Vladimir Georgiev, Atanas Stefanov and Mirko Tarulli

Strichartz Estimates for the Schrödinger Equation with small Magnetic Potential

Strichartz Estimates for the Schrödinger Equation with small Magnetic Potential

Vladimir Georgiev   Atanas Stefanov   Mirko Tarulli

1. Introduction and statement of results

In this work we study the dispersive properties of the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta_A u = F(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where

$$\Delta_A = \sum_j (\partial_j + iA_j)^2 = \Delta + 2iA\nabla + idiv(A) - \left(\sum_j A_j^2\right)$$

and

$A = (A_1(t,x), \cdots, A_n(t,x)), x \in \mathbb{R}^n, n \geq 3$ is a magnetic potential, such that $A_j(t,x), j = 1, \cdots, n$, are real valued functions.

More precisely, we plan to establish Strichartz and smoothing estimates for (1.1), when the vector potential A is small in certain sense. In fact, we aim at obtaining global scale invariant Strichartz and smoothing estimates, under appropriate scale invariant smallness assumptions on A.

In the "free" case $A = 0$, there exists vast literature on the subject. Introduce the mixed space-time norms

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

We say that a pair of exponents (q,r) is Strichartz admissible, if $2 \leq q, r \leq \infty$, $2/q + n/r = n/2$ and $(q,r,n) \neq (2,\infty,2)$. Then, by result of Strichartz, Ginibre-

Velo, and Keel-Tao,

$$\|e^{it\Delta}f\|_{L_t^q L_x^r} \le C\|f\|_{L^2} \tag{1.2}$$

$$\left\|\int e^{is\Delta}F(s,\cdot)ds\right\|_{L_x^2} \le C\|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{1.3}$$

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds\right\|_{L_t^q L_x^r} \le C\|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{1.4}$$

where $(\tilde{q}, \tilde{r})$ is another Strichartz admissible pair and $q' = q/(q-1)$. Note that for $n \ge 3$ the set of admissible pairs (q, r) can be represented equivalently as $(1/q, 1/r) \in AB$, where AB is the segment with end points $A(0, 1/2)$, $B(1/2, 2n/(n-2))$ and we can rewrite the estimate (1.3) as

$$\left\|\int e^{is\Delta}F(s,\cdot)ds\right\|_{L_x^2} \le C\left(\inf_{F=F_1+F_2}\|F_1\|_{L_t^1 L_x^2} + \|F_2\|_{L_t^2 L_x^{2n/(n+2)}}\right). \tag{1.5}$$

On the other hand, the smoothing estimates were established by Kenig-Ponce-Vega in the seminal paper, [11], see also Ruiz-Vega [16] . These were later extended to more general second order Schrödinger equations in [12]. Some possible scale and rotation invariant smoothing estimates similar to (1.2), (1.3) and (1.4) can be written as (see Corollary 1 below)

$$\sup_{m\in\mathbb{Z}}\left(2^{-m/2}2^{k/2}\left\|e^{it\Delta}f_k\right\|_{L_t^2 L^2(|x|\sim 2^m)}\right) \le C\|f_k\|_{L^2}, \tag{1.6}$$

$$\left\|\int e^{is\Delta}F_k(s,\cdot)ds\right\|_{L_x^2} \le C\left(\sum_{m\in\mathbb{Z}} 2^{m/2}2^{-k/2}\|F_k\|_{L_t^2 L^2(|x|\sim 2^m)}\right). \tag{1.7}$$

$$\sup_{m\in\mathbb{Z}}\left(2^{-m/2}2^{k/2}\left\|\int_0^t e^{i(t-s)\Delta}F_k(s,\cdot)ds\right\|_{L_t^2 L^2(|x|\sim 2^m)}\right) \le \tag{1.8}$$
$$\le C\left(\sum_{m\in\mathbb{Z}} 2^{m/2}2^{-k/2}\|F_k\|_{L_t^2 L^2(|x|\sim 2^m)}\right),$$

where k is any integer, $\phi_k := P_k\phi$ is the kth Littlewood-Paley piece of ϕ (see Section 2.1 below).

Motivated by these estimates, given any integer $k \in \mathbb{Z}$ introduce the spaces Y_k, defined by the norms[1]

$$\|\phi\|_{Y_k} = 2^{-k/2}\sum_m 2^{m/2}\|\phi_k\|_{L_t^2 L^2(|x|\sim 2^m)}.$$

---

[1]The expressions $\phi \to \|\phi\|_{Y_k}$ are not faithfull norms, in the sense that may be zero, even for some $\phi \ne 0$. On the other hand, they satisfy all the other norm requirements and $\phi \to (\sum_k \|\phi_k\|_{Y_k}^2)^{1/2}$ is a norm!

Now we can define the Banach spaces Y as a closure of the functions

$$\phi(t, x) \in C_0^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus 0))$$

with respect to the norm

$$\|\phi\|_Y := \left(\sum_k \|\phi\|_{Y_k}^2 \right)^{1/2}. \tag{1.9}$$

Its dual space Y' consists of tempered distributions $S'(\mathbb{R} \times \mathbb{R}^n)$, having finite norm

$$\|\phi\|_{Y'} := \left(\sum_k \|\phi\|_{Y_k'}^2 \right)^{1/2},$$

where

$$\|\phi\|_{Y_k'} = 2^{k/2} \sup_m 2^{-m/2} \|\phi_k\|_{L_t^2 L^2(|x| \sim 2^m)}.$$

Then the smoothing estimates (1.6), (1.7) and (1.8) read

$$\left\| e^{it\Delta} f \right\|_{Y'} \leq C \|f\|_{L^2}, \quad \left\| \int e^{is\Delta} F(s, \cdot) ds \right\|_{L_x^2} \leq C \|F\|_Y. \tag{1.10}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds \right\|_{Y'} \leq C \|F\|_Y. \tag{1.11}$$

