Billiards and boundary traces of eigenfunctions

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Abstract

This is a report on recent results with A. Hassell on quantum ergodicity of boundary traces of eigenfunctions on domains with ergodic billiards, and of work in progress with Hassell and Sogge on norms of boundary traces. Related work by Burq, Grieser and Smith-Sogge is also discussed.

1. Introduction

This is a report on recent work (and work in progress) with A. Hassell and C. Sogge on boundary traces of eigenfunctions of the Laplacian on bounded domains and their relations to the dynamics of the billiard map. Boundary traces of eigenfunctions arise when one makes a reduction of the interior/exterior eigenvalue problem or wave equation of a domain to its boundary. In the classical approach to boundary problems of Neumann and Fredholm, the boundary reduction is made via layer potentials and associated boundary integral operators. Our results are based on an analysis of these operators as semiclassical Fourier integral operators.

We work on a Lipschitz domain $\Omega \subset \mathbb{R}^n$ which is assumed to be compact and piecewise smooth with corners. We give a precise definition below. The interior eigenvalue problem is:

$$\begin{cases} -\Delta u_j = \lambda_j^2 u_j \ \text{ in } \ \Omega, \quad \langle u_j, u_k \rangle = \delta_{jk} \\ \\ Bu_j = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where

$$Bu_j = \begin{cases} u_j|_{\partial\Omega} \text{ (Dirichlet), resp.} \\ \\ \partial_\nu u_j + Ku|_{\partial\Omega}, \ \text{ (Neumann, Robin)} \end{cases} \tag{1.2}$$

Here, ∂_ν denotes the unit interior normal derivative.

By the boundary trace u_j^b of an interior eigenfunction u_j, we mean the complementary operator to B, namely

$$u_j^b(q) = \partial_\nu u_j|_{\partial\Omega} \ \text{ (Dirichlet), resp. } u_j^b(q) = u_j|_{\partial\Omega} \quad \text{ (Neumann, Robin).} \tag{1.3}$$

These boundary traces are a reduction to the boundary of the eigenfuncions, and they become eigenfunctions of a reduction to the boundary of the resolvent, given by the boundary integral operator F_λ on $\partial\Omega$ defined by

$$F_\lambda f(y) = \begin{cases} -2 \int_{\partial\Omega} \frac{\partial}{\partial \nu_{y'}} G_0(y, y', \lambda) f(y') dA(y'), & Neumann \\[2mm] 2 \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} G_0(y, y', \lambda) f(y') dA(y'), & Dirichlet \end{cases} \tag{1.4}$$

where $dA(y)$ is the induced boundary area form, and where

$$
\begin{aligned}
G_0(z, z', \lambda) &= (2\pi)^{-n} \lambda^{n-2} \int e^{i\lambda(z-z')\cdot\xi} \frac{1}{|\xi|^2 - 1 - i0} \, d\xi \\[2mm]
&= C\lambda^{n-2}(\lambda|z - z'|)^{-(n-2)/2} \operatorname{Ha}^{(1)}_{n/2-1}(\lambda|z - z'|).
\end{aligned}
$$

is the free outgoing Green's function $\mathbb{R}^n$. Here, $\operatorname{Ha}^{(1)}_{n/2-1}$, is the Hankel function. Thus, the Dirichlet boundary integral operator has kernel

$$F_\lambda(y, y') = (2\pi)^{-n} \lambda^{n-1} \int e^{i\lambda(y-y')\cdot\xi} \frac{\xi \cdot \nu_z}{|\xi|^2 - 1 - i0} \, d\xi.$$

where ν_z is the unit inward pointing normal at z. It is elementary and classical (from Green's formula) that

$$F_{\lambda_j} u_j^b = \begin{cases} u_j^b, & Neumann \\[2mm] -u_j^b, & Dirichlet. \end{cases} \tag{1.5}$$

One may think of F_λ as a transfer operator or effective Hamiltonian in the reduction to the boundary. More precisely, as we will explain in Section 3, F_h is the quantization of the billiard map β on the ball bundle $B^*\partial\Omega$, and is thus the reduction to the boundary of the quantum dynamics of the wave group in the interior. Yet it is clearly a more elementary operator than the Dirichlet and Neumann wave groups.

Our interest is in the relation between the dynamics of β on $B^*\partial\Omega$ and the asymptotics of the boundary traces u_j^b. In particular we study:

- Asymptotics of matrix elements $\langle A(h_j) u_j^b, u_j^b \rangle$;

- Asymptotics of L^p norms: $||u_j^b||_{L^p}$.

- Asymptotics of ratios $||u_j^b||_{L^p}/||u_j^b||_{L^2}$

The problems and results we present are to some extent analogues for boundary traces of results of [SZ, ZZw] on eigenfunctions on boundaryless manifolds or on interior eigenfunctions of manifolds with boundary. In the next section, we state the main results at this time.

Besides A. Hassell and C. Sogge, the author would like to thank M. Zworski for helpful discussions about non-convex domains and ghost orbits, and D. Tataru for checking the statements of his results in Section 5 (they are stated differently in [T]).

2. Statement of results

Let Ω be a bounded subdomain of $\mathbb{R}^n$ with closure $\overline{\Omega}$.

Definition: We say that $\Omega \subset \mathbb{R}^n$ is a piecewise smooth manifold if the boundary $Y = \partial\Omega$ is strongly Lipschitz, and can be written as a disjoint union

$$Y := \partial\Omega = H_1 \cup \cdots \cup H_m \cup \Sigma,$$

where each H_i is an open, relatively compact subset of a smooth embedded hypersurface S_i, and Σ is a closed set of $(n-1)$-measure zero.

The sets H_i are called boundary hypersurfaces of Ω. We call Σ the singular set, and write $Y^o = Y \setminus \Sigma$ for the regular part of the boundary.

2.1. Statement of results on quantum ergodicity

Our main results concern the quantum ergodicity of boundary traces. To state the results, we need some notation. In the table, κ denotes a C^∞ function on Y while k is the principal symbol of the operator $K \in \Psi^1(Y)$, and $d\sigma$ is the natural symplectic volume measure on B^*Y. We also define the function $\gamma(q)$ on B^*Y by

$$\gamma(q) = \sqrt{1 - |\eta|^2}, \quad q = (y, \eta). \tag{2.1}$$

Boundary Values			
B	Bu	u^b	$d\mu_B$
Dirichlet	$u|_Y$	$\partial_\nu u|_Y$	$\gamma(q)d\sigma$
Neumann	$\partial_\nu u|_Y$	$u|_Y$	$\gamma(q)^{-1}d\sigma$
Robin	$(\partial_\nu u - \kappa u)|_Y$	$u|_Y$	$\gamma(q)^{-1}d\sigma$
Ψ^1-Robin	$(\partial_\nu u - Ku)|_Y$	$u|_Y$	$\dfrac{\gamma(q)d\sigma}{\gamma(q)^2 + k(q)^2}$

THEOREM **1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded piecewise smooth manifold (see Definition 2) with ergodic billiard map. Let $\{u_j^b\}$ be the boundary traces f the eigenfunctions $\{u_j\}$ of Δ_B on $L^2(\Omega)$ in the sense of the table above. Let A_h be a semiclassical operator of order zero on Y. Then there is a subset S of the positive integers, of density one, such that*

$$\lim_{j \to \infty, j \in S} \langle A_{h_j} u_j^b, u_j^b \rangle = \omega_B(A), \quad B = \text{ Neumann, Robin or } \Psi^1\text{-Robin},$$

$$\lim_{j \to \infty, j \in S} \lambda_j^{-2} \langle A_{h_j} u_j^b, u_j^b \rangle = \omega_B(A), \quad B = \text{ Dirichlet}, \tag{2.2}$$

where $h_j = \lambda_j^{-1}$ and $\omega_B(A) = \int_{B^*\Omega} \sigma_A d\mu_B$.

