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Dynamics of a small rigid body in a perfect incompressible fluid

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Abstract

We consider a solid in a perfect incompressible fluid in dimension two. The fluid is driven by the classical Euler equation, and the solid evolves under the influence of the pressure on its surface. We consider the limit of the system as the solid shrinks to a point. We obtain several different models in the limit, according to the asymptotics for the mass and the moment of inertia, and according to the geometrical situation that we consider. Among the models that we get in the limit, we find Marchioro and Pulvirenti's vortex-wave system and a variant of this system where the vortex, placed in the point occupied by the shrunk body, is accelerated by a lift force similar to the Kutta-Joukowski force. These results are obtained in collaboration with Christophe Lacave (Paris-Diderot), Alexandre Munnier (Nancy) and Franck Sueur (Bordeaux).

1. Introduction

In this paper, I describe several results obtained in collaboration with Christophe Lacave (Université Paris-Diderot), Alexandre Munnier (Université de Lorraine) and Franck Sueur (Université de Bordeaux).

These studies concern the asymptotic behaviour of a solid immersed in a perfect incompressible fluid, when the solid shrinks to a point. We obtain a simple particle in the limit whose trajectory we can characterize in various situations.

We begin by introducing the model of fluid/rigid body interaction.

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1.1. Presentation of the model: a rigid body immersed in an incompressible perfect fluid

We consider the motion of a rigid body immersed in an incompressible perfect fluid in a regular domain $\Omega \subset \mathbb{R}^2$ where $\Omega = \mathbb{R}^2$ or is a bounded domain, such as described for instance in Figure 1.1.

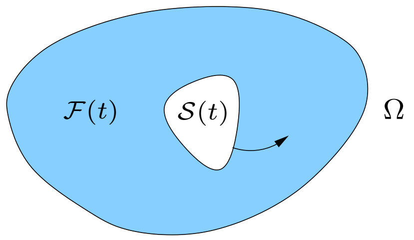

Figure 1.1. A solid immersed in an incompressible perfect fluid

The solid occupies at each instant $t \geq 0$ a subset $\mathcal{S}(t) \subset \Omega$, and the fluid occupies $\mathcal{F}(t) := \Omega \setminus \mathcal{S}(t)$. We take the convention that $\mathcal{S}(t)$ is a closed subset in $\mathbb{R}^2$ and consequently $\mathcal{F}(t)$ is an open subset of the plane. Since the solid is rigid, $\mathcal{S}(t)$ is obtained at each time t as the image of $\mathcal{S}(0)$ by a rigid movement, that is, the composition of a translation and a rotation.

Now, let us be more specific about the equations that drive the fluid and the solid.

Fluid equation. In $\mathcal{F}(t)$, the fluid satisfies the *incompressible Euler equation*:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad t \in [0,T], \ x \in \mathcal{F}(t), \\ \operatorname{div} u = 0 \qquad t \in [0,T], \ x \in \mathcal{F}(t). \end{cases}$$

At the boundaries, the fluid satisfies the *no-penetration/slip condition*:

$$u \cdot n = 0 \text{ for } x \in \partial\Omega,$$
$$u \cdot n = [h'(t) + \vartheta'(t)(x - h(t))^{\perp}] \cdot n \text{ for } x \in \partial\mathcal{S}(t).$$

Here:

- $u = u(t,x) : \mathcal{F}(t) \to \mathbb{R}^2$ is the fluid velocity, $p = p(t,x) : \mathcal{F}(t) \to \mathbb{R}$ the pressure,
- n is the normal to the boundaries $\partial\Omega$ and $\partial\mathcal{S}(t)$, pointing outside $\mathcal{F}(t)$,
- $h(t)$ is the position of the center of mass of the solid (we will take the convention that $h(0) = 0$), ϑ is the angle with respect to the initial position (so $\vartheta(0) = 0$). Of course, one can deduce $\mathcal{S}(t)$ from $h(t)$, $\vartheta(t)$ and $\mathcal{S}(0)$.

When $\Omega = \mathbb{R}^2$, one should add a condition at infinity, typically $\lim_{|x| \to +\infty} u(x) = 0$.

The main difference with the usual setting of Euler's equation in a bounded domain is that here the fluid domain $\mathcal{F}(t)$ actually depends on time. But should

$\mathcal{F}(t)$ be given in advance (which is not the case), many classical results could be transposed in this situation.

Dynamics of the solid. The dynamics of the solid is driven by the action of the pressure on its surface, and obeys Newton's law of classical mechanics:

$$m\,h''(t) = \int_{\partial\mathcal{S}(t)} p\,n\;ds,$$

$$\mathcal{J}\,\vartheta''(t) = \int_{\partial\mathcal{S}(t)} p\,(x - h(t))^{\perp}\cdot n\;ds,$$

where

- $m > 0$ is the mass of the body,
- $\mathcal{J} > 0$ denotes its moment of inertia.

We could naturally add a force such as gravity in the right hand side.

We can see in these equations that the coupling between the solid and the fluid is reciprocal: the solid influences the fluid through the moving domain and the boundary conditions on $\partial\mathcal{S}(t)$, and the fluid influences the solid through the pressure appearing in the right hand side of the equations.

Initial data. We prescribe as initial data the following:

- $\mathcal{S}(0) = \mathcal{S}_0$, with $\mathcal{S}_0 \subset \Omega$ a smooth closed subset of Ω,
- $u|_{t=0} = u_0$, for $x \in \mathcal{F}_0 := \Omega \setminus \mathcal{S}_0$,
- $(h'(0), \vartheta'(0)) = (h'_0, \vartheta'_0)$, with $(h'_0, \vartheta'_0, u_0)$ satisfying

$$\operatorname{div}(u_0) = 0 \text{ in } \mathcal{F}_0, \quad u_0\cdot n = (h'_0 + \vartheta'_0 \times (x - x_0))\cdot n \text{ on } \partial\mathcal{S}_0,$$

$$u_0 \cdot n = 0 \text{ on } \partial\Omega \quad \text{or} \quad \lim_{|x|\to+\infty} u_0(x) = 0.$$

Given the whole set of these data, it is reasonable to expect a good Cauchy theory. We will discuss this a bit in a moment.

Two remarks. Before giving a few words on the Cauchy problem, two remarks are in order.