Motivated by the Strichartz estimates and these Besov versions of "local smoothing" norms, introduce the spaces

$$X = L_t^1 L_x^2 + L_t^2 L_x^{2n/(n+2)} + Y$$

with norm

$$\|F\|_X = \inf_{F = F^{(1)} + F^{(2)} + F^{(3)}} \left\| F^{(1)} \right\|_{L_t^1 L_x^2} + \left\| F^{(2)} \right\|_{L_t^2 L_x^{2n/(n+2)}} + \left\| F^{(3)} \right\|_Y.$$

The dual to X space is X' and the norm in this space is defined in similar way:

$$\|\phi\|_{X'} := \left(\sum_k \|\phi\|_{X_k'}^2 \right)^{1/2}, \tag{1.12}$$

where

$$\|\phi\|_{X_k'} = \sup_{(q,r) - Str.} \|\phi_k\|_{L_t^q L_x^r} + 2^{k/2} \sup_m 2^{-m/2} \|\phi_k\|_{L_t^2 L^2(|x| \sim 2^m)}.$$

The main result of this work is

Theorem 1.1. *If $n \geq 3$, then one can find a positive number $\varepsilon > 0$ so that for any (vector) potential $A = A(t, x)$ satisfying*

$$\|\nabla A\|_{L^\infty L^{n/2}} + \sup_k (\sum_m 2^m \|A_{<k}\|_{L^\infty L^\infty(|x| \sim 2^m)}) \leq \varepsilon, \qquad (1.13)$$

there exists $C > 0$, such that for any $F(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$ we have the estimate

$$\left\| \int_{t-s>0} e^{i(t-s)\Delta_A} F(s, \cdot) ds \right\|_{X'} \leq C \|F\|_X.$$

In particular, the solutions to (1.1) satisfy the smoothing - Strichartz estimate

$$\|u\|_{X'} \lesssim \|f\|_{L^2} + \|F\|_X. \qquad (1.14)$$

Remark 1. *The estimate (1.14) implies various interesting inequalities. For example we have the classical Strichartz estimate*

$$\sup_{(q,r)-Str.} \|u\|_{L^q L^r} \lesssim \|f\|_{L^2} + \inf_{F=F_1+F_2} \|F_1\|_{L_t^1 L_x^2} + \|F_2\|_{L_t^2 L_x^{2n/(n+2)}}$$

as well as the smoothing - Strichartz estimates

$$\|u\|_{Y'} \lesssim \|f\|_{L^2} + \inf_{F=F_1+F_2} \|F_1\|_{L_t^1 L_x^2} + \|F_2\|_{L_t^2 L_x^{2n/(n+2)}},$$

$$\sup_{(q,r)-Str.} \|u\|_{L^q L^r} \lesssim \|f\|_{L^2} + \|F\|_Y.$$

The main idea to prove this Theorem is to apply appropriate scale invariant estimate for the free Schrödinger equation involving Strichartz and smoothing type norms.

Estimates of this type have been obtained earlier in [15] and [16] with Strichartz type norms of the form $\|F\|_{L_x^{2n/(n+2)} L_t^2}$. Recently, we found (the authors are grateful to Luis Vega for pointing them this recent work) similar estimate in the work [9] and this estimate has the form

$$\|D_x^{1/2} \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds\|_{L_{x_1}^\infty L_{x'}^2 L_t^2} \leq C \|F\|_{L_t^2 L_x^{2n/(n+2)}}. \qquad (1.15)$$

On one hand, this estimate can be used to derive the Strichartz estimate for the perturbed Schrödinger equation provided its (formally) "dual" version

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds \right\|_{L_t^2 L_x^{2n/(n-2)}} \leq C \|D_x^{-1/2} F\|_{L_{x_1}^1 L_{x'}^2 L_t^2}. \qquad (1.16)$$

is verified. We apply (1.16) and show that (1.14) is satisfied for the free Schrödinger equation. Once (1.14) is established for the free case we show that these estimates are stable under small magnetic perturbations satisfying (1.13).

2. Preliminaries

2.1. Fourier transform and Littlewood-Paley projections

The kth Littlewood-Paley projection is defined as a multiplier type operator by $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$, where φ is smooth, non - negative function, supported in the annulus $1/2 \leq |\xi| \leq 2$ and $\sum_{k \in \mathcal{Z}}(2^{-k}\xi) = 1$ for all $\xi \neq 0$. Note that the kernel of P_k is integrable, smooth and real valued for every k. In particular, it is bounded on every $L^p : 1 \leq p \leq \infty$ and it commutes with differential operators. Another helpful observation is that for the differential operator D_x^s defined via the multiplier $|\xi|^s$, one has

$$D_x^s P_k u = 2^{ks}\tilde{P}_k u,$$

where $\tilde{P}_k$ is given by the multipleir $\tilde{\varphi}(2^{-k}\xi)$, where $\tilde{\varphi}(\xi) = \varphi(\xi)|\xi|^s$. One can construct φ as follows: take a positive, decreasing, smooth away from zero function $\chi : \mathbf{R}_+^1 \to \mathbf{R}_+^1$, supported in $\{\xi : 0 \leq \xi \leq 2\}$ and $\chi(\xi) = 1$, $\forall 0 \leq \xi \leq 1$. Define $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$.
We also consider $P_{<k} := \sum_{l<k} P_l$, which essentially restricts the Fourier transform to frequencies $\lesssim 2^k$.

