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Monge-Ampère Equations, Geodesics and Geometric Invariant Theory

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Abstract

Existence and uniqueness theorems for weak solutions of a complex Monge-Ampère equation are established, extending the Bedford-Taylor pluripotential theory. As a consequence, using the Tian-Yau-Zelditch theorem, it is shown that geodesics in the space of Kähler potentials can be approximated by geodesics in the spaces of Bergman metrics. Motivation from Donaldson's program on constant scalar curvature metrics and Yau's strategy of approximating Kähler metrics by Bergman metrics is also discussed.

1. A complex Monge-Ampère equation

The equation of primary interest in this lecture is the following Dirichlet problem for a completely degenerate complex Monge-Ampère equation: Let $\bar{M}$ be a complex manifold of dimension m with smooth boundary $\partial\bar{M}$, Ω_0 a smooth $(1,1)$-form with $\Omega_0^m = 0$. Then for any $\phi \in C^0(\partial\bar{M})$, find $\Phi \in PSH(\bar{M}, \Omega_0)$ so that

$$(\Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Phi)^m = 0, \qquad \Phi_{|\partial\bar{M}} = \phi. \tag{1}$$

Here the space $PSH(\bar{M}, \Omega_0)$ is the space of Ω_0-plurisubharmonic functions on $\bar{M}$ defined in section §7, and the equation is to hold in the generalized sense also defined there. We shall be especially interested in general existence and uniqueness theorems for (1), as well as in the construction of explicit solutions when $\bar{M}$ is of the form $\bar{M} = X \times A$, where X is a closed Kähler manifold equipped with a positive line bundle L, $A = \{w \in \mathbf{C}; 1 \le |w| \le e\}$ is an annulus in $\mathbf{C}$, and ϕ is a boundary value function which is invariant with respect to the rotations of A. Our main results [25] are the general existence and uniqueness theorems for the Monge-Ampère equation (1) stated in Theorems 2-5, and, when $\bar{M} = X \times A$, the explicit construction in Theorem 1, which says in particular that the solutions of (1) can be approximated by geodesic paths of Bergman metrics.

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The solutions of (1) are central to Donaldson's program [12], relating the geometry of the space of Kähler potentials on X to the existence of metrics of constant scalar curvature. The approximations by Bergman metrics have long been advocated by Yau [31, 32] as a means of reducing differential geometric questions to finite-dimensional algebraic-geometric ones. Some motivation from geometric invariant theory and related topics is provided in sections §3 and §4.

2. Geodesics in the space $\mathcal{K}$ of Kähler potentials

Our set-up is a compact complex manifold X of dimension n without boundary, together with a positive holomorphic line bundle L over X. Recall that a metric h on L is a smooth positive section of $L^{-1}\otimes\bar{L}^{-1}$. The corresponding metric connection is given by $\nabla_{\bar{j}}\psi = \partial_{\bar{j}}\psi$ and $\nabla_j\psi = h^{-1}\partial_j(h\psi)$ for ψ a local section of L. The curvature is given by $[\nabla_j, \nabla_{\bar{k}}]\psi = F_{\bar{k}j}\psi$, $F_{\bar{k}j} = -\partial_j\partial_{\bar{k}}\log h$. The line bundle L is said to be positive if there exists a metric h_0 on L whose curvature form $\omega_0 = \frac{\sqrt{-1}}{2}F_{\bar{k}j}dz^j\wedge d\bar{z}^k = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log h_0$ is a strictly positive $(1,1)$-form.

Given (X, L), the space $\mathcal{K}$ of Kähler potentials is defined by

$$\begin{aligned}\mathcal{K} &= \{\text{metrics } h \text{ on } L \text{ of positive curvature}\}\\ &= \{\phi\in C^\infty(X); \omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi > 0\},\end{aligned}\tag{1}$$

where in the second line, the metric h has been written as $h = e^{-\phi}h_0$ and identified with the scalar function ϕ. The tangent space $T_\phi(\mathcal{K})$ can then be identified with $C^\infty(X)$ and carries a natural metric $||\psi||^2 = \int_X|\psi|^2\omega_\phi^n$, which, by the work of Donaldson [12], Mabuchi [20], and Semmes [27], makes $\mathcal{K}$ into an infinite-dimensional symmetric space of negative curvature. The geodesics in $\mathcal{K}$ are the paths $t\to\phi(t)$ satisfying the following equation

$$\ddot{\phi} - g^{j\bar{k}}\partial_{\bar{k}}\dot{\phi}\partial_j\dot{\phi} = 0,\tag{2}$$

where $g_{j\bar{k}}$ is the metric corresponding to the Kähler form ω_ϕ, $\omega_\phi = \frac{\sqrt{-1}}{2}g_{\bar{k}j}dz^j\wedge d\bar{z}^k$. A key observation of [12],[20],[27] is that this geodesic equation can be re-written as a Monge-Ampère equation (1) on $\bar{M} = X\times A$, through the correspondence

$$C^\infty(X\times[0,1])\ni\phi\leftrightarrow\Phi(z,w) = \phi(z,\log|w|)\in C^\infty(X\times A).\tag{3}$$

Indeed, if we view $\omega_0 = \Omega_0$ as a $(1,1)$-form on $\bar{M}$ independent of w, and set

$$\Omega_\Phi = \Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Phi,\tag{4}$$

then in local coordinates $v = \log w$ for A, z^j for X with $\omega_0 = \frac{\sqrt{-1}}{2}(g_0)_{\bar{k}j}dz^j\wedge d\bar{z}^k$, we have

$$\Omega_\Phi = \begin{pmatrix}(g_0)_{\bar{k}j} + \partial_j\partial_{\bar{k}}\phi & \frac{1}{2}\partial_{\bar{k}}\dot{\phi}\\ \frac{1}{2}\partial_j\dot{\phi} & \frac{1}{4}\ddot{\phi}\end{pmatrix}.\tag{5}$$

Diagonalizing the matrix $g_{\bar{k}j} = (g_0)_{\bar{k}j} + \partial_j\partial_{\bar{k}}\phi$, we find at once $\Omega_\Phi^{n+1} = \frac{1}{4}(\ddot{\phi} - g^{j\bar{k}}\partial_{\bar{k}}\dot{\phi}\partial_j\dot{\phi})\omega_\phi^n \wedge (\frac{\sqrt{-1}}{2}dv \wedge d\bar{v})$, whence the equivalence between the geodesic equation in $\mathcal{K}$ and the degenerate complex Monge-Ampère equation $\Omega_\Phi^{n+1} = 0$ in $\bar{M}$. Barrier methods for Monge-Ampère boundary value prolem have been developed by a number of authors, including Caffarelli, Kohn, Nirenberg, Spruck [5], [6], [7], and Guan [17]. Using such a barrier method, together with Yau's estimates in [30], X.X. Chen [9] was able to establish the existence of unique $C^{1,1}$ geodesics joining any two points $h_0, h_1 \in \mathcal{K}$.