We note that the boundary traces are only normalized by $||u_j||_{L^2(\Omega)} = 1$. We obtain new results on L^2 norms by taking $A = I$. For the Neumann case, we have

$$\lim_{j\to\infty, j\in S} ||u_j^b||^2_{L^2(Y)} = \frac{2\operatorname{vol}(Y)}{\operatorname{vol}(\Omega)}, \tag{2.3}$$

while for the Dirichlet boundary condition they imply

$$\lim_{j\to\infty, j\in S} \lambda_j^{-2} ||u_j^b||^2_{L^2(Y)} = \frac{2\operatorname{vol}(Y)}{n\operatorname{vol}(\Omega)}. \tag{2.4}$$

To our knowledge, the only prior results on L^2 norms of boundary traces are the upper and lower bounds of [BLR, HT] in the Dirichlet case, and the bounds implicit in [T] in the Neumann case. We review the latter in the last section.

2.2. Statement of results on L^p norms of boundary traces

We now consider L^p norms of boundary traces for general domains. We state a number of results of work in progress [HSZ]. For simplicity of exposition, we only state the results for L^∞ norms.

First we consider estimates of boundary traces and state general L^∞ bounds analogous to those which follow from the remainder estimate of Avakumovic-Levitan for the spectral function of the Laplacian on a boundaryless manifold (see [SZ] for references). We denote by

$$E^b_{[a,b]}(x,x) = \sum_{j:\lambda_j \le \lambda \in [a,b]} |u_j^b(q)|^2 \tag{2.5}$$

the boundary trace of spectral projections kernel (on the diagonal) for $\sqrt{\Delta}$ corresponding to the interval $[a,b]$.

PROPOSITION **2.1.** *We have:*

$$\sum_{j:\lambda_j \le \lambda} |u_j^b(q)|^2 = \begin{cases} C\lambda^{n+2} + O(\lambda^{n+1}), & Dirichlet \\ \lambda^n + O(\lambda^{n-1}), & Neumann. \end{cases}$$

Hence,

$$||E^b_{[\lambda,\lambda+1]}(x,x)||_{L^\infty} = \begin{cases} O(\lambda^{n+1}), & Dirichlet \\ O(\lambda^{n-1}), & Neumann. \end{cases}$$

Of course, the estimate holds for each term $||u_j^b(q)|^2||_{L^\infty(\partial\Omega)}$ separately. We state it in the above form because it is sharper than the statement for boundary traces of eigenfunctions: for instance, the results implies the same bound for boundary traces ψ_j^b of quasimodes of order 0 satisfying $||(\Delta - \lambda_j^2)\psi_j|| = O(1)$. We also wish to emphasize that our estimates on boundary traces of eigenfunctions cannot be better than estimates for the above spectral projections.

Even for boundary traces of eigefunctions, the above estimate on L^∞ norms is sharp among all compact Riemannian domains with boundary. Indeed, they are achieved on the northern hemisphere of the standard S^n by zonal spherical harmonics with pole on the boundary: let x_0 be a point on the boundary of the northern hemisphere (i.e. the equator), and let $Y_0^{Nx_0}$ be the L^2-normalized zonal spherical harmonic of degree N with pole at x_0, i.e. the spherical harmonic which is invariant under rotations fixing x_0. Then $\frac{1}{2}[Y_0^{Nx_0}(x) + Y_0^{Nx_0}(x^*)]$ is a Neumann eigenfunction which is essentially L^2-normalized and which has the same L^∞ norm as $Y_0^{Nx_0}$. Here, x^* is the reflection of x through the equation, i.e. $(x_1, \ldots, x_n)^* = (x_1, \ldots, x_{n-1}, -x_n)$. Taking the normal derivative of the odd part of Y_1^N under the reflection produces a similar sharp example in the Dirichlet case. Eucliean examples will be discussed in the final section.

However, our main result on norms of boundary traces, as in the interior case studied in [SZ], shows that this result is generically not sharp. In the following, the symbol $f(x) = \Omega(g(x))$ is the negative of $f(x) = o(g(x))$, i.e. $f(x)$ grows at least as fast as $g(x)$ along a subsequence. In the following, $E_{[\lambda, \lambda + o(1)]}$ is short for $E_{[a,b]}$ where a, b are the endpoints of a sequence of shrinking intervals satisfying $a = \lambda, b = \lambda + \epsilon(\lambda)$ where $0 < \epsilon(\lambda) = o(1)$.

THEOREM 2. *Suppose that*

$$||u_j^b||_{L^\infty} = \begin{cases} \Omega(\lambda_j^{\frac{n+1}{2}}), & \text{Dirichlet} \\ \\ \Omega(\lambda_j^{\frac{n-1}{2}}), & \text{Neumann}. \end{cases}$$

or more generally that

$$||E_{[\lambda, \lambda + o(1)]}(x,x)||_{L^\infty} = \begin{cases} \Omega(\lambda^{\frac{n+1}{2}}), & \text{Dirichlet} \\ \\ \Omega(\lambda^{\frac{n-1}{2}}). & \text{Neumann}. \end{cases}$$

Then there exists a recurrent point $q \in \partial\Omega$, i.e. there exists $T > 0$ and a positive measure set of directions $\xi \in S_q^*\Omega$ such that the billiard orbit $\exp_q(t\xi)$ returns to q at time T.

To put this another way, if the set of looping directions at q has measure zero for all $q \in \partial\Omega$, then for any $\epsilon > 0$,

$$\limsup_{\lambda \to \infty} \lambda^{-\frac{n+1}{2}} ||E_{[\lambda, \lambda + \epsilon]}(q,q)||_{L^\infty} \to 0 \ \text{ as } \ \epsilon \to 0.. \tag{2.6}$$

As pointed out in Section 5.4, a recurrent point in the sense above cannot occur in a convex analytic Euclidean domain. Hence,

COROLLARY **2.2.** *If $\Omega \subset \mathbb{R}^n$ is a convex analytic domain with Euclidean metric, then*

$$||u_j^b||_{L^\infty} = \begin{cases} o(\lambda_j^{\frac{n+1}{2}}), & \text{Dirichlet} \\ \\ o(\lambda_j^{\frac{n-1}{2}}), & \text{Neumann}. \end{cases}$$

The hypothesis in the Corollary can probably be weakened quite a bit. For instance, it seems unlikely that real analytic Euclidean domains of any kind can have recurrent points in our sense, and the same may be true for smooth Euclidean domains (see also [SV] for related conjectures).

