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From classical mechanics to kinetic theory and fluid dynamics

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From classical mechanics to kinetic theory and fluid dynamics

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Abstract

In these notes we report on a work in collaboration with Thierry Bodineau and Laure Saint-Raymond, where we show how the heat equation can be obtained from a deterministic system of hard spheres when the number of particles goes to infinity while their radius simultaneously goes to zero. As suggested by Hilbert in his sixth problem, the kinetic theory of Boltzmann is used as an intermediate level of description.

1. Introduction

In these notes we report on the recent studies [4] and [7], in which an analysis is performed, of the passage from microscopic to macroscopic dynamics, in some specific frameworks. In particular the microscopic dynamics is described by a very simple process of hard spheres moving in straight lines between two elastic collisions, or particles interacting via a compactly supported, repulsive potential – here we shall only discuss the hard-spheres case which is more simple to analyze. The aim of the theory is to derive, in the limit of an infinite number of particles (of size going to zero), and in the limit of a vanishing mean free path, fluid equations such as compressible Euler, or incompressible Navier-Stokes equations. This question was raised by D. Hilbert [8] at the second International Congress of Mathematicians held in Paris in 1900: Hilbert’s sixth problem consists indeed in understanding whether or not the different models describing the dynamics of fluids are consistent, and more precisely “*to develop mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua*”. He also suggests using Boltzmann’s equation, which is set at a mesoscopic scaling, as an intermediate level of description.

In [7] we give a complete proof of the derivation of the Boltzmann equation starting from a system of hard spheres, and in [4] we study the very particular case of a tagged particle initially in a background of particles at equilibrium: we show that in that case the density distribution of the tagged particle solves asymptotically the heat equation. This is to our knowledge the first instance when a deterministic

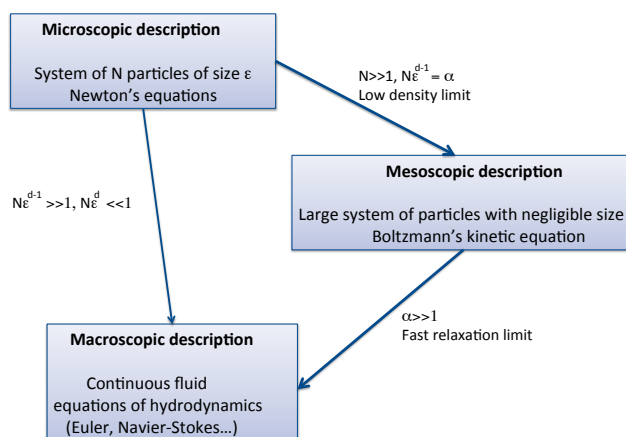


Figure 1.1: The different levels of description of a rarefied gas

system of particles is shown to converge to a model in fluid dynamics (and actually the motion of the tagged particle is shown to converge towards a Brownian motion).

A large number of previous works deal with related results, either in the study of the Particle-to-Fluid limit (in the context when some noise is added to the particle system) or for the Boltzmann-to-Fluid limit. We refer for instance to [13] or [12] for references on those limits, as well as to the references provided in [4] and [7]. For the Particle-to-Boltzmann limit, references will be given below.

In these notes we do not intend to give the details of the proofs of those results, but we describe the main steps and difficulties in the analysis. We refer to [4] and [7] for all the details (see also [3] for a short presentation).

The plan of these notes is as follows. In Section 2 we present the formal asymptotics leading from particles to Boltzmann, and from Boltzmann to the heat equation. We also state the main results. Section 3 is then devoted to the presentation of the main features of the proofs.

2. Formal asymptotics and statement of the result

In this section we start by showing formally, in Paragraph 2.2, why the nonlinear Boltzmann equation is a good approximation to the one-particle distribution function for a system of hard spheres; this system is presented in Paragraph 2.1. Paragraph 2.3 is then devoted to the formal asymptotics of the linear Boltzmann equation to the heat equation, and Paragraph 2.4 states the main results presented in this text.

2.1. The microscopic model and the one-particle distribution function

We consider a system of N hard spheres of diameter ε in the phase-space $\mathbb{T}^{dN} \times \mathbb{R}^{dN}$, where \mathbb{T}^d denotes the torus in dimension d . We denote the inverse of the mean free path by $\alpha := N\varepsilon^{d-1}$. We assume the particles move in straight lines between collisions

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as } |x_i(t) - x_j(t)| > \varepsilon \quad \text{for } 1 \leq i \neq j \leq N, \quad (2.1)$$

and at a collision the velocities of the colliding particles are modified according to the following rules:

$$\begin{aligned} v_i(t^-) &= v_i(t^+) - \omega^{i,j} \cdot (v_i(t^+) - v_j(t^+)) \omega^{i,j} \\ v_j(t^-) &= v_j(t^+) + \omega^{i,j} \cdot (v_i(t^+) - v_j(t^+)) \omega^{i,j}, \end{aligned} \quad (2.2)$$

where $\omega^{i,j} := (x_i - x_j)/|x_i - x_j|$. Note that one can prove (see for instance [1], [7])