Our first main theorem is the following explicit construction of geodesics in $\mathcal{K}$:

Theorem 1 *Let $h_0, h_1 \in \mathcal{K}$, and let $\{s_j^\alpha(z)\}_{j=0}^{N_k}$ be orthonormal bases for $H^0(X, L^k)$ with respect to the L^2 metrics on $\Gamma(X, L^k)$ induced respectively by h_α, $\alpha = 0, 1$, $N_k + 1 = \dim H^0(X, L^k)$. Without loss of generality, we may assume that $s_j^{(1)} = e^{\lambda_j} s_j^{(0)}$, $0 \leq j \leq N_k$. Then the $C^{1,1}$ geodesic $\phi(t)$ joining h_0 to h_1 is given by*

$$\phi(t) = \lim_{\ell \to \infty} \Big(\sup_{k \geq \ell} \phi(t; k) \Big)^* \text{ uniformly as } \ell \to \infty, \tag{6}$$

where the $\phi(t;k)$'s are defined by

$$\phi(t; k) = \frac{1}{k} \log \Big(\sum_{j=0}^{N_k} e^{\lambda_j t} |s_j^{(0)}(z)|^2_{h_0^k} \Big) - n \frac{\log k}{k}, \quad 0 \leq t \leq 1. \tag{7}$$

and u^* denotes the smallest lower semi-continuous function which is greater or equal to u. In particular, $\phi(t) = \lim_{\ell\to\infty} \sup_{k\geq\ell} \phi(t;k)$ pointwise almost everywhere.

3. The Kodaira imbedding and the spaces $\mathcal{K}_k$

Theorem 1 has an important geometric interpretation, which says that geodesic segments in the infinite-dimensional symmetric space $\mathcal{K}$ can be uniformly approximated by geodesics in the finite-dimensional symmetric spaces $\mathcal{K}_k = GL(N_k+1)/U(N_k+1)$.

3.1. The Kodaira imbedding

First, recall that $\mathbf{CP}^N = \{Z \in \mathbf{C}^{N+1} \setminus 0\}/\mathbf{C}^\times$, so that $\mathbf{CP}^N$ is equipped with a natural line bundle, whose fiber above an equivalence class $[Z] \in \mathbf{CP}^N$ consists of all vectors Z in $[Z]$ together with 0. Its dual is the hyperplane bundle $O(1) \to \mathbf{CP}^N$. The global sections of $O(1)$ are then the space of linear functions $p(Z) = \sum_{j=0}^N p_j Z_j$, and a natural metric on $O(1)$ is the Fubini-Study metric $|p(Z)|^2_{h_{FS}} = |\sum_{j=0}^N p_j Z_j|^2 / \sum_{j=0}^N |Z_j|^2$. If we denote $|p(Z)|^2_{h_{FS}}$ by $h_{FS} p\bar{p}$, then h_{FS} and its curvature $\omega_{FS} = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log h_{FS}$ are given by

$$h_{FS} = \frac{1}{\sum_{j=0}^N |Z_j|^2}, \quad \omega_{FS} = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\frac{1}{\sum_{j=0}^N |Z_j|^2}. \tag{1}$$

Thus the curvature of h_{FS} is the Fubini-Study metric on $\mathbf{CP}^N$, and $O(1)$ is positive.

Next, consider the pair (X,L), with L a positive line over X. Then, for each fixed basis $s=\{s_j(z)\}_{j=0}^N$ of $H^0(X,L^k)$ [1], X can be mapped into $\mathbf{CP}^N$ by

$$\iota_s \ : \ X \ni z \to \iota_s(z) = [s_0(z),\cdots,s_N(z)] \in \mathbf{CP}^N \tag{2}$$

The Kodaira imbedding theorem guarantees that this is well-defined and an imbedding for $k >> 1$. Under this imbedding, $O(1)$ pulls back to L^k, since its sections $p(Z)=\sum_{j=0}^N p_j Z_j$ pull back to $\sum_{j=0}^N p_j s_j(z)$, which are sections of L^k. Thus we have the diagram

$$\begin{array}{ccc} L^k & \longrightarrow & O(1) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{CP}^N \end{array} \tag{3}$$

The metrics h_{FS} and ω_{FS} over $O(1)$ and $\mathbf{CP}^N$ pull back to the metrics $\iota_s^*(h_{FS})$ and $\iota_s^*(\omega_{FS})$ on L^k and X given by

$$\iota_s^*(h_{FS}) = \frac{1}{\sum_{j=0}^N |s_j(z)|^2}, \quad \iota_s^*(\omega_{FS}) = -\frac{\sqrt{-1}}{2}\partial\bar\partial \log \frac{1}{\sum_{j=0}^N |s_j(z)|^2}. \tag{4}$$

3.2. The Tian-Yau-Zelditch theorem

Let h_0 be a positive metric on L, ω_0 the corresponding Kähler form, and let $s=\{s_j^{(0)}(z)\}_{j=0}^N$ be a basis of $H^0(X,L^k)$ which is orthonormal with respect to the L^2 inner product on $\Gamma(X,L^k)$ induced by h_0. Define the "density of states" $\rho^{(0)}(z)$ by

$$\rho^{(0)}(z) = \sum_{j=0}^N |s_j^{(0)}(z)|^2 h_0^k \equiv \sum_{j=0}^N |s_j^{(0)}(z)|^2_{h_0^k}. \tag{5}$$

