

and moreover $\rho(t) := \frac{1}{3}\mathrm{tr}\, R$ is a function of t only and satisfies

$$|R(x, t) - \rho(t)\mathrm{Id}| < r_0\rho(t) \quad \text{ for all } (x, t). \tag{4.17}$$

Here r_0 is the radius from Lemma 3.2 for $N = 2$. We make two remarks about this definition.

(i) Since $r_0 < 1$, the inequality (4.17) implies that $R(x, t)$ is positive definite, in agreement with (2.10). Furthermore, setting $\bar{e}(x, t) := \frac{1}{2}(\rho(t) + |v|^2)$ and

$$u = R - \frac{2}{3}\bar{e}\,\mathrm{Id} + v \otimes v, \quad q = p + \frac{2}{3}\bar{e}$$

we obtain a subsolution (v, u, q) with kinetic energy density $\bar{e}$. Therefore the notion of strong subsolution is a strenthening of the notion of subsolution from Definition 2.5. This definition also agrees with the notion of strong subsolution introduced in [27].

(ii) If $(v_q, p_q, \mathring{R}_q)$ is a solution of (3.1) and R_q is defined by (3.11), then (v_q, p_q, R_q) will be a strong subsolution provided the constants in (3.12) and (3.6) are chosen appropriately, depending on $\min_t e(t)$ and r_0.

Carrying out the construction with all estimates sketched in the previous sections (in particular Section 4.2), we arrive at the following (c.f. Sections 2-5 in [20]) proposition:

Proposition 4.4. *Fix $b > 1, \theta < 1/5$ and let $(\delta_q, \lambda_q)_{q\in\mathbb{N}}$ be a (b, θ)-admissible sequence. Let (v_q, p_q, R_q) be a smooth strong subsolution on $\mathbb{T}^3 \times (0, T)$. Furthermore, let $M_0 > 0$ be such that*

$$\|v_q\|_1 \leq M_0\delta_q^{1/2}\lambda_q, \quad \|p_q\|_1 \leq M_0\delta_q\lambda_q$$

and

$$\rho_q(t) \leq \delta_{q+1}, \quad \|R_q\|_1 \leq M_0\delta_{q+1}\lambda_q, \quad \|(\partial_t + v_q \cdot \nabla)R_q\|_0 \leq M_0\delta_{q+1}\delta_q^{1/2}\lambda_q.$$

Then, for any $\varepsilon > 0$ there exists smooth solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ of (3.1) on $\mathbb{T} \times (0, T)$ such that

$$\|v_{q+1} - v_q\|_0 \leq M\delta_{q+1}^{1/2}, \qquad \|v_{q+1} - v_q\|_1 \leq M\delta_{q+1}^{1/2}\lambda_{q+1}$$

$$\|p_{q+1} - p_q\|_0 \leq M^2\delta_{q+1}, \qquad \|p_{q+1} - p_q\|_1 \leq M^2\delta_{q+1}\lambda_{q+1}$$

[4] In fact a better choice of right-inverse div^{-1} than the one in Lemma 3.1 can be used to avoid this loss, see [58]

with

$$M = \overline{M} + C\lambda_{q+1}^{-\beta}$$

and

$$\|\mathring{R}_{q+1}\|_0 + \frac{1}{\lambda_{q+1}}\|\mathring{R}_{q+1}\|_1 + \frac{1}{\delta_{q+1}^{1/2}\lambda_{q+1}}\|(\partial_t + v_{q+1}\cdot\nabla)\mathring{R}_{q+1}\|_0 \leq C\delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{\varepsilon-1/2}.$$

In the above $C = C(b, M_0, \varepsilon)$*,* $\beta = \beta(b)$ *and* $\overline{M}$ *is a universal constant.*

In fact $\overline{M}$ is determined by the C^0 norm of the Beltrami flow W in (4.4), hence determined by the number of modes in $\Lambda_{even} \cup \Lambda_{odd}$ and the C^0 norms of the functions $\gamma_k^{(j)}$ from Lemma 3.2.