A remark on D'Alembert's paradox. It is worth mentioning that D'Alembert's paradox does not apply here, because it concerns fluids which are potential in $\mathbb{R}^2$, stationary and constant at infinity. In that case (only), D'Alembert's paradox states that the fluid does not influence the dynamics of the solid.

Alternative formulations. One can also describe the system with a bit different formulations. The first one is the formulation using the vorticity $\omega = \operatorname{curl} u$, which is scalar in dimension two. The fluid part of the system can indeed be written in the form

$$\partial_t\omega + (u\cdot\nabla)\omega = 0 \;\text{ in }\; \mathcal{F}(t),$$

and

$$
(1.1)\quad \begin{cases} \operatorname{curl} u = \omega \ \text{ in } \ \mathcal{F}(t), \\ \operatorname{div} u = 0 \ \text{ in } \ \mathcal{F}(t), \\ u \cdot n = (h' + \vartheta'(x - h(t))^{\perp}) \cdot n \ \text{ on } \ \partial\mathcal{S}(t), \\ u \cdot n = 0 \text{ on } \partial\Omega \ \text{ or } \ \lim_{|x|\to+\infty} u(t,x) = 0, \\ \oint_{\partial\mathcal{S}(t)} u \cdot \tau \, ds = \oint_{\partial\mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma \ \text{ (Kelvin's law).} \end{cases}
$$

Here τ is the unit tangent to $\partial\mathcal{S}(t)$, such that (τ, n) is direct.

And another formulation of the problem is more geometrical. As we showed in collaboration with F. Sueur [12], the complete system can be indeed viewed as an *equation of geodesics* on an infinite dimensional Riemannian manifold, in the spirit of Arnold's work [1] for the Euler equation alone, see also Ebin-Marsden [7]. The main difference here is that the infinite dimensional Riemannian manifold has no longer a group structure and hence cannot be seen as a Lie group.

Cauchy problem. Let us give a few references about the Cauchy problem concerning this system. In the context of regular solutions (say at least C^1) with finite energy, the problem has been considered by Ortega, Rosier and Takahashi [25] in the full plane, by Rosier and Rosier [26] in the full space and by Houot, San Martin and Tucsnak [16] in a bounded domain (see also G., Sueur and Takahashi [14]).

Concerning weak solutions (solutions à la Yudovich [31] or à la DiPerna-Majda [6]) the problem has been studied by Sueur and the author [11, 13] (with possibly infinite energy) and by Wang and Xin [30] (in a finite energy setting but with less restrictions on the support of the vorticity.)

Now we give the statement on the Cauchy problem that we will use in the sequel. This result is the equivalent of Yudovich's theorem in the context of the fluid-body system.

Theorem 1 (G.-Sueur [13]). *Let $\mathcal{S}_0$ be a smooth, bounded domain in $\Omega \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$. For any $u_0 \in C^0(\overline{\mathcal{F}}_0; \mathbb{R}^2)$, $(h'_0, \vartheta'_0) \in \mathbb{R}^3$ such that*

$$
\mathit{div}\, u_0 = 0, \quad \mathit{curl}\, u_0 = \omega_0 \in L^\infty_c(\overline{\mathcal{F}_0}), \tag{1.2}
$$

$$
u_0 \cdot n = (h'_0 + \vartheta'_0(x - h_0)^{\perp}) \cdot n \ \text{on } \partial\mathcal{S}_0, \quad \lim_{|x|\to+\infty} |u_0| = 0 \ \text{or } u_0 \cdot n = 0 \ \text{on } \partial\Omega, \tag{1.3}
$$

there exists a unique maximal solution

$$
(h, \vartheta, u) \in C^2([0, T^*); \mathbb{R}^3) \times L^\infty_{loc}([0, T^*); \mathcal{LL}(\mathcal{F}(t))),
$$

where $T^* \in (0, +\infty]$ is the first meeting time between $\mathcal{S}(t)$ and $\partial\Omega$.

Here, we denoted by $\mathcal{LL}$ the usual space of log-Lipschitz functions:

$$
\mathcal{LL}(U) := \Big\{ f \in C^0(U) \,/\, \exists C > 0,\ \forall x, y \in U,\ |f(x) - f(y)| \leq C|x-y|(1 + \ln^- |x-y|) \Big\}.
$$

We made a slight abuse of notations in "$L^\infty_{loc}([0, T^*); \mathcal{LL}(\mathcal{F}(t)))$" since the space domain depends on time. This space refers to functions that belong to $\mathcal{LL}(\mathcal{F}(t))$ for almost every t, with uniform norm on compacts subsets of $[0, T^*)$ (but for a negligible set).

The main difference of Theorem 1 with Yudovich's result is that here the time of existence of a solution can be limited by the possible encounter of the solid and the outer boundary (when $\partial\Omega \neq \emptyset$).

Remark 1. *When $\Omega = \mathbb{R}^2$, in general $u(t, \cdot) \notin L^2(\mathcal{F}(t); \mathbb{R}^2)$. Finite energy solutions would be too particular in the sequel...*

1.2. The problem of a small body

We are now in position to explain the main problem under view, that is to determine the behaviour of the solutions of this system when the body becomes very small.

Let us be more specific. We suppose that we are given $\mathcal{S}_0$ smooth and h'_0, ϑ'_0, γ, $\omega_0 \in L^\infty_c(\mathbb{R}^2)$ fixed as above. The question is the following: what can be said for as the size of solid goes to zero, that is, how behaves the solution $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$ corresponding to the initial position of the solid:

$$\mathcal{S}_0^\varepsilon := \varepsilon \mathcal{S}_0,$$

as ε goes to 0^+?

We note in particular that for fixed $\varepsilon > 0$, we can reconstruct from the data above the initial velocity field u_0^ε via the system (1.1) and apply Theorem 1 to get the solution $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$. In the absence of an outer boundary $\partial\Omega$, this solution is global in time; in the presence of a boundary, on the contrary, the question of getting a uniform time of existence is quite important.

Prescribing the initial vorticity rather than the initial velocity gives the advantage that it can be given on $\mathbb{R}^2$ rather than on a domain depending on ε, in such a way that the compatibility conditions (1.2)-(1.3) are automatically satisfied.

We will give answers to these questions in various situations which we now describe.

