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Bernard Helffer

**On spectral problems related to a time dependent model in  
superconductivity with electric current**

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# On spectral problems related to a time dependent model in superconductivity with electric current

Bernard Helffer

## Abstract

This lecture is mainly inspired by a paper of Y. Almog appearing last year at Siam J. Math. Anal. Our goal here is first to discuss in detail the simplest models which we think are enlightning for understanding the role of the pseudospectra in this question and secondly to present proofs which will have some general character and will for example apply in a more physical model, for which we have obtained recently results together with Y. Almog and X. Pan.

## 1. Introduction

We would like to understand the following problem coming from superconductivity. We consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state. As described for example in our recent book with S. Fournais [FoHel], this kind of question, without magnetic fields, can be treated by using fine results on the spectral theory of Schrödinger operator with magnetic field starting with the analysis of the case with constant magnetic field in the whole space and in the half-space. So we would like to start an analogous analysis when an electric current is considered.

This lecture is mainly inspired by a paper of Y. Almog [Alm2]. Our main goal here is to discuss in details the simplest models which we think are enlightning for understanding the role of the pseudospectra in this question. In the second part, we will present proofs which have some general character and for example apply in a more physical model, involving for example the non self-adjoint operator  $-\partial_x^2 - (\partial_y - i\frac{x^2}{2})^2 + iy$  on  $\mathbb{R}_{x,y}^2$ , for which we have obtained recently results together with Y. Almog and X. Pan [AHP].

After a presentation in the next section of the general problems and of our main results, we will come back to Almgren's analysis and will start from a fine "pseudo-spectral" analysis for the complex Airy operator on  $-\partial_x^2 + ix$  on the line or on  $\mathbb{R}^+$  and make a survey of what is known.

We hope also that this will illustrate some aspects discussed in the lectures of J. Sjöstrand at this conference. They are also related to recent results of Gallagher-Gallay-Nier [GGN] and to results on the Fokker-Planck equation obtained by Helffer, Hérau, Nier ([HelNi] and references therein) or Villani [Vil].

### Acknowledgements.

During the preparation of these notes we have benefitted, in addition to the collaboration with Y. Almgren and X. Pan, of discussions with many colleagues including W. Bordeaux-Montrieux, B. Davies, P. Gérard, C. Han, F. Hérau, J. Martinet, F. Nier and J. Sjöstrand. In addition, W. Bordeaux-Montrieux and K. Pravda-Starov provide us also with numerical computations which were sometimes confirming and sometimes predicting interesting properties.

## 2. The model in superconductivity

### 2.1. General context

Coming back to the physical motivation, let us consider a two-dimensional superconducting sample capturing the entire  $xy$  plane. We can assume also that a magnetic field of magnitude  $\mathcal{H}^e$  is applied perpendicularly to the sample. Denote the Ginzburg-Landau parameter of the superconductor by  $\kappa$  ( $\kappa > 0$ ) and the normal conductivity of the sample by  $\sigma$ .

The physical problem is posed in a domain  $\Omega$  with specific boundary conditions. We will only analyze here limiting situations where the domain possibly after a blowing argument becomes the whole space (or the half-space). We will mainly work in dimension 2 for simplification.

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in  $]0, T[ \times \Omega$  :

$$\begin{cases} \partial_t \psi + i\kappa \Phi \psi = -\Delta_{\kappa \mathbf{A}} \psi + \kappa^2(1 - |\psi|^2)\psi, \\ \kappa^2 \operatorname{curl}^2 \mathbf{A} + \sigma(\partial_t \mathbf{A} + \nabla \Phi) = \kappa \operatorname{Im}(\bar{\psi} \nabla_{\kappa \mathbf{A}} \psi) + \kappa^2 \operatorname{curl} \mathcal{H}^e, \end{cases} \quad (2.1)$$

where  $\psi$  is the order parameter,  $\mathbf{A}$  the magnetic potential,  $\Phi$  the electric potential,  $\nabla_{\kappa \mathbf{A}} = \nabla + i\kappa \mathbf{A}$  and  $-\Delta_{\kappa \mathbf{A}}$  is the magnetic Laplacian associated with magnetic potential  $\kappa \mathbf{A}$ .

In addition  $(\psi, \mathbf{A}, \Phi)$  satisfies an initial condition at  $t = 0$ .

In order to solve this equation, one should also define a gauge (Coulomb, Lorentz,...). The orbit of  $(\psi, \mathbf{A}, \Phi)$  by the gauge group is

$$\{(\exp i\kappa q \psi, \mathbf{A} + \nabla q, \Phi - \partial_t q) \mid q \in \mathcal{Q}\},$$

where  $\mathcal{Q}$  is a suitable space of regular functions of  $(x, t)$ . We refer to [BJP] (Paragraph B in the introduction) for a discussion of this point. We will choose the Coulomb gauge which reads that we can add the condition  $\operatorname{div} \mathbf{A} = 0$  for any  $t$ . Another possibility could be to take  $\operatorname{div} \mathbf{A} + \sigma \Phi = 0$ . A solution  $(\psi, \mathbf{A}, \Phi)$  is called a normal state solution if  $\psi = 0$ .