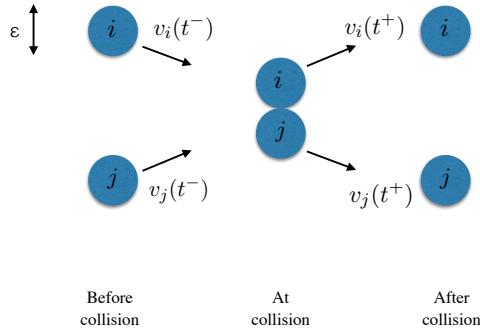


Figure 2.1: The microscopic model

that pathological situations such as collisions involving three or more particles, or for which there is a clustering of collision times, may be neglected as the set of corresponding initial data is of measure zero in phase space. Similarly we shall not discuss the wellposedness of this system.

The system can be described by the distribution function $f_N(t, Z_N)$, writing $Z_N := (X_N, V_N)$ with $X_N := (x_1, \dots, x_N)$ and $V_N := (v_1, \dots, v_N)$, which satisfies the Liouville equation

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0 \quad \text{in } \mathcal{D}_N^\varepsilon \times \mathbb{R}^{dN}, \quad (2.3)$$

with $\mathcal{D}_N^\varepsilon := \{X_N \in \mathbb{T}^{dN}, \forall i \neq j, |x_i - x_j| > \varepsilon\}$, and with a specular reflection on the boundary: if X_N satisfies $|x_i - x_j| = \varepsilon$, and for all (k, ℓ) in $[1, N] \setminus \{i, j\}$ there holds $|x_k - x_\ell| > \varepsilon$, with moreover $(v_i - v_j) \cdot (x_i - x_j) < 0$, then

$$f_N(t, Z_N^{\text{out}}(i, j)) = f_N(t, Z_N^{\text{in}}(i, j))$$

with $Z_N^{in}(i, j) := Z_N$, $X_N^{out}(i, j) := X_N$, $v_k^{out}(i, j) := v_k$ for all $k \in [1, N] \setminus \{i, j\}$ and as in (2.2),

$$\begin{aligned} v_i^{out}(i, j) &:= v_i^{in} - \frac{1}{\varepsilon^2}(v_i^{in} - v_j^{in}) \cdot (x_i - x_j)(x_i - x_j) \\ v_j^{out}(i, j) &:= v_j^{in} + \frac{1}{\varepsilon^2}(v_i^{in} - v_j^{in}) \cdot (x_i - x_j)(x_i - x_j). \end{aligned}$$

In the limit when N goes to infinity, one is no longer interested in the function f_N but rather in the one-particle distribution $f(t, x, v)$, describing the distribution of particles having position x and velocity v at time t . Assuming f_N is unchanged under relabeling of particles, this amounts to studying the limiting behaviour of the first marginal of f_N , namely

$$f_N^{(1)}(t, z_1) := \int f_N(t, Z_N) dz_2 \dots dz_N.$$

2.2. Formal derivation of the Boltzmann equation

Let us derive formally the equation satisfied by $f_N^{(1)}$. Integrating (2.3) over z_2, \dots, z_N , a formal computation based on Green's formula leads to

$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_N^{(1)}(t, z_1) = \alpha (C_{1,2} f_N^{(2)})(t, z_1) \quad (2.4)$$

where

$$f_N^{(2)}(t, Z_2) := \int f_N(t, Z_N) dz_3 \dots dz_N$$

and

$$(C_{1,2} f_N^{(2)})(z_1) := (N-1) \varepsilon^{d-1} \alpha^{-1} \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon \omega, v_2) \left((v_2 - v_1) \cdot \omega \right) d\omega dv_2,$$

where \mathbb{S}^{d-1} denotes the unit sphere in \mathbb{R}^d . Using the boundary condition, one can transform this integral into

$$\begin{aligned} (C_{1,2} f_N^{(2)})(z_1) &= (N-1) \varepsilon^{d-1} \alpha^{-1} \\ &\times \left(\int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v'_1, x_1 + \varepsilon \omega, v'_2) \left((v_2 - v_1) \cdot \omega \right)_+ d\omega dv_2 \right. \\ &\quad \left. - \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon \omega, v_2) \left((v_2 - v_1) \cdot \omega \right)_- d\omega dv_2 \right) \end{aligned}$$

with

$$v'_1 := v_1 - (v_1 - v_2) \cdot \omega \omega, \quad v'_2 := v_2 + (v_1 - v_2) \cdot \omega \omega.$$

In order to obtain the equation satisfied by the limit f of $f_N^{(1)}$ (supposing such a limit exists), we pass formally to the limit $N \rightarrow \infty$, $\varepsilon \rightarrow 0$ under the Boltzmann-Grad scaling $N \varepsilon^{d-1} = \alpha$: assuming moreover that the limit of $f_N^{(2)}(Z_2)$ may be written $f(z_1) f(z_2)$ (meaning that correlations disappear in the limit), then one finds that f should satisfy the Boltzmann equation