Asymptotic regimes. We will be interested in the following *two particular regimes* concerning the mass and the moment of inertia:

- *A massive point in the limit:*

$$m_\varepsilon = m_1 \quad \text{and} \quad \mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J}_1, \tag{1.4}$$

- *A constant density:*

$$m_\varepsilon = \varepsilon^2 m_1 \quad \text{and} \quad \mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1, \tag{1.5}$$

where m_1 and $\mathcal{J}_1$ are fixed positive constants.

Two geometric situations. We have obtained results in that direction in *two situations*:

- *Situation* 1: $\Omega = \mathbb{R}^2$ and $\omega_0 \in L^\infty_c(\mathbb{R}^2 \setminus \{0\})$. We note that a consequence of this constraint on the support of ω_0 is that for small $\varepsilon > 0$,

$$\operatorname{dist}\Big(\operatorname{Supp}(\omega_0), \mathcal{S}_0^\varepsilon\Big) > 0 \ldots$$

- *Situation* 2: Ω is a smooth bounded domain and the fluid is *irrotational*, that is, $\omega_0 = 0$.

Hence we have two regimes and two geometrical situations; this gives four possibilities which we describe in the following table. The goal of this text is mainly to explain how to fill the cells.

	Situation 1 : $\Omega = \mathbb{R}^2$ $\omega_0 \in L^\infty_c(\mathbb{R}^2 \setminus \{0\})$	Situation 2 : Ω *bounded domain* $\omega_0 = 0$
Massive particle : $m_\varepsilon = m_1$ $\mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J}_1$		
Light particle: $m_\varepsilon = \varepsilon^2 m_1$ $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$		

Table 1.1. Two regimes, two geometrical situations, four possibilities

Motivations. Our motivations to study the limits of a small body are of several nature.

- Obtain *simplified fluid-solid* models. As we will see, we will obtain in the limit models which are much simpler, in particular because they take place in a fixed domain. Hence at the computational level, when studying these models, one does not have to adapt the grid to the moving domain. This would be particularly interesting in the case of many bodies (for which no rigorous derivation is obtained yet, but the study of the limit of a single body is an important step toward this goal).

- Give another *justification of classical vortex models.* As we will see, some of the models that we get in the limit exist in the literature but were obtained in another way (as limits for a singular initial vorticity for instance), or formally. Our studies give a rigorous justification of these models as limits of rigid/body systems.

- One would like to study *control problems* associated to a fluid-rigid body system. A problem for instance would be to understand how one can control the trajectory of the solid by acting on the fluid, or by using the deformation of the body. A possibility to tackle such problems would be to prove a control result on the simplified limit model (for instance by using tools of geometric control theory) and then to use a perturbation argument to obtain a result on the original one.

2. Results

Now let us describe the results that we obtain in the various situations mentioned above.

2.1. First case: a massive small solid in $\mathbb{R}^2$

We start with the case where we have a non trivial vorticity distribution in $\mathbb{R}^2$ (compactly supported away from 0) and where the inertia regime is given by (1.4).

Notations. Let us recall the Biot-Savart formula in $\mathbb{R}^2$: $K[\omega]$ describes the velocity generated in $\mathbb{R}^2$ by the vorticity $\omega \in L^\infty_c(\mathbb{R}^2)$:

$$K[\omega] := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)\, dy,$$

so that

$$\operatorname{curl} u = \omega, \quad \operatorname{div} u = 0 \ \text{ in } \mathbb{R}^2, \quad \lim_{|x|\to+\infty} u(x) = 0.$$

Note however that in general $u \notin L^2(\mathbb{R}^2)$ (except in the case where ω has zero mass). This explains why we wanted a result on the Cauchy problem in the infinite-energy case.

We also recall that a *point vortex* located at h with intensity γ corresponds to a Dirac mass at h with mass γ in the vorticity, generating the velocity field:

$$K[\gamma\delta_h] := \frac{\gamma}{2\pi} \frac{(x-h)^\perp}{|x-h|^2}.$$

Our main result in this situation is the following one.

Theorem 2 (G.-Lacave-Sueur [8])**.** *Let $m_1 > 0$, $\mathcal{J}_1 > 0$, $h'_0 \in \mathbb{R}^2$, ϑ'_0, $\gamma \in \mathbb{R}$ and $\omega_0 \in L^\infty_c(\mathbb{R}^2 \setminus \{0\})$ be fixed. Let $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$ the maximal solution on $[0,+\infty)$ of the fluid-solid system with solid of size ε, with m_ε and $\mathcal{J}_\varepsilon$ given by (1.4), and corresponding to ω_0, h'_0, ϑ'_0 and γ. Let $T > 0$. Up to a subsequence, one has:*

- $h^\varepsilon \rightharpoonup h$, $\varepsilon\vartheta^\varepsilon \rightharpoonup 0$ *weakly-$\star$ in* $W^{2,\infty}(0,T)$,
- $\omega^\varepsilon \longrightarrow \omega$ *in* $C^0([0,T]; L^\infty(\mathbb{R}^2) - w\star)$,
- $u^\varepsilon \longrightarrow K\left[\omega + \gamma\delta_{h(t)}\right]$ *in* $C^0([0,T]; L^q_{loc}(\mathbb{R}^2))$, $q < 2$.

Moreover, one has in the limit

$$\frac{\partial \omega}{\partial t} + \mathit{div}\left(K\left[\omega + \gamma\delta_{h(t)}\right]\omega\right) = 0 \quad \mathit{in} \quad [0,T]\times\mathbb{R}^2,$$

$$mh''(t) = \gamma\Big(h'(t) - K[\omega(t,\cdot)](h(t))\Big)^\perp,$$

$$\omega|_{t=0} = \omega_0, \ h(0) = 0, \ h'(0) = h'_0.$$

In this statement, we made a small abuse of notations: we actually "complete" the functions defined in $\mathcal{F}^\varepsilon(t)$ (that is, u^ε and ω^ε) by 0 in $\mathcal{S}^\varepsilon(t)$, in order that the convergences can take place in $\mathbb{R}^2$.