Define also the function $\psi(\xi) = \chi(\xi/4) - \chi(4\xi)$. Note that ψ has similar support properties as φ and $\psi(\xi)\varphi(\xi) = \varphi(\xi)$. Thus, we may also define the operators Z_k by $\widehat{Z_k f}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi)$. By the construction, $Z_k P_k = P_k$ and $Z_k = P_{k-2} + \ldots + P_{k+1}$. Recall a version of the Calderón commutator estimate (see for example Lemma 2.1 in the work of Rodnianski and Tao, [14]), which reads

$$\|[P_k, f]g\|_{L^r} \leq C 2^{-k}\|\nabla f\|_{L^q}\|g\|_{L^p},$$

whenever $1 \leq r, p, q \leq \infty$ and $1/r = 1/q + 1/p$.
Also of interest will be the properties of products under the action of P_k. Starting with the relations

$$P_k(fg) = \sum_{\ell,m} P_k(f_\ell g_m),$$

$$P_k(f_\ell g_m) = \int\limits_{\mathbf{R}^n} \int\limits_{\mathbf{R}^n} P_k(\xi)P_\ell(\xi-\eta)\hat{f}(\xi-\eta)P_m(\eta)\hat{g}(\eta)e^{2\pi i x \cdot \xi}d\xi d\eta,$$

we exploit the property supp $P_k(\xi) \subseteq \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ and see that the sum can be restricted to the set

$$\{|\ell - m| \geq 2 + N_0, \ |\max(\ell, m) - k| \leq 3\} \cup \{|\ell - m| \leq 1 + N_0, \ k \leq \max(\ell, m) + 3\},$$

where $N_0 \geq 1$ is arbitrary number. This domain can be enlarged slightly using the inequality $\max(\ell, m) \leq \ell + 1 + N_0$ provided $|\ell - m| \leq 1 + N_0$. So we can restrict the sum over the union of the following sets (the first two are disjoint for $N_0 \geq 5$, while the third one can overlap with them)

$$\{m \leq k - N_0 + 1, \ |\ell - k| \leq 3\}, \{\ell \leq k - N_0 + 1, \ |m - k| \leq 3\}$$

and

$$\{|\ell - m| \leq 1 + N_0, \ \ell \geq k - N_0 - 4\}.$$

In conclusion, for any two (Schwartz) functions f, g we have the pointwise estimate

$$|P_k(fg)(x)| \leq \sum_{l \geq k-N_0-4} \sum_{|m-\ell| \leq 1+N_0} |P_k(f_l g_m)(x)| +$$
$$+ |P_k(f_{\leq k-N_0+1} \, g_{k-3 \leq \cdot \leq k+3})(x)| +$$
$$+ |P_k(f_{k-3 \leq \cdot \leq k+3} \, g_{\leq k-N_0+1})(x)|$$

Taking for determinacy $N_0 = 7$, we get

$$|P_k(fg)(x)| \leq |f_{\leq k-6}(x) g_k(x)| + |[P_k, f_{\leq k-6}] g_{k-3 \leq \cdot \leq k+3}(x)| +$$
$$+ |P_k(f_{k-3 \leq \cdot \leq k+3} \, g_{\leq k-6})(x)| + \sum_{l \geq k-11} \sum_{|m-\ell| \leq 8} |P_k(f_l g_m)(x)|$$

In particular, we need an appropriate (product like!) expression for $P_k(A \nabla u)$. The main term is clearly when ∇u is in high frequency mode, while $\vec{A}$ is low frequency. More precisely, according to our considerations above,

$$P_k(A \nabla u) = A_{\leq k-6} \nabla u_k + E^k,$$

where $E^k(x)$ satisfies the pointwise estimate

$$|E^k(x)| \leq |[P_k, A_{\leq k-6}] \nabla u_{k-3 \leq \cdot k+3}(x)| + \tag{2.1}$$
$$\sum_{l \geq k-11} \sum_{|m-\ell| \leq 8} |P_k(A_l \cdot \nabla u_m)(x)| + |P_k(A_{k-3 \leq \cdot \leq k+3} \cdot \nabla u_{\leq k-6})(x)|$$

Note that in terms of L^p behavior and Littlewood-Paley theory, one treats these error terms as if they were in the form $(\partial_x A) u$.

2.2. Besov spaces versions of the "local smoothing space"

The space Y was introduced as the closure of $S(\mathbb{R} \times \mathbb{R}^n)$ with respect to the norm in (1.9), where

$$\|\phi\|_{Y_k} = 2^{k/2} \sum_m 2^{-m/2} \|P_k \phi\|_{L_t^2 L^2(|x| \sim 2^m)}. \tag{2.2}$$

We can replace $\|F\|_{L^2(|x| \sim 2^m)}$ by the comparable expression $\|\varphi(2^{-m} \cdot) F\|_{L^2}$. This will be done frequently (and without much discussions) in the sequel in order to make use of the Plancherel's theorem, which is of course valid only in the global L^2 space. We mention also that the norm $\|\phi\|_Y$ is scale invariant for rescale factors any diadic number.

We show that the the "local smoothing space" defined as a closure of Schwartz functions ϕ with respect to the "local smoothing norms"

$$\sum_m 2^{m/2} \|D_x^{-1/2} \phi(t, x)\|_{L_t^2 L^2(|x| \sim 2^m)}$$

can be embedded in Y.

Lemma 2.1. *There is a constant $C = C(n)$, so that for every Schwartz function ϕ we have*

$$\|\phi\|_Y \leq C \sum_m 2^{m/2} \left\|D_x^{-1/2}\phi(t,x)\right\|_{L_t^2 L^2(|x| \sim 2^m)}. \tag{2.3}$$

Proof. Taking into account the definition of the space Y, it is sufficient to establish the estimate

$$\|\phi\|_{Y_k} \leq C \sum_m 2^{m/2} \left\|D_x^{-1/2}\phi_k(t,x)\right\|_{L_t^2 L^2(|x| \sim 2^m)}$$

for any integer k. Using the scale invariance of the estimate we see that we lose no generality taking $k = 0$. Thus, we have to verify the estimate

$$\sum_m 2^{m/2} \left\|\varphi(2^{-m}\cdot)P_0\phi\right\|_{L_t^2 L_x^2} \leq C \sum_m 2^{m/2} \left\|\varphi(2^{-m}\cdot)D_x^{-1/2}\phi_0(t,x)\right\|_{L_t^2 L_x^2}.$$

