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Eigenvalue asymptotics for randomly perturbed non-self adjoint operators

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The following is based on joint work with Johannes Sjöstrand ([1]), to which we refer for references and details that had to be omitted here.

We will examinate the distribution of eigenvalues of non-selfadjoint h-pseudodifferential operators, perturbed by a random operator, in the limit as $h \to 0$.

Let us start by recalling that for $h > 0$, we associate to a symbol $p : \mathbf{R^{2n}} \to \mathbf{C}$ in an appropriate class the h-Weyl quantization $P = p^w$, where

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\eta} p(\frac{x+y}{2}, \eta; h) u(y) dy d\eta, \tag{1}$$

and consider this operator in $L^2(\mathbf{R^n})$. For example, if $p(x, \xi) = \xi^2 + cx^2$, for $c \in \mathbf{C}$, we obtain the harmonic oscillator $P = (hD_x)^2 + cx^2$. If c is real, this operator is selfadjoint and has real spectrum. As has been found by Weyl, and generalized by Helffer-Robert and Ivrii, the number of eigenvalues inside an interval $I \subset \mathbf{R}$ can be expressed in terms of a classical quantity, namely a volume depending only on the symbol p of the operator:

$$N(P, I) = \frac{1}{(2\pi h)^n}(\mathrm{vol}\,(p^{-1}(I)) + o(1)),\ \ h \to 0. \tag{2}$$

This Weyl-law gives us a nice description of the eigenvalue asymptotics as $h \to 0$.

Consider now the case $\mathrm{Im}\, c \neq 0$: the situation changes dramatically as the operator is non-selfadjoint and could a priori have spectrum anywhere in the complex plane. For the harmonic oscillator, it is known to lie on a line. But as p is complex-valued now, $\Sigma(p) = \overline{p(\mathbf{R^2})}$ is a cone (and not a line anymore), and it is not clear if there is any relation of the spectrum of P to p. Actually, everywhere in the interior of $\Sigma(p)$, the norm of the resolvent $(P - z)^{-1}$, $z \in \mathbf{C} \setminus \mathrm{Spec}\,(\mathbf{P})$, is very large. In particular, if one adds a very small perturbation to the operator, the eigenvalues could lie anywhere inside $\Sigma(p)$ as $h \to 0$, possibly far away from the line to which the spectrum of the unperturbed operator is confined. Given all this, it is really surprising that we obtain a Weyl-law for the eigenvalues of the randomly perturbed operator asymptotically almost surely.

Before stating our result, let us indicate the consequences of these phenomena. For example, if one wishes to perform numerical calculations involving non-selfadjoint operators (or matrices), rounding errors might introduce small perturbations to the operator, meaning that the eigenvalues computed numerically could actually be far away from the true ones (this has been observed in particular by M. Zworski). Also in stability studies for non-linear problems where the linearized operator is non-selfadjoint, spectral properties are of great importance.

Let us now introduce the frame of our theorem, and begin with the symbols we consider. Let $m \geq 1$ be an order function, i.e.

$$\exists C_0 \geq 1,\ N_0 > 0 \text{ such that } m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu),$$
$$\forall \rho, \mu \in \mathbf{R}^{2n},\ \langle \rho - \mu \rangle = \sqrt{1 + |\rho - \mu|^2}.$$

Let p be in the corresponding symbol space

$$S(\mathbf{R}^{2n}, m) = \{a \in C^\infty(\mathbf{R}^{2n});\ |\partial_\rho^\alpha a(\rho)| \leq C_\alpha m(\rho),\ \rho \in \mathbf{R}^{2n},\ \alpha \in \mathbf{N}^{2n}\}, \tag{3}$$

and assume that $\exists\ z_0 \in \mathbf{C}$, $C_0 > 0$ such that

$$|p(\rho) - z_0| \geq m(\rho)/C_0,\ \ \rho \in \mathbf{R}^{2n}. \tag{4}$$

Then for any simply connected $\Omega \Subset \mathbf{C} \setminus \Sigma_\infty$, where Σ_∞ is the set of accumulation points of p at ∞, that is not entirely contained in Σ, we know that $\mathrm{Spec}(P) \cap \Omega$ is discrete for $h > 0$ small enough. Here the natural domain of P is, as first introduced by Bony-Chemin, $H(m) \equiv (P - z_0)^{-1}(L^2(\mathbf{R}^n))$.