Clearly, $\int_X \rho^{(0)}\omega_0^n = N+1$, whence the terminology for $\rho^{(0)}(z)$. The formulas (4) for $\iota_s^*(h_{FS})$ and $\iota_s^*(\omega_{FS})$ can be rewritten as

$$\begin{aligned} \log \frac{\left(\iota_s^*(h_{FS})\right)^{1/k}}{h_0} &= -\frac{1}{k}\log\rho^{(0)}(z) \\ \omega_0 - \frac{1}{k}\iota_s^*(\omega_{FS}) &= -\frac{1}{k}\frac{\sqrt{-1}}{2}\partial\bar\partial\log\rho^{(0)}(z). \end{aligned} \tag{6}$$

The Tian-Yau-Zelditch theorem [31, 28, 33] (see also Catlin [8]) asserts the following asymptotic expansion for the density of states

$$\rho^{(0)}(z) = k^n + A_1(z)k^{n-1} + A_2(z)k^{n-2} + \cdots . \tag{7}$$

This asymptotic expansion implies in turn the following approximations

$$|| \log \frac{\left(\iota_s^*(h_{FS})\right)^{1/k}}{h_0} + n\frac{\log k}{k} ||_{C^\infty} \leq O(\frac{1}{k^2}), \quad ||\omega_0 - \frac{1}{k}\iota_s^*(\omega_{FS})||_{C^\infty} \leq O(\frac{1}{k^2}). \tag{8}$$

[1] To lighten the notation, we denote the dimension N_k of $H^0(X,L^k)$ just by N.

The possibility that metrics on L and X can be approximated in this canonical way by pull backs from the Kodaira imbedding was originally formulated by Yau [31]. The C^2 convergence was established by Tian [28]. The precise C^∞ convergence stated here as well as the expansion of $\rho^{(0)}(z)$ were established by Zelditch [33], using the asymptotic expansion for the Bergman and Szegö kernels on strongly pseudoconvex domains obtained earlier by Fefferman [16] and Boutet de Monvel-Sjöstrand [4]. Several of the leading coefficients in the asymptotic expansion for $\rho^{(0)}(z)$ have been determined by Lu [19]. The underlying structure is the Grauert correspondence between a line bundle L on X equipped with a metric $h_0(z)$ and the unit ball $B = \{(z,\zeta); \zeta \in L_z^{-1}, h_0(z)^{-1}\zeta\bar{\zeta} < 1\}$ in the dual bundle L^{-1}. The ball B is strongly pseudoconvex if the metric h_0 has positive curvature. The space $L^2(\partial B)$ can be identified with $\oplus_{k=1}^{\infty} L^2(X, L^k)$, and the Fefferman-Boutet de Monvel-Sjöstrand asymptotic expansion for the Szegö kernel corresponds to the expansion for the density of states $\rho^{(0)}(z)$ (cf. [33]).

3.3. The symmetric spaces $\mathcal{K}_k = GL(N_k+1)/U(N_k+1)$

The spaces $\mathcal{K}_k$ of all metrics on L arising from Kodaira imbeddings by arbitrary bases of $H^0(X, L^k)$ can thus be expressed as

$$\mathcal{K}_k = \left\{ \frac{1}{(\sum_{j=0}^{N_k} |s_j(z)|^2)^{1/k}} ; \{s_j(z)\}_{j=0}^{N_k} \text{ basis of } H^0(X, L^k) \right\} \subset \mathcal{K}. \tag{9}$$

The spaces $\mathcal{K}_k$ can be identified with $GL(N_k+1)/U(N_k+1)$, since a change of bases is given by an element of $GL(N_k+1)$, and unitary elements leave the corresponding metric on L unchanged. As such, $\mathcal{K}_k$ are finite-dimensional symmetric spaces. Their geodesics are given explicitly by expressions of the form

$$t \to \frac{1}{(\sum_{j=0}^{N_k} e^{\lambda_j t} |s_j(z)|^2)^{1/k}}. \tag{10}$$

In this framework, the Tian-Yau-Zelditch theorem just says that $\mathcal{K} = \lim_{k\to\infty} \mathcal{K}_k$. Theorem 1 states then that the $C^{1,1}$ geodesics in $\mathcal{K}$ can be approximated by the geodesics in $\mathcal{K}_k$, in other words, that as $\mathcal{K}_k$ fill up $\mathcal{K}$, they also become geodesically flat.

4. Constant scalar curvature metrics and geometric invariant theory

The approximation of geodesics in $\mathcal{K}$ by geodesics in $\mathcal{K}_k$ is an attractive geometric phenomenon, but it takes on a special significance in the context of a criterion for the existence of constant scalar curvature Kähler metrics in geometric invariant theory.

4.1. The conjecture of Yau

A central problem in complex differential geometry is to determine, for given (X, L), when there exists a metric h in $\mathcal{K}$ whose curvature ω is a metric on X of constant scalar curvature R. In the particular case where $L = \wedge^{max} T^{1,0}(X) = K(X)^{-1}$, where $K(X)$ is the canonical bundle of X, the condition of constant scalar curvature is equivalent to the condition that ω be Kähler-Einstein, that is, $R_{\bar{k}j} = \mu g_{\bar{k}j}$ with $\mu = constant$ [2]. A classic conjecture of Yau asserts that the existence of a constant scalar curvature metric in $\mathcal{K}$ should be equivalent to the "stability of (X, L) in the sense of geometric invariant theory".

4.2. Stability in geometric invariant theory

As was seen earlier, the Kodaira imbedding associates to the pair (X, L) a submanifold $\iota_s(X)$ of $\mathbf{CP}^N$. However, $\iota_s(X)$ depends on the choice of basis $s = \{s_j(z)\}$ for $H^0(X, L^k)$ and only the orbit of $\iota_s(X)$ under $GL(N_k + 1)$ is intrinsically associated to (X, L).