Choice of parameters. Next, observe that, due to (4.14), we have that $\delta_q^{1/2}\lambda_q \sim \lambda_q^{1-\theta}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \dots$. Hence, by choosing λ_1 sufficiently large, we can ensure that

$$C\lambda_q^{-\beta} \leq \overline{M} \quad \text{ and } \quad \sum_{j=1}^{q}\delta_j^{1/2}\lambda_j \leq 2\delta_q^{1/2}\lambda_q,$$

where $C = C(b, 4\overline{M}, \varepsilon)$. This will ensure that $M \leq 2\overline{M}$ and hence that (v_{q+1}, p_{q+1}) will satisfy the same estimates with $q+1$ as (v_q, p_q), with constant $M_0 = 4\overline{M}$. In order to close the induction it remains then to verify that

$$C\delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{\varepsilon-1/2} \leq \delta_{q+2}.$$

Fixing $\delta_q > 0$ and $\lambda_q \in \mathbb{N}$ as in (4.15), we see that this inequality is equivalent to

$$C \leq a^{\frac{1}{4}b^q(1+3b-2cb+(2c-4-4\varepsilon c)b^2)},$$

which, since $b > 1$, is satisfied for all $q \geq 1$ for a sufficiently large fixed constant $a \gg 1$, provided

$$\left(1 + 3b - 2cb + (2c - 4 - 4\varepsilon c)b^2\right) > 0.$$

Factorizing, we obtain the inequality $(b-1)((2c-4)b-1) - 4\varepsilon cb^2 > 0$. It is then easy to see that for any $b > 1$ and $c > 5/2$ there exists $\varepsilon > 0$ so that this inequality is satisfied. In this way we can choose $\varepsilon > 0$ depending on b and c. Consequently, this choice will determine $C = C(b, M_0, \varepsilon)$ and, as argued above, we then choose $a \gg 1$ (and hence $\lambda_1 \gg 1$) sufficiently large. This way the induction step can be closed, leading to a valid iteration for the construction of the sequence $(v_q, p_q, \mathring{R}_q)$. In the limit we obtain $v = \lim_{q\to\infty} v_q \in C([0,T]; C^\theta(\mathbb{T}^3))$.

4.4. Time-dependent estimates

So far all estimates used in the construction, including the inductive estimates (3.2)–(3.8) in Section 3.1 and the stationary phase estimate (3.19) have been understood to be *uniform in time*, although in principle we could treat them as being spatial norms depending on time t, e.g.

$$\|v_q(t)\|_0 := \sup_{x\in\mathbb{T}^3}|v_q(x,t)|, \quad \|v_q(t)\|_1 := \sup_{x\in\mathbb{T}^3}\left(|v_q(x,t)| + |Dv_q(x,t)|\right),.$$

The time dependence as an additional degree of freedom was exploited in [19, 18] and subsequently in [21] to show that, after minor modifications of the construction presented above, one obtains weak solutions of (W) which have better, Onsager-critical spatial regularity at the expense of a worse temporal dependence of the spatial norms. More precisely, statement (A) in Theorem 1.8, namely the existence of (nontrivial) weak solutions with compact support in time was obtained, so that the solution v satisfies (i) $[v(t)]_{C^{1/3-\varepsilon}(\mathbb{T}^3)} < \infty$ for almost every t [19], and (ii) $t \mapsto [v(t)]_{C^{1/3-\varepsilon}(\mathbb{T}^3)}$ is integrable [21]. In this final section we would like to give the basic idea in the proof of these results.

$1/3$-Hölder continuity for almost every time t. Let us return to the construction in Section 4.2 and the form of the new Reynolds error term in (4.12). In deriving (4.12), only spatial norms and estimates have been used. Consequently, let us from now on fix λ_q, $q \in \mathbb{N}$, as before, but treat $\delta_q = \delta_q(t)$ as functions of time in estimates (3.2)–(3.8) in Section 3.1. Observe that in (4.12) the term

$$\frac{\delta_{q+1}\mu_q}{\lambda_{q+1}}$$

arises from the time derivative of the cut-off functions χ_j. The basic idea is to treat these "cut-off regions" as bad regions and the complement, where χ_j is constant, as "good regions". Thus, instead of optimizing in μ_q for the expression in (4.12), we will fix μ_q so that (4.7) is satisfied for all t (remember that $\delta_q = \delta_q(t)$ now), and try to minimize the size of "bad regions" by choosing steeper cut-off functions. More precisely, introduce another large parameter $\eta_q \gg 1$, with (super-)exponential growth as $q \to \infty$, such that the partition of unity in Section 4.2 is carried out with cut-off functions $(\chi_j)_j$ such that

- each χ_j is supported on an interval of length μ_q^{-1};
- the derivative χ_j' is supported on an interval of length $(\mu_q\eta_q)^{-1}$;
- $\sup_t |\chi_j'(t)| \lesssim \mu_q\eta_q$.