Next, we introduce the type of sets $\Gamma \subset \Omega$ inside of which we will count the eigenvalues, where Ω is as before. We assume that $\partial\Gamma \in C^\infty$, and that $\forall z \in \partial\Gamma$, $p^{-1}(z)$ is a smooth submanifold of T^*R^n on which dp, $d\overline{p}$ are linearly independant (at every point). This means that p has no critical value on $\partial\Gamma$. In the example, this assumption is fulfilled for any $\Gamma \Subset \mathrm{int}(\Sigma)$.

Finally, the type of random perturbations we consider is as follows. Let $0 < \widetilde{m}, \widehat{m} \leq 1$ be square integrable order functions on $\mathbf{R}^{2n}$ such that $\widetilde{m}$ or $\widehat{m}$ is integrable, and let $\widetilde{S} \in S(\widetilde{m})$, $\widehat{S} \in S(\widehat{m})$ be elliptic symbols. Let $\widetilde{e}_1, \widetilde{e}_2, \ldots$, and $\widehat{e}_1, \widehat{e}_2, \ldots$ be orthonormal bases for $L^2(\mathbf{R}^n)$. Our random perturbation will be

$$Q_\omega = \widehat{S} \circ \sum_{j,k} \alpha_{j,k}(\omega) \widehat{e}_j \widetilde{e}_k^* \circ \widetilde{S}, \tag{5}$$

where $\alpha_{j,k}$ are independent complex $\mathcal{N}(0,1)$ random variables, and $\widehat{e}_j \widetilde{e}_k^* u = (u|\widetilde{e}_k)\widehat{e}_j$, $u \in L^2$. As is shown in [1], under these assumptions $\|Q\|_{HS} \leq C_1 h^{-n}$ (hence also $\|Q\| \leq Ch^{-n}$) and $\|Q\|_{tr} \leq C_2 h^{-\frac{3n}{2}}$ asymptotically almost surely (a.a.s.), where a.a.s. means with probability tending to 1 as $h \to 0$.

Our result is the following.

Theorem 1. *For every $0 < \epsilon \ll 1$, for $0 < h \ll 1$ small enough, and $\delta > 0$ such that*

$$e^{-\frac{\epsilon}{h}} < \delta \ll h^{3n+1/2},$$

the number of eigenvalues of $P + \delta Q$ inside Γ is asymptotically almost surely given by

$$N(P + \delta Q, \Gamma) = \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma)) + O(\frac{\sqrt{\epsilon}}{h^n}). \tag{6}$$

This means that we have obtained a Weyl-law with an error depending on the small parameter ϵ. Remark that the volume on the right-hand side is now implying a "bidimensional" distribution of the eigenvalues of the perturbed harmonic oscillator (whose spectrum before perturbation was confined to a line).

Actually, we have tried here to state the simplest special case of the general result in [1], but some of the assumptions can be relaxed very much. There, we do also obtain that the probability is actually bounded below by $1 - \frac{C}{\sqrt{\epsilon}} e^{-\frac{\epsilon/2}{(2\pi h)^n}}$.

Let us now give a short sketch of the proof. For all details, we refer to [1].