Broadly speaking, geometric invariant theory associates to a geometric structure such as (X, L), an orbit under a non-compact group G such as $GL(N_k + 1)$. The space, or "moduli", of such geometric structures is identified with the space of orbits. Stability conditions are conditions which ensure that the space of orbits be a well-behaved space, and particularly that it be Hausdorff. Since the group G is not compact, the major difficulties arise from the behavior of the orbits near infinity. The Hilbert-Mumford numerical criterion roughly states that one does not need to check the Hausdorff property of the orbits under the full group G, but only of the orbits of one-parameter subgroups of G (see [21] and references therein). Thus one-parameter subgroups, which define geodesics on $G = GL(N_k+1)$, are of special interest.

There are two equivalent ways of formulating stability conditions. The first is in terms of energy functionals: Let $I(\omega_0, \phi)$ be a functional on the space of all potentials ϕ in $\mathcal{K}$ (typically, the functionals of interest also depend on a reference metric ω_0, which is what is indicated in the notation for I). Let $s = \{s_j(z)\}$ be a fixed basis of $H^0(X, L^k)$, and let ω_0 be the pull-back of the Fubini-Study metric on $\mathbf{CP}^{N_k}$ by the Kodaira imbedding defined by the basis $\{s_j(z)\}$. For each one-parameter subgroup $\sigma : \mathbf{C}^\times \ni t \to \sigma(t) \in SL(N_k + 1)$, we can then construct a function $I(t)$ by

$$I(t) = I(\omega_0, \phi_{\sigma(t)}) \tag{1}$$

where we have set

$$\phi_\sigma(z) = \log \frac{|\sigma \cdot Z|^2}{|Z|^2} = \log \frac{|\sigma \cdot s(z)|^2}{|s(z)|^2}. \tag{2}$$

Note that $\omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi_\sigma = \iota_{\sigma\cdot s}(\omega_{FS})$ is just the pull-back of the Fubini-Study metric by the Kodaira imbedding defined by the basis $\sigma \cdot s$. The pair (X, L) is then said to be I-stable if for any reference basis $\{s_j^{(0)}(z)\}$ and any one-parameter subgroup σ, we have

$$I(t) = \mu_\sigma \log \frac{1}{|t|} + O(1) \quad \text{as} \quad t \to 0, \quad \text{with } \mu_\sigma > 0. \tag{3}$$

[2]Since $R_{\bar{k}j}$ is always in $c_1(X)$, and since $\mu g_{\bar{k}j}$ is assumed to be in $c_1(L) = c_1(X)$ in this case, it follows that $R_{\bar{k}j} = \mu g_{\bar{k}j} + \partial_j \partial_{\bar{k}} F$ for some smooth scalar function F. In particular $R = \mu n + \Delta F$, and F must be constant if R is constant.

In practice, the choice of reference basis is immaterial.

The notions of Chow-Mumford stability [21] and K-stability [29, 15] arise respectively from the following two choices for the functional $I(\omega_0, \phi)$,

$$
\begin{aligned}
F^0(\omega_0, \phi) &= -\frac{1}{c_1(L)^n} \int_X \phi \sum_{i=0}^{n} \omega_\phi^{n-i} \omega_0^i, \\
\nu(\omega_0, \phi) &= \frac{1}{V} \int_X \log\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_\phi^n - \phi(Ric(\omega_0) \sum_{i=0}^{n-1} \omega_0^i \omega_\phi^{n-1-i} - \frac{\mu}{n+1} \sum_{i=0}^{n} \omega_0^i \omega_\phi^{n-i}) \qquad (4)
\end{aligned}
$$

Both functionals are motivated by the variational approach to the problem of constant scalar curvature metrics. The Euler-Lagrange equation for $F^0(\omega_0, \phi)$ subjected to the constraint $\frac{1}{V}\int_X e^{f-\phi}\omega_0^n = 1$, where f is defined by $\frac{\sqrt{-1}}{2}\partial\bar\partial f = Ric(\omega_0) - \omega_0$, gives the Monge-Ampère equation $\omega_\phi^n = e^{f-\phi}\omega^n$ which implies that the metric ω_ϕ is Kähler-Einstein. The Euler-Lagrange equation for $\nu(\omega_0, \phi)$ is precisely the equation $R = \mu n$ of constant scalar curvature. The functional $\nu(\omega_0, \phi)$ is the Mabuchi energy functional (originally denoted by K, hence the name K-stability), and we shall return to it below.

Another way of introducing stability is via a choice of line bundle $\mathcal{L}$ over the Hilbert scheme and a choice of norm $||\cdot||$ on $\mathcal{L}$.

The Chow line bundle $\mathcal{L}^{Ch}$ is defined over the Hilbert scheme of all projective subvarieties V with fixed Hilbert polynomial as follows: For each V of dimension n and degree d, let

$$
Z = \{\ell \in Gr(N-n-1, \mathbf{CP}^N); \ell \cap V \neq \emptyset\}, \qquad (5)
$$

where Gr denotes the Grassmannian of $N-n-1$ planes in $\mathbf{CP}^N$. The variety Z is of codimension 1 in $Gr(N-n-1, \mathbf{CP}^N)$, and so defines a line in $H^0(Gr(N-n-1), O(d))$ (called the Chow point of V) consisting of sections f of minimal degree with the property: $Z = \{\ell; f(\ell) = 0\}$. These lines fit together to form $\mathcal{L}^{Ch}$.

Next a norm $||\cdot||$ for $\mathcal{L}^{Ch}$ is defined as follows [34],

$$
\log ||f||^2 = \int_{Gr} \log \frac{|f(\ell)|^2}{|\ell|^{2d}} \omega_{Gr}^m \qquad (6)
$$

where $|\ell|$ is the norm of the Plücker coordinate of $\ell \in Gr$ and ω_{Gr} is the pull back of Fubini-Study with respect to the Plücker imbedding, and $m = \dim Gr$. We can now define a numerical invariant μ_σ by the asymptotics

$$
\log \frac{||f^\sigma||^2}{||f||^2} = \mu_\sigma \log \frac{1}{|t|} + O(1), \qquad (7)
$$

where f is a representative of the Chow point of $X \subseteq \mathbf{CP}^N$, and f^σ the result of σ acting on f. We also define a corresponding algebraic-geometric notion of stability by the requirement that $\mu_\sigma > 0$ for all one-parameter subgroups σ. It is a basic result of Zhang [34] that the two definitions (3) and (7) of μ_σ and hence of Chow-Mumford stability agree (see also [22] for a more recent and different proof). Thus we have, schematically,

$$\text{Chow} - \text{Mumford stability} \leftrightarrow \{\text{Line bundle } \mathcal{L}^{Ch} \ + \ \text{Norm } ||\cdot||\}. \tag{8}$$