Remark 2. *The vorticity ω is transported by the total velocity field $K[\omega + \gamma\delta_{h(t)}]$, but of course $\delta_{h(t)}$ follows h. Hence it is possible that the vortex point located at $h(t)$ enters in finite time the support of vorticity, even if it is not the case at initial time. This explains why we do not know how to prove uniqueness in the limit, and hence, why this result is stated "up to a subsequence". See Marchioro-Pulvirenti* [22] *and Lacave-Miot* [19] *for related questions of uniqueness on fluid/vortex models.*

The force appearing in the equation of the point in the limit

$$F := \gamma\Big(h'(t) - K[\omega(t,\cdot)](h(t))\Big)^{\perp},$$

is similar to the *Kutta-Joukowski lift force* of the *irrotational* theory (see e.g. Lamb [20]): the force applied to a body at speed v, immersed in an irrotational fluid, with

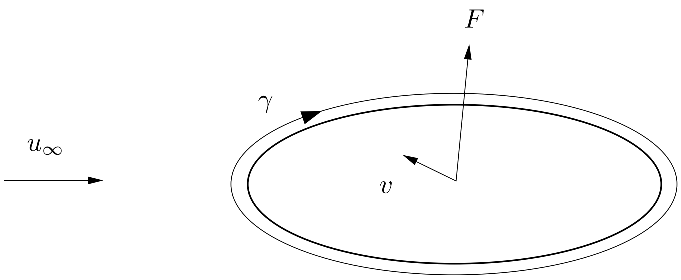

Figure 2.1. The Kutta-Joukowski force

fluid velocity u_∞ at infinity and circulation γ around the body (see Figure 2.1) is given by

$$F = \gamma(v - u_\infty)^{\perp}.$$

Remarks.

- In the case of a small *fixed obstacle*, a similar result was obtained by Iftimie, Lopes Filho and Nussenzveig Lopes [17]. Of course in that case, the vortex in the limit is δ_0.

- An analogous system was introduced and studied in the *irrotational case with several solids* by Grotta Raggazzo, Koiller and Oliva [15]:

$$m_j h_j'' = \gamma_j \left(h_j' - \sum_{i \neq j} K\left[\gamma_i \delta_{h_i}\right] \right)^{\perp},$$

  see also Vankerschaver, Kanso and Marsden [29]. Our result above gives a rigorous derivation of their system in the “single-body + vorticity” case.

2.2. A light small solid in Ω with $\omega_0 = 0$

We now turn to the case where Ω is a smooth bounded domain, where the flow is irrotational, i.e. $\omega_0 = 0$, and where the inertia regime is given by (1.5). We begin by introducing the limit system. We let $\psi(h,\cdot)$ be the solution of the following Dirichlet problem (from which $\mathcal{S}_0$ is absent!):

$$\Delta\psi = 0 \text{ in } \Omega, \quad \psi(h,\cdot) = G(\cdot - h) \text{ on } \partial\Omega, \text{ where } G(r) := -\frac{1}{2\pi}\ln|r| \text{ for } r \in \mathbb{R}^2.$$

The *Kirchhoff-Routh* stream function ψ_Ω/velocity u_Ω is defined as

$$\psi_\Omega(x) := \frac{1}{2}\psi(x,x) \text{ and } u_\Omega := \nabla^{\perp}\psi_\Omega.$$

The limit system here reads as follows

$$h' = \gamma u_\Omega(h) \quad \text{for } t \in [0,\infty), \quad h(0) = 0. \tag{2.1}$$

This ODE describes the dynamics of a single vortex point in a bounded domain. It has solutions which can be obtained as limits of regular solutions of the Euler equation where the vorticity concentrates to a point (see Turkington [28]). The solutions of this system are known to be global in time when $\gamma \neq 0$ (cf. Ibid.)

Our result in this situation is as follows.

Theorem 3 (G.-Munnier-Sueur [10]). *Let $m_1 > 0$, $\mathcal{J}_1 > 0$, $h_0' \in \mathbb{R}^2$, $\vartheta_0' \in \mathbb{R}$ and $\gamma \neq 0$ be fixed. Let h the maximal solution on $[0,+\infty)$ of* (2.1)*. Let $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$ the maximal solution on $[0, T^\varepsilon)$ of the fluid-solid system with solid of size ε, with m_ε and $\mathcal{J}_\varepsilon$ given by* (1.5)*, corresponding to $\omega_0 = 0$, h_0', ϑ_0' and γ. Then one has as $\varepsilon \to 0^+$,*

- $\lim T^\varepsilon = +\infty$*,*
- $h^\varepsilon \rightharpoonup h$ *in* $W^{1,\infty}([0,T];\mathbb{R}^2)$ *weak-$\star$ for all $T > 0$,*
- $\varepsilon \vartheta^\varepsilon \rightharpoonup 0$ *in* $W^{1,\infty}([0,T];\mathbb{R})$ *weak-$\star$ for all $T > 0$.*

We note that, with respect to Theorem 2, the uniqueness of the solutions in the limit ensures that the whole family (h^ε) converges to the solution of the particle system (2.1), and not merely a subsequence.

2.3. Remaining cases

We have "filled" two diagonally opposite cells in Table 1.1. The two remaining ones are obtained by retaining features of the above two cases, according to their column and their row.

A massive small solid in a bounded Ω. We begin with the case where Ω is a smooth bounded domain, the flow is irrotational but the inertia regime is given by (1.4). In that case, we obtain the following result which mixes Theorems 2 and 3 in the sense that the equation for the limit particle is second-order, but with a reference velocity field coming from the Kirchhoff-Routh potential.

Theorem 4 (G.-Munnier-Sueur [10]). *Let $m_1 > 0$, $\mathcal{J}_1 > 0$, h_0' in $\mathbb{R}^2$, ϑ_0' and γ in $\mathbb{R}$ be fixed. Let h the maximal solution on $[0, T^*)$ of the ordinary differential equation:*

$$m h'' = \gamma \Big(h' - \gamma u_\Omega(h) \Big)^\perp \quad \text{for } t \in [0, T^*),$$
$$h(0) = 0 \text{ and } h'(0) = h_0'.$$

Let $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$ the maximal solution on $[0, T^\varepsilon)$ of the fluid-solid system with the solid of size ε, with m_ε and $\mathcal{J}_\varepsilon$ given by (1.4)*, $\omega_0 = 0$, h_0', ϑ_0' and γ. Then one has as $\varepsilon \to 0^+$,*

- $\liminf T^\varepsilon \geq T^*$*,*
- $h^\varepsilon \rightharpoonup h$ *in* $W^{2,\infty}([0,T];\mathbb{R}^2)$ *weak-$\star$ for all $T \in (0, T^*)$,*
- $\varepsilon \vartheta^\varepsilon \rightharpoonup 0$ *in* $W^{2,\infty}([0,T];\mathbb{R})$ *weak-$\star$ for all $T \in (0, T^*)$.*

A light small solid in $\mathbb{R}^2$. The last situation concerns the case where $\Omega = \mathbb{R}^2$, the vorticity is not trivial, and the inertia regime is given by (1.5). In that case, we obtain the following result.