We note that although the result on boundary traces appears similar to that of [SZ] on eigenfunctions on manifolds without boundary, there is a serious difference in that the estimate in the boundaryless case is on the ratio $||u_j||_{L^p}/||u_j||_{L^2}$ whereas in the case of boundary traces, it is the interior eigenfunction and not the boundary trace which is L^2 normalized. In the Dirichlet case, $||u_j^b||_{L^2}$ is known to be bounded above and below by a constant times λ_j [BLR, HT], so one could restate the result in terms of the ratio $||u_j^b||_{L^\infty}/||u_j^b||_{L^2}$. In the Neumann case, $||u_j^b||_{L^2}$ can vary with the domain and metric.

2.3. Remarks on methods and related results

As mentioned above, our approach to boundary problems is to work entirely on the boundary. In a recent article [Bu], Burq takes the opposite route of 'reducing' the study of ergodicity of bounday traces to the interior. He extends the methods of Gerard-Leichtnam [GL] to reduce the proof of ergodicity of boundary traces to the results of Zelditch-Zworski [ZZw] on ergodicity of interior eigenfunctions. Both approaches have their advantages, to which we devote a few remarks.

Our motivation for working entirely on the boundary comes from several sources. First, it seems to us natural to prove results about boundary traces by working with quantum dynamics on the boundary, i.e. by making a semiclassical analysis of the operator F_h. We also hope (and believe) that the semiclassical analysis of F_h and related operators has an independent interest and other applications, for instance to inverse spectral theory (see [Z1]).

A second motivation is that the use of semiclassical asymptotics of the resolvent and of F_h to relate spectrum and billiards is now the standard approach in the physics literature on eigenvalues of bounded domains. The method originated in the articles of Balian-Bloch [BB1, BB2] and has been developed in, for instance, the articles [AG, B, BS, BFSS, GP, THS, THS2, TV]. A key reason for the wide use of the boundary reduction (as explained to the author by A. Backer and R. Schubert) is that boundary traces of eigenfunctions on smooth domains are much easier to compute numerically than interior eigenfunctions. See the expository article [B] by A. Backer (especially sections (3.3.1) - (3.3.7) and figure 20) for a discussion of the numerical methods, for further references, and for pictures of boundary traces of eigenfunctions. Despite the classical nature of layer potentials and the induced boundary integral operators, and their common use in physics and engineering, we are not aware of prior mathematical studies of their semiclassical properties.

The boundary integral method based on F_h seems to us more elementary than the interior methods, but it does have the disadvantage that the phase of F_h, given by the distance function between boundary points, is the generating function of both the interior and exterior billiard maps. Hence one has to deal with the so-called ghost billiard orbits which have links outside the domain (see Section 4.1.4). In the original version of [HZ], we restricted to convex domains where ghost orbits do not arise.

Using his interior method, Burq [B] then proved a more general ergodicity result for non-convex as well as convex domains. After that, in the second version of [HZ], we extended our methods and results to general (possibly non-convex) domains. In the interim, discussions with M. Zworski were very helpful. The additional complication of ghost orbits of non-convex domains does not in the end turn out to be serious.

3. Boundary integral operators as semiclassical Fourier integral operators

We begin by explaining the sense in which F_h is the boundary reduction or effective Hamiltonian for the resolvent. It is a classical result of potential theory.

We recall that the single, resp. double layer potentials of a domain $\Omega \subset \mathbb{R}^n$ are the operators

$$\begin{cases} \mathcal{S}\ell(\lambda)f(x) = \int_{\partial\Omega} G_0(x, q, \lambda)f(q)dA(q), \\[2mm] \mathcal{D}\ell(\lambda)f(x) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_y}G_0 x, q, \lambda)f(q)dA(q), \end{cases} \tag{3.1}$$

where $dA(q)$ is the surface area measure on $\partial\Omega$, where ν is the interior unit normal to Ω, and where $\partial_\nu = \nu \cdot \nabla$. They induce the boundary integral operators (1.4).

Denote by $R_\Omega^D(\lambda)$ (resp. $R_{\Omega^c}^N(\lambda)$) the resolvent of the interior Dirichlet Laplacian (resp. the exterior Neumann Laplacian). Then the Fredholm-Neumann reduction takes the form:

$$\begin{cases} R_\Omega^D(\lambda) = 1_\Omega R_0(\lambda)1_\Omega + 1_\Omega \mathcal{D}\ell(\lambda)(I + F_\lambda)^{-1}\mathcal{S}(\lambda)^{tr}1_\Omega, \\[2mm] R_{\Omega^c}^{Ntr}(\lambda) = 1_{\Omega^c} R_0^{tr}(\lambda)1_{\Omega^c} + 1_{\Omega^c} \mathcal{D}\ell(\lambda)(I + F_\lambda)^{-1}\mathcal{S}\ell(\lambda)^{tr}1_{\Omega^c} \end{cases} \tag{3.2}$$

This formula has the form of a Grushin reduction of the Laplacians to operators on the boundary in the sense of Sjostrand (see e.g. [DS]). Although we are not studying trace formula here, we note that the combination of these two formula implies that (in the sense of distributions),

$$Tr_{\mathbb{R}^2}[R_D^{\Omega^c}(\lambda) \oplus R_N^\Omega - R_{0\rho}(\lambda)] = \tfrac{d}{d\lambda}\log\det(I + F_\lambda), .$$

where $\det(I + F_\lambda)$ is the Fredholm determinant. This formula is often used to determine the eigenvalues and resonances of domains.

It should be observed that the Dirichlet resolvent has the complexity of the Dirichlet wave group, while F_λ is no more complicated that the free resolvent. It is $(I + F_\lambda)^{-1}$ which has the complexity of the Dirichlet resolvent or the Dirichlet-to-Neumann operator. The relative simplicity of F_λ is one of the attractions of the boundary reduction.

3.1. Quantized billiard map: Neumann

Since F_λ is a reduction to the boundary of the resolvent, it should be a kind of quantization of the billiard map. We now explain in what sense this is true.