## 2.2. Stationary normal solutions

From (2.1), we see that if  $(0, \mathbf{A}, \Phi)$  is a time-independent normal state solution, then  $(\mathbf{A}, \Phi)$  satisfies the equality

$$\kappa^2 \operatorname{curl}^2 \mathbf{A} + \sigma \nabla \Phi = \kappa^2 \operatorname{curl} \mathcal{H}^e, \quad \operatorname{div} \mathbf{A} = 0 \quad \text{in } \Omega. \quad (2.2)$$

(Note that if one identifies  $\mathcal{H}^e$  to a function  $h$ , then  $\operatorname{curl} \mathcal{H}^e = (-\partial_y h, \partial_x h)$ .) Interpreting these equations as the Cauchy-Riemann equations, this can be rewritten as the property that

$$\kappa^2 (\operatorname{curl} \mathbf{A} - \mathcal{H}^e) + i \sigma \Phi,$$

is an holomorphic function in  $\Omega$ . In particular, if  $\sigma \neq 0$ ,  $\Phi$  and  $\operatorname{curl} \mathbf{A} - \mathcal{H}^e$  are harmonic.

**Special situation:  $\Phi$  affine.** As simplest natural example, we observe that, if  $\Omega = \mathbb{R}^2$ , (2.1) has the following stationary normal state solution

$$\mathbf{A} = \frac{1}{2J} (Jx + h)^2 \hat{i}_y, \quad \Phi = \frac{\kappa^2 J}{\sigma} y. \quad (2.3)$$

Note that

$$\operatorname{curl} \mathbf{A} = (Jx + h) \hat{i}_z,$$

that is, the induced magnetic field equals the sum of the applied magnetic field  $h \hat{i}_z$  and the magnetic field produced by the electric current  $Jx \hat{i}_z$ .

For this normal state solution, the linearization of (2.1) with respect to the order parameter is

$$\partial_t \psi + \frac{i \kappa^3 J y}{\sigma} \psi = \Delta \psi - \frac{i \kappa}{J} (Jx + h)^2 \partial_y \psi - \left( \frac{\kappa}{2J} \right)^2 (Jx + h)^4 \psi + \kappa^2 \psi. \quad (2.4)$$

Applying the transformation  $x \rightarrow x - J/h$ , the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial \psi}{\partial t} + i \frac{J}{\sigma} y \psi = \Delta \psi - i J x^2 \frac{\partial \psi}{\partial y} - \left( \frac{1}{4} J^2 x^4 - \kappa^2 \right) \psi. \quad (2.5)$$

Rescaling  $x$  and  $t$  by applying

$$t \rightarrow J^{2/3} t; \quad (x, y) \rightarrow J^{1/3} (x, y), \quad (2.6)$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u, \quad (2.7)$$

where, with  $D_x = -i \partial_x$ ,  $D_y = -i \partial_y$ ,

$$\mathcal{A}_{0,c} := D_x^2 + \left( D_y - \frac{1}{2} x^2 \right)^2 + i c y, \quad (2.8)$$

and

$$c = 1/\sigma; \quad \lambda = \frac{\kappa^2}{J^{2/3}}; \quad u(x, y, t) = \psi(J^{-1/3} x, J^{-1/3} y, J^{-2/3} t).$$

Our main problem will be to analyze the long time property of the attached semi-group.

We now apply the transformation

$$u \rightarrow u e^{i c y t}$$

to obtain

$$\partial_t u = - \left( D_x^2 u + \left( D_y - \frac{1}{2} x^2 - ct \right)^2 u - \lambda u \right). \quad (2.9)$$

Note that considering the partial Fourier transform, we obtain

$$\partial_t \hat{u} = -D_x^2 \hat{u} - \left[ \left( \frac{1}{2} x^2 + (ct - \omega) \right)^2 - \lambda \right] \hat{u}. \quad (2.10)$$

This can be rewritten as the analysis of a family (depending on  $\omega \in \mathbb{R}$ ) of time-dependent problems on the line

$$\partial_t \hat{u} = -\mathcal{L}_{\beta(t, \omega)} \hat{u} + \lambda \hat{u}, \quad (2.11)$$

with  $\mathcal{L}_\beta$  being the well-known anharmonic oscillator (or Montgomery operator) :

$$\mathcal{L}_\beta = D_x^2 + \left( \frac{1}{2} x^2 + \beta \right)^2, \quad (2.12)$$

and

$$\beta(t, \omega) = ct - \omega.$$

Note that in this point of view, we can after a change of time look at the family of problems

$$\partial_\tau v(x, \tau) = -(\mathcal{L}_{c\tau} v)(x, \tau) + \lambda v(x, \tau), \quad (2.13)$$

the initial condition at  $t = 0$  becoming at  $\tau = -\frac{\omega}{c}$ .

### 2.3. Recent results by Almog-Helffer-Pan [AHP]

The main point concerning the previously defined operator is to obtain results which are quite close to the Airy operator on the line.