Theorem 5 (G.-Lacave-Sueur [9])**.** *Let $m_1 > 0$, $\mathcal{J}_1 > 0$, $h_0' \in \mathbb{R}^2$, $\vartheta_0' \in \mathbb{R}$, $\gamma \neq 0$ and $\omega_0 \in L_c^\infty(\mathbb{R}^2 \setminus \{0\})$ be fixed. Let $(h^\varepsilon, \vartheta^\varepsilon, u^\varepsilon)$ the maximal solution on $[0, T^\varepsilon)$ of the fluid-solid system with the solid of size ε, with m_ε and $\mathcal{J}_\varepsilon$ given by (1.5), and with initial conditions ω_0, h_0', ϑ_0' and γ. Then for all $T > 0$ one has as $\varepsilon \to 0^+$,*

- $h^\varepsilon \rightharpoonup h, \ \ \varepsilon\vartheta^\varepsilon \rightharpoonup 0$ *weakly-$\star$ in* $W^{1,\infty}(0,T)$,
- $\omega^\varepsilon \longrightarrow \omega$ *in* $C^0([0,T]; L^\infty(\mathbb{R}^2) - w\star)$,
- $u^\varepsilon \longrightarrow K[\omega + \gamma\delta_{h(t)}]$ *in* $C^0([0,T]; L^q_{loc}(\mathbb{R}^2))$, $q < 2$,

Moreover one has in the limit

$$\frac{\partial \omega}{\partial t} + div\Big(K[\omega + \gamma\delta_{h(t)}]\omega\Big) = 0 \quad in \quad [0,T] \times \mathbb{R}^2,$$
$$h'(t) = K[\omega(t,\cdot)](h(t)),$$
$$\omega|_{t=0} = \omega_0, \ \ h(0) = h_0.$$

The limit system here is known as Marchioro and Pulvirenti's *wave/vortex system.* It can be obtained as limits of regular solutions of the Euler equation (as a part of the vorticity concentrates in h_0). If $h_0 \notin \operatorname{Supp}\omega_0$, then one has global existence and uniqueness, and $h(t) \notin \operatorname{Supp}\omega(t)$ for all t. We refer here to Marchioro-Pulvirenti [21], Lacave-Miot [19] and Bjorland [3].

2.4. Summary

The above results can be summarized in the following table.

	Situation 1 : $\Omega = \mathbb{R}^2$ $\omega_0 \in L_c^\infty(\mathbb{R}^2 \setminus \{0\})$	Situation 2 : Ω *bounded domain* $\omega_0 = 0$
Massive particle : $m_\varepsilon = m_1$ $\mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J}_1$	$mh'' = \gamma(h' - \tilde{u})^\perp$ $\tilde{u} = K[\omega(t,\cdot)](h(t))$	$mh'' = \gamma(h' - \tilde{u})^\perp$ $\tilde{u} = \gamma u_\Omega(h(t))$
Light particle: $m_\varepsilon = \varepsilon^2 m_1$ $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$	$h' = \tilde{u}$ $\tilde{u} = K[\omega(t,\cdot)](h(t))$ $\partial_t\omega + \operatorname{div}(K[\omega + \gamma\delta_{h(t)}]\omega) = 0$	$h' = \tilde{u}$ $\tilde{u} = \gamma u_\Omega(h(t))$

Table 2.1. The filled table

3. Several ideas on the proofs

We finish this note by giving a few ideas about the proofs.

3.1. Difficulties

Let us first mention the difficulties that we have to face. We focus on the the solid equations:

$$m_\varepsilon\, h''(t) = \int_{\partial \mathcal{S}^\varepsilon(t)} p\, n\; ds, \quad \mathcal{J}_\varepsilon\, \vartheta''(t) = \int_{\partial \mathcal{S}^\varepsilon(t)} p\, (x - h(t))^\perp \cdot n\; ds.$$

- First, as is clear in these equations, we have to study the *pressure* in detail. In many works on incompressible fluid dynamics, the pressure is more or less ignored: one uses the vorticity formulation or the Leray projector to get rid of the pressure from the equations. Here this is no longer possible, since the pressure plays a central role in the dynamics of the solid.

- The problem is *singular in space* since the solid domain $\mathcal{S}^\varepsilon$ shrinks to a point and the circulation remains constant. This clearly means that the velocity u^ε becomes singular on the solid boundary as $\varepsilon \to 0^+$.

- The problem is also *singular in time* when $m_\varepsilon = \varepsilon^2 m_1$ and $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$. In particular, it is conspicuous in the case of a light particle that one passes from a second order equation for the dynamics of the solid, to a first order equation for the dynamics of the limit particle.

- The *energy* is not finite (in the case $\Omega = \mathbb{R}^2$); hence one cannot rely on the standard energy estimate to get bounds on the solid velocity. Moreover, even when considering a "desingularized" energy, this does not give a strong control when $m_\varepsilon = \varepsilon^2 m_1$ and $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$. Even when assuming that the energy is finite, this tells us that $\varepsilon (h^\varepsilon)'$ and $\varepsilon^2 (\vartheta^\varepsilon)'$ are bounded, which is a very poor information...

3.2. Some ideas of the proof (light particles)

Let us give some ideas in the case where $m_\varepsilon = \varepsilon^2 m_1$ and $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$. In some sense, this case is the most singular, since it gives the poorest information when performing energy estimates.

As one can guess, the main difficulty consists in obtaining uniform estimates as $\varepsilon \to 0^+$. Even if we assume the total energy to be finite, this merely gives

$$\|(h^\varepsilon)'\|_{L^\infty} = \mathcal{O}(1/\varepsilon) \ \text{ and } \ \|(\vartheta^\varepsilon)'\|_{L^\infty} = \mathcal{O}(1/\varepsilon^2) \ldots$$

We begin to describe our general strategy, and then explain several steps of its implementation.

General strategy. The general principle of the proof is as follows.

- Find a *modulated energy* which gives a better a priori estimate on $(h^\varepsilon)'$ (cf. Brenier [4]).

- For that purpose, find a *normal form* for the equation of the solid, with "*modulated unknowns*".

- This equation in normal form will look like an equation of *geodesics* but with a right hand side.