The notion of K-stability was originally defined by Tian using energy functionals [29] (see also Donaldson [15]). But the formulas obtained in [24] show that, as in the case for Chow-Mumford stability, it can be recast in the formalism of norms and line bundles over the Hilbert scheme as well: Schematically, we have in this case

$$\text{K} - \text{stability} \leftrightarrow \{\text{Line bundle } \mathcal{L}^{Ch}\otimes\mathcal{L}^{Ch}_s \ + \ \text{Norm } ||\cdot||\times||\cdot||_{\#}\}. \tag{9}$$

for a suitable line bundle $\mathcal{L}^{Ch}_s$ and norm $||\cdot||_{\#}$. To construct $\mathcal{L}^{Ch}_s$, we set for each variety V in the Hilbert scheme

$$Z_s = \text{singular locus of the Chow variety Z} \subset Gr \subset \mathbf{CP}^{N'}, \tag{10}$$

where we imbed the Grassmannian into projective space by using Plücker coordinates. Introduce also the norm

$$\log||f||^2_{\#} = a\int_Z \log\Big(\frac{\omega^m_{Gr}\wedge\partial\bar\partial\frac{|f(\ell)|^2}{|\ell|^{2d}}}{\omega^{m+1}_{Gr}}\Big)\omega^m_{Gr} + b\int_{Gr}\log\frac{|f(\ell)|^2}{|\ell|^{2d}}\omega^m_{Gr}. \tag{11}$$

where a,b are given by: $a = \frac{(m+1)}{D(m+2)(d-1)}$ and $b = \frac{d-m-2}{(m+1)(d-1)}$ with $m+1 = \dim Gr$, $D = \int_{Gr}\omega^{m+1}_{Gr}$ and d is the degree of X in $\mathbf{CP}^N$. The line bundle $\mathcal{L}^{Ch}\otimes\mathcal{L}^{Ch}_s$ can now be equipped with the norm $|||\cdot||| = ||\cdot||\times||\cdot||_{\#}$, and a numerical invariant μ_σ can be defined as in (7), with f now a section of $\mathcal{L}^{Ch}\otimes\mathcal{L}^{Ch}_s$, and the norm $||\cdot||$ replaced by $|||\cdot|||$. Then [24] shows that this algebraic-geometric notion of stability coincides with the notion of K-stability.

In [23], it has also been shown that K-stability coincides with CM-stability, which is yet another algebraic-geometric notion of stability introduced in [29].

4.3. Donaldson's program

In [12], Donaldson has laid out a long-range program, partly motivated from symplectic geometry, which would relate directly the problems of existence and uniqueness of constant scalar curvature metrics to the differential geometry of the infinite-dimensional symmetric space $\mathcal{K}$. For example, the uniqueness of constant scalar curvature metrics can be related to the existence of C^2 geodesics in $\mathcal{K}$ by the following formula for the second variation of the Mabuchi energy functional,

$$\ddot\nu = \int_X|\nabla_{\bar k}\nabla_{\bar j}\dot\phi|^2\omega^n_\phi - \int_X(\ddot\phi - g^{j\bar k}\partial_{\bar k}\dot\phi\partial_j\dot\phi)(R-\mu n)\omega^n_\phi, \tag{12}$$

which holds along any C^2 path in $\mathcal{K}$. In particular, along geodesics in $\mathcal{K}$, the second integral vanishes, and the Mabuchi energy functional is convex. Since $\nabla_{\bar k}\nabla_{\bar j}\dot\phi = 0$

if and only if $g^{p\bar{k}}\nabla_{\bar{k}}\dot{\phi}$ is a holomorphic vector field on X, the functional ν is strictly convex along geodesics of $\mathcal{K}$ if the automorphism group of X is assumed to be discrete. This would rule out the existence of any two critical points of ν which can be joined by a C^2 geodesic, and hence establish the uniqueness of constant scalar curvature metrics if C^2 geodesics always exist. (Recently, the uniqueness of constant scalar curvature metrics was established in [14] and [10] by other methods.) On the sufficiency side, Donaldson has conjectured that there is no constant scalar curvature metrics in $\mathcal{K}$ if and only if there exists an infinitely extended geodesic $t \to \phi(t) \in \mathcal{K}$ satisfying

$$\int_X \dot{\phi}(R - \mu n)\omega_\phi^n < 0, \quad \text{for all } t. \tag{13}$$

Since $R - \mu n = 0$ is precisely the Euler-Lagrange equation for ν and the above left hand side its variational derivative, this condition is clearly an analogue of the condition for K-stability, with the geodesics of $\mathcal{K}$ replacing the one-parameter subgroups of $GL(N_k + 1)$.

4.4. Finite-dimensional and infinite-dimensional geometric invariant theory

If classical geometric invariant theory is viewed as a finite-dimensional theory, suitable for the methods and objectives of algebraic geometry, then the Donaldson program can be viewed as an infinite-dimensional analogue. We have now seen some elements of the correspondence between these finite-dimensional and infinite-dimensional versions

Finite-dimensional GIT	Infinite-dimensional GIT
Geometry of $\mathcal{K}_k$	Geometry of $\mathcal{K}$
Conjecture of Yau	Conjecture of Donaldson
Geodesics in $\mathcal{K}_k$	Geodesics of $\mathcal{K}$

Table 1: *Geometric Invariant Theory.*

Theorem 1 can be viewed as a first step in establishing a precise correspondence between the two theories in the last row. But clearly, much remains to be discovered or fleshed out, and a major problem is still to establish the sufficiency of either finite-dimensional or infinite-dimensional stability conditions for the existence of constant scalar curvature metrics. A relation between stability and the convergence of the Kähler-Ricci flow can be found in [26].