Since

$$P_0\phi = \sum_{|k| \leq 2} P_0 D_x^{1/2} D_x^{-1/2} P_k \phi = \sum_{|k| \leq 2} \sum_{\ell \in \mathbb{Z}} \widetilde{P}_0 \varphi(2^{-\ell}\cdot)D_x^{-1/2}P_k\phi,$$

we can apply the triangle inequality, and reduce the proof to the following estimate

$$\sum_m 2^{m/2} \sum_{\ell \in \mathbb{Z}} \left\|\varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)D_x^{-1/2}P_k\phi\right\|_{L_t^2 L_x^2} \leq$$

$$\leq C \sum_\ell 2^{\ell/2} \left\|\varphi(2^{-\ell}\cdot)D_x^{-1/2}\phi_k(t,x)\right\|_{L_t^2 L_x^2},$$

where $k \in \mathbb{Z}, |k| \leq 2$. This estimate follows easily from

$$\left\|\varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f\right\|_{L_x^2} \leq C\|f\|_{L_x^2}, \tag{2.4}$$

$$\left\|\varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f\right\|_{L_x^2} \leq C2^{-m}\|f\|_{L_x^2}, \ m \geq \ell + 2 \tag{2.5}$$

and the obvious observation that

$$\sum_{m \leq \ell+1} 2^{m/2} + \sum_{m \geq \ell+2} 2^{m/2}2^{-m} \lesssim 2^{\ell/2}.$$

The estimate (2.4) is obvious, while the proof of (2.5) follows from

$$\varphi(2^{-m}\cdot)\widetilde{P}_0\varphi(2^{-\ell}\cdot)f = [\varphi(2^{-m}\cdot), \widetilde{P}_0]\varphi(2^{-\ell}\cdot)f, \ m \geq \ell + 2$$

and the Calderón estimate

$$\left\|[\varphi(2^{-m}\cdot), \widetilde{P}_0]g\right\|_{L_x^2} \leq C2^{-m}\|g\|_{L_x^2}.$$

This completes the proof of the Lemma. $\qquad \square$

Remark 2. *Note that the argument in the proof of this lemma implies also the estimates*

$$\|P(D)f_k\|_Y \lesssim \|f_k\|_Y = \|f\|_{Y_k}, \ \forall k \in \mathbb{Z} \tag{2.6}$$

for any pseudodiferential operator with symbol $P(\xi) \in C_0^\infty(\mathbb{R}^n)$.

We have also the estimate (dual to (2.3))

Lemma 2.2. *There is a constant $C = C(n)$, so that for every Schwartz function $\phi \in Y'$, we have*

$$\sup_m 2^{-m/2} \left\| D_x^{1/2} \phi(t, x) \right\|_{L_t^2 L^2(|x| \sim 2^m)} \leq C_n \|\phi\|_{Y'}. \tag{2.7}$$

Remark 3. *Some generalizations of the previous two Lemmas can be seen in Theorem 1.6 and Theorem 1.7 in* [6].

3. Estimates for the bilinear form $Q(F, G)$

The sesquilinear form

$$Q(F, G) = \int \int_{t > s} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)} ds dt$$

was used in [13] to derive Strichartz estimates (with endpoint) and this estimates can be expressed in terms of Q

$$|Q(F, G)| \leq C \|F\|_{L_t^{q_1'} L_x^{r_1'}} \|G\|_{L_t^{q_2'} L_x^{r_2'}}, \tag{3.1}$$

for all Strichartz pairs (q_1, r_1), (q_2, r_2).

We have the following estimate that can be obtained by applying Lemma 3 from the work of Ionescu-Kenig [9].

Theorem 3.1. *There exists a constant $C = C(n)$ so that for any integer k, any $F(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$ and $G(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$*

$$|Q(F_k, G_k)| \leq C \left(\sum_{m \in \mathbb{Z}} 2^{m/2} 2^{-k/2} \|\varphi(2^{-m} \cdot) F_k\|_{L_t^2 L_x^2} \right) \|G_k\|_{L_t^2 L_x^{2n/(n+2)}}. \tag{3.2}$$

We have also the following energy-smoothing estimate.

Theorem 3.2. *There exists a constant $C = C(n)$ so that for any integer k, any $F(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$ and $G(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$*

$$|Q(F_k, G_k)| \leq C \left(\sum_{m \in \mathbb{Z}} 2^{m/2} 2^{-k/2} \|\varphi(2^{-m} \cdot) F_k\|_{L_t^2 L_x^2} \right) \|G_k\|_{L_t^1 L_x^2}. \tag{3.3}$$

Before proving these Theorems, we recall some of the smoothing estimates used in this work.

3.1. Estimates in the local smoothing space

For $n = 1$ we have the following smoothing estimates (see Kenig, Ponce, Vega [10])

$$2^{k/2}\|e^{-it\Delta}f_k\|_{L_x^\infty L_\gamma^2} \leq C\|f_k\|_{L^2}, \tag{3.4}$$

$$2^{k/2}\left\|\int_{s<t} e^{-i(t-s)\Delta}F_k(s)ds\right\|_{L_x^\infty L_t^2} \leq C\|F_k\|_{L_x^1 L_t^2}, \tag{3.5}$$

as well as

$$2^{k/2}\left\|\int_\gamma e^{-it\Delta}F_k(t)dt\right\|_{L^2} \leq C\|F_k\|_{L_x^1 L_\gamma^2} \tag{3.6}$$

for any interval $\gamma \subseteq \mathbb{R}_t$. Here $C > 0$ is a constant independent of f, F, γ.

For $n > 1$ we may assume

$$\text{supp } \widehat{f}(\xi) \subseteq \{|\xi'| \leq \xi_1/10, \quad \xi' = (\xi_2, \cdots, \xi_n)\}. \tag{3.7}$$

Then we have the representation

$$\left(e^{-it\Delta}f\right)(x_1, x') = \tag{3.8}$$

$$= c\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{it|\xi'|^2 + i(x'-y')\xi'}\left(e^{-it\Delta_1}f\right)(x_1, y') \, d\xi'dy',$$

where $\Delta_1 = \partial_{x_1}^2$.