The first step is to introduce a well-defined determinant whose zeros determine the spectrum of our operator. Here we construct an auxiliary operator $\tilde{P}$ with symbol $\tilde{p} = \kappa \circ p$ such that $(\tilde{P} - z)^{-1}$ exists $\forall z \in \tilde{\Omega}$, $\Gamma \Subset \tilde{\Omega} \Subset \Omega$, and $P(z) = (\tilde{P} - z)^{-1}(P - z) = 1 + K(z)$, where $K(z)$ is trace class. The spectrum of P inside of Γ is given by the zeros of the well-defined determinant $\det(P(z))$. Finally, the spectrum of $P + \delta Q$ is a.a.s. given by the zeros of $\det P^\delta(z)$, where $P^\delta(z) = (\tilde{P} - z)^{-1}(P + \delta Q - z)$, and it is holomorphic in $z \in \Omega$.

We would now like to use a proposition about the zeros of a holomorphic function, if we are given an upper bound and lower bounds pointwise on its modulus.

To this end, we set up a Grushin problem, leading to the following result (similar to the Schur complementation formula):

$$|\det P| = |\det \mathcal{P}||\det E_{-+}|, \tag{7}$$

where for $0 < h < \alpha \ll 1$,

$$|\det \mathcal{P}|^2 = \alpha^{N(\alpha)} \det 1_\alpha(P^*P), \quad 1_\alpha(t) = \max\{\alpha, t\} \ , \tag{8}$$

and $N(\alpha)$ is the number of eigenvalues of P^*P less than α. Here E_{-+} is a certain matrix of size $N(\alpha)$, and we have written P for $P(z)$.

We will first study $N(\alpha)$ and $1_\alpha(P^*P)$ using the functional calculus. We introduce a C^∞-function χ such that $1_\alpha(Q) \sim Q + \alpha\chi(\frac{Q}{\alpha})$, for $Q = P^*P \geq 0$ selfadjoint. Given our assumptions on p, we can relate $\ln\det(Q + \alpha\chi(\alpha^{-1}Q))$ to $\iint \ln q dx d\xi$, and obtain the estimate $N(\alpha) = O(\alpha h^{-n})$. Consider now the perturbed operator. A.a.s., we can show using the "smallness" of the perturbation and our previous result that

$$\ln|\det P^\delta| \leq \frac{1}{(2\pi h)^n}\left(\iint \ln|p - z| dx d\xi + O(h \ln \frac{1}{h})\right), \tag{9}$$

where we have chosen $\alpha = Ch$. For a lower bound, we use the Grushin-problem:

$$|\det P^\delta| = |\det \mathcal{P}^\delta||\det E^\delta_{-+}|, \tag{10}$$

where we can again relate $\ln|\det \mathcal{P}^\delta|$ to $\iint \ln|p - z| dx d\xi$.

Hence it remains to obtain a lower bound on $\ln|\det E^\delta_{-+}|$. Here we can take advantage of a perturbative expansion and need to elaborate estimates on determinants of random matrices, showing finally that a.a.s. it is bounded below by $-\frac{C\epsilon}{h^n}$.

As a conclusion, we obtain in addition to (9) that a.a.s.

$$\ln|\det P^\delta| \geq \frac{1}{(2\pi h)^n}\left(\iint \ln|p - z| dx d\xi - \frac{\epsilon}{C}\right). \tag{11}$$

Applying the proposition about the number of zeros of such a holomorphic function as mentionned above gives us that a.a.s.,

$$
\begin{aligned}
N(P+\delta Q,\Gamma) &= \#\{z\in\Gamma; \det(P^\delta(z))=0\} \\
&= \frac{1}{(2\pi h)^n}\int_\Gamma \frac{1}{2\pi}\Delta_z\left(\iint \ln|p-z|dxd\xi\right)\mathcal{L}(dz) + O(\frac{\sqrt{\epsilon}}{h^n}) \\
&= \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma)) + O(\frac{\sqrt{\epsilon}}{h^n}),
\end{aligned}
$$

which is the result of our theorem.

References

[1] M. Hager, J. Sjöstrand, Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators, to appear in Mathematische Annalen.

http://xxx.lanl.gov/pdf/math/0601381