5. Estimates for $\Phi(k)$ and $\Omega^{n+1}_{\Phi(k)}$

We sketch now the proof of Theorem 1. It consists of two parts. In the first part, we obtain estimates on the functions $\phi(t;k)$ and their Monge-Ampère determinants which suggest that they should converge to a weak solution of the desired Dirichlet problem. In the second part, we formulate and establish convergence and uniqueness theorems for the Dirichlet problem that show that we indeed have convergence, and that the resulting weak solution coincides with the $C^{1,1}$ geodesic in $\mathcal{K}$.

Recall the notation $\phi(t;k) = \Phi(k)$ and $\Omega_{\Phi(k)} = \Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar\partial\Phi(k)$, where $\Phi(k)$ is viewed as a function on $\bar M = X \times A$. Then $\Omega_{\Phi(k)}$ satisfies

$$\begin{aligned} \Omega_{\Phi(k)} &\geq 0, \\ \int_{\bar M} \Omega_{\Phi(k)}^{n+1} &\leq C\frac{1}{k}, \end{aligned} \tag{1}$$

while the $\Phi(k)$'s themselves satisfy

$$|\Phi(k)| + |\dot\Phi(k)| \leq C. \tag{2}$$

To establish these estimates, we need to establish first the following important estimate for the eigenvalues λ_j

$$|\lambda_j| \leq C\,k. \tag{3}$$

This can be done using the Tian-Yau-Zelditch theorem. The bounds for $|\lambda_j|$ imply readily the desired bounds for $|\Phi(k)|$ and $|\dot\Phi(k)|$. The estimates for $\Omega_{\Phi(k)}^{n+1}$ follow next from the following basic identity

$$\int_{\bar M} \Omega_{\Phi(k)}^{n+1} = \int_X \dot\phi(1)\omega_{\phi(1)}^n - \int_X \dot\phi(0)\omega_{\phi(0)}^n. \tag{4}$$

Explicitly, denoting by $\omega_\alpha(k)$ the metric on X corresponding to the potentials $\phi(\alpha;k)$, $\alpha = 0,1$, and recalling that we pass from the orthonormal basis $\{s_j^{(0)}(z)\}$ with respect to h_0 to the orthonormal basis $\{s_j^{(1)}(z)\}$ with respect to h_1 by the diagonal matrix with entries e^{λ_j}, we can re-write the right hand side as

$$\frac{1}{k}\int_X \frac{\sum \lambda_j |s_j^{(1)}(z)|^2}{\sum |s_j^{(1)}(z)|^2}\omega_1(k)^n - \frac{1}{k}\int_X \frac{\sum \lambda_j |s_j^{(0)}(z)|^2}{\sum |s_j^{(0)}(z)|^2}\omega_0(k)^n \tag{5}$$

In terms of the densities of states $\rho^{(\alpha)}$, this is

$$\frac{1}{k}\sum \lambda_j \int_X \frac{|s^{(1)}(z)|_{h_1}^2}{\rho^{(1)}(z)}\omega_1^n \frac{\omega_1(k)^n}{\omega_1^n} - \frac{1}{k}\sum \lambda_j \int_X \frac{|s^{(0)}(z)|_{h_0}^2}{\rho^{(0)}(z)}\omega_0^n \frac{\omega_0(k)^n}{\omega_0^n}. \tag{6}$$

Applying again the Tian-Yau-Zelditch theorem, we see that the leading terms cancel between the two expressions, resulting in the desired bound for $\Omega_{\Phi(k)}^{n+1}$.

6. Weak solutions of the complex Monge-Ampère equation

The estimates for $\Phi(k)$ and $\Omega_{\Phi(k)}^{n+1}$ obtained in the previous section suggest that the functions $\Phi(k)$ should converge weakly to a generalized solution of the Monge-Ampère equation. We begin with a review of the theory of generalized solutions of the complex Monge-Ampère equation.

6.1. The Monge-Ampère operator

Let D be a bounded open domain in $\mathbf{C}^m$. A (p,p)-current $T = \sum_{\bar{J}I} T_{\bar{J}I} dz^I \wedge d\bar{z}^J$ is said to be positive if $\int T \wedge \phi \geq 0$, for all $C_0^\infty(D)$ $(m-p, m-p)$ forms ϕ of the form $\phi = \prod_{j=1}^{m-p}(i\alpha_j \wedge \bar{\alpha}_j)$. A key consequence of the positivity of a current T is that all its coefficients $T_{\bar{J}I}$ must be given by complex measures.

Next, a function $u : D \to [-\infty, \infty)$ is said to be plurisubharmonic (PSH) if it is upper semi-continuous, and $u_{|D\cap L}$ is subharmonic for any complex line L. The plurisubharmonicity of u implies that either $u \equiv -\infty$, or $u \in L^1_{loc}(D)$ and $i\partial\bar{\partial}u$ is a positive $(1,1)$-current.

The positivity property allows to define the product $i\partial\bar{\partial}u \wedge T$ for any locally bounded, PSH function u and closed positive (p,p)-current T, even though we cannot in general take the product of two measures. It suffices to set

$$(i\partial\bar{\partial}u) \wedge T = i\partial\bar{\partial}(uT) \tag{1}$$

Taking successively $T = (i\partial\bar{\partial}u)^p$, $p = 1, 2, \cdots$, we obtain $(i\partial\bar{\partial}u)^m$ for any locally bounded, PSH function u, which is the definition of the Monge-Ampère operator.