The *billiard map* $\beta : B^*Y^o \to \tilde{T}^*Y$ is defined on the open ball bundle B^*Y^o as follows: given $(y, \eta) \in T^*Y$, with $|\eta| < 1$ we let $(y, \zeta) \in S^*\Omega$ be the unique inward-pointing unit covector at y which projects to (y, η) under the map $T^*_{\partial\Omega}\overline{\Omega} \to T^*Y$. Then we follow the geodesic (straight line) determined by (y, ζ) to the first place it intersects the boundary again; let $y' \in Y$ denote this first intersection. If $y' \in \Sigma$ then we define $\beta(y, \eta) = y'$. Otherwise, let η' be the projection of ζ to $T^*_{y'}Y$. Then we define

$$\beta(y, \eta) = (y', \eta').$$

The map $\beta_- : B^*Y^o \to \tilde{T}^*Y$ is defined similarly, following the backward billiard trajectory (that is, the straight line with initial condition $(y, 2(\zeta \cdot \nu_y)\nu_y - \zeta)$).

The graph

$$C_{\text{bill}} = \text{graph}\,\beta \equiv \{(\beta(q), q) \mid q \in \mathcal{R}^1\}. \tag{3.3}$$

of β is a smooth Lagrangian submanifold of $B^*Y^o \times B^*Y^o$. In a neighbourhood of $(y_0, \eta_0, y_0', \eta_0')$ it is given by

$$C_{\text{bill}} = \{(y, -\nabla_y d(y, y'), y', \nabla_{y'} d(y, y')), \tag{3.4}$$

where $d(y, y')$ is the Euclidean distance function. For strictly convex Ω it is given globally by (3.4), for $y, y' \in Y^o$, but this is not true in general. This causes extra difficulties for nonconvex domains, namely it introduces spurious billiard orbits (known as ghost orbits in the physics literature) which do not remain entirely in the domain, and which have to be shown to be irrelevant.

We see directly from (1.4) that F_h is an oscillatory integral operator with phase equal to the generating function $d(y, y')$ of C_{bill}. This is especially clear in dimension 3 when

$$G(x, x', \lambda) = \frac{1}{4\pi} \frac{e^{i\lambda|x-x'|}}{|x-x'|}.$$

However, it is also clear that $F_h(y, y')$ has the homogeneous singularity on the diagonal of a pseudodifferential operator of order -1, and this is how it is usually described. At least in the convex case, one may extend β to the boundary $|\eta| = 1$ of B^*Y by fixing S^*Y pointwise. Then the diagonal of $B^*Y \times B^*Y$ and C_{bill} intersect in the diagonal of $S^*Y \times S^*Y$, so one may view the wave front set of F_h as being the union of two intersecting Lagrangeans. $F_h(y, y')$ thus has similarities to the the oscillatory integral operator kernels associated to two intersecting Lagrangeans of Melrose-Uhlmann. However, the intersection occurs at the boundary of both Lagrangeans and is thus outside the scope of the class of Melrose-Uhlmann operators, presumably explaining the unusual composition law described in [Z1]. Due to the explicit nature of our problem, we carry out the analysis by hand without making use of general operator theories. The main point to observe is that the Fourier integral part is of order 0 while the pseudodifferential part is of order -1, so the Fourier integral part dominates.

PROPOSITION **3.1.** *Assume that $\partial\Omega$ is smooth. Let U be any neighbourhood of $\Delta_{S^*\partial\Omega}$. Then there is a decomposition of F_λ as*

$$F_\lambda = F_{1,\lambda} + F_{2,\lambda} + F_{3,\lambda},$$

where F_1 is a Fourier Integral operator of order zero associated with the canonical relation $C_{\text{bill}} = graph(\beta)$, F_2 is a pseudodifferential operator of order -1 and F_3 has operator wavefront set contained in U.

This implies an Egorov type result for the operator F_λ:

PROPOSITION **3.2.** *Let $A_h = \text{Op}(a_h)$ be a zeroth order operator whose symbol $a(y, \eta, 0)$ at $h = 0$ is supported away from $|\eta| = 1$. Put $h = \lambda^{-1}$ and let γ be as above, and let β denote the billiard ball map on B^*Y. Then*

$$F_\lambda^* A_h F_\lambda = \tilde{A}_h + S_h,$$

where $\tilde{A}_h$ is a zeroth order pseudodifferential operator and $\|S_h\|_{L^2 \to L^2} \le Ch$. The symbol of $\tilde{A}_h$ is

$$\tilde{a} = \begin{cases} \gamma(q)[\gamma^{-1}(\beta(q))a(\beta(q))], & q \in B^*Y \\ 0, & q \notin B^*Y. \end{cases} \tag{3.5}$$

This is a rigorous version of the statement that F_h quantizes the billiard ball map.

4. Boundary quantum ergodicity

Quantum ergodicity is concerned with quantizations of classically ergodic Hamiltonian systems. It is essentially a *convexity* result relating the *time average*

$$\langle A \rangle := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} U_t A U_t^* dt,$$

and the (constant) *space average*,

$$\omega(A)\, I$$

of an observable. Here, we use the notation of the interior problem where U_t is the wave group. Also, $\omega(A)$ is an invariant state on the algebra of observables, which arises from the local Weyl law:

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \langle A u_j, u_j \rangle = \omega(A). \tag{4.1}$$

The system is *quantum ergodic* if

$$\langle A \rangle = \omega(A)\, I + K, \quad \text{with} \quad \frac{1}{N(\lambda)} \|\Pi_\lambda K \Pi_\lambda\|_{HS} \to 0, \tag{4.2}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm and where Π_λ is the spectral projection onto the span of eigenfunctions of $\sqrt{\Delta}$ of eigenvalue $\le \lambda$. In terms of the L^2-normalized eigenfunctions u_j, QE says:

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \le \lambda} |\langle A u_j, u_j \rangle - \omega(A)|^2 \to 0,$$

for any observable A. Following the work of Schnirelman, Colin de Verdiere, Zelditch in the boundaryless case, it was proved by Gerard-Leichtnman [GL] ($C^{1,1}$ domains with Dirichlet boundary conditions) and Zelditch-Zworski [ZZw] (general case with corners) that domains with ergodic billiards are quantum ergodic.

4.1. Proof of Theorem 1

The main steps in the proof are:

- The local Weyl law;

- Analysis of the classical limit state and the L^2 ergodic theorem on the classical level.

- A convexity inequality to convert quantum ergodicity to the classical L^2 ergodic theorem.

4.1.1. Local Weyl law

It states:

PROPOSITION **4.1.** *Let A_h be a semiclassical operator of order zero on $\partial\Omega$. Then for any of the above boundary conditions B, we have:*

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A_{h_j} u_j^\flat, u_j^\flat \rangle = \omega_B(A), \quad h_j = \lambda_j^{-1}, \tag{4.3}$$

where ω_B is the state defined in the table in Section 4.2.

In the case of multiplication operators, the Weyl law was first proved by S. Ozawa [O].

4.1.2. Mean ergodic theorem for the classical limit states

It is obvious that the states

$$\rho_j^\flat(A) := \langle A_{h_j} u_j^\flat, u_j^\flat \rangle \tag{4.4}$$

are invariant for F_{λ_j}:

$$\rho_j^\flat(F_{\lambda_j}^* A F_{\lambda_j}) = \rho_j^\flat(A) \tag{4.5}$$

Here, as above, $h_j = \lambda_j^{-1}$.