This representation and one dimensional estimates (3.4), (3.5) and (3.6) lead to the following.

Lemma 3.1. *There exists a constant C depending only on the dimension, so that for any $f \in S(\mathbb{R}^n), F \in S(\mathbb{R} \times \mathbb{R}^n)$, satisfying (3.7) and*

$$\text{supp}_\xi \, \widehat{F}(t, \xi) \subseteq \{|\xi'| \leq \xi_1/10, \quad \xi' = (\xi_2, \cdots, \xi_n)\} \tag{3.9}$$

we have

$$2^{k/2}\|e^{-it\Delta}f_k\|_{L_{x_1}^\infty L_{x'}^2 L_\gamma^2} \leq C\|f_k\|_{L_{x'}^2 L_{x_1}^2} \tag{3.10}$$

$$2^{k/2}\left\|\int_{s<t} e^{-i(t-s)\Delta}F_k(s)ds\right\|_{L_{x_1}^\infty L_{x',t}^2} \leq C\|F_k\|_{L_{x_1}^1 L_{x',t}^2}, \tag{3.11}$$

and

$$2^{k/2}\left\|\int_\gamma e^{-it\Delta}F_k(t)dt\right\|_{L_{x_1}^2 L_{x'}^2} \leq C\|F_k\|_{L_{x_1}^1 L_{x'}^2 L_\gamma^2} \tag{3.12}$$

for any interval $\gamma \subseteq \mathbb{R}_t$.

Applying the Hölder inequalities

$$\|g\|_{L_{x_1}^1} \lesssim \sum_{m\in\mathbb{Z}} 2^{m/2}\|g\|_{L_{x_1}^2(|x|\sim 2^m)}, \quad \sup_{m\in\mathbb{Z}} 2^{-m/2}\|g\|_{L_{x_1}^2(|x|\sim 2^m)} \leq \|g\|_{L_{x_1}^\infty},$$

we obtain

Corollary 1. *The smoothing estimates* (1.6), (1.7), (1.8) *are satisfied.*

By Corollary 1 one gets

$$|Q(F_k, G_k)| \le C_n (\sum_m 2^{-k/2} 2^{m/2} \|F_k\|_{L_t^2 L^2(|x| \sim 2^m)}) \times$$
$$\times (\sum_m 2^{-k/2} 2^{m/2} \|G_k\|_{L_t^2 L^2(|x| \sim 2^m)}) \tag{3.13}$$

After this prepartion, we turn to

3.2. Proof of Theorem 3.1: Bilinear smoothing-Strichartz estimate

The estimate (3.2) is scale invariant and for this we can take $k = 0$. We have the relation

$$Q(F, G) = \int \int_{\mathbb{R}^2} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)} ds dt -$$

$$- \int \int_{t<s} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)} ds dt$$

For the form

$$Q_0(F, G) = \left\langle \int_{\mathbb{R}} ds \ e^{-is\Delta} F(s), \int_{\mathbb{R}} dt \ e^{-it\Delta} G(t) \right\rangle_{L^2(\mathbb{R}^n)}$$

we can apply the Cauchy inequality and via (1.3) and (1.7) we get

$$|Q_0(F_0, G_0)| \le C \left(\sum_{m \in \mathbb{Z}} 2^{m/2} \|\varphi(2^{-m}\cdot) F_0\|_{L_t^2 L_x^2} \right) \|G_0\|_{L_t^2 L_x^{2n/(n+2)}}.$$

Hence it remains to evaluate the form

$$Q^*(F, G) = \int \int_{t<s} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)}$$

and verify the inequality

$$|Q^*(F_0, G_0)| \le C \left(\sum_{m \in \mathbb{Z}} 2^{m/2} \|\varphi(2^{-m}\cdot) F_0\|_{L_t^2 L_x^2} \right) \|G_0\|_{L_t^2 L_x^{2n/(n+2)}}. \tag{3.14}$$

To prove (3.14) it is sufficient to consider F with

$$\text{supp}_\xi \ \widehat{F}(t, \xi) \subseteq \{|\xi'| \le \xi_1/10, \ \xi' = (\xi_2, \cdots, \xi_n)\}. \tag{3.15}$$

Also, note that

$$Q^*(F, G) = \int_{\mathbb{R} \times \mathbb{R}^n} F(s, y) \overline{u(s, y)} \ ds dy,$$

where u is a solution to the free Schrödinger equation $i\partial_t u + \Delta u = G$ having initial data identically 0.

With (3.15) in mind, apply Lemma 3 in Ionescu-Kenig [9]. We get

$$\|D_{x_1}^{1/2}u\|_{L_{x_1}^\infty L_{x',t}^2} \lesssim \|G\|_{L_t^2 L_x^{2n/(n+2)}} \tag{3.16}$$

Here and below we use the notations $x = (x_1, x')$, $x' = (x_2, \cdots, x_n)$. So we have

$$|Q^*(F,G)| \le C\left(\|D_{x_1}^{-1/2}F\|_{L_{x_1}^1 L_{x',t}^2}\right)\|G\|_{L_t^2 L_x^{2n/(n+2)}}. \tag{3.17}$$

Thus, we need to establish the inequality

$$\|D_{x_1}^{-1/2}F_0\|_{L_{x_1}^1 L_{x',t}^2} \lesssim \|F_0\|_{Y_0} = \sum_{m\in\mathbb{Z}} 2^{m/2}\|\varphi(2^{-m}\cdot)F_0\|_{L_t^2 L_x^2}.$$

For the purpose it is sufficient to apply (2.6), the Hölder inequality

$$\|g\|_{L_{x_1}^1} \lesssim \sum_{m\in\mathbb{Z}} 2^{m/2}\|g\|_{L_{x_1}^2},$$

and note that

$$D_{x_1}^{-1/2}F_0 = P(D)F_0,$$

for some $P(\xi) \in C_0^\infty(\mathbb{R}^n)$ due to our assumption (3.15). This completes the proof of the Theorem.

