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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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We present the results of [8] and [9] concerning the Gross-Pitaevskii equation with a repulsive delta function potential. We show that a high velocity incoming soliton is split into a transmitted component and a reflected component. The transmitted mass (L^2 norm squared) is shown to be in good agreement with the quantum transmission rate of the delta function potential. We further show that the transmitted and reflected components resolve into solitons plus dispersive radiation, and quantify the mass and phase of these solitons.

More precisely we consider

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - q\delta_0(x)u + u|u|^2 = 0 \\ u(x,0) = u_0(x)\,, \end{cases} \tag{1}$$

with $q > 0$. As initial data we take a fast soliton approaching the impurity from the left:

$$u_0(x) = e^{ivx}\operatorname{sech}(x - x_0)\,, \quad v \gg 1\,, \quad x_0 \ll 0. \tag{2}$$

Because of the homogeneity of the problem this covers the case of the general soliton profile $A\operatorname{sech}(Ax)$. The quantum transmission rate at velocity v is given by the square of the absolute value of the transmission coefficient, see (10) below,

$$T_q(v) = |t_q(v)|^2 = \frac{v^2}{v^2 + q^2}\,. \tag{3}$$

For the soliton scattering the natural definition of the transmission rate is given by

$$T_q^{\mathrm{s}}(v) = \lim_{t\to\infty} \frac{\|u(t)\!\restriction_{x>0}\|_{L^2}^2}{\|u(t)\|_{L^2}^2} = \frac{1}{2}\lim_{t\to\infty}\|u(t)\!\restriction_{x>0}\|_{L^2}^2\,, \tag{4}$$

provided that the limit exists. We expect that it *does* and that for fixed q/v, there is a $\sigma > 0$ such that

$$T_q^{\mathrm{s}}(v) = T_q(v) + \mathcal{O}(v^{-\sigma}), \quad \text{as } v \to +\infty\,. \tag{5}$$

Based on the comparison with the linear case and the numerical evidence [9] we expect (5) with $\sigma = 2$. Towards this heuristic claim we have

Theorem 1. *Let δ satisfy $\frac{2}{3} < \delta < 1$. If $u(x,t)$ is the solution of (1) with initial condition (2) and $x_0 \le -v^{1-\delta}$, then for fixed q/v,*

$$\frac{1}{2}\int_{x>0}|u(x,t)|^2\,dx = \frac{v^2}{v^2 + q^2} + \mathcal{O}(v^{1-\frac{3}{2}\delta}), \quad \text{as } v \to +\infty\,, \tag{6}$$
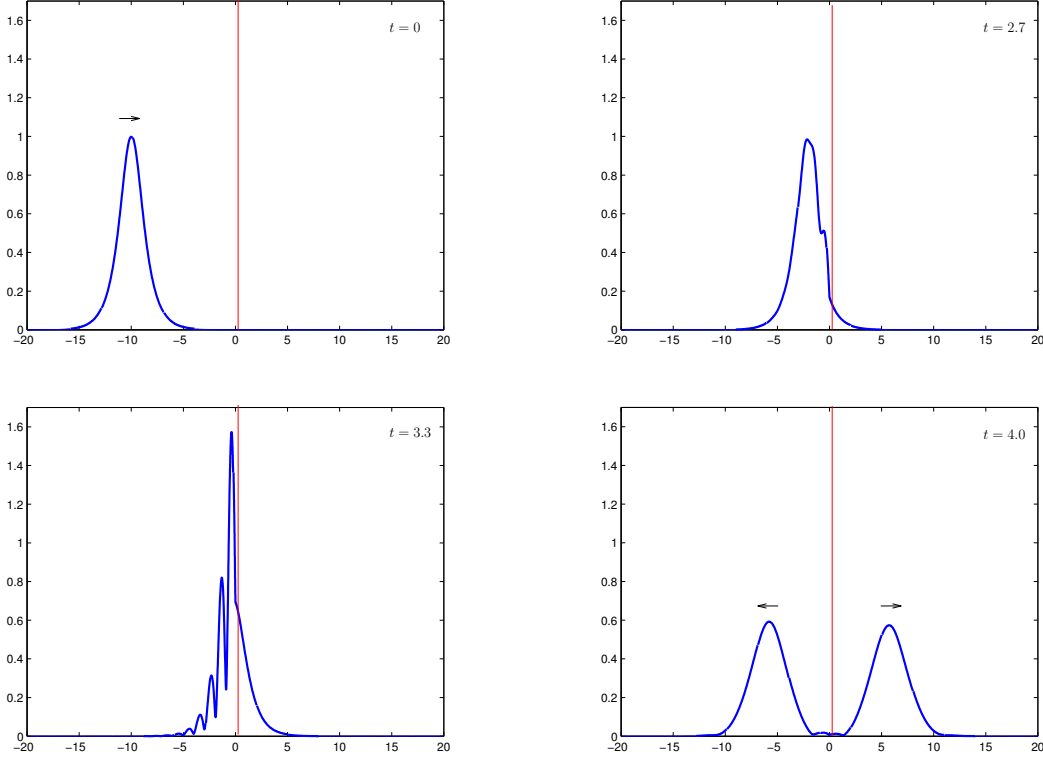
Figure 1: Numerical simulation of the case $q = v = 3$, $x_0 = -10$, at times $t = 0.0, 2.7, 3.3, 4.0$. Each frame is a plot of amplitude $|u|$ versus x.