6.2. The Bedford-Taylor pluripotential theory

A classic inequality in the theory of the Monge-Ampère operator is the Chern-Levine-Nirenberg inequality which says that for any compact $K \subseteq D$,

$$|\int_K \phi(i\partial\bar{\partial}u)^m| \leq C_K ||\phi||_{C^0}(\sup_D|u|)^m, \quad \phi \in C_0^\infty(K), u \ PSH. \tag{2}$$

This inequality leads a uniqueness and weak convergence theory for the Monge-Ampère operator in the class of continuous and uniformly convergent plurisubharmonic functions. However, these conditions are not satisfied in our case, and we shall require the much stronger theory developed by Bedford and Taylor [1, 2] (see also the exposition in [3]). Two of the key theorems which they obtained are as follows:

Bedford-Taylor convergence theorem: Let u_k, u be PSH functions on $D \subset \mathbf{C}^m$. If
(a) either $u_k \to u$ uniformly on compact subsets,
(b) or $u_k \to u$ pointwise and $\{u_k\}$ is a decreasing sequence,
then $(i\partial\bar{\partial}u_k)^m \to (i\partial\bar{\partial}u)^m$ weakly, that is,

$$\int_D \phi(i\partial\bar{\partial}u_k)^m \to \int_D \phi(i\partial\bar{\partial}u)^m, \quad \phi \in C_0^\infty(D). \tag{3}$$

Bedford-Taylor uniqueness theorem: Let u, v be bounded PSH functions such that
(a) $\mathrm{limsup}_{\zeta\to\partial D}|u(\zeta) - v(\zeta)| = 0$;
(b) $\int_{u\leq v}(i\partial\bar{\partial}u)^m = 0$.
Then $u \geq v$ in D. In particular, if $u = v$ on ∂D and $(i\partial\bar{\partial}u)^m = (i\partial\bar{\partial}v)^m = 0$, then $u = v$ in D.

7. New convergence and uniqueness theorems for weak solutions

The Bedford-Taylor theory does not apply directly to our situation, since the functions $\Phi(k)$ neither converge uniformly nor are decreasing. Furthermore, our setting of a complex manifold with boundary instead of domains in $\mathbf{C}^m$ leads to some subtle difficulties due to the absence of smooth regularizations of PSH functions. Some of the difficulties in extending the theory from domains in $\mathbf{C}^m$ to complex manifolds are discussed in [18] and [11]. However, in our situation, we have the key uniform bound $|\dot{\Phi}(k)| \leq C$, and it turns out that the following convergence and uniqueness theorems hold. They suffice to prove Theorem 1 from the estimates for $\Phi(k)$ and $\Omega_{\Phi(k)}$ obtained in §5, and may also be interesting in their own right.

Let $\bar{M} = \cup_{\alpha=1}^N U_\alpha$ be a compact smooth manifold with boundary, with coordinate charts U_α, and let Ω_0 be a fixed C^∞ closed $(1,1)$-form, $\Omega_0 = \frac{1}{2} i\partial\bar{\partial}\Psi_\alpha$ on U_α. Set

$$PSH(M, \Omega_0) = \{\Phi; \Psi_\alpha + \Phi \text{ is } PSH \text{ on } U_\alpha\} \tag{1}$$

Let ϕ be a continuous function on ∂D. A function $\Phi : \bar{M} \to \mathbf{R}$ is said to be a solution of the Dirichlet problem with boundary value ϕ if $\Phi \in PSH(M, \Omega_0)$, Φ is continuous at all $p \in \partial\bar{M}$, $\Phi_{|\partial\bar{M}} = \phi$, and $(\Omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi)^m = 0$ in M in the sense of (1).

Theorem 2 *Assume that* $\Omega_0^m = 0$*, and that* $\Phi(k) \in PSH(M, \Omega_0) \cap C^\infty(\bar{M})$ *satisfies*
(a) $||\Phi(k)||_{C^0(\bar{M})} \leq C$*;*
(b) $||\Phi(k) - \phi||_{C^0(\partial\bar{M})}$ *decreases to* 0 *and* $\sum_k ||\Phi(k) - \phi||_{C^0(\partial\bar{M})} < \infty$*;*
(c) $\lim_{k\to\infty} \int_M (\Omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi(k))^m \to 0$.
Assume further that there exists a vector field Y *transversal to* $\partial\bar{M}$ *and a neighborhood* U *of* $\partial\bar{M}$ *so that*
(d) $||Y(\Phi(k))||_{C^0(U)} \leq C$.
Then $\Phi = \lim_{k\to\infty}[\sup_{\ell\geq k}\Phi(\ell)]^*$ *is a solution of the Dirichlet problem with boundary value* ϕ*.*

To formulate the uniqueness theorems, we require the Bedford-Taylor notion of capacity defined as follows for domains $E \subset U \subset \mathbf{C}^m$

$$c(E, U) = \sup\{\int_E (i\partial\bar{\partial}v)^m; v \in PSH(U), 0 \leq v \leq 1\}. \tag{2}$$

We can now introduce the notion of "nearly continuous" function on the complex manifold $\bar{M} = \cup_{\alpha=1}^N U_\alpha$. First, if $E \subset M$, we say that $c(E, M) = 0$ if for any $\epsilon > 0$, we can write $E = \cup_{\alpha=1}^N E_\alpha$, with $E_\alpha \subset U_\alpha$, and $\sum_\alpha c(E_\alpha, U_\alpha) < \epsilon$. Then a function $v : \bar{M} \to \mathbf{R}$ is said to be "nearly continuous" if
(a) There exists v_0 lower semi-continuous on $\bar{M}$ with $v = v_0^*$;
(b) $\{v_0 < v\}$ has capacity $c(\{v_0 < v\}, M) = 0$;
(c) $v = v_0$ on ∂M.
With this notion, we have the following uniqueness theorems:

Theorem 3 *Assume that* $\Omega_0^m = 0$*. Let* $u, v \in PSH(M, \Omega_0) \cap L^\infty$ *satisfy*
(a) $(u - v)_* \geq 0$ *on* $\partial\bar{M}$*;*

(b) u is continuous;
(c) There exists a decreasing sequence of functions $v_k \in PSH(M, \Omega_0)$ which are nearly continuous and converges to v;
(d) For any $\delta > 0$, there exists a compact set $K \subset M$ with $v_k < v + \delta$ on $M \setminus K$ for $k >> 1$.
Then

$$\int_{u<v} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}v)^m \leq \int_{u<v} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}u)^m. \tag{3}$$

Theorem 4 *Assume that $\Omega_0^m = 0$. Let $u, v \in PSH(M, \Omega_0) \cap L^\infty$ satisfy*
(a) $(u - v)_* \geq 0$ on $\partial\bar{M}$;
(b) v is continuous;
(c) There exists a decreasing sequence of functions $u_k \in PSH(M, \Omega_0)$ which are nearly continuous and converges to u;
Then

$$\int_{u<v} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}v)^m \leq \int_{u<v} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}u)^m. \tag{4}$$

Theorems 3 and 4 imply the following theorem, which applies readily to our situation:

Theorem 5 *Assume that $\Omega_0^m = 0$. Let $u, v \in PSH(M, \Omega_0) \cap L^\infty$ satisfy*
(a) u is continuous;
(b) There exists a decreasing sequence of functions $v_k \in PSH(M, \Omega_0)$ which are nearly continuous and converges to v;
(c) For any $\delta > 0$, there exists a compact set $K \subset M$ with $v_k < v + \delta$ on $M \setminus K$ for $k >> 1$.
Assume that $(u - v)_* = (u - v)^* = 0$ on ∂M and $(\Omega_0 + \frac{i}{2}\partial\bar{\partial}u)^m = (\Omega_0 + \frac{i}{2}\partial\bar{\partial}v)^m = 0$. Then $u = v$.