3.3. Proof of Theorem 3.2: bilinear energy − smoothing estimate

The proof follows the same line of the proof of Theorem 3.1 with the following changement: in the place of Ionescu-Kenig inequality (3.16) we use

$$\sup_t \left\|\int_0^t e^{i(t-s)\Delta}F_k(s,\cdot)ds\right\|_{L_x^2} \le C\left(\sum_{m\in\mathbb{Z}} 2^{m/2}2^{-k/2}\|F_k\|_{L_t^2 L^2(|x|\sim 2^m)}\right). \tag{3.18}$$

This estimate is trivial, since by the L^2 energy conservation, the left-hand side of this inequality is equal to

$$\sup_t \left\|\int_0^t e^{-is\Delta}F_k(s,\cdot)ds\right\|_{L_x^2}$$

and applying the estimate (1.7), we can finish the proof as before.

4. Proof of Theorem 1.1

We start by some reductions of the problem. First, note that (1.1) is in the form

$$\begin{cases} \partial_t u - i\Delta u + 2A\nabla u = \tilde{F}(t,x) \\ u(0,x) = f(x), \end{cases} \tag{4.1}$$

where $\tilde{F} = F - div(A)u - i(\sum_j A_j^2)u$. We claim that suffices to prove

$$\|u\|_{X'} \leq C_n(\|f\|_{L^2} + \|\tilde{F}\|_X), \tag{4.2}$$

for the solutions of (4.1). Indeed, assuming the validity of (4.2) and since by our assumptions and Sobolev embedding $\|\nabla A\|_{L_t^\infty L_x^{n/2}} + \|A\|_{L_t^\infty L_x^n} \leq C\|\nabla A\|_{L_t^\infty L_x^{n/2}} \leq C\varepsilon$. We have

$$\|u\|_{X'} \leq C\|f\|_{L^2} + C\|\tilde{F}\|_X \leq$$
$$\leq C\|f\|_{L^2} + C\|F\|_X + C(\|\nabla A\|_{L_t^\infty L_x^{n/2}} + \|A\|_{L_t^\infty L_x^n}^2)\|u\|_{L^2 L^{2n/(n-2)}}$$
$$\leq C_n\|f\|_{L^2} + C_n\|F\|_X + C_n\varepsilon\|u\|_{L^2 L^{2n/(n-2)}} \leq$$
$$\leq C_n\|f\|_{L^2} + C_n\|F\|_X + C_n\varepsilon\|u\|_{X'}.$$

It follows that

$$\|u\|_{X'} \leq C\|f\|_{L^2} + C\|F\|_X,$$

as claimed, as long as $\varepsilon : C_n\varepsilon < 1/2$.

Thus, we concentrate on showing (4.2) for the solutions of (4.1), where we denote the right hand side by F again.

Next, we take a Littlewood-Paley projection of (4.1). We get

$$\partial_t u_k - i\Delta u_k = F_k - 2A_{<k-6}\nabla u_k - 2E^k := H_k,$$

where E^k is the error term $E^k = P_k(A\nabla u) - A_{\leq k-6}\nabla u_k$ given by (2.1).

We will show that the solution to $\partial_t u_k - i\Delta u_k = H_k$ with initial data $u_k(0,x) = f_k$, satisfies the estimate

$$\|u_k\|_{X'} \leq C\|f_k\|_{L^2} + C\|H_k\|_X \tag{4.3}$$

We will show first how (4.3) implies Theorem 1.1 and then we proceed to show (4.3).

4.1. (4.3) implies Theorem 1.1

Apply (4.3) to u_k. We have

$$\|u_k\|_{X'} \leq C\|f_k\|_{L^2} + C(\|F_k\|_X + \|E^k\|_{L^2 L^{2n/(n+2)}} + $$
$$+ C\sum_m 2^{m/2} 2^{-k/2}\|A_{<k-6}\nabla u_k\|_{L^2 L^2(|x|\sim 2^m)}) \tag{4.4}$$

We will need the following estimates.

$$(\sum_k \|E^k\|_{L^2 L^{2n/(n+2)}}^2)^{1/2} \leq C_n\varepsilon(\sum_k \|u\|_{L^2 L^{2n/(n-2)}}^2)^{1/2} \leq C_n\varepsilon\|u\|_{X'} \tag{4.5}$$

$$\sum_m 2^{m/2} 2^{-k/2}\|A_{<k-6}\nabla u_k\|_{L^2 L^2(|x|\sim 2^m)} \leq C_n\varepsilon\|u_k\|_{X_k'}. \tag{4.6}$$

Let us show first how based on (4.5) and (4.6), we finish the proof of Theorem 1.1. Plugging in these estimates in (4.4), using the definition (1.12) of X' and square summing in k yields

$$\|u\|_{X'} = \left(\sum_k \|u_k\|_{X'_k}^2\right)^{1/2} \leq C_n(\|f\|_{L^2} + \|F\|_X) + C_n\varepsilon\|u\|_{X'},$$

whence since $\varepsilon : C_n\varepsilon < 1/2$,

$$\|u\|_{X'} \leq C_n(\|f\|_{L^2} + \|F\|_X).$$

Thus, for this section, remains to see (4.5) and (4.6).