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \alpha Q(f, f) \\ Q(f, f)(v) &:= \iint_{\mathbb{S}^{d-1} \times \mathbb{R}^d} [f(v') f(v'_1) - f(v) f(v_1)] \left((v - v_1) \cdot \omega \right)_+ dv_1 d\omega \quad (2.5) \\ v' &= v + \omega \cdot (v_1 - v) \omega, \quad v'_1 = v_1 - \omega \cdot (v_1 - v) \omega. \end{aligned}$$

The justification of this limit is a difficult task, and is in general not known except for very small times (of the order of $1/\alpha$, which makes it then impossible to take

the $\alpha \rightarrow \infty$ limit in order to recover a fluid equation). More precisely the following theorem can be proved, which goes back to Lanford..

Theorem 1 ([9],[5],[6],[7]). *Consider a system of N particles interacting as hard-spheres of diameter ε . Let $f_0 : \mathbb{T}^d \times \mathbb{R}^d \mapsto \mathbb{R}^+$ be a continuous density of probability such that*

$$\left\| f_0 \exp\left(\frac{\beta}{2}|v|^2\right) \right\|_{L^\infty(\mathbb{T}_x^d \times \mathbb{R}_v^d)} \leq \exp(-\mu)$$

for some $\beta > 0, \mu \in \mathbb{R}$. Assume that the N particles are initially distributed according to f_0 and “independent”. Then, there exists some $T^* > 0$ (depending only on β and μ) such that, in the Boltzmann-Grad limit $N \rightarrow \infty, \varepsilon \rightarrow 0, N\varepsilon^{d-1} = \alpha$, the distribution function of the particles converges uniformly on $[0, T^*/\alpha] \times \mathbb{T}^d \times \mathbb{R}^d$ to the solution of the Boltzmann equation (2.5).

Here, by “independent”, we mean that the initial N -particle distribution satisfies a chaos property, namely that the correlations vanish asymptotically. Typically

$$f_N^0(Z_N) = \mathcal{Z}_N^{-1} f_0^{\otimes N}(Z_N) \mathbb{1}_{\mathcal{D}_\varepsilon^N}(X_N),$$

with

$$f_0^{\otimes N}(x_1, v_1, \dots, x_N, v_N) := \prod_{i=1}^N f_0(x_i, v_i),$$

while \mathcal{Z}_N normalizes the integral of f_N^0 to 1.

Theorem 1 holds also if the N particles interact via a compactly supported, repulsive potential satisfying some additional assumption (namely the fact that the scattering of particles can be parametrized by their deviation angle); for more details we refer to [7], [11].

2.3. Formal derivation of the heat equation starting from the linear Boltzmann equation

The Boltzmann equation is not known to have global in time, unique solutions in general. As our aim is to study the limit $\alpha \rightarrow \infty$ in a diffusive scaling in time, meaning at times rescaled by α , we need to find a setting in which global solutions do exist. One such setting is that of the linear Boltzmann equation, consisting in replacing, in (2.5), $f(v'_1)$ and $f(v_1)$ by the stationary solution

$$M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{\beta}{2}|v|^2\right), \quad \beta > 0.$$

The equation under study from now on is therefore the following:

$$\partial_t g_\alpha + v \cdot \nabla_x g_\alpha = \alpha \iint [g_\alpha(v') M_\beta(v'_1) - g_\alpha(v) M_\beta(v_1)] \left((v - v_1) \cdot \omega\right)_+ dv_1 d\omega. \quad (2.6)$$

As suggested above we choose a diffusive scaling in time, namely we want to study the asymptotic behaviour of $\tilde{g}_\alpha(\tau) := g_\alpha(\alpha\tau)$. It turns out to be more convenient to define

$$\tilde{\varphi}_\alpha(\tau, x, v) := M_\beta(v) \tilde{g}_\alpha(\tau, x, v),$$

which satisfies

$$\begin{aligned} \partial_\tau \tilde{\varphi}_\alpha + \alpha v \cdot \nabla_x \tilde{\varphi}_\alpha &= -\alpha^2 L \tilde{\varphi}_\alpha \\ L \tilde{\varphi}_\alpha(v) &:= \iint [\tilde{\varphi}_\alpha(v) - \tilde{\varphi}_\alpha(v')] M_\beta(v_1) \left((v - v_1) \cdot \omega \right)_+ dv_1 d\omega. \end{aligned} \quad (2.7)$$

Next assume that

$$\tilde{\varphi}_\alpha(\tau, x, v) = \tilde{\rho}_0(\tau, x, v) + \frac{1}{\alpha} \tilde{\rho}_1(\tau, x, v) + \frac{1}{\alpha^2} \tilde{\rho}_2(\tau, x, v) + \dots$$