- These additional terms in the right hand side will not all be conservative, but will give a "reasonable" contribution to the modulated energy. In particular it is useful to put them in *electromagnetic form*, and to use the analogy with the dynamics of a charged particle in a strong electromagnetic field (cf. Berkowitz-Gardner [2]).

Now we describe important steps in the proof.

1. The added mass effect. A first step to reformulate the equation, which is important for this study as well as for the Cauchy problem, is to identify the "added mass effect". This effect is quite intuitive: a solid immersed in a fluid acts as if it had a larger mass, because moving it requires to give some energy to the fluid as well. Mathematically speaking, this can be described as follows.

Suppose for instance that we are in the case $\Omega = \mathbb{R}^2$. We consider equations in the body frame. For that we introduce

$$\begin{cases} v = R(\vartheta)^T u(t, R(\vartheta)x + h(t)), \\ q = p(t, R(\vartheta)x + h(t)), \\ \ell = R(\vartheta)^T h', \end{cases}$$

with $R(\vartheta)$ the rotation of angle $\vartheta(t)$. The equations of the fluid/body system become

$$\begin{gathered} \partial_t v + \Big[(v - \ell - \vartheta' x^\perp) \cdot \nabla\Big] v + \vartheta' v^\perp + \nabla q = 0 \text{ for } x \in \mathcal{F}_0, \\ \operatorname{div} v = 0 \text{ for } x \in \mathcal{F}_0, \\ m\ell'(t) = \int_{\partial \mathcal{S}_0} qn \, ds - m\vartheta' \ell^\perp \\ \mathcal{J}\vartheta''(t) = \int_{\partial \mathcal{S}_0} x^\perp \cdot qn \, ds. \end{gathered}$$

The great advantage is that the domain is now fixed. The price to pay is singular terms (such as $\vartheta'(x^\perp \cdot \nabla)v$) appearing in the equation, but it is worth it. (This possibility makes the analysis simpler in the case $\Omega = \mathbb{R}^2$.)

Now one introduces *Kirchhoff's potentials* Φ_1, Φ_2, Φ_3 as the solutions of the following Neumann problems:

$$\Delta \Phi_i = 0 \text{ in } \mathcal{F}_0, \qquad \nabla \Phi_i \underset{\infty}{\longrightarrow} 0, \tag{3.1}$$

$$\partial_n \Phi_i = \begin{cases} n_i \ \ (i = 1, 2), \\ x^\perp \cdot n \ \ (i = 3), \end{cases} \quad \text{on } \partial \mathcal{S}_0, \tag{3.2}$$

where denoted by n_i the i-th component of the normal n.

The solid equations then become after an integration by parts:

$$\begin{bmatrix} m \, \mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} \begin{bmatrix} \ell \\ \vartheta' \end{bmatrix}' = \left[\int_{\mathcal{F}_0} \nabla q \cdot \nabla \Phi_i \, dx \right]_{i=1,2,3} - \begin{bmatrix} m\vartheta' \ell^\perp \\ 0 \end{bmatrix}.$$

Let P the Leray projector, that is, the orthogonal projection in $L^2(\mathcal{F}_0; \mathbb{R}^2)$ on tangent divergence-free vector fields (parallel to gradient fields). The pressure is decomposed as follows:

$$\nabla q = \underbrace{(I-P)(-\partial_t v)}_{=:\nabla\varphi} + \underbrace{(I-P)(-(v-\ell-\vartheta' x^\perp)\cdot\nabla v - \vartheta' v^\perp)}_{=:\nabla\mu}.$$

Using that $\partial_t v$ is already divergence-free, one easily deduces that

$$\nabla\varphi = -\begin{pmatrix}\ell\\ \vartheta'\end{pmatrix}'\cdot\big(\nabla\Phi_i\big)_{i=1,2,3}.$$

We end up with this new equation for the solid:

$$\mathcal{M}\begin{bmatrix}\ell\\ \vartheta'\end{bmatrix}' = \begin{bmatrix}mr\ell^\perp\\ 0\end{bmatrix} + \left[\int_{\mathcal{F}_0}\nabla\mu\cdot\nabla\Phi_i\,dx\right]_{i=1,2,3}, \tag{3.3}$$

where

$$\mathcal{M} := \underbrace{\begin{bmatrix}m\,\mathrm{Id}_2 & 0\\ 0 & \mathcal{J}\end{bmatrix}}_{=:\mathcal{M}_g} + \underbrace{\left[\int_{\mathcal{F}_0}\nabla\Phi_i\cdot\nabla\Phi_j\;dx\right]_{i,j=1,2,3}}_{=:\mathcal{M}_a}.$$

The matrix $\mathcal{M}_a$ is a matrix of *added inertia*, expressing how the fluid opposes the movement of the solid. It is *positive*, and even *positive definite* when $\mathcal{S}_0$ is not a disk, as a Gram matrix of independent functions. The matrix $\mathcal{M}_g$ of *genuine* inertia is positive definite and independent of the position of the solid.

In this form, the equation of the solid (3.3) has no more terms involving h'' or ϑ'' in the right hand side.

This argument of added mass can be performed when Ω is a bounded domain as well, despite the absence of a rigid change of variables transferring the problem in a fixed domain. In that case, the added mass matrix $\mathcal{M}_a$ depends on the relative position of the solid and the outer boundary (and not merely on the rotation matrix).

2. Reformulation of the solid equation (irrotational case). We consider the irrotational case, that is the case where $\omega = 0$. For $\varepsilon = 1$, using arguments of Lagrangian mechanics and shape derivative, we can prove that the unknown

$$q := (\vartheta, h_1, h_2),$$

satisfies the following ODE

$$\mathcal{M}(q)q'' + \langle\Gamma(q), q', q'\rangle = F(q,q'), \tag{3.4}$$

where

- $\mathcal{M}(q) = \mathcal{M}_g + \mathcal{M}_a(q)$ (genuine inertia + *added inertia*),
- $\Gamma(q)$ contains the *Christoffel symbols* associated to the metric $\mathcal{M}(q)$,
- $F(q,q')$ is a *Lorentz-type force*, of the form:

$$F(q,q') := \gamma^2 E(q) + \gamma\, q'\times B(q),$$

with strong conditions on E.