4.1.1. Proof of (4.6)

Let $\widetilde{k}$ be integer with $|k - \widetilde{k}| \leq 3$. We have

$$\sum_m 2^{m/2}2^{-k/2}\big\|A_{<k-6}\nabla u_{\widetilde{k}}\big\|_{L^2L^2(|x|\sim 2^m)} \lesssim$$

$$\lesssim \left(\sum_m 2^m\|A_{<k-5}\|_{L^\infty L^\infty(|x|\sim 2^m)}\right)\sup_m 2^{-m/2}2^{-k/2}\big\|\nabla u_{\widetilde{k}}\big\|_{L^2L^2(|x|\sim 2^m)} \leq$$

$$\leq C_n\varepsilon \sup_m 2^{-m/2}2^{-k/2}\big\|\nabla u_{\widetilde{k}}\big\|_{L^2L^2(|x|\sim 2^m)}.$$

This last expression is very similar to $\big\|u_{\widetilde{k}}\big\|_{X'}$. We will show that it is controlled by it, which of course is enough to establish (4.6).
Fix an m. Then

$$2^{-m/2}2^{-k/2}\big\|\nabla u_{\widetilde{k}}\big\|_{L^2L^2(|x|\sim 2^m)} \lesssim 2^{-m/2}2^{k/2}\big\|\varphi(2^{-m}\cdot)Q_{\widetilde{k}}u_{\widetilde{k}}\big\|_{L^2L^2},$$

where Q_k acts as a (vector) multiplier $\psi(2^{-k}\xi)2^{-k}\xi$. We have by the Calderón commutator estimate[2] and the Bernstein inequality

$$2^{-m/2}2^{k/2}\big\|\varphi(2^{-m}\cdot)Q_k u_k\big\|_{L^2L^2} \leq 2^{-m/2}2^{k/2}\big\|Q_k(\varphi(2^{-m}\cdot)u_k)\big\|_{L^2L^2} +$$

$$+2^{-m/2}2^{k/2}\big\|[Q_k,\varphi(2^{-m}\cdot)]u_k\big\|_{L^2L^2} \lesssim 2^{-m/2}2^{k/2}\big\|\varphi(2^{-m}\cdot)u_k\big\|_{L^2L^2} +$$

$$+2^{-k/2}\|u_k\|_{L_t^2 L^{2n/(n-3)}} \lesssim 2^{-m/2}2^{k/2}\big\|\varphi(2^{-m}\cdot)u_k\big\|_{L^2L^2} + \|u_k\|_{L_t^2 L^{2n/(n-2)}}$$

$$\leq C_n\|u_k\|_{X'_k}.$$

---

[2]We are using the particular form $\|[Q_k,\varphi(2^{-m}\cdot)]u_k\|_{L^2L^2} \lesssim$
$2^{-k}2^{-m}\|(\nabla\varphi)(2^{-m}\cdot)\|_{L^{2n/3}}\|u_k\|_{L^{2n/(n-3)}} = 2^{-k}2^{m/2}\|u_k\|_{L^{2n/(n-3)}}$

4.1.2. Proof of (4.5)

We treat E^k on a term-by-term basis in (2.1). For the first term, by Calderón commutators,

$$(\sum_k \|[P_k, A_{<k-6}]\nabla u_k\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim$$

$$\lesssim (\sum_k \|\nabla A_{<k-6}\|^2_{L^\infty L^{n/2}} \|u_{k-3\leq\cdot k+3}\|^2_{L^2 L^{2n/(n-2)}})^{1/2} \lesssim$$

$$\lesssim \sup_k \|\nabla A_{<k-6}\|_{L^\infty L^{n/2}} (\sum_k \|u_{k-3\leq\cdot k+3}\|^2_{L^2 L^{2n/(n-2)}})^{1/2} \lesssim$$

$$\lesssim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{X'}.$$

For the second term, we have by standard Littlewood-Paley theory

$$(\sum_k \|P_k G\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim \|G\|_{L^2 L^{2n/(n+2)}},$$

$$\|(\sum_l |g_l|^2)^{1/2}\|_{L^p} \sim \|g\|_{L^p} \qquad \text{for all} \quad 1 < p < \infty.$$

whence with $m, \ell \in \mathbb{Z}$ with $|m - \ell| \leq 8$ we have

$$(\sum_k \|P_k(\sum_{|\ell-m|\leq 8} A_\ell \cdot \nabla u_m)\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \sim$$

$$\sim \|\sum_{|\ell-m|\leq 8} A_\ell \cdot \nabla u_m\|_{L^2 L^{2n/(n+2)}} \lesssim$$

$$\lesssim \|(\sum_\ell 2^{2l}|A_\ell|^2)^{1/2}\|_{L^\infty L^{n/2}} \|(\sum_m |\tilde{P}_m u|^2)^{1/2}\|_{L^2 L^{2n/(n-2)}} \sim$$

$$\sim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon \|u\|_{X'}.$$

For the third term in (2.1), observe that since for all $1 \leq p \leq 2$,

$$(\sum_k \|G^k\|^2_{L^p})^{1/2} \leq C_n \|(\sum_k |G^k|^2)^{1/2}\|_{L^p}$$

we can estimate by

$$(\sum_k \|P_k(A_{k-3\leq\cdot\leq k+3} \cdot \nabla u_{<k-5})\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim$$

$$\lesssim (\sum_k 2^{2k}\|A_{k-3\leq\cdot\leq k+3}\tilde{P}_{<k-5}u\|^2_{L^2 L^{2n/(n+2)}})^{1/2} \lesssim$$

$$\lesssim \|(\sum_k 2^{2k}|A_{k-3\leq\cdot\leq k+3}|^2|\tilde{P}_{<k-5}u|^2)^{1/2}\|_{L^2 L^{2n/(n+2)}} \lesssim$$

$$\lesssim \|(\sum_k 2^{2k}|A_{k-3\leq\cdot\leq k+3}|^2)^{1/2}\|_{L^\infty L^{n/2}} \|\sup_k |\tilde{P}_{<k-5}u|\|_{L^2 L^{2n/(n-2)}} \lesssim$$

$$\lesssim \|\nabla A\|_{L^\infty L^{n/2}} \|u\|_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon \|u\|_{X'},$$

IV–14

Here, we have used the pointwise estimate(see section 6.1, Chapter I, [18]) $\sup_k |\tilde{P}_{<k-5}u|(x) \leq CM(u)(x)$, where $M(u)$ is the Hardy-Littlewood maximal function and therefore

$$\|\sup_k |\tilde{P}_{<k-5}u|\|_{L^p} \leq C\|u\|_{L^p}$$

for all $1 < p < \infty$.