Given the whole iteration for obtaining the sequence v_q as above, the set of "good times" will be defined as the set of times $t \in [0,T]$, which are included in a cut-off region for only a finite number of iteration steps. That is, the set of "good times" is the complement of

$$V := \bigcap_{q\in\mathbb{N}} \bigcup_{q'\geq q} \Big\{ \text{ cut-off region in step } q' \Big\}.$$

Lemma 4.5. *The set $V \subset [0,T]$ has zero Lebesgue measure*[5].

Proof. Observe that the cut-off region at a single step q' is the union of $\sim 2\mu_{q'}$ intervals of size $\sim (\mu_{q'}\eta_{q'})^{-1}$. Hence this set has measure $\sim \eta_{q'}^{-1}$. Since we have chosen the sequence $\eta_q \to \infty$ to have as least exponential growth, we obtain

$$\left| \bigcup_{q'\geq q} \Big\{ \text{ cut-off region in step } q' \Big\} \right| \lesssim \sum_{q'\geq q} \frac{1}{\eta_{q'}} \sim \frac{1}{\eta_q}.$$

This implies that V has zero measure. □

In other words almost every $t \in [0,T]$ is a good time. For each such $t \notin V$ there exists a step $q_0 = q_0(t)$ such that, for every later step $q \geq q_0$ the time t does not belong to the cut-off region. Consequently

$$\|\mathring{R}_{q+1}(t)\|_0 \lesssim \frac{\delta_{q+1}(t)^{1/2}\delta_q(t)^{1/2}\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1}(t)\delta_q(t)^{1/2}\lambda_q}{\mu_q}$$

for all $q \geq q_0$. The two terms on the right hand side are not balanced. However, assuming that the second term is larger for some step q, the iteration will give

$$\delta_{q+2}(t) \sim \frac{\delta_{q+1}(t)\delta_q(t)^{1/2}\lambda_q}{\mu_q}. \tag{4.18}$$

If μ_q is chosen so that $\mu_q \gg \max_t \delta_q(t)^{1/2}\lambda_q$ (such a choice is possible, as shown in Section 4.2 just below (4.12)), then (4.18) will lead to rapid (super-exponential) decay since the right-hand side in (4.18) is superlinear in δ. Consequently, eventually the first term will be dominating. On the other hand the iteration

$$\delta_{q+2}(t) \sim \frac{\delta_{q+1}(t)^{1/2}\delta_q(t)^{1/2}\lambda_q}{\lambda_{q+1}}$$

is consistent with Hölder-exponent 1/3, c.f. (3.23) in Section 3.3. Thus v is Hölder continuous with exponent 1/3 for almost every time (more precisely $1/3 - \varepsilon$, taking into account the corrections in Section 4.3).

Convergence in $L^1(0,T;C^{1/3-\varepsilon}(\mathbb{T}^3))$.

In order to obtain an estimate on the Hölder seminorm $[v(t)]_{C^{1/3-\varepsilon}(\mathbb{T}^3)}$, we need to know $q_0(t)$ for "good times" t, i.e. the step when t was part of a cut-off region for the last time. This requires a more detailed book-keeping of the whole iteration and a partition of time into intervals whose length also depends on time.

[5]In fact one can also show that $\dim_{\mathcal{H}} V < 1$, see [19].

It will be convenient to record all parameters δ_q, μ_q and η_q in terms of powers of λ_q. Therefore we will fix a sequence λ_q which satisfies

$$\lambda_{q+1} \sim \lambda_q^b \quad \text{for all } q$$

for some $b > 1$ and write

$$\delta_q(t) = \lambda_q^{-2\beta_q(t)}.$$

Thus, $\beta_q(t)$ denotes the "local Hölder exponent" at iteration step q, c.f. (3.24), (4.3) and (4.13). Setting

$$\mu_q = \lambda_{q+1}^{\gamma}, \quad \eta_q = \lambda_{q+1}^{\omega}$$

for some constants $\gamma, \omega > 0$ leads to the estimates in (4.12), as before. Since the three terms are in general not balanced, each of the three terms gives rise to an inequality relating β_{q+2} to β_{q+1} and β_q, corresponding to the inductive inequality $\|\mathring{R}_{q+1}(t)\|_0 \leq \delta_{q+2}$. By taking a logarithm of base λ_q we arrive at the three inequalities

$$\beta_{q+2} \leq \frac{1}{2b}\beta_{q+1} + \frac{1}{2b^2}\beta_q + \frac{1}{2b^2}(b-1) \tag{4.19}$$

$$\beta_{q+2} \leq \frac{1}{2b}\beta_{q+1} + \frac{1}{2b}(1-\gamma-\omega) \tag{4.20}$$

$$\beta_{q+2} \leq \frac{1}{b}\beta_{q+1} + \frac{1}{2b^2}\beta_q - \frac{1}{2b^2}(1-b\gamma). \tag{4.21}$$