**Theorem 2.1.**

*If  $c \neq 0$ ,  $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$  has compact resolvent, empty spectrum, and there exists  $C > 0$  such that*

$$\| \exp(-t\mathcal{A}) \| \leq \exp\left( -\frac{2\sqrt{2c}}{3} t^{3/2} + Ct^{3/4} \right), \quad (2.14)$$

*for any  $t \geq 1$  and*

$$\| (\mathcal{A} - \lambda)^{-1} \| \leq \exp\left( \frac{1}{6c} \operatorname{Re} \lambda^3 + C \operatorname{Re} \lambda^{3/2} \right), \quad (2.15)$$

*for all  $\lambda$  such that  $\operatorname{Re} \lambda \geq 1$ .*

Here we can no longer use the explicit properties of the Airy function but a semi-classical analysis of the operator  $\mathcal{L}_\beta$  as  $|\beta| \rightarrow +\infty$  plays an important role. We refer to [AHP] for details.

### 3. A simplified model : no magnetic field

We assume, following Almog, that a current of constant magnitude  $J$  is being flown through the sample in the  $x$  axis direction, and  $h = 0$ . Then (2.1) has (in some asymptotic regime) the following stationary normal state solution

$$\mathbf{A} = 0, \quad \Phi = Jx. \quad (3.1)$$

For this normal state solution, the linearization of (2.1) gives

$$\partial_t \psi + iJx\psi = \Delta_{x,y} \psi + \psi, \quad (3.2)$$

whose analysis is (see ahead) strongly related to the Airy equation.

### 3.1. The complex Airy operator in $\mathbb{R}$

This operator can be defined as the closed extension  $\mathcal{A}$  of the differential operator on  $C_0^\infty(\mathbb{R})$   $\mathcal{A}_0^+ := D_x^2 + ix$ . We observe that  $\mathcal{A} = (\mathcal{A}_0^-)^*$  with  $\mathcal{A}_0^- := D_x^2 - ix$  and that its domain is

$$D(\mathcal{A}) = \{u \in H^2(\mathbb{R}), xu \in L^2(\mathbb{R})\}.$$

In particular  $\mathcal{A}$  has compact resolvent.

It is also easy to see that

$$\operatorname{Re} \langle \mathcal{A}u | u \rangle \geq 0. \quad (3.3)$$

Hence  $-\mathcal{A}$  is the generator of a semi-group  $S_t$  of contraction,

$$S_t = \exp -t\mathcal{A}. \quad (3.4)$$

Hence all the results of this theory can be applied.

In particular, we have, for  $\operatorname{Re} \lambda < 0$

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re} \lambda|}. \quad (3.5)$$

One can also show that the operator is maximally accretive.

A very special property of this operator is that, for any  $a \in \mathbb{R}$ ,

$$T_a \mathcal{A} = (\mathcal{A} - ia)T_a, \quad (3.6)$$

where  $T_a$  is the translation operator  $(T_a u)(x) = u(x - a)$ .

As immediate consequence, we obtain that the spectrum is empty and that the resolvent of  $\mathcal{A}$ , which is defined for any  $\lambda \in \mathbb{C}$  satisfies

$$\|(\mathcal{A} - \lambda)^{-1}\| = \|(\mathcal{A} - \operatorname{Re} \lambda)^{-1}\|. \quad (3.7)$$

One can also look at the semi-classical question, i.e. look

$$\mathcal{A}_h = h^2 D_x^2 + ix, \quad (3.8)$$

and observe that it is the toy model for some results of Dencker-Sjöstrand-Zworski [DSZ]. Of course in such an homogeneous situation one can go from one point of view to the other but it is sometimes good to look at what each theory gives on this very particular model. This for example interacts with the first part of the lectures by J. Sjöstrand [Sjö2].

The most interesting property is the control of the resolvent for  $\operatorname{Re} \lambda \geq 0$ .

**Proposition 3.1.**

*There exist two positive constants  $C_1$  and  $C_2$ , such that*

$$C_1 |\operatorname{Re} \lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \operatorname{Re} \lambda^{\frac{3}{2}} \leq \|(\mathcal{A} - \lambda)^{-1}\| \leq C_2 |\operatorname{Re} \lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \operatorname{Re} \lambda^{\frac{3}{2}}, \quad (3.9)$$

(see Martinet [Mart] for this fine version). Note that W. Bordeaux-Montrieux and J. Sjöstrand<sup>1</sup> have obtained a better result.

The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + ix)\mathcal{F}^{-1} = \xi^2 + \frac{d}{d\xi}. \quad (3.10)$$

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<sup>1</sup>Personnal communication

Then we have

$$\mathcal{F}S_t\mathcal{F}^{-1}v = \exp(-\xi^2t + \xi t - \frac{t^3}{3})v(\xi - t), \quad (3.11)$$

and this implies immediately

$$\|S_t\| = \exp \max_{\xi}(-\xi^2t + \xi t - \frac{t^3}{3}) = \exp(-\frac{t^3}{12}). \quad (3.12)$$

Then one can get an estimate of the resolvent by using, for  $\lambda \in \mathbb{C}$ , the formula

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A} - \lambda) dt. \quad (3.13)$$

For a closed accretive operator, (3.13) is standard when  $\operatorname{Re} \lambda < 0$ , but estimate (3.12) on  $S_t$  gives immediately an holomorphic extension of the right hand side to the whole space, showing independently that the spectrum is empty (see Davies [Dav]) and giving for  $\lambda > 0$  the estimate

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \int_0^{+\infty} \exp(\lambda t - \frac{t^3}{12}) dt. \quad (3.14)$$

The asymptotic behavior as  $\lambda \rightarrow +\infty$  of this integral is immediately obtained by using the Laplace method and the dilation  $t = \lambda^{\frac{1}{3}}s$  in the integral.