A Lagrangian approach to the dynamics of a solid immersed in a perfect fluid was already performed by Munnier [24] when $\gamma = 0$ (that is, in the *potential* case). In that case, no electromagnetic force appears, and in particular (3.4) is an actual geodesic equation associated to the metric $\mathcal{M}(q)$. Let us underline that it is not quite the same result as to see the whole system as a geodesic equation in the spirit of Arnold (see [1, 12]), since here the configuration space contains merely the position of the solid.

Equation (3.4) is the starting point towards a normal form.

A important particular case: without outer boundary. In the particular case where $\Omega = \mathbb{R}^2$, the coefficients above can be identified (depending on $\mathcal{S}_0$), and turn out to be useful in the general case.

To be more precise, when $\Omega = \mathbb{R}^2$, the ODE becomes:

$$\mathcal{M}_{\partial\!\!\!\Omega}(\vartheta)\, q'' + \langle \Gamma_{\partial\!\!\!\Omega}(\vartheta), q', q' \rangle = F_{\partial\!\!\!\Omega}(\vartheta, q'),$$

with

$$F_{\partial\!\!\!\Omega}(\vartheta, q') = \gamma \begin{pmatrix} R(\vartheta)\zeta \cdot h' \\ (h')^\perp - \vartheta' R(\vartheta)\zeta \end{pmatrix} = \gamma q' \times B_{\partial\!\!\!\Omega}(\vartheta), \tag{3.5}$$

where

$$B_{\partial\!\!\!\Omega}(\vartheta) = \begin{pmatrix} -1 \\ R(\vartheta)\zeta^\perp \end{pmatrix},$$

and ζ is a geometrical constant (known as the conformal center of $\mathcal{S}_0$) depending on $\mathcal{S}_0$ only; it could be described in terms of a certain complex integral. Moreover, one can simply describe the total mass matrix as follows

$$\mathcal{M}_{\partial\!\!\!\Omega}(\vartheta) = \mathcal{M}_g + \mathcal{M}_{a,\partial\!\!\!\Omega}(\vartheta) = \mathcal{M}_g + Q(\vartheta)\mathcal{M}_{a,\partial\!\!\!\Omega}(0)Q(\vartheta)^T = Q(\vartheta)\mathcal{M}_{\partial\!\!\!\Omega}(0)Q(\vartheta)^T, \tag{3.6}$$

with $Q(\vartheta)$ the 3×3 matrix acting as a rotation of angle ϑ on the (h_1, h_2) variables.

3. A normal form (Ω bounded, irrotational case). The next step towards a normal form and a modulated energy estimate is to introduce a new unknown. Here we consider the case of an irrotational fluid in a bounded domain.

We consider the *modulated velocity*

$$\tilde{p} = \Big(\varepsilon\vartheta', h' - \gamma u_\Omega(h) - \varepsilon\gamma u_c(\vartheta, h)\Big),$$

where u_c is explicit and depends merely on Ω, $\mathcal{S}_0$ and $q = (\vartheta, h)$. Note that $\gamma u_\Omega(h)$ is the expected limit of h'; the subprincipal term at order ε that we add will be useful to get rid of some singular terms in the equations.

Then one shows that in terms of the new unknown the ODE can be put in the following *normal form*:

$$\varepsilon^2\Big(\mathcal{M}^1_g + \mathcal{M}^1_{a,\partial\!\!\!\Omega}(\vartheta)\Big)\tilde{p}' + \varepsilon\langle \Gamma^1_{\partial\!\!\!\Omega}(\vartheta), \tilde{p}, \tilde{p}\rangle = F^1_{\partial\!\!\!\Omega}(\vartheta, \tilde{p}) + \varepsilon\gamma^2 G(q) + \mathcal{O}(\varepsilon^2), \tag{3.7}$$

where $G(q)$ is *weakly gyroscopic* in the sense that it satisfies:

$$\Big| \int_0^t \tilde{p} \cdot G(q) \Big| \leq \varepsilon K (1 + t + \int_0^t |\tilde{p}|^2_{\mathbb{R}^3}).$$

One recognizes the coefficients corresponding to the case with outer boundary. Here the exponent 1 means that the quantity is computed for $\varepsilon = 1$, and hence does no longer depend directly on ε (but is applied to ϑ or $\tilde{p}$ which of course depend on ε). In other words, these quantities are of order 1.

Using this normal form, one is in position to win a factor ε in the energy estimate. Recall that $F_{\partial\Omega}$ given in (3.5) is purely magnetic, and hence does not contribute to the energy!

4. To get to the normal form. The main difficulty is to get to the normal form (3.7). The recipe consists in:

- developing in powers of ε the Kirchhoff potentials Φ_i^ε,
- developing in powers of ε the "circulation potential" ψ^ε satisfying

$$\Delta\psi^\varepsilon = 0 \text{ in } \mathcal{F}(t), \quad \psi^\varepsilon = 0 \text{ on } \partial\Omega, \quad \psi^\varepsilon = C^\varepsilon \text{ on } \partial\mathcal{S}^\varepsilon(t), \quad \int_{\partial\mathcal{S}^\varepsilon(t)} \partial_n \psi^\varepsilon \, ds = 1,$$

  where C^ε is some constant fixed by the latter constraint,
- decompose the fluid velocity in terms of these potentials and inject the developments in the velocity decomposition and in the coefficients of the equation (that is, the inertia matrix, the Christoffel symbols and the electromagnetic field),
- make computations (using the so-called Lamb's lemma [20]) and use some *cancellations*...

Expansions of the potentials. To expand the potentials in powers of ε, one uses an iterative procedure of successive corrections, relying on potential and Fredholm theories. One "ignores" alternatively the solid $\mathcal{S}_0$ and the outer boundary $\partial\Omega$, and then introduces the corresponding corrector.

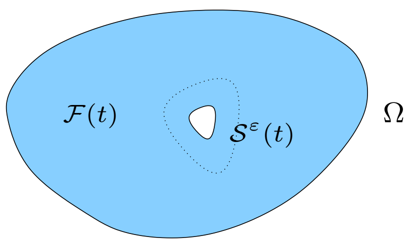

Figure 3.1. The shrinking solid

To state it informally, one can describe the procedure as:

$$\begin{aligned}\text{Fluid state in } \Omega \setminus \mathcal{S}^\varepsilon(t) = {} & \text{Fluid state as if there were no } \partial\Omega \\ & + \text{ Correction from } \partial\Omega \text{ as if there were no } \mathcal{S}^\varepsilon(t) \\ & + \text{ Correction(Correction) on } \partial\mathcal{S}^\varepsilon(t) \text{ as if there were no } \partial\Omega \\ & + \dots\end{aligned}$$

Once these expansions are performed, one can inject them in the equation. To obtain the normal form mentioned above, we then rely on key cancellations when regrouping certain terms. The property that the subprincipal term in (3.7) (that is, the term of order ε in the right hand side) is weakly gyroscopic is in particular a crucial phenomenon.