Since any average or limit of averages of these states will be invariant, the local Weyl law implies:

COROLLARY **4.2.** *The states ω_B are invariant under F_λ: $\omega_B(F_\lambda^* A F_\lambda) = \omega_B(A)$.*

The formulae for the limit states in the table in Section 2.1 are found by explicit calculation of traces, e.g. of the dual heat or wave traces. The limit measures may be understood as follows:

- The Neumann limit measure $\frac{d\sigma}{\gamma(q)}$ is the projection to $B^*\Omega$ of the interior Liouville measure restricted to the set $S^*_{\in}\partial\Omega$ of inward pointing unit vectors to Ω along $\partial\Omega$ under the map $\pi(x, \xi) = (x, \xi^T)$ taking a (co)-vector to its tangential component.

- The Dirichlet limit measure $\gamma(q)d\sigma$ arises because boundary values of eigenfunctions involve the normal derivative in both factors before restricting to the boundary. The symbol of $h\partial_\nu$, restricted to the spherical normal bundle, and then projected to $B^*\partial\Omega$ is equal to γ, so we expect to get the square of this factor in the Dirichlet case compared to the Neumann case.

We now consider the mean ergodic theorem for these classical limit states. Let us first recall the result for the standard 'Koopman' operator associated to the billiard map:

$$T : L^2(B^*\partial\Omega, d\sigma) \to L^2(B^*\partial\Omega, d\sigma), \;\; Tf(\zeta) = f(\beta(\zeta)).$$

From the invariance it follows that T is a unitary operator. When β is ergodic, the unique invariant L^2-normalized eigenfunction is a constant c, and one has the mean ergodic theorem

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} T_F^n(f) - \langle f, c\rangle|| \to 0. \tag{4.6}$$

But the transformation provided by Proposition 3.2 (Egorovs theorem) is defined by:

$$Tf(\zeta) = \frac{\gamma(\zeta)}{\gamma(\beta(\zeta))} f(\beta(\zeta)).$$

This T is *not* unitary on $L^2(B^*\partial\Omega, d\sigma)$. The invariance $\omega_B(A) = \omega_B(F_h^* A F_h)$ implies that the associated measure $d\mu_B$ is invariant under

$$T^*f(\zeta) = \frac{\gamma(\beta(\zeta))}{\gamma(\zeta)} f(\beta(\zeta)).$$

Simple calculations show:

- (i) The unique positive T^*-invariant density is given by $\gamma^{-1}d\mu$. The unique positive T-invariant density is given by $\gamma d\mu$.

- (ii) T is unitary relative to the inner product $\langle\langle,\rangle\rangle$ on $B^*\partial\Omega$ defined by the measure $d\nu = \gamma^{-2}d\mu$.

When β is ergodic, the orthogonal projection P onto the T-invariant L^2-eigenvectors has the form

$$P(f) = \langle\langle f, \gamma\rangle\rangle = [\int_{B^*\partial\Omega} f\gamma\gamma^{-2}d\mu_B]\gamma$$

Thus, $P(\sigma_A) = [\int_{B^*\partial\Omega} \sigma_A\gamma^{-1}d\mu_B]\gamma = \omega_B(A)\gamma$.

The mean ergodic theorem thus says:

$$\frac{1}{N}\sum_{k=0}^{N-1} T^k\sigma_A \to \omega_B(A)\gamma.$$

4.1.3. Convexity

To show that

$$\langle Au_j^\flat, u_j^\flat \rangle \to \omega_B(A),$$

along a density one subsequence of integers j is essentially to show that

$$\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A)) u_j^\flat, u_j^\flat \rangle \right|^2 = 0. \tag{4.7}$$

Due to the novel form of the local Weyl law and the Egorov theorem, we first prove an auxiliary result of this kind for the quantization of the invariant symbol

$$\sigma_R(q) \sim c\gamma(q) = c(1 - |\eta|^2)^{1/2}, \tag{4.8}$$

with c a normalizing constant.

LEMMA **4.3.** *For all $\epsilon > 0$, there exists a pseudodifferential operator R of the form (4.8) such that*

$$\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A) \, R) u_j^\flat, u_j^\flat \rangle \right|^2 < \epsilon.$$

We prove this intermediate step by the usual time-average + convexity:

$$\frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A) \, R) u_j^\flat, u_j^\flat \rangle \right|^2$$

$$= \sum_{\lambda_j < \lambda} \left| \langle (\langle A_h \rangle_N - \omega_B(A) \, R) u_j^\flat, u_j^\flat \rangle \right|^2,$$

$$\leq C \sum_{\lambda_j < \lambda} \langle |\langle A_h \rangle_N - \omega_B(A) \, R|^2 u_j^b, u_j^b \rangle,$$

where $\langle A_h \rangle_N = \frac{1}{N} \sum_{k=1}^N ((F_h^k)^* A_h F_h^k$. By the local Weyl law, the limit equals $\omega_B((A_N - \omega_B(A)R)^2)$, or

$$= \int_{B^*\partial\Omega} \left(\langle \sigma_A \rangle_N - \omega_B(A)\sigma(R) \right)^2 d\mu_B. \tag{4.9}$$

By Proposition 3.2, the symbol $\sigma_{\langle A \rangle_N}$ of $\langle A \rangle_N$ is $\frac{1}{N} \sum_{k=0}^N T_F^k(\sigma_A)$. By the mean ergodic theorem, this converges to

$$P(\sigma_A) = c\gamma(q) \times \int \sigma_A \gamma^{-1} \, d\sigma = c \, \omega_B(A) \, \gamma(q) .$$

This is approximately equal to the symbol of $\omega_B(A)R$. Thus,

$$\int_{B^*\partial\Omega} \left(\sigma_{A_N} - \omega_B(A)\sigma(R) \right)^2 d\mu_B$$

becomes small as $N \to \infty$. Thus, $\frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A) \, R) u_j^\flat, u_j^\flat \rangle \right|^2$ is arbitrarily small as $N \to \infty$, proving Lemma 4.3.

Finally, we need to go from

$$\frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A)\, R) u_j^\flat, u_j^\flat \rangle \right|^2$$

to

$$\frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (A - \omega_B(A)\, I) u_j^\flat, u_j^\flat \rangle \right|^2.$$

It suffices to show that $\frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \left| \langle (I - R) u_j^\flat, u_j^\flat \rangle \right|^2$ is arbitrarily small. But this is the case $A = I$ above, since $\omega_B(I) = 1$.

4.1.4. Non-convex domains

We now adapt the argument to nonconvex domains. The problem with the argument above is that the canonical relation of F_λ is larger than the billiard relation, since it relates points on the boundary which are connected by a straight line even if the line passes outside the domain (ghost orbits). However, one can modify F_λ so that it leaves the boundary traces invariant and so that its wave front relation has an arbitrarily small measure of ghost orbits. That this should be possible was emphasized to us by M. Zworski.