The proof (see [Mart]) of the lower bound is obtained by constructing quasimodes for the operator  $(\mathcal{A} - \lambda)$  in its Fourier representation. We observe (assuming  $\lambda > 0$ ), that

$$\xi \mapsto u(\xi; \lambda) := \exp\left(-\frac{\xi^3}{3} + \lambda\xi - \frac{2}{3}\lambda^{\frac{3}{2}}\right) \quad (3.15)$$

is a solution of

$$\left(\frac{d}{d\xi} + \xi^2 - \lambda\right)u(\xi; \lambda) = 0. \quad (3.16)$$

Multiplying  $u(\cdot; \lambda)$  by a cut-off function  $\chi_\lambda$  with support in  $] -\sqrt{\lambda}, +\infty[$  and  $\chi_\lambda = 1$  on  $] -\sqrt{\lambda} + 1, +\infty[$ , we obtain a very good quasimode, concentrated as  $\lambda \rightarrow +\infty$ , around  $\sqrt{\lambda}$ , with an error term giving almost<sup>2</sup> the announced lower bound for the resolvent.

Of course this is a very special case of a result on the pseudo-spectra but this leads to an almost optimal result.

### 3.2. The complex Airy operator in $\mathbb{R}^+$

Here we mainly describe some results presented in [Alm2], who refers to [IvKo]. We can then associate the Dirichlet realization  $\mathcal{A}^D$  of the complex Airy operator  $D_x^2 + ix$  on the half-line, whose domain is

$$D(\mathcal{A}^D) = \{u \in H_0^1(\mathbb{R}^+), x^{\frac{1}{2}}u \in L^2(\mathbb{R}^+), (D_x^2 + ix)u \in L^2(\mathbb{R}^+)\}, \quad (3.17)$$

and which is defined (in the sense of distributions) by

$$\mathcal{A}^D u = (D_x^2 + ix)u. \quad (3.18)$$

Moreover, by construction, we have

$$\operatorname{Re} \langle \mathcal{A}^D u | u \rangle \geq 0, \quad \forall u \in D(\mathcal{A}^D). \quad (3.19)$$

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<sup>2</sup>One should indeed improve the cut-off for getting an optimal result

Again we have an operator, which is the generator of a semi-group of contraction, whose adjoint is described by replacing in the previous description  $(D_x^2 + ix)$  by  $(D_x^2 - ix)$ , the operator is injective and as its spectrum contained in  $\operatorname{Re} \lambda > 0$ . Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([Alm2]) that the spectrum of  $\mathcal{A}^D$   $\sigma(\mathcal{A}^D)$  is given by that

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty} \{\lambda_j\} \quad (3.20)$$

with

$$\lambda_j = \exp i \frac{\pi}{3} \mu_j, \quad (3.21)$$

the  $\mu_j$ 's being real zeroes of the Airy function satisfying

$$0 < \mu_1 < \dots < \mu_j < \mu_{j+1} < \dots \quad (3.22)$$

It is also shown in [Alm2] that the vector space generated by the corresponding eigenfunctions is dense in  $L^2(\mathbb{R}^+)$ .

We arrive now to the analysis of the properties of the semi-group and the estimate of the resolvent.

As before, we have, for  $\operatorname{Re} \lambda < 0$ ,

$$\|(\mathcal{A}^D - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Re} \lambda|}, \quad (3.23)$$

If  $\operatorname{Im} \lambda < 0$  one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set  $\operatorname{Re} \lambda \geq 0$ ,  $\operatorname{Im} \lambda \geq 0$ , which corresponds to the numerical range of the symbol.

We recall that for any  $\epsilon > 0$ , we define the  $\epsilon$ -pseudospectra by

$$\Sigma_\epsilon(\mathcal{A}^D) = \{\lambda \in \mathbb{C} \mid \|(\mathcal{A}^D - \lambda)^{-1}\| > \frac{1}{\epsilon}\}, \quad (3.24)$$

with the convention that  $\|(\mathcal{A}^D - \lambda)^{-1}\| = +\infty$  if  $\lambda \in \sigma(\mathcal{A}^D)$ .