uniformly for

$$\frac{|x_0|}{v} + v^{-\delta} \le t \le (1 - \delta) \log v$$

We see that by taking δ very close to 1, we obtain an asymptotic rate just shy of $v^{-1/2}$. More precisely, we show that there exists

$$v_0 = v_0(q/v, \delta),$$

diverging to $+\infty$ as $\delta \uparrow 1$ and $q/v \to +\infty$, such that for fixed q/v, if $v \ge v_0$, then

$$\left| \frac{1}{2} \int_{x>0} |u(x,t)|^2 dx - \frac{v^2}{v^2 + q^2} \right| \le c v^{1 - \frac{3}{2}\delta}.$$

The constant c appearing here is independent of all parameters (q, v, and δ).

We have conducted a numerical verification of Theorem 1 – see Fig. 2. It shows that the approximation given by (6) is very good even for velocities as low as ~ 3, at least for

$$0.6 \le \alpha \overset{\text{def}}{=} q/v \le 1.4.$$

A more elaborate numerical analysis will appear in our forthcoming paper [9].

Our second result shows that the scattered solution is given, on the same time scale, by a sum of a reflected and a transmitted soliton, and of a time decaying (radiating) term – see the fourth frame of Fig. 1. This is further supported by a

forthcoming numerical study [9]. In previous works in the physics literature (see for instance [2]) the resulting waves were only described as "soliton-like".

Theorem 2. *Under the hypothesis of Theorem 1 and for*

$$\frac{|x_0|}{v} + 1 \le t \le (1 - \delta) \log v,$$

we have, as $v \to +\infty$,

$$u(x,t) = u_T(x,t) + u_R(x,t) + \mathcal{O}_{L_x^\infty}\left((t - |x_0|/v)^{-1/2}\right) + \mathcal{O}_{L_x^2}(v^{1 - \frac{3}{2}\delta}),$$

$$u_T(x,t) = e^{i\varphi_T} e^{ixv + i(A_T^2 - v^2)t/2} A_T \operatorname{sech}(A_T(x - x_0 - tv)), \qquad (7)$$

$$u_R(x,t) = e^{i\varphi_R} e^{-ixv + i(A_R^2 - v^2)t/2} A_R \operatorname{sech}(A_R(x + x_0 + tv)),$$

where $A_T = (2|t_q(v)| - 1)_+$, $A_R = (2|r_q(v)| - 1)_+$, and

$$\varphi_T = \arg t_q(v) + \varphi_0(|t_q(v)|) + (1 - A_T^2)|x_0|/2v,$$

$$\varphi_R = \arg r_q(v) + \varphi_0(|r_q(v)|) + (1 - A_R^2)|x_0|/2v, \qquad (8)$$

$$\varphi_0(\omega) = \int_0^\infty \log\left(1 + \frac{\sin^2 \pi\omega}{\cosh^2 \pi\zeta}\right) \frac{\zeta}{\zeta^2 + (2\omega - 1)^2} \, d\zeta.$$

Here $t_q(v)$ and $r_q(v)$ are the transmission and reflection coefficients of the delta-potential (see (10)). When $2|t_q(v)| = 1$ or $2|r_q(v)| = 1$ the first error term in (7) is modified to $\mathcal{O}_{L_x^\infty}((\log(t - |x_0|/v))/(t - |x_0|/v)^{\frac{1}{2}})$.