As is easy to see, the property $F_{\lambda_j} u_j^b = u_j^b$ (which follows from Green's formula) is valid for any choice of Green's function on a neighborhood of Ω. So we modify the metric outside the domain, while keeping it Euclidean inside. Let b be a smooth, compactly supported nonnegative function on $\mathbb{R}^n$ which vanishes on $\overline{\Omega}$. Consider the metric

$$g_s = (1 + sb) g_{\text{Euclidean}} \quad \text{on } \mathbb{R}^n.$$

For sufficiently small s, no geodesics of g_s starting at a point in $\overline{\Omega}$ have conjugate points in $\overline{\Omega}$. Let $G_s(z, z, \lambda) = (\Delta_s - (\lambda + i0)^2)^{-1}(z, z')$ denote the kernel of the outgoing resolvent of the Laplacian on $\mathbb{R}^n$ with respect to the metric g_s. It has the parametrix

$$G_t(z, z', \lambda) = \lambda^{n-2} e^{i\lambda \operatorname{dist}_s(z, z')} a(z, z', \lambda), \quad z \neq z'.$$

We then define:

$$F_\lambda^s(y, y') = 2 \frac{\partial}{\partial \nu_y} G_s(y, y', \lambda), \quad y, y' \in Y,$$

and average over s to obtain the operator

$$\tilde{F}_\lambda = \int_0^1 \chi(s) F_\lambda^s \, ds. \tag{4.10}$$

Here χ is smooth, supported in $(0, \delta)$, and nonnegative with integral 1. This averaging removes all but an arbitrarily small measure of ghost orbits or spurioius points from the $\operatorname{WF}'(F_\lambda)$, i.e. points (y, η, y', η'), with $\eta = d_y |y - y'|$, $\eta' = -d_{y'} |y - y'|$ such that the line $\overline{yy'}$ leaves $\overline{\Omega}$. We still have

$$\tilde{F}_{\lambda_j} u_j^b = u_j^b.$$

and in addition the averaged operator has, as in the convex case, a decomposition

$$\tilde{F}_\lambda = \tilde{F}_{1,\lambda} + \tilde{F}_{2,\lambda} + \tilde{F}_{3,\lambda},$$

where $\tilde{F}_1$ is a Fourier Integral operator of order zero associated with the canonical relation C_{bill}, $\tilde{F}_2$ is a pseudodifferential operator of order -1 and $\tilde{F}_3$ has operator wavefront set contained in U. Moreover, the symbol of $\tilde{F}_{1,\lambda}$ is the same as for F_λ. Thus, the proof of quantum ergodicity in the non-convex case now runs as in the convex case, with this averaged operator in place of F_λ.

5. Norm estimates of boundary traces

We now turn to the results on L^p norms of boundary traces. They are derived from an analysis of the singularities of the boundary trace

$$E_B^b(t, q, q) = \sum_{j=1}^{\infty} e^{it\lambda_j} |u_j^b(q)|^2 \tag{5.1}$$

of the kernel of the wave operator $\cos t\sqrt{\Delta_B}$ on Ω with boundary condition B. The nice feature of the boundary trace of $E_B(t, x, x)$ is that the singularity at $t = 0$ becomes uniformly isolated from other singularities, while the interior kernel $E(t, x, x)$ has singularities at $t = 2d(x)$ arbitrarily close to $t = 0$. We note that $E_B^b(t, q, q')$ is a spectral transform of the Dirichlet-to-Neumann kernel.

We recall that the boundary traces u_j^b are normalized by $||u_j||_{L^2(\Omega)}$, so that it is of interest even to obtain estimates on $||u_j^b||_{L^2(\partial\Omega)}$. In (2.3)-(2.4) we obtained asymptotics of a density one sequence of boundary traces in the ergodic case. In general, the results are:

- Under a non-trapping assumption, $C_1\lambda \leq ||u_j^b||_{L^2(\partial\Omega)}/||u_j||_{L^2(\Omega)} \leq C_2\lambda$ in the Dirichlet case [BLR, HT];

- For any smooth domain and metric, $||u_j^b||_{L^2(\partial\Omega)}/||u_j||_{L^2(\Omega)} = O(\lambda_j^{1/3})$ in the Neumann case (Tataru, [T], Theorem 3);

- For a Sinai billiard on a torus (the exterior of a convex smooth domain in the torus), $||u_j^b||_{L^2(\partial\Omega)}/||u_j||_{L^2(\Omega)} = O(\lambda_j^{1/6})$ in the Neumann case ([T], Theorem 5).

We introduce some notation. Given $(q, \eta) \in B_q^*\partial\Omega$, we let $\xi(q, \eta) \in S_{in,q}^*\Omega$ be the inward pointing unit vector to Ω obtained from (q, η) by adding a multiple of the unit normal. We further denote by Φ^t the broken bicharacteristic flow of the wave group, i.e. the flow which carries singularities of the solution of the wave equation. Finally, we let v^T denote the tangential projection of $v \in S_q^*\Omega$ at $q \in \partial\Omega$. Finally, we denote by γ_q^B the boundary trace in the q variable corresponding to the boundary condition B.

5.1. Singularity of boundary trace of wave kernel at $t = 0$

Isolation of the singularity at $t = 0$ of $E_B^b(t, q, q)$ follows from simple wave front set considerations. We first note that

$$WF(\gamma_q^B \gamma_{q'}^B E(t, q, q')) \subset \{(t, \tau, q, \eta, q', \eta') : [\Phi^t(q, \xi(q, \eta))]^T = (q', \eta'), \ \tau = -|\xi|\},$$

which follows from propagation of singularities for the wave equation. Composition with the boundary trace just pulls back (i.e. restricts) this wave front relation to the boundary.

It follows that

$$WF(\gamma_q^B \gamma_{q'}^B E(t, q, q)) \subset \{(t, \tau, q, \eta, q, \eta') : [\Phi^t(q, \xi(q, \eta))]^T = (q, \eta'), \ \tau = -|\xi(q, \eta)|\}.$$

Thus, the singularities of the boundary trace $\gamma_q^B \gamma_{q'}^B E(t, q, q)$ at $q \in \partial\Omega$ to broken bicharacteristic loops based at q in $\overline{\Omega}$. Let us first consider loops in convex domains. They are either:

- closed geodesics on $Y = \partial\Omega$;

- m-link transversal reflecting rays with m vertices in Y, one of which is at q and which satisfy Snell's law of equal angles except possibly at q.

In the non-convex case, the description is more complicated since there exist additional gliding rays and since boundary geodesics need not carry singularities. For instance, for Euclidean plane domains in dimension 2, bicharacteristics enter and exit the boundary at inflection points and glide only over the convex part of the boundary.