We have

$$\cap_{\epsilon > 0} \Sigma_\epsilon(\mathcal{A}^D) = \sigma(\mathcal{A}^D). \quad (3.25)$$

We define, for any accretive closed operator, for  $\epsilon > 0$ ,

$$\hat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \operatorname{Re} z. \quad (3.26)$$

We also define

$$\hat{\omega}_0(\mathcal{A}) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\exp -t\mathcal{A}\| \quad (3.27)$$

$$\hat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \operatorname{Re} z. \quad (3.28)$$

**Theorem 3.2** (Gearhart-Prüss).

*Let  $\mathcal{A}$  be a densely defined closed operator in an Hilbert space  $X$  such that  $-\mathcal{A}$  generates a contraction semi-group and let  $\hat{\alpha}_\epsilon(\mathcal{A})$  and  $\hat{\omega}_0(\mathcal{A})$  denote the  $\epsilon$ -pseudospectral abscissa and the growth bound of  $\mathcal{A}$  respectively. Then*

$$\lim_{\epsilon \rightarrow 0} \hat{\alpha}_\epsilon(\mathcal{A}) = -\hat{\omega}_0(\mathcal{A}). \quad (3.29)$$

We refer to [EN] for a proof.

This theorem is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the  $\epsilon$ -spectra of the operator. We apply this theorem to our operator  $\mathcal{A}_D$  and our main theorem is



**Theorem 3.3.**

$$\widehat{\omega}_0(\mathcal{A}_D) = -\operatorname{Re} \lambda_1. \quad (3.30)$$

This statement was established by Almgren [Alm2] in a much weaker form. Using the first eigenfunction it is easy to see that

$$\|\exp -t\mathcal{A}^D\| \geq \exp -\operatorname{Re} \lambda_1 t. \quad (3.31)$$

Hence we have immediately

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\operatorname{Re} \lambda_1. \quad (3.32)$$

To prove that  $-\operatorname{Re} \lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$ , it is enough to show the following lemma.

**Lemma 3.4.**

*For any  $\alpha < \operatorname{Re} \lambda_1$ , there exists a constant  $C$  such that, for all  $\lambda$  s.t.  $\operatorname{Re} \lambda \leq \alpha$*

$$\|(\mathcal{A}^D - \lambda)^{-1}\| \leq C. \quad (3.33)$$

**Proof :** We know that  $\lambda$  is not in the spectrum. Hence the problem is just a control of the resolvent as  $|\operatorname{Im} \lambda| \rightarrow +\infty$ . The case, when  $\operatorname{Im} \lambda < 0$  has already been considered. Hence it remains to control the norm of the resolvent as  $\operatorname{Im} \lambda \rightarrow +\infty$  and  $\operatorname{Re} \lambda \in [-\alpha, +\alpha]$ .

This is indeed a semi-classical result ! The main idea is that when  $\operatorname{Im} \lambda \rightarrow +\infty$ , we have to invert the operator

$$D_x^2 + i(x - \operatorname{Im} \lambda) - \operatorname{Re} \lambda.$$

If we consider the Dirichlet realization in the interval  $]0, \frac{\operatorname{Im} \lambda}{2}[$  of  $D_x^2 + i(x - \operatorname{Im} \lambda) - \operatorname{Re} \lambda$ , it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse  $R_1(\lambda)$  satisfies

$$\|R_1(\lambda)\| \leq \frac{2}{\operatorname{Im} \lambda}.$$

Far from the boundary, we can use the resolvent of the problem on the line for which we have a uniform control of the norm for  $\operatorname{Re} \lambda \in [-\alpha, +\alpha]$ .

**Application.**

Coming back to the application in superconductivity, one is looking at the semigroup associated with  $\mathcal{A}_J := D_x^2 + iJx - 1$  (where  $J \geq 0$  is a parameter). The stability analysis leads to a critical value

$$J_c = (\operatorname{Re} \lambda_1)^{-\frac{3}{2}}, \quad (3.34)$$

such that :

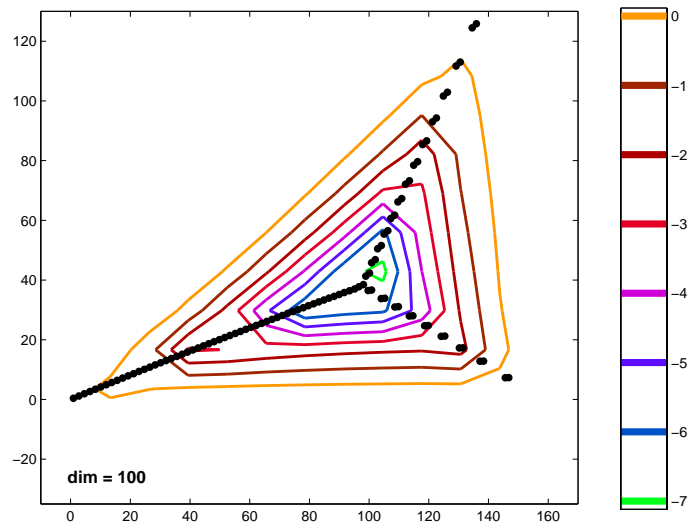
- For  $J \in [0, J_c[$ ,  $\|\exp -t\mathcal{A}_J\| \rightarrow +\infty$  as  $t \rightarrow +\infty$ .
- For  $J > J_c$ ,  $\|\exp -t\mathcal{A}_J\| \rightarrow 0$  as  $t \rightarrow +\infty$ .

This improves Lemma 2.4 in Almgren [Alm2], who gets only this decay for  $\|\exp -t\mathcal{A}_J\psi\|$ , with  $\psi$  in a specific dense space.