Let us denote the right hand sides of the previous inequalities as $N(\beta_q, \beta_{q+1})$, $C(\beta_q, \beta_{q+1})$ and $Q(\beta_q, \beta_{q+1})$, so that N, C, Q are functions of β_q and β_{q+1}. The iteration on the set of estimates is as follows: given a partition of unity $\{\chi_j\}$ in time, there will be a cut-off region, defined by the set of times $\bigcup_j \operatorname{supp} \chi'$, and the complement, referred to as the set of good times. If the cut-off functions χ_j are supported on intervals of length μ_q^{-1} and χ_j' supported on intervals of length $(\mu_q \eta_q)^{-1}$, then we essentially obtain

$$\beta_{q+2}(t) \sim \begin{cases} C(\beta_q, \beta_{q+1}) & \text{if } t \text{ is in the cut-off region,} \\ \min\{N(\beta_q, \beta_{q+1}), Q(\beta_q, \beta_{q+1})\} & \text{otherwise.} \end{cases}$$

In other words we can view the inductive estimates as a dynamical system on functions of t of the type

$$T : (\beta_q, \beta_{q+1}) \mapsto \beta_{q+2}.$$

We are interested in the integrability of the spatial $1/3$-Hölder norm, i.e. in

$$\int_0^1 \lambda_q^{1/3-\beta_q(t)}\, dt \tag{4.22}$$

in the limit $q \to \infty$. In order to explain how one should proceed, let us simplify further, and consider the dynamical system given by

$$T : \beta_q \mapsto \beta_{q+1} = \begin{cases} C(\beta_q) & \text{if } t \text{ is in the cut-off region,} \\ \min\{N(\beta_q), Q(\beta_q)\} & \text{otherwise,} \end{cases}$$

where N, C, Q are the linear functions

$$\begin{aligned} N(\beta) &:= \frac{1}{2b}\beta + \frac{1}{2b^2}\beta + \frac{1}{2b^2}(b-1) \\ C(\beta) &= \frac{1}{2b}\beta + \frac{1}{2b}(1-\gamma-\omega) \\ Q(\beta) &= \frac{1}{b}\beta + \frac{1}{2b^2}\beta - \frac{1}{2b^2}(1-b\gamma). \end{aligned}$$

The map T can be best explained in Figure 4.1 below.

In fact T really acts on the distribution function of $\beta_q(t)$, that is, on $m_q(\beta)$ defined by

$$\lambda_q^{-m_q(\beta)} := \big|\{t \in (0,T) : \beta_q(t) < \beta\}\big|.$$

In terms of $m_q(\beta)$ the integral (4.22) can be written as

$$\int_0^1 \lambda_q^{1/3-\beta_q(t)}\, dt = \int_{\beta_{min}}^{1/3} \lambda_q^{1/3-m_q(\beta)-\beta}\, d\beta.$$

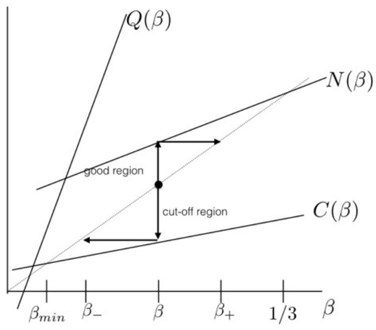

Figure 4.1: The temporal region $\{t : \beta_q(t) = \beta\}$ is split into a cut-off region of size η_q^{-1} where $\beta_{q+1} := \beta_-$ and the complement (the "good region"), where $\beta_{q+1} := \beta_+$.

Thus, essentially we would like to verify

$$m_q(\beta) > \frac{1}{3} - \beta \quad \text{for all } q. \tag{4.23}$$

One can proceed inductively, based on the rule induced by T above.

Of course, as in Section 3 we have performed many simplifications here which cannot be justified, and ignored several details. Moreover, it turns out that (4.23) fails if N, C, Q are defined as above, i.e. with constant γ, ω. In fact we need to allow for cut-off functions, whose parameters depend on t, in the form $\gamma = \gamma(\beta)$ and $\omega = \omega(\beta)$. This in turn leads to a much more complicated partition of unity $\{\chi_j\}$ on $(0, T)$. However, the details of the proof exceed the scope of these notes, and we refer to the paper [21].

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Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04109 Leipzig
Germany
laszlo.szekelyhidi@math.uni-leipzig.de