By definition of normal singularity, there exists $r \in R_+$ such that $t^r E^b(t, q, q) \in C^\infty$ near $t = 0$. Thus, there exist coefficients $a_j(q)$ such that

$$E^b(t, q, q) \sim t^{-r} \sum_{j=0}^{\infty} a_j(q) t^j \tag{5.2}$$

The next step is to calculate the coefficients of the singularity.

Using a method of Ivrii, we first prove:

PROPOSITION **5.1.** *The singularity of $E^b(t, x, x)$ at $t = 0$ is a classical conormal singularity.*

Granted the proposition, we can calculate the coefficients (and the order) of the singularity using Hadamard type variational formulae for eigenvalues:

$$\delta\lambda_j = \int_Y \rho |u_j^b(y)|^2 dA(y),$$

where δ denotes a variation of either the boundary or the boundary conditions and ρ represents the tangent vector to the variation. Thus, we can determine the integrals $\int_{\partial\Omega} \rho(q) a_j(q) dA(q)$ and hence the coefficients $a_j(q)$ asymptotically by considering the variation of the wave trace

$$\frac{1}{it} \delta \sum_j e^{it\lambda_j} = \sum_j (\delta\lambda_j) e^{it\lambda_j} = \sum_j e^{it\lambda_j} \int_Y \rho |u_j^b(y)|^2 dA(y). \tag{5.3}$$

5.1.1. Dirichlet

In this case, we vary the boundary in the normal direction with variation vector field $\rho\nu$. The wave trace formula has the form:

$$\sum_j e^{it\lambda_j} = C_n Vol_n(\Omega)(t+i0)^{-n} + C_{n-1}Vol_{n-1}(\partial\Omega)(t+i0)^{-n+1} + a_1(t+i0)^{-n+2} + \cdots$$

where $\cdots$ represents lower order terms (cf. [I], Theorem 2.1 or [M], Corollary (3.6)). It follows that

$$\sum_j e^{it\lambda_j} \int_Y \rho|u_j^b(y)|^2 dA(y) = C_n\delta Vol_n(\Omega)(t+i0)^{-n-1} + C_{n-1}\delta Vol_{n-1}(\partial\Omega)(t+i0)^{-n} + \cdots$$

$$\tag{5.4}$$

Here, we divided by the coefficient it in (5.3). The variation of $Vol(\Omega)$ is non-zero, so we get a $(t+i0)^{-n-1}$ term on the right.

5.1.2. Neumann boundary conditions

In this case, we vary the boundary conditions, and therefore consider more general boundary conditions of the form

$$\partial_\nu \phi(q) + \kappa(q)\phi(q) = 0, \ q \in \partial\Omega.$$

We denote by δ the first variation relative to a change in the boundary condition from $\kappa \to \kappa + \epsilon\rho$. By the Hadamard variational formula (studied by S. Ozawa for general boundary conditions), the jth Neumann eigenvalue μ_j has the variation

$$\delta\mu_j = \int_{\partial\Omega} |u_j^b(q)|^2 \rho(q) d\sigma(q).$$

We now consider the first variation of the Neumann wave trace under variations of pseudodifferential boundary conditions. We obtain:

$$\delta TrE(t) = it\sum_j \delta\mu_j e^{it\mu_j}. \tag{5.5}$$

The ρ term only influences the third term of the singularity trace expansion at $t = 0$ for Neumann-Robin boundary conditions (see [GM] for the calculation) Hence the variation has the form

$$\delta TrE(t) = C_n(\int_{S^*\partial\Omega} \rho d\mu)(t+i0)^{-n+2} + \cdots , \tag{5.6}$$

from which we conclude

$$\sum_j e^{it\mu_j}[\int_{\partial\Omega} \rho|u_j^b|^2 dA] = C_n(\int_{S^*\partial\Omega} \rho d\mu)(t+i0)^{-n+1} + \cdots \tag{5.7}$$

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5.2. Spectral asymptotics of boundary traces

We apply standard Tauberian theorems to obtain spectral asymptotics of boundary traces from the singularity of the boundary trace of the wave kernel.

PROPOSITION **5.2.** *We have:*

$$\sum_{j:\lambda_j\leq\lambda}|u_j^b(q)|^2 = \begin{cases} C\lambda^{n+2}+O(\lambda^{n+1}), & Dirichlet \\ \\ C\lambda^n+O(\lambda^{n-1}), & Neumann. \end{cases}$$

Proof. It follows by Proposition 5.1 and a standard Tauberian argument that there exists a two-term asymptotic expansion for some principal coefficient. We may determine it from an integrated version of the asymptotics where we integrate the left side against a smooth function ρ on $\partial\Omega$. The integrated version follows from (5.4) in the Dirichlet case and (5.7) in the Neumann case.

LEMMA **5.3.** *For any smooth ρ on $\partial\Omega$,*

$$\sum_{j:\lambda_j\leq\lambda}\int_Y \rho|u_j^b(q)|^2 dA = \begin{cases} \int_{\partial\Omega}\rho dA\lambda^{n+2}+O(\lambda^{n+1}), & Dirichlet \\ \\ \int_{\partial\Omega}\rho dA\lambda^n+O(\lambda^{n-1}), & Neumann. \end{cases}$$

$\square$

We now improve the result in the generic case:

THEOREM **5.4.** *Suppose that the set of loops at q has measure 0 in $B_q^*\partial\Omega$. Then*

$$\sum_{j:\lambda_j\leq\lambda}|u_j^b(q)|^2 = \begin{cases} C\lambda^{n+2}+o(\lambda^{n+1}), & Dirichlet \\ \\ C\lambda^n+o(\lambda^{n-1}), & Neumann. \end{cases}$$

It follows that $||u_j^b||_{L^\infty(\partial\Omega)}=o(\lambda^{\frac{n+1}{2}})$ in the Dirichlet case, resp. $o(\lambda^{\frac{n-1}{2}})$ in the Neumann case. These results are sharp.

Proof. The proof follows the outline of that in [SZ]. We write the left side as the boundary trace $E_{[0,\lambda]}^b(q,q)$ of the spectral projections, and define the remainders in the local Weyl laws by

$$E_{[0,\lambda]}^b(q,q) = \begin{cases} C\lambda^{n+2}+R_D(\lambda,q), & Dirichlet \\ \\ C\lambda^n+R_N(\lambda,q), & Neumann. \end{cases}$$

We first show that if the set of billiard loops at $q\in\partial\Omega$ has measure zero in $B_q^*\partial\Omega$, then given $\varepsilon>0$, we can find a ball B centered at q and a $\Lambda<\infty$ so that for $\lambda\geq\Lambda$,

$$\begin{cases} |R_D(\lambda,q)|\leq\varepsilon\lambda^{n+1}, & q\in B, \\ \\ |R_N(\lambda,q)|\leq\varepsilon\lambda^{n-1}, & q\in B. \end{cases} \tag{5.8}$$

To prove this, we need some more notation. The loop-length function on $B^*\partial\Omega$ is the lower semi-continuous funciton defined by

$$L^*(q,\eta) = \begin{cases} \inf\{n > 0 : \pi \circ \beta^n(q,\eta) = q\}, \\ \\ \infty, \quad \text{if no such } n \text{ exists.} \end{cases} \tag{5.9}$$