Here and later we use the standard notation

$$a_+^k = \begin{cases} a^k & a \ge 0, \\ 0 & a < 0. \end{cases} \qquad (9)$$

This asymptotic description holds for v greater than some threshold depending on q/v and δ, as in Theorem 1. The implicit constant in the $\mathcal{O}_{L_x^2}$ error term is entirely independent of all parameters (q, v, and δ), although the implicit constant in the $\mathcal{O}_{L_x^\infty}$ error term depends upon q/v, or more precisely, the proximity of $|t_q(v)|$ and $|r_q(v)|$ to $\frac{1}{2}$.

A comparison of the transmission and reflection coefficients (3) of the δ potential, and of the soliton transmission and reflections coefficients, A_T and A_R, appearing in (7), is shown in Figure 3.

To present the definitions of the transmission and reflection coefficients, t_q and r_q respectively, we recall the definition of the scattering matrix for the Hamiltonian

$$H_q = -\frac{1}{2}\frac{d^2}{dx^2} + q\,\delta_0(x).$$

For that we consider solutions of $(H_q - \lambda^2)u = 0$ and their coefficients:

$$u(x) = A_\pm e^{-i\lambda x} + B_\pm e^{i\lambda x}, \quad \pm x > 0.$$

The matrix defined by

$$S(\lambda) \; : \; \begin{bmatrix} A_+ \\ B_- \end{bmatrix} \longmapsto \begin{bmatrix} A_- \\ B_+ \end{bmatrix},$$

is called the *scattering matrix*. In our simple case it is easily computed:

$$S(\lambda) = \begin{bmatrix} t_q(\lambda) & r_q(\lambda) \\ r_q(\lambda) & t_q(\lambda) \end{bmatrix}, \quad t_q(\lambda) = \frac{i\lambda}{i\lambda - q}, \quad r_q(\lambda) = \frac{q}{i\lambda - q}. \qquad (10)$$
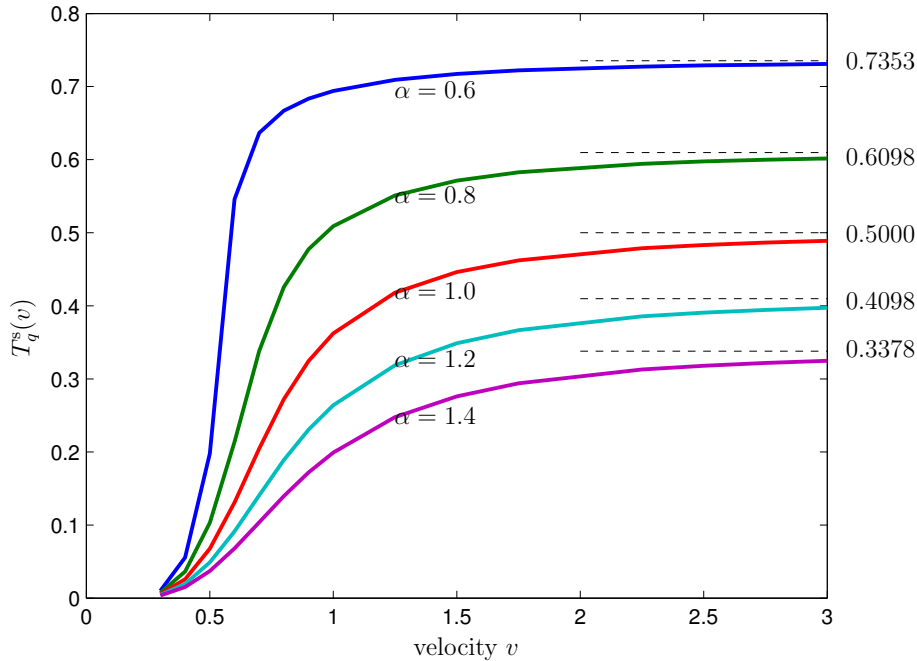
Figure 2: A plot of the numerically obtained transmission $T_q^{\mathrm{s}}(v)$ versus velocity v for five values of $\alpha = q/v = 0.6, 0.8, 1.0, 1.2, 1.4$. The dashed lines are the corresponding theoretical $v \to +\infty$ asymptotic values given by $1/(1 + \alpha^2)$.