This uniqueness theorem allows us to conclude that, in the case of $\bar{M} = X \times A$, the solution Φ constructed as $\Phi = \lim_{k\to\infty}[\sup_{\ell\geq k}\Phi(\ell)]^*$ must coincide with the solution obtained from the $C^{1,1}$ geodesic in the space $\mathcal{K}$ of Kähler potentials.

References

[1] Bedford, E. and B.A. Taylor, "The Dirichlet problem for a complex Monge-Ampère equation", Inventiones Math. **37** (1976) 1-44.

[2] Bedford, E. and B.A. Taylor, "A new capacity theory for plurisubharmonic functions", Acta Math. **149** (1982) 1-40.

[3] Blocki, Z., "The complex Monge-Ampère operator and pluripotential theory", lecture notes available from the author's website.

[4] Boutet de Monvel, L. and J. Sjöstrand, "Sur la singularité des noyaux de Bergman et de Szegö" Journées: Equations aux Dérivées Partielles de Rennes (1975), 123-164. Asterisque, No. 34-35, Soc. Math. France, Paris, 1976.

[5] Caffarelli, L., L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation". Comm. Pure Appl. Math. 37 (1984), no. 3, 369–402.

[6] Caffarelli, L., L. Nirenberg, L., J.J. Kohn, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equationsv". Comm. Pure Appl. Math. 38 (1985), no. 2, 209–252.

[7] Caffarelli, L., L. Nirenberg, and J. Spruck, "The Dirichlet problem for the degenerate Monge-Ampère equation", Rev. Mat. Iberoamericana 2 (1986), no. 1-2, 19–27.

[8] Catlin, D., "The Bergman kernel and a theorem of Tian", *Analysis and geometry in several complex variables* (Katata, 1997), 1-23, Trends Math., Birkhäuser Boston, Boston, MA, 1999.

[9] Chen, X.X., "The space of Kähler metrics", J. Differential Geom. **56** (2000), 189-234.

[10] Chen, X.X. and G. Tian, "Geometry of Kähler metrics and foliations by discs", arXiv: math.DG / 0409433.

[11] Demailly, J.P., "*Complex analytic and algebraic geometry*", book available from the author's website.

[12] Donaldson, S.K., "Symmetric spaces, Kähler geometry, and Hamiltonian dynamics", Amer. Math. Soc. Transl. **196** (1999) 13-33.

[13] Donaldson, S.K., "Scalar curvature and projective imbeddings II", arXiv: math.DG / 0407534.

[14] Donaldson, S.K., "Scalar curvature and projective embeddings. I", J. Diff. Geometry **59** (2001) 479-522.

[15] Donaldson, S.K., "Scalar curvature and stability of toric varieties", J. Diff. Geometry **62** (2002) 289-349.

[16] Fefferman, C., "The Bergman kernel and biholomorphic mappings of pseudoconvex domains", Invent. Math. 26 (1974), 1–65.

[17] Guan, B., "The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function", Comm. Anal. Geom. 6 (1998), no. 4, 687–703.

[18] Guedj, V. and A. Zeriahi, "Intrinsic capacities on compact Kähler manifolds", arXiv: math.CV / 0401302.

[19] Lu, Z., "On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch", Amer. J. Math. **122** (2000) 235-273.

[20] Mabuchi, T., "Some symplectic geometry on compact Kähler manifolds", Osaka J. Math. **24** (1987) 227-252.

[21] Mumford, D., J. Fogarty, and F. Kirwan, "*Geometric invariant theory*" Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer-Verlag, Berlin, 1994.

[22] Paul, S., "Geometric analysis of Chow Mumford stability", Adv. Math. 182 (2004), no. 2, 333–356.

[23] Paul, S. and G. Tian, "Algebraic and analytic stability", arXiv: math.DG/0405530.

[24] Phong, D.H. and J. Sturm, "Stability, energy functionals, and Kähler-Einstein metrics", Comm. Anal. Geometry **11** (2003) 563-597, arXiv: math.DG / 0203254.

[25] Phong, D.H. and J. Sturm, "The complex Monge-Ampère operator and geodesics in the space of Kähler metrics", arXiv: math.DG/0504157.

[26] Phong, D.H. and J. Sturm, "On stability and the convergence of the Kähler-Ricci flow", arXiv: math.DG / 0412185.

[27] Semmes, S., "Complex Monge-Ampère and symplectic manifolds", Amer. J. Math. **114** (1992) 495-550.

[28] Tian, G., " On a set of polarized Kähler metrics on algebraic manifolds", J. Diff. Geom. **32** (1990) 99-130.

[29] Tian, G., "Kähler-Einstein metrics with positive scalar curvature", Inventiones Math. **130** (1997) 1-37.

[30] Yau, S.T., "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I", Comm. Pure Appl. Math. **31** (1978) 339-411.

[31] Yau, S.T., " Nonlinear analysis in geometry", Enseign. Math. (2) **33** (1987), no. 1-2, 109–158.

[32] Yau, S.T., "Open problems in geometry", Proc. Symp. Pure Math. **54** (1993) 1-28.

[33] Zelditch, S., "The Szegö kernel and a theorem of Tian", Int. Math. Res. Notices **6** (1998) 317-331.

[34] Zhang, S., "Heights and reductions of semi-stable varieties", Compositio Math. **104** (1996) 77-105.

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