The set of loop directions at $q \in \partial\Omega$ is defined by:

$$\mathcal{L}_q = \{\eta \in B_q^*\partial\Omega \, 1/L^*(q,\eta) \neq 0\}. \tag{5.10}$$

We construct two semiclassical pseudodifferential operators $b(q,D), B(q,D) = I - b(x,D)$ on $L^2(\partial\Omega)$ with the property that B is microsupported in the set where $L^*(q,\eta) >> T$. We then study $E_{[0,\lambda]}^b(q,q)$ by writing it in the form:

$$E_{[0,\lambda]}^b(q,q,) = [(B+b)E_\lambda^b(B+b)](q,q). \tag{5.11}$$

We choose $b \in C^\infty(S^{n-1})$ so that

$$\int_{S^{n-1}} b(\eta)d\sigma(\eta) \leq 1/T^2, \tag{5.12}$$

and

$$|L^*(q,\eta)| \leq 1/T, \quad \text{on } \mathcal{N} \times \text{supp } B,$$

where $\mathcal{N}$ is a neighborhood of q. We then put

$$B(\xi) = 1 - b(\xi).$$

By construction,

$$E^b(t)B^*(q,q), \quad BE^b(t,q,q) \in C^\infty(0,T). \tag{5.13}$$

A calculation using the conormal singularity at $t = 0$ of $BE^b(t,q,q), bE^b(t,q,q)$ shows that, for $\lambda \geq 1$,

$$\left|R_N(\lambda,q)\right| \leq CT^{-1}\lambda^{n-1} + C_T\lambda^{n-2}, \tag{5.14}$$

where C_T depends on T but C does not. This of course yields (5.8). The argument is similar for the Dirichlet case.

$\square$

5.3. Applications to eigenfunctions

We now use that bounds on eigenfunctions (or for spectral projections for intervals of shrinking width) can be obtained from the jump in the remainder:

$$\sum_{j:\lambda_j=\lambda} |u_j^b(q)| = \sqrt{R(\lambda,q) - R(\lambda - 0, q)}. \tag{5.15}$$

To complete the proof, we observe that for fixed $q \in \partial\Omega$ and any $\varepsilon > 0$ then one can find a neighborhood $\mathcal{N}_\varepsilon(q)$ of q and an $\Lambda_\varepsilon(q)$ so that when $\lambda \geq \Lambda_\varepsilon(xq)$ and $y \in \mathcal{N}_\varepsilon(q)$ we have $|R(\lambda, y)| \leq \varepsilon \lambda^{n-1}$. This implies that $|u_j^b(y)| \leq \varepsilon \lambda^{(n-1)/2}$ if $y \in \mathcal{N}_\varepsilon(q)$ and $\lambda \geq \Lambda_\varepsilon(q)$. Since M is compact and since the $\mathcal{N}_\varepsilon(q)$ form open cover of M, we may choose a finite subcover and extract the largest $\Lambda_\varepsilon(q)$. For this Λ_ε, we get

$$|u_j^b(y)| \leq \varepsilon \lambda^{(n-1)/2}, \quad \lambda \geq \Lambda_\varepsilon,$$

The $o(\lambda^{(n-1)/2})$ bound follows since Λ_ε depends only on ε.

5.4. Examples

For generic metrics and/or boundaries, there are no recurrent points [SZ]. Moreover, convex analytic domains in $\mathbb{R}^n$ never have have recurrent points on the boundary. Suppose to the contrary that there exists $x_0 \in \partial\Omega$ such that a positive measure of geodesic rays starting at x_0 return to x_0 at the same time. By analyticity, all rays starting at x_0 return to x_0, and they must all return at the same time. In particular, boundary geodesics have to return to x_0. So do creeping rays which make a small angle to the boundary, and which accumulate at boundary geodesics. But the creeping rays which approach a boundary geodesic are strictly shorter than the limiting boundary geodesic in the Euclidean metric, so the return time could not be the same. Therefore, the L^∞ estimate on boundary traces of eigenfunctions or spectral projections is never sharp for convex analytic Euclidean domains.

On the other hand, the L^∞ bound is sharp for the Euclidean half-circle or for any sector of a circle. Indeed, the invariant Neumann eigenfunctions (under rotations) of the disc achieve the interior L^∞ bound at the center. On a half-circle or sector, they remain Neumann eigenfunctions and their boundary traces achieve the maximal L^∞ bound above. In the case of Dirichlet boundary conditions, the bounds are also saturated on such domains (by taking the boundary trace of the imaginary part of the eigenfunction transforming by $e^{i\theta}$ under rotation by angle θ).

Above, we have mainly considered simply connected domains, but the same questions may be posed for multiply connected plane domains (for instance). It seems likely that any analytic (or even piecewise analytic) domain which achieves the maximum L^∞ bound must be simply connected. This would generalize the topological result of [SZ] that the maximum L^∞ bound can only be achieved for metrics on the sphere.

The question thus arises to find the maximal growth rate of L^∞ norms of boundary traces of eigenfunctions of smooth Euclidean domains, (or of convex analytic Euclidean domains), and which domains achieve the bounds? In fact, the answer depends on whether one studies relative sup-norms $||u_j^b||_{L^\infty(\partial\Omega)}/||u_j^b||_{L^2(\partial\Omega)}$ or absolute sup-norms $||u_j^b||_{L^\infty(\partial\Omega)}$. Moreover, the sharpness depends on whether we consider individual eigenfunctions or spectral projections for shrinking intervals.

We do not know the answer to the question, but offer some speculations. By recent results of Smith-Sogge [SS], following earlier results of Grieser (in his unpublished PhD thesis and in [G]; (see also [S])), it is known that whispering gallery Neumann modes of two-dimensional convex Euclidean domains saturate the L^p norms with $2 \leq p \leq 8$ because they live in thin $\lambda^{-2/3}$ layers around the boundary and

therefore are of size $\lambda^{1/3}$ there. The analogous problem in higher dimensions is still open.

However, they are not extremal for relative sup norms because are spread out all over the boundary. For instance, in the case of the disc, the relative sup norm $||u_j^b||_{L^\infty(\partial\Omega)}/||u_j^b||_{L^2(\partial\Omega)}$ is equal to one. A better guess that boundary traces of modes (or quasimodes) associated to stable elliptic orbits are extremals for the relative sup norm $||u_j^b||_{L^\infty(\partial\Omega)}/||u_j^b||_{L^2(\partial\Omega)}$. They live in $\lambda^{-1/2}$ tubes around the orbits and therefore the modes have size $\lambda^{(n-1)/4}$ along the orbit in dimension n. The boundary trace is therefore concentrated in small balls of radius $\lambda^{-1/2}$ around the bounce points of the orbit. So the L^2 norm of the boundary trace should be ~ 1.

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