Plugging that expansion in Equation (2.7), and canceling successively all the powers of α leads to the following set of equations (where we have considered only the $O(1)$, $O(\alpha)$ and $O(\alpha^2)$ terms):

$$\begin{aligned} L \tilde{\rho}_0 &= 0, \\ v \cdot \nabla_x \tilde{\rho}_0 + L \tilde{\rho}_1 &= 0, \\ \partial_\tau \tilde{\rho}_0 + v \cdot \nabla_x \tilde{\rho}_1 + L \tilde{\rho}_2 &= 0. \end{aligned} \quad (2.8)$$

In order to find the expressions for $\tilde{\rho}_1$ and $\tilde{\rho}_2$, as well as the equation on $\tilde{\rho}_0$ (which we expect to be the heat equation), it is necessary to be able to invert the operator L . It can be shown that this is possible on the set of functions orthogonal to M_β in L^2 . Moreover the kernel of L is made of functions independent of v so $\tilde{\rho}_0$ does not depend on v . We then define the vector $b(v) := L^{-1}v$ and returning to (2.8), we get

$$\tilde{\rho}_1(\tau, x, v) = -b(v) \cdot \nabla_x \tilde{\rho}_0(\tau, x) + \bar{\rho}_1(\tau, x), \quad \bar{\rho}_1 \in \text{Ker } L.$$

Next we consider the third equation in (2.8) and we notice that for $\tilde{\rho}_2$ to exist it is necessary for $\partial_\tau \tilde{\rho}_0 + v \cdot \nabla_x \tilde{\rho}_1$ to belong to the range of L . Since $\tilde{\rho}_0$ does not depend on v , this means that

$$\partial_\tau \tilde{\rho}_0 + \int_{\mathbb{R}^d} v \cdot \nabla_x \tilde{\rho}_1(\tau, x, v) M_\beta(v) dv = 0.$$

We then define the diffusion coefficient

$$\kappa_\beta := \int_{\mathbb{R}^d} v L^{-1} v M_\beta(v) dv, \quad (2.9)$$

and an easy computation shows that

$$\partial_\tau \tilde{\rho}_0 - \kappa_\beta \Delta_x \tilde{\rho}_0 = 0,$$

which means that the heat equation is indeed the limit of the linear Boltzmann equation in a diffusive scaling in time, in the limit of a vanishing mean free path.

It is not difficult to make the above arguments rigorous and hence to prove the following result.

Theorem 2 (From Linear Boltzmann to the heat equation). *Let ρ^0 be a bounded function and let ρ be the unique, bounded solution to*

$$\partial_\tau \rho - \kappa_\beta \Delta_x \rho = 0 \quad \text{in } \mathbb{T}^d \times \mathbb{R}^d, \quad \rho|_{\tau=0} = \rho^0. \quad (2.10)$$

Let g_α be the unique solution to (2.6) with initial data $g_\alpha|_{t=0} = M_\beta \rho^0$. Then for all $T > 0$ there is a constant $C_T > 0$ such that

$$\sup_{\tau \in [0, T]} \sup_{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d} \left| g_\alpha(\alpha\tau, x, v) - \rho(\tau, x) M_\beta(v) \right| \leq C_T \alpha^{-1/2}.$$

2.4. Statement of the main results

Our goal is to combine the Particle-to-Boltzmann limit described in Theorem 1 with the Boltzmann-to-Fluid limit described in Theorem 2, which means in particular that we are interested in the particular case when the limiting Boltzmann equation is no longer the nonlinear equation (2.5) but rather the linear equation (2.6). Actually what determines the form of the limit equation is the initial data to the system of ODEs satisfied by the system of particles, which in our hard-spheres setting is (2.1). In order to retrieve the linear equation in the limit, one should take for initial data a perturbation of the equilibrium density

$$M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N} \left(\frac{\beta}{2\pi} \right)^{\frac{dN}{2}} \exp \left(-\beta \sum_{i=1}^N |v_i|^2 \right) \mathbb{1}_{\mathcal{D}_\varepsilon^N}(X_N) = \frac{1}{\mathcal{Z}_N} \mathbb{1}_{\mathcal{D}_\varepsilon^N}(X_N) M_\beta^{\otimes N}(V_N)$$

where this perturbation acts only with respect to the position of a tagged particle (labeled 1 in the following). To this end, consider ρ^0 a continuous density of probability on \mathbb{T}^d and define

$$f_N^0(Z_N) := M_{N,\beta}(Z_N) \rho^0(x_1). \quad (2.11)$$

Note that the distribution f_N^0 is normalized by 1 in $L^1(\mathbb{T}^{dN} \times \mathbb{R}^{dN})$ thanks to the translation invariance of \mathbb{T}^d and to the fact that $\int_{\mathbb{T}^d} \rho^0(x) dx = 1$.

The main result of our study is the following statement.