4.2. The proof of (4.3)

The nontrivial part of (4.3) is the case when u_k is the solution to $\partial_t u_k - i\Delta u_k = H_k$ with zero initial data $u_k(0,x) = 0$. Then the fact that the norm of X_k has three components implies that the inequality

$$\|u_k\|_{X'} \leq C\|H_k\|_X$$

is equivalent to the following nine inequalities

$$\|u_k\|_{L^2 L^{2n/(n-2)}} \leq C\|H_k\|_{L_t^2 L^{2n/(n+2)}} \tag{4.7}$$

$$\|u_k\|_{L^2 L^{2n/(n-2)}} \leq C\|H_k\|_{L_t^1 L^2} \tag{4.8}$$

$$\|u_k\|_{L^2 L^{2n/(n-2)}} \leq C\sum_m 2^{m/2}2^{-k/2}\|H_k\|_{L_t^2 L^2(|x|\sim 2^m)}, \tag{4.9}$$

$$\|u_k\|_{L^\infty L^2} \leq C\|H_k\|_{L_t^2 L^{2n/(n+2)}} \tag{4.10}$$

$$\|u_k\|_{L^\infty L^2} \leq C\|H_k\|_{L_t^1 L^2} \tag{4.11}$$

$$\|u_k\|_{L^\infty L^2} \leq C\sum_m 2^{m/2}2^{-k/2}\|H_k\|_{L_t^2 L^2(|x|\sim 2^m)}, \tag{4.12}$$

$$2^{k/2}\sup_m 2^{-m/2}\|u_k\|_{L_t^2 L^2(|x|\sim 2^m)} \leq C\|H_k\|_{L_t^2 L^{2n/(n+2)}}, \tag{4.13}$$

$$2^{k/2}\sup_m 2^{-m/2}\|u_k\|_{L_t^2 L^2(|x|\sim 2^m)} \leq C\|H_k\|_{L_t^1 L^2}, \tag{4.14}$$

and

$$\begin{aligned}2^{k/2}\sup_m 2^{-m/2}\|u_k\|_{L_t^2 L^2(|x|\sim 2^m)} \leq \\ \leq C\sum_m 2^{m/2}2^{-k/2}\|H_k\|_{L_t^2 L^2(|x|\sim 2^m)}.\end{aligned} \tag{4.15}$$

The estimates (4.7), (4.8), (4.10) and (4.11) are Strichartz inequalities (see (1.4) for general case).

The estimate (4.15) is smoothing - smoothing estimate established in Corollary 1(actually they follow from the bilinear estimate (3.13)) .

The estimates (4.9), (4.13) are smoothing - endpoint Strichartz inequalities following from bilinear estimate of Theorem 3.1.

Finally, the estimates (4.12), (4.14) are smoothing - energy inequalities following from bilinear estimate of Theorem 3.2.

The first inequality is the usual Strichartz estimate, while the second one is equivalent to (3.2).

This completes the proof of the inequality (4.3) and Theorem 1.1.

References

[1] S. Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(2):151–218, 1975.

[2] P. Alsholm and G. Schmidt. Spectral and scattering theory for Schrödinger operators. *Arch. Rational Mech. Anal.*, 40:281–311, 1970/1971.

[3] A. A. Balinsky, W. D. Evans, R. T. Lewis, On the number of negative eigenvalues of Schrödinger operators with an Aharonov-Bohm magnetic field. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), no. 2014, 2481–2489.

[4] J. A. Barcelo, A. Ruiz, and L. Vega. Weighted estimates for the Helmholtz equation and some applications. *J. Funct. Anal.* **150** (1997), 356–382.

[5] J. Bergh and J. Löfström, Interpolation spaces, *Springer* Berlin, Heidelberg, New York, 1976.

[6] V. Georgiev and M. Tarulli. Scale invariant energy smoothing estimates for the Scrödinger Equation with small Magnetic Potential. Preprint Universitá di Pisa, 2005.

[7] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.*, **133(1)** (1995) 50–68.

[8] L. Hörmander, *The analysis of linear partial differential operators. II. Differential operators with constant coefficients.* Fundamental Principles of Mathematical Sciences, 257. Springer-Verlag, Berlin, 1983.

[9] A. Ionescu, C. Kenig, *Well - posedness and local smoothing of solutions of Schrödinger equations* preprint 2005.

[10] C. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, *Indiana Univ. Math. J.*, **40** (1991), 33–69.

[11] C. Kenig, G. Ponce, L. Vega, *Small solutions to nonlinear Schrödinger equations.*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **10** (1993), no. 3, 255–288.

[12] C. Kenig, G. Ponce, L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations.*, *Invent. Math.*, **134** (1998), no. 3, 489–545.

[13] M. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[14] I. Rodnianski, T. Tao, *Global regularity for the Maxwell-Klein-Gordon equation with small critical Sobolev norm in high dientions. Comm. Math. Phys.* 2005.

[15] A. Ruiz, L. Vega *On local regularity of Schrödinger equations. Int. Math. Research Notes* **1**, 1993, 13 − 27 .

[16] A. Ruiz, L. Vega *Local regularity of solutions to wave equations with time−dependent potentials. Duke Math. Journal* **76, 1**, 1994, 913 − 940.

[17] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients.* Comm. Partial Differential Equations 27 (2002), no. 7-8, 1337–1372.

[18] E. Stein, Harmonic Analysis. Princeton Mathematical Series, Princeton Univ. Press, Princeton.

[19] A. Stefanov *Strichartz estimates for the magnetic Schrödinger equation* preprint 2004.

[20] M. Tarulli. *Smoothing Estimates for Scalar Field with Electromagnetic Perturbation.* EJDE. Vol. 2004(2004), No. 146, pp. 1-14.