### 3.3. Numerical computations

Here we reproduce in Figure 3.3 the classical picture due to Trefethen of the pseudo-spectra of the Davies operator  $D_x^2 + ix^2$  on the line and in Figure 3.3 the corresponding picture realized with numerical computations for us by W. Bordeaux Montrieux for the case of Airy. W. Bordeaux Montrieux is using eigtool<sup>3</sup>. These figures give the level-curves of the norm of the resolvent  $\|(A - z)^{-1}\| = \frac{1}{\epsilon}$  corresponding to the boundary of the  $\epsilon$ -pseudospectra. The right column gives the correspondence between the color and  $\log_{10}(\epsilon)$ .

Figure 3.1: Davies operator: pseudospectra



As usual for this kind of computation for non self adjoint operators, we observe on both figures, in addition to the (discrete) spectrum lying on the half-line of argument  $\frac{\pi}{4}$  (resp.  $\frac{\pi}{3}$ ), an unexpected spectrum starting from the fifteenth eigenvalue. This was already observed by B. Davies for the complex harmonic oscillator  $D_x^2 + ix^2$ . This is immediately connected with the accuracy of the computations of Maple.

The computation for Figure 3.3 is done on an interval  $[0, L]$  with Dirichlet conditions at 0 and  $L$  using 400 “grid points”. The figure gives the level-curves of the norm of the resolvent  $\|(A - z)^{-1}\| = \frac{1}{\epsilon}$  corresponding for each  $\epsilon$  to the boundary of the  $\epsilon$ -pseudospectrum. The right column gives the correspondence between the color and  $\log_{10}(\epsilon)$ .

In the upper part of the Airy-picture in Figure 3.3, these level-curves become asymptotically vertical lines corresponding to the fact that each  $\epsilon$ -pseudospectrum of the Airy operator is a left-bounded half-plane.

The first zoom in Figure 3.3 below shows that for  $\epsilon = 10^{-1}$ , the  $\epsilon$ -pseudo-spectrum has two components, the bounded one containing the first eigenvalue. For  $\epsilon = 10^{-2}$ , the  $\epsilon$ -pseudo-spectrum has three components, each bounded one containing one

<sup>3</sup>see <http://www-pnp.physics.ox.ac.uk/~stokes/courses/scicomp/eigtool/html/eigtool/documentation/menus/airy-demo.html> and <http://www.comlab.ox.ac.uk/pseudospectra/eigtool/>.

Figure 3.2: Airy with Dirichlet condition : pseudospectra

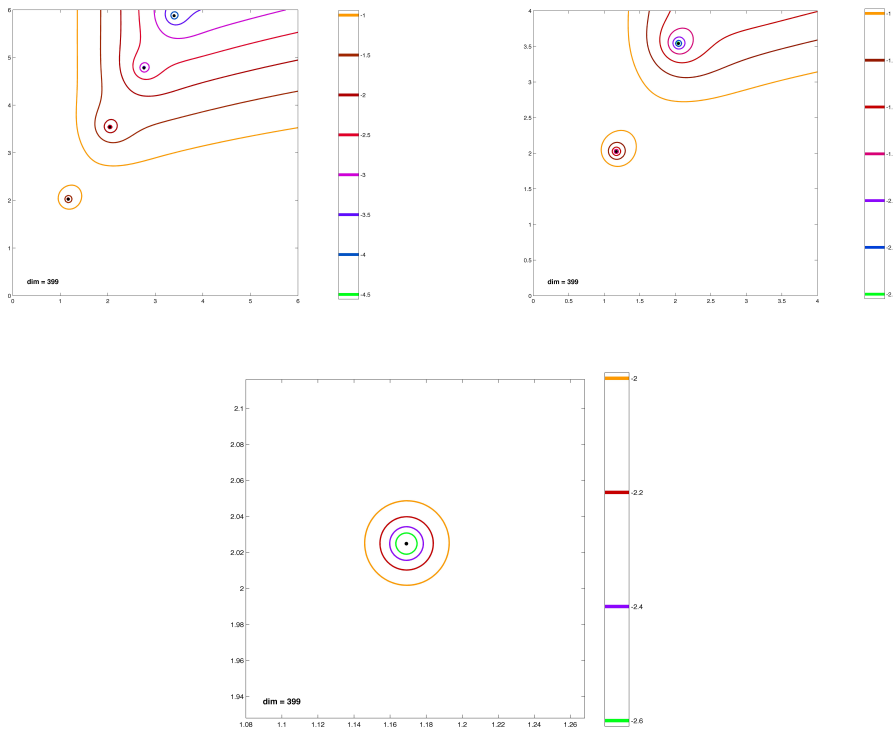
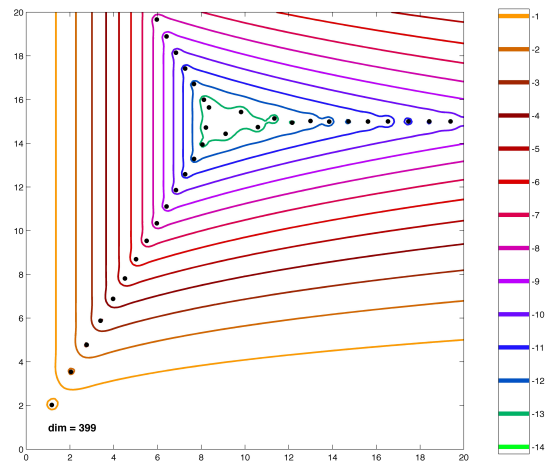


Figure 3.3: Zooms

eigenvalue. The second and third zoom illustrate the property that, for a given  $k$ , as  $\epsilon \rightarrow 0$ , the component of the  $\epsilon$ -pseudospectrum containing one eigenvalue  $\mu_k$  becomes asymptotically a disk centered at  $\mu_k$ .