Theorem 3 ([4]). *Consider the initial distribution f_N^0 defined in (2.11). Then the distribution $f_N^{(1)}$ of the tagged particle is close to $M_\beta(v) \varphi_\alpha(t, x, v)$, where $\varphi_\alpha(t, x, v)$ is the solution of the linear Boltzmann equation (2.6) with initial data $M_\beta(v) \rho^0(x)$. More precisely, for all $t > 0$ and all $\alpha > 1$, in the limit $N \rightarrow \infty$, $N\varepsilon^{d-1} \alpha^{-1} = 1$, one has*

$$\left\| f_N^{(1)}(t, x, v) - M_\beta(v) \varphi_\alpha(t, x, v) \right\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C \left[\frac{t\alpha}{(\log \log N)^{\frac{A-1}{A}}} \right]^{\frac{A^2}{A-1}},$$

where $A \geq 2$ can be taken arbitrarily large, and C depends on A, β, d and $\|\rho^0\|_{L^\infty}$.

In [2, 10], the linear Boltzmann equation was derived for any time $t > 0$ (independent of N). In comparison, our approach leads to quantitative estimates on the convergence up to times diverging when $N \rightarrow \infty$. This is the key to derive the diffusive limit described in the following result.

Theorem 4 ([4]). *Consider N hard spheres on the space $\mathbb{T}^d \times \mathbb{R}^d$, initially distributed according to f_N^0 defined in (2.11). Assume that ρ^0 belongs to $C^0(\mathbb{T}^d)$. Then the distribution $f_N^{(1)}(\alpha\tau, x, v)$ remains close for the L^∞ -norm to $\rho(\tau, x) M_\beta(v)$ where $\rho(\tau, x)$ is the solution of the linear heat equation (2.10) and the diffusion coefficient κ_β is given by (2.9). More precisely,*

$$\left\| f_N^{(1)}(\alpha\tau, x, v) - \rho(\tau, x) M_\beta(v) \right\|_{L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)} \rightarrow 0$$

in the limit $N \rightarrow \infty$, with $\alpha = N\varepsilon^{d-1}$ going to infinity much slower than $\sqrt{\log \log N}$.

Note that one can prove that in the same asymptotic regime, the process $\Xi(\tau) = x_1(\alpha\tau)$ associated with the tagged particle converges in law towards a Brownian motion of variance κ_β , initially distributed under the measure ρ^0 .

3. Elements of proofs

This section is devoted to the presentation of the proof of Theorem 3. We shall only give some elements of the proof and refer to [7] for all the details. We shall moreover not prove Theorem 4, as it is rather classical and simple, by making rigorous the formal asymptotics presented in Paragraph 2.3.

In order to prove Theorem 3, the strategy consists in going back to the proof of the general convergence result of particles to Boltzmann provided in Theorem 1 (see [7] for a complete proof), and in seeing where the specificity of the tagged particle framework comes in. As we shall see this comes from a global comparison principle with the invariant measure.

3.1. The BBGKY and Boltzmann hierarchies

As noticed in (2.4), the equation on the first marginal $f_N^{(1)}$ involves the second marginal $f_N^{(2)}$, so a control on $f_N^{(1)}$ requires a control on $f_N^{(2)}$. Similarly controlling $f_N^{(2)}$ implies controlling $f_N^{(3)}$, so finally we are led to writing the full hierarchy of equations on each marginal

$$f_N^{(s)}(t, Z_s) := \int f_N(t, Z_N) dz_{s+1} \dots dz_N, \quad 1 \leq s \leq N.$$

Note that in the case under study, namely when the particle labeled 1 is tagged and initially in a background of particles at equilibrium, the one-particle distribution $f_N^{(1)}$ is exactly the distribution of the tagged particle, and $f_N^{(s)}$ is the correlation between this tagged particle and $(s-1)$ particles of the background.

Let us write down the equation satisfied by $f_N^{(s)}$. A formal computation based again on Green's formula leads to the following BBGKY hierarchy for $s < N$

$$\left(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}\right) f_N^{(s)}(t, Z_s) = \alpha \left(C_{s,s+1} f_N^{(s+1)}\right)(t, Z_s)$$

on $\mathcal{D}_\varepsilon^s \times \mathbb{R}^{ds}$ with specular boundary condition. The collision term is defined by

$$\begin{aligned} & \left(C_{s,s+1} f_N^{(s+1)}\right)(Z_s) := (N-s) \varepsilon^{d-1} \alpha^{-1} \\ & \times \left(\sum_{i=1}^s \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v'_i, \dots, x_i + \varepsilon \omega, v'_{s+1}) \left((v_{s+1} - v_i) \cdot \omega \right)_+ d\omega dv_{s+1} \right. \\ & \left. - \sum_{i=1}^s \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon \omega, v_{s+1}) \left((v_{s+1} - v_i) \cdot \omega \right)_- d\omega dv_{s+1} \right). \end{aligned}$$