5. Passage to the limit. Once the normal form is obtained, one can achieve a modulated energy estimate implying that, as long as $\mathcal{S}^\varepsilon(t)$ is at a minimal distance from the boundary, one has

$$\|(h^\varepsilon)'\|_{L^\infty} = \mathcal{O}(1) \quad \text{and} \quad \|(\vartheta^\varepsilon)'\|_{L^\infty} = \mathcal{O}(1/\varepsilon).$$

Next it remains to pass to the limit in

$$\varepsilon^2 \Big(\mathcal{M}_g^1 + \mathcal{M}_{a,\partial\Omega}^1(\vartheta)\Big)\tilde{p}' + \varepsilon\langle \Gamma_{a,\partial\Omega}^1(\vartheta), \tilde{p}, \tilde{p}\rangle = F_{\partial\Omega}^1(\vartheta, \tilde{p}) + \varepsilon\gamma^2 G(q) + \mathcal{O}(\varepsilon^2),$$

with

$$F_{\partial\Omega}^1(\vartheta, \tilde{p}) = \gamma \begin{pmatrix} R(\vartheta)\zeta \cdot \Big[h' - \gamma u_\Omega(h) - \varepsilon\gamma u_c(\vartheta, h))\Big] \\ \Big[h' - \gamma u_\Omega(h) - \varepsilon\gamma u_c(\vartheta, h)\Big]^\perp - \varepsilon\vartheta' R(\vartheta)\zeta \end{pmatrix}$$

This can be done by arguments of compactness and weak/strong convergence. Note in particular that the left hand side converges to 0, as well as the last two terms in the right hand side.

6. In the context $\Omega = \mathbb{R}^2$ and $\omega_0 \neq 0$. Let us give a few words about the case with non trivial vorticity, which in our studies is restricted to $\Omega = \mathbb{R}^2$.

Here, on the one hand, we can change the frame in order to work in $\mathbb{R}^2 \setminus \mathcal{S}_0^\varepsilon$, but on the other hand, we know the vorticity with little precision. However, using $\mathrm{dist}\Big[\mathcal{S}^\varepsilon(t), \mathrm{Supp}\,\Big(\omega^\varepsilon(t)\Big)\Big] > 0$, we can reach here the following *normal form*:

$$\varepsilon^2 \Big[\mathcal{M}_g^1 + \mathcal{M}_a^1\Big]\,\tilde{p}' + \varepsilon\langle \Gamma, \tilde{p}, \tilde{p}\rangle = \gamma\,\tilde{p} \times B + \varepsilon\gamma G(\varepsilon, t) + \mathcal{O}(\varepsilon^2), \tag{3.8}$$

where

- the modulated velocity takes the following form:

$$\tilde{p} := \Big(\varepsilon\vartheta', R(\vartheta)^T \Big(h'(t) - K[\omega(t,)](h(t)) - \varepsilon DK[\omega^\varepsilon(t,\cdot)](0)\cdot\zeta\Big)\Big),$$

- the inertia matrices $\mathcal{M}_g^1$ and $\mathcal{M}_a^1 = \mathcal{M}_{a,\partial\Omega}^1(0)$ are the same as in (3.6),
- Γ generates a gyroscopic (skew-symmetric) term,
- B is given by $B = B_{\partial\Omega}(0)$ and the term $\gamma\,\tilde{p} \times B$ is gyroscopic as well,
- $G(\varepsilon, t)$ is weakly gyroscopic, in the same sense as above, and depends on the vorticity.

Approximation of the velocity on the solid's boundary. Here to get to the normal form (3.8), and hence to evaluate the pressure force on the solid's boundary, one computes an approximation of the velocity on the boundary of the solid. This approximation uses:

- Kirchhoff's potentials Φ_i^ε (defined as (3.1)-(3.2) in the domain $\mathcal{F}_0^\varepsilon$) and the harmonic field H^ε defined by

$$\operatorname{div} H^\varepsilon = 0 \text{ in } \mathcal{F}_0^\varepsilon, \ \operatorname{curl} H^\varepsilon = 0 \text{ in } \mathcal{F}_0^\varepsilon, \ H^\varepsilon \cdot n = 0 \text{ on } \partial\mathcal{S}_0^\varepsilon,$$
$$\int_{\partial\mathcal{S}_0^\varepsilon} H^\varepsilon \cdot \tau \, ds = 1, \ \lim_{|x|\to+\infty} H^\varepsilon(x) = 0.$$

  When $\Omega = \mathbb{R}^2$, the scaling of these objects with respect to ε is not difficult to determine.

- The Taylor first order approximation of the fluid contribution to this velocity:

$$K[\omega]_{|x=0} + DK[\omega]_{|x=0} \cdot x.$$

- New harmonic potentials (to approach the latter) in the same spirit as Kirchhoff potentials, for instance with $x \cdot n$ as Neumann boundary conditions.

We are led to compute many integrals on $\partial\mathcal{S}_0^\varepsilon$. Using *Blasius' lemma*, these are transformed into complex integrals that one can compute using basic complex analysis (Laurent series and so on). Here again we rely again on many *cancellations* to reach the above normal form. Again this leads to a modulated energy estimate and to weak convergence arguments.

3.3. Perspectives

Let us give as a final word a few perspectives connected to these studies.

- It would be natural to try to get an *asymptotic expansion* in ε of the solution. This would involve multiple scales in time.

- Clearly, it would be satisfying to get a *unified treatment* with a boundary and vorticity at the same time.

- The *many-bodies systems* would be a natural and important generalization. From a computational viewpoint, the complexity grows importantly as the number of solids increases; hence the simplified models become more and more interesting. This could be a good starting point to study the limit of infinitely many particles, which are important for spray models, see in particular Moussa-Sueur [23].

- The cases of dimension 3 and of a viscous fluid leave many open questions (see however Dashti-Robinson [5], Silvestre-Takahashi [27]).

- Finally, many control questions related to these systems remain open!

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