### 3.4. Higher dimension problems relative to Airy

Here we follow (and extend) [Alm2] Almgog.

#### 3.4.1. The model in $\mathbb{R}^2$

We consider the operator

$$\mathcal{A}_2 := -\Delta_{x,y} + ix. \quad (3.35)$$

**Proposition 3.5.**

$$\sigma(\mathcal{A}_2) = \emptyset. \quad (3.36)$$

**Proof :** After a Fourier transform in the  $y$  variable, it is enough to show that

$$(\widehat{\mathcal{A}}_2 - \lambda)$$

is invertible with

$$\widehat{\mathcal{A}}_2 = D_x^2 + ix + \eta^2. \quad (3.37)$$

We have just to control for a given  $\lambda \in \mathbb{C}$ ,  $(D_x^2 + ix + \eta^2 - \lambda)^{-1}$  (whose existence is given by the 1D result) uniformly in  $\mathcal{L}(L^2(\mathbb{R}))$  uniformly with respect to  $\eta$ .

#### 3.4.2. The model in $\mathbb{R}_+^2$ : perpendicular current.

Here it is useful to reintroduce the parameter  $J$ , which is assumed to be positive. Hence we consider the Dirichlet realization

$$\mathcal{A}_2^{D,\perp} := -\Delta_{x,y} + iJx, \quad (3.38)$$

in  $\mathbb{R}_+^2 = \{x > 0\}$ .

**Proposition 3.6.**

$$\sigma(\mathcal{A}_2^{D,\perp}) = \cup_{r \geq 0, j \in \mathbb{N}^*} (\lambda_j + r). \quad (3.39)$$

**Proof :** For the inclusion

$$\cup_{r \geq 0, j \in \mathbb{N}^*} (\lambda_j + r) \subset \sigma(\mathcal{A}_2^{D,\perp}),$$

we can use  $L^\infty$  eigenfunctions in the form

$$(x, y, z) \mapsto \exp i(y\eta + z\zeta)u_j(x)$$

where  $u_j$  is the eigenfunction associated to  $\lambda_j$ . We have then to use the fact that  $L^\infty$ -eigenvalues belong to the spectrum. This can be formulated in the following proposition.

**Proposition 3.7.**

Let  $\Psi \in L^\infty(\mathbb{R}_+^2) \cap H_{loc}^1(\mathbb{R}_+^2)$  satisfying, for some  $\lambda \in \mathbb{C}$ ,

$$-\Delta_{x,y}\Psi + iJx\Psi = \lambda\Psi \quad (3.40)$$

in  $\mathbb{R}_+^2$  and

$$\Psi_{x=0} = 0. \quad (3.41)$$

Then either  $\Psi = 0$  or  $\lambda \in \sigma(\mathcal{A}_3^{D,\perp})$ .

For the opposite inclusion, we observe that we have to control uniformly

$$(\mathcal{A}^D - \lambda + \eta^2)^{-1}$$

with respect to  $\eta$  under the condition that

$$\lambda \notin \cup_{r \geq 0, j \in \mathbb{N}^*} (\lambda_j + r).$$

It is enough to observe the uniform control as  $\eta^2 \rightarrow +\infty$  which results of (3.23).

### 3.4.3. The model in $\mathbb{R}_2^+$ : parallel current

Here the models are the Dirichlet realization in  $\mathbb{R}_+^2$  :

$$\mathcal{A}_2^{D,\parallel} = -\Delta_{x,y} + i J y, \quad (3.42)$$

or the Neumann realization

$$\mathcal{A}_2^{N,\parallel} = -\Delta_{x,y} + i J y. \quad (3.43)$$

Using the reflexion (or antireflexion) trick we can see the problem as a problem on  $\mathbb{R}^2$  restricted to odd (resp. even) functions with respect to  $(x, y) \mapsto (-x, y)$ . It is clear from Proposition 3.5 that in this case the spectrum is empty.

## 4. A few theorems for more general situations

### 4.1. Other models

The goal is to treat more general situations where we no more know explicitly the spectrum like for complex Airy or complex harmonic oscillator. At least for the case without boundary this is close to the problematic of the lectures of J. Sjöstrand. The operators we have in mind are (see [AHP])

$$D_x^2 + (D_y - \frac{1}{2}x^2)^2 + i c y, \quad (4.1)$$

and the next one could be

$$(D_x + \frac{x^3}{3})^2 + (D_y - x^2 y)^2 + i c(x^2 - y^2). \quad (4.2)$$

More generally :

$$B(x, y) = \operatorname{Re} \psi(z), \quad \Phi(x, y) = c \operatorname{Im} \psi(z), \quad (4.3)$$

with  $\psi$  holomorphic will work.