The starting point in Lanford's proof consists in writing the iterated Duhamel formula

$$\begin{aligned} f_N^{(s)}(t) = & \sum_{n=0}^{N-s} \alpha^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \\ & \dots \mathbf{S}_{s+n}(t_n) f_N^{(s+n)}(0) dt_n \dots dt_1, \end{aligned} \quad (3.1)$$

where \mathbf{S}_s denotes the group associated to free transport in $\mathcal{D}_\varepsilon^s \times \mathbb{R}^{ds}$ with specular reflection on the boundary. To simplify notations, we define the operators $Q_{s,s}(t) =$

$\mathbf{S}_s(t)$ and for $n \geq 1$

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \cdots \\ \cdots \mathbf{S}_{s+n}(t_n) dt_n \cdots dt_1$$

so that

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s} \alpha^n Q_{s,s+n}(t) f_N^{(s+n)}(0).$$

Lanford's idea is to interpret the term inside the integral in (3.1) as a "pseudo-trajectory" followed backwards in time, where at time t one starts with s particles following the s -particle hard spheres flow, and at each time step t_1, \dots, t_{n-1} a new particle is adjoined to the system (see Figure 3 for an example with $s = 1$).

To obtain the Boltzmann hierarchy we compute the formal limit of the collision operators when ε goes to 0: we define

$$\left(C_{s,s+1}^0 g^{(s+1)} \right) (Z_s) := \sum_{i=1}^s \int g^{(s+1)}(\dots, x_i, v'_i, \dots, x_i, v'_{s+1}) \left((v_{s+1} - v_i) \cdot \omega \right)_+ d\omega dv_{s+1} \\ - \sum_{i=1}^s \int g^{(s+1)}(\dots, x_i, v_i, \dots, x_i, v_{s+1}) \left((v_{s+1} - v_i) \cdot \omega \right)_- d\omega dv_{s+1}$$

and

$$Q_{s,s+n}^0(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbf{S}_s^0(t-t_1) C_{s,s+1}^0 \mathbf{S}_{s+1}^0(t_1-t_2) C_{s+1,s+2}^0 \cdots \mathbf{S}_{s+n}^0(t_n) dt_n \cdots dt_1$$

where \mathbf{S}_s^0 denotes the free flow of s particles on $\mathbb{T}^{ds} \times \mathbb{R}^{ds}$. Then the iterated Duhamel formula for the Boltzmann hierarchy takes the form

$$\forall s \geq 1, \quad g_\alpha^{(s)}(t) = \sum_{n \geq 0} \alpha^n Q_{s,s+n}^0(t) g^{(s+n)}(0).$$

3.2. A priori and continuity estimates

3.2.1. The initial data

For the initial data f_N^0 in (2.11), the marginal of order s is $f_N^{0(s)}(Z_s) = \rho^0(x_1) M_{N,\beta}^{(s)}(Z_s)$, where

$$M_{N,\beta}^{(s)}(Z_s) := \int M_{N,\beta}(Z_N) dz_{s+1} \cdots dz_N.$$

It is not difficult to see that there is a constant $C > 0$ such that as $N \rightarrow \infty$ in the scaling $N\varepsilon^{d-1} = \alpha \ll 1/\varepsilon$,

$$\left| \left(f_N^{0(s)} - g^{0(s)} \right) \mathbb{1}_{\mathcal{D}_\varepsilon^s} \right| \leq C^s \varepsilon \alpha M_\beta^{\otimes s} \|\rho^0\|_{L^\infty},$$

where $g^{0(s)}$ is defined by

$$g_0^{(s)}(Z_s) := \rho^0(x_1) M_\beta^{\otimes s}(V_s). \quad (3.2)$$

Then the family $(g_\alpha^{(s)})_{s \geq 1}$ defined by

$$g_\alpha^{(s)}(t, Z_s) := \varphi_\alpha(t, z_1) M_\beta^{\otimes s}(V_s) \quad (3.3)$$

is a solution to the Boltzmann hierarchy with initial data $g_0^{(s)}$, where $M_\beta(v_1) \varphi_\alpha(t, z_1)$ satisfies the linear Boltzmann equation (2.6) with initial data $M_\beta(v_1) \rho^0(x_1)$.

A uniqueness result on the hierarchy, which will follow from the a priori estimates below, therefore implies that it is enough to prove the convergence of the hierarchies to prove that the one-particle distribution converges to $M_\beta(v_1)\varphi_\alpha(t, z_1)$.