The scattering coefficients satisfy two equations, one standard (unitarity) and one due to the special structure of the potential:

$$|t_q(\lambda)|^2 + |r_q(\lambda)|^2 = 1 \,, \quad t_q(\lambda) = 1 + r_q(\lambda) \,.$$

Scattering of solitons by delta impurities is a natural model explored extensively in the physics literature – see for instance [2],[7], and references given there. The heuristic insight that at high velocities "linear scattering" by the external potential should dominate the partition of mass is certainly present there. In the mathematical literature the dynamics of solitons in the presence of external potentials has been studied in high velocity or semiclassical limits following the work of Floer and Weinstein [5], and Bronski and Jerrard [1] – see [6] for recent results and a review of the subject. Roughly speaking, the soliton evolves according to the classical motion of a particle in the external potential. That is similar to the phenomena in other settings, such as the motion of the Landau-Ginzburg vortices.

The possible novelty in (6) and (7) lies in seeing *quantum* effects of the external potential strongly affecting soliton dynamics. As shown in Fig. 2, Theorem 1 gives a very good approximation to the transmission rate already at low velocities. Fig. 1 shows time snapshots of the evolution of the soliton, and the last frame suggests the soliton resolution (7). We should stress that the asymptotic solitons are resolved at a much larger time – see [9].

The proof of the two theorems proceeds by approximating the solution during the "interaction phase" (the interval of time during which the solution significantly
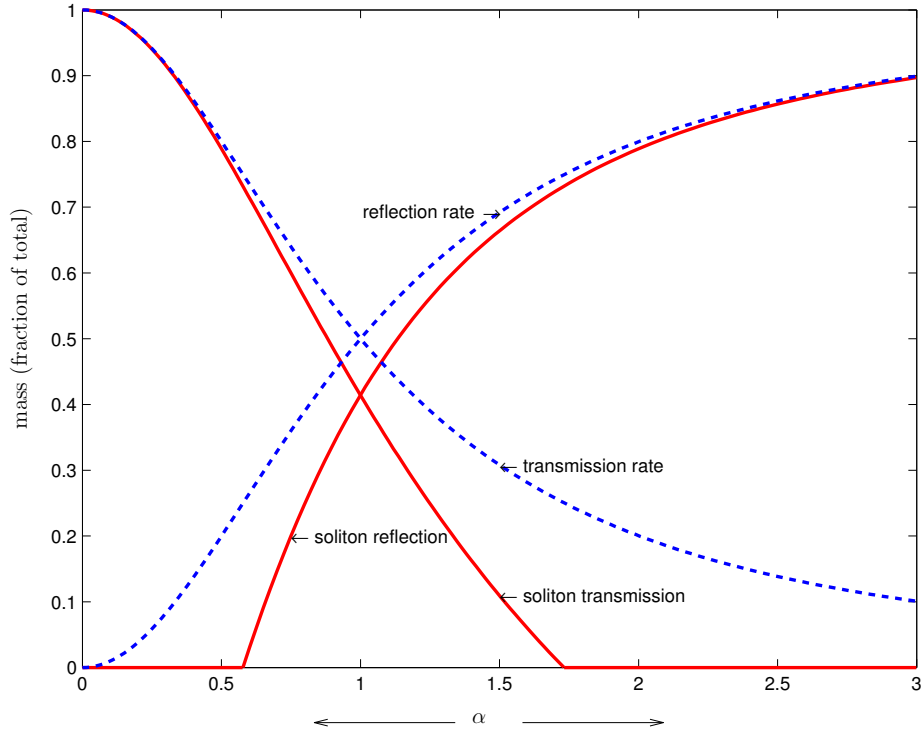
Figure 3: Comparison of linear and nonlinear scattering coefficients as functions of $\alpha \stackrel{\text{def}}{=} q/v$.

interacts with the delta potential at the origin) by the corresponding linear flow. This approximation is achieved, uniformly in q, by means of Strichartz estimates. The crucial observation is that for $q \geq 0$ the constants in the Strichartz estimates do not depend on q. The use of the Strichartz estimates as an approximation device, as opposed to say energy estimates, is critical since the estimates obtained depend only upon the L^2 norm of the solution, which is conserved and *independent* of v. Thus, v functions as an asymptotic parameter; larger v means a shorter interaction phase and a better approximation of the solution by the linear flow. Theorem 2 combines this analysis with the inverse scattering method [11],[4],[3],[10]. The delta potential splits the incoming soliton into two waves which become single *solitons*.

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