If  $\psi$  is a non constant polynomial and  $c \neq 0$ , then one can prove that the operator will have compact resolvent (see Theorem 4.1 below).

### 4.2. Maximal accretivity

All the operators considered before can be placed in the following more general context. We consider in  $\mathbb{R}^n$  (or in an open set  $\Omega \subset \mathbb{R}^n$ )

$$P_{A,V} := -\Delta_A + V, \quad (4.4)$$

with

$$\operatorname{Re} V \geq 0 \text{ and } V \in C^\infty(\overline{\Omega}), \mathbf{A} \in C^\infty(\overline{\Omega}, \mathbb{R}^n).$$

Then it is interesting to observe that when  $\Omega = \mathbb{R}^n$ , then the operator is maximally accretive (see [HelNi] for the definition). The proof which is given in [AHP] is close to the proof (see for example [FoHel]) of the fact that if  $V$  is in addition real  $-\Delta_A + V$  is essentially self-adjoint.

### 4.3. A criterion for compactness of the resolvent

All the results of compact resolvent stated in this paper can be proved in an unified way. Here we follow the proof of Helffer-Mohamed [HelMo], actually inspired by Kohn's proof in subellipticity (see [HelNi] for a presentation). We will analyze the problem for the family of operators  $P_{A,V}$ , where the electric potential has in addition the form :

$$V(x) = \left( \sum_j V_j(x)^2 \right) + iQ(x),$$

with  $V_j$  and  $Q$  in  $C^\infty$ .

We note also that it has the form :

$$P_{A,V} = \sum_{j=1}^{n+p} X_j^2 = \sum_{j=1}^n X_j^2 + \sum_{\ell=1}^p Y_\ell^2 + iX_0,$$

with

$$X_j = (D_{x_j} - A_j(x)), \quad j = 1, \dots, n, \quad Y_\ell = V_\ell, \quad \ell = 1, \dots, p, \quad X_0 = Q.$$

In particular, the magnetic field is recovered by observing that

$$B_{jk} = \frac{1}{i} [X_j, X_k] = \partial_j A_k - \partial_k A_j, \quad \text{for } j, k = 1, \dots, n.$$

We now introduce the quantities :

$$m_q(x) = \sum_\ell \sum_{|\alpha|=q} |\partial_x^\alpha V_\ell| + \sum_{j < k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| + \sum_{|\alpha|=q-1} |\partial_x^\alpha Q|. \quad (4.5)$$

It is easy to reinterpret this quantity in terms of commutators of the  $X_j$ 's. When  $q = 0$ , the convention is that

$$m_0(x) = \sum_\ell |V_\ell(x)|. \quad (4.6)$$

Let us also introduce

$$m^r(x) = 1 + \sum_{q=0}^r m_q(x). \quad (4.7)$$

Then the criterion is

#### Theorem 4.1.

*Let us assume that there exists  $r$  and a constant  $C$  such that*

$$m_{r+1}(x) \leq C m^r(x), \quad \forall x \in \mathbb{R}^n, \quad (4.8)$$

*and*

$$m^r(x) \rightarrow +\infty, \quad \text{as } |x| \rightarrow +\infty. \quad (4.9)$$

*Then  $P_{A,V}(h)$  has a compact resolvent.*

#### 4.4. About the $L^\infty$ -spectrum

We already met this question in Proposition 3.7. This proposition is actually a particular case of the more general statement [AHP], which can be seen as giving a comparison between the  $L^\infty$ -spectrum and the  $L^2$ -spectrum.

**Proposition 4.2.**

*We assume that  $V \in C^0$  and  $\operatorname{Re} V \geq 0$ . If  $(\psi, \lambda)$  satisfies*

$$(P_{\mathbf{A},V} - \lambda)\psi = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n),$$

*with  $\psi \in L^\infty$  (or  $L^2$ ), then either  $\psi = 0$  or  $\lambda$  is in the spectrum of  $\mathcal{P} = \overline{P_{\mathbf{A},V}}$ .*

The proof is reminiscent of the so-called Schnol's theorem.

### 5. Conclusion

What we plan to continue in collaboration with Y. Almog and X. Pan is to analyze

$$\frac{1}{t} \ln \|\exp -t\overline{P_{\mathbf{A},V}}\|,$$

as  $t \rightarrow +\infty$  for our specific examples

1. in the case of the whole space,
2. in the case of the half space
3. and apply these results to the stability question of problem with boundary in various asymptotic limits.

This should lead to the introduction of critical fields like in the standard superconductivity theory (see [FoHel]). One of the difficulties for these generalizations is that we have no longer the explicit knowledge of the spectrum like for the complex Airy operator. We do not know for example if the spectrum of the Dirichlet realization in the half-space associated with  $P_{\mathbf{A},V}$  is non empty.

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MATHÉMATIQUES, BAT. 425, UNIVERSITÉ PARIS-SUD 11, 91405 ORSAY  
CEDEX, UMR 8628 DU CNRS  
The author was also visiting the Erwin Schrödinger Institute in Vienna when writing this paper.