3.2.2. A global a priori estimate

It is not difficult to prove a global a priori estimate on the sequence of solutions $f_N^{(s)}$ thanks to the maximum principle. Indeed it is obvious that

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\rho^0(x_1) \leq M_{N,\beta}(Z_N)\|\rho^0\|_{L^\infty}.$$

Since the maximum principle holds for the Liouville equation (2.3), and as the Gibbs measure $M_{N,\beta}$ is a stationary solution, we get for all $t \geq 0$

$$f_N(t, Z_N) \leq M_{N,\beta}(Z_N)\|\rho^0\|_{L^\infty}.$$

By integration one obtains rather easily the following bound for any $s \geq 1$:

$$\sup_t f_N^{(s)}(t, Z_s) \leq M_{N,\beta}^{(s)}(Z_s)\|\rho^0\|_{L^\infty} \leq C^s M_\beta^{\otimes s}(V_s)\|\rho^0\|_{L^\infty}, \quad (3.4)$$

for some $C > 0$, provided that $\alpha\varepsilon \ll 1$.

3.2.3. Continuity estimates for the collision operator

For $\beta > 0$ and $k \in \mathbb{N}^*$, we define $X_{\varepsilon,k,\beta}$ the space of measurable functions f_k defined almost everywhere on $\mathcal{D}_\varepsilon^k$ such that

$$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k \times \mathbb{R}^{dk}} \left| f_k(Z_k) \exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right) \right| < \infty,$$

and similarly $X_{0,k,\beta}$ is the space of continuous functions g_k defined on $\mathbb{T}^{dk} \times \mathbb{R}^{dk}$ such that

$$\|g_k\|_{0,k,\beta} := \sup_{Z_k \in \mathbb{T}^{dk} \times \mathbb{R}^{dk}} \left| g_k(Z_k) \exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right) \right| < \infty.$$

Using the arguments of [7] one sees that

$$\left\| Q_{s,s+n}(t)f_{s+n} \right\|_{\varepsilon,s,\frac{\beta}{2}} \leq e^{s-1} \left(\frac{C_d t}{\beta^{\frac{d+1}{2}}} \right)^n \|f_{s+n}\|_{\varepsilon,s+n,\beta}$$

and for all g_{s+n} in $X_{0,s+n,\beta}$,

$$\left\| Q_{s,s+n}^0(t)g_{s+n} \right\|_{0,s,\frac{\beta}{2}} \leq e^{s-1} \left(\frac{C_d t}{\beta^{\frac{d+1}{2}}} \right)^n \|g_{s+n}\|_{0,s+n,\beta},$$

where C_d denotes generically a constant depending only on dimension. With these estimates and using a Cauchy-Kowalevskaya type argument, it is possible (see [7]) to prove the wellposedness of both hierarchies in this functional framework, on a time interval of the order of $1/\alpha$ (this is due to the fact that these estimates imply a linear loss in time in the parameter β , of the order of αt).

Notice that in the case of the tagged particle, one can compute thanks to (3.4) that

$$\begin{aligned} \|f_N^{(k)}(t)\|_{\varepsilon,k,\beta} &= \sup_{Z_k \in \mathcal{D}_\varepsilon^k \times \mathbb{R}^{dk}} \left| f_N^{(k)}(t, Z_k) \exp\left(\frac{\beta}{2}|V_k|^2\right) \right| \\ &\leq \sup_{Z_k \in \mathcal{D}_\varepsilon^k \times \mathbb{R}^{dk}} \left(M_{N,\beta}^{(k)}(Z_k) \exp\left(\frac{\beta}{2}|V_k|^2\right) \right) \|\rho^0\|_{L^\infty} \\ &\leq C^k \sup_{Z_k \in \mathcal{D}_\varepsilon^k \times \mathbb{R}^{dk}} \left(M_\beta^{\otimes k}(V_k) \exp\left(\frac{\beta}{2}|V_k|^2\right) \right) \|\rho^0\|_{L^\infty}. \end{aligned}$$

It follows that for all $t \in \mathbb{R}$,

$$\|f_N^{(k)}(t)\|_{\varepsilon,k,\beta} \leq C^k \left(\frac{\beta}{2\pi}\right)^{kd/2} \|\rho^0\|_{L^\infty}. \quad (3.5)$$

Similarly for the initial data for the Boltzmann hierarchy defined in (3.2), the solution (3.3) of the evolution is bounded by

$$\|g_\alpha^{(k)}(t)\|_{0,k,\beta} \leq \left(\frac{\beta}{2\pi}\right)^{kd/2} \|\rho^0\|_{L^\infty}.$$

This is in remarkable contrast with the case of general initial data, since one no longer is faced with the linear in time loss in the weight β mentioned above. This explains why the case of the tagged particle is much more favorable to hope for a long-time convergence.

3.3. Pruning of collision trees

Let us fix a (small) parameter $h > 0$ and a sequence $\{n_k\}_{k \geq 1}$ of integers which we choose of the type $n_k = A^k$ where A is a (large) constant to be fixed later. We also fix a large integer K and $t := Kh$, and split the interval $[0, t]$ into K intervals of size h . We define collision trees “of controlled size” by the condition that they have strictly less than n_k branch points on the interval $[t - kh, t - (k - 1)h]$.

More precisely we write

$$f_N^{(1)}(t) = f_N^{(1,K)}(t) + R_N^K(t),$$

where denoting $J_0 := 1$ and $J_k := 1 + j_1 + \dots + j_k$,

$$f_N^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \alpha^{J_{K-1}} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) f_N^{0(J_K)} \quad (3.6)$$

with

$$\begin{aligned} R_N^K(t) &:= \sum_{k=1}^K \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \alpha^{J_{k-1}-1} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}}(h) \\ &\quad R_{J_{k-1},n_k}(t - kh, t - (k - 1)h), \end{aligned}$$

and

$$R_{k,n}(t', t) := \sum_{p=n}^{N-k} \alpha^p Q_{k,k+p}(t - t') f_N^{(k+p)}(t').$$

We truncate in an identical way the Boltzmann hierarchy:

$$g_\alpha(t) = g_\alpha^{(1,K)}(t) + R_\alpha^{0,K}(t),$$

where with notation (3.2) and (3.3),

$$g_\alpha^{(1,K)}(t) := \sum_{j_1=0}^{n_1-1} \dots \sum_{j_K=0}^{n_K-1} \alpha^{J_K-1} Q_{1,J_1}^0(h) Q_{J_1,J_2}^0(h) \dots Q_{J_{K-1},J_K}^0(h) g_\alpha^{0(J_K)} \quad (3.7)$$

with

$$R_\alpha^{0,K}(t) := \sum_{k=1}^K \sum_{j_1=0}^{n_1-1} \dots \sum_{j_{k-1}=0}^{n_{k-1}-1} \alpha^{J_{k-1}-1} Q_{1,J_1}^0(h) \dots Q_{J_{k-2},J_{k-1}}^0(h) R_{J_{k-1},n_k}^0(t - kh, t - (k-1)h),$$

$$R_{k,n}^0(t', t) := \sum_{p \geq n} \alpha^p Q_{k,k+p}^0(t - t') g_\alpha^{(k+p)}(t').$$

The main result of this section, whose proof is omitted here (see [4]), states that the remainders $R_N^K(t)$ and $R_\alpha^{0,K}(t)$ are small.

Proposition 1. *Under the assumptions of Theorem 3, the following holds. Let $A \geq 2$ be given and define $n_k := A^k$, for $k \geq 1$. Then there exist $c, C, \gamma_0 > 0$ depending on d, A and β such that for any $t > 1$ and any $\gamma \leq \gamma_0$, choosing*

$$h \leq \frac{c\gamma}{\alpha^{A/(A-1)} t^{1/(A-1)}} \quad \text{and} \quad K = t/h \text{ integer}$$

we get

$$\|R_N^K(t)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} + \|R_\alpha^{0,K}(t)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C\gamma^A \|\rho^0\|_{L^\infty}.$$

Roughly speaking, that result is true thanks to the fact that the solution is controlled exactly by the same bound at each time step due to the maximum principle (3.5). It follows that the error made on each time step when cutting off super-exponential trees is of the type $(C\alpha h)^{n_k} (C\alpha t)^{n_{k-1}-1}$, which one can sum over k as soon as αh is small enough.

3.4. Termwise convergence and end of the proof

The last part of the proof consists in checking the termwise convergence of the (truncated) series $f_N^{(1,K)}(t)$ and $g_\alpha^{(1,K)}(t)$ defined in (3.6) and (3.7) respectively. This is the most technical part of the proof, and it relies in an essential way on the arguments developed in [7] for the proof of the short time convergence result for general initial data. Indeed it turns out that the main obstacle to convergence is the occurrence of recollisions in the trajectory between two times t_i and t_{i+1} , in the iterated Duhamel formula: recall that

$$Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) f_N^{0(J_K)}$$

$$= \int_{\mathcal{T}_J(h)} \mathbf{S}_1(t - t_1) C_{1,2} \mathbf{S}_2(t_1 - t_2) C_{2,3} \dots \mathbf{S}_{J_K}(t_{J_{K-1}}) f_N^{0(J_K)} dT$$

where the time integral is over the collision times $T = (t_1, \dots, t_{J_{K-1}})$ taking values in

$$\mathcal{T}_J(h) := \left\{ T = (t_1, \dots, t_{J_{K-1}}) \mid t_i < t_{i-1} \text{ and } (t_{J_k}, \dots, t_{J_{k-1}+1}) \in [t - kh, t - (k-1)h] \right\}.$$

The main issue is described in Figure 3, taken from [4]. It turns out recollisions can be avoided thanks to a geometrical analysis of trajectories, which shows essentially that the velocities of any two colliding particles may be chosen in such a way that after scattering, the particles in consideration will stay further away than some prescribed

distance from all the others (which are of finite number, of the order of A^k at time t_k) as soon as time has evolved somewhat, and provided velocities are not too large; it is actually not difficult to truncate high energies thanks to the weighted norms, nor is it difficult to remove very close collision times by Lebesgue's theorem. The error may actually be explicitly controlled as a function of ε . Optimizing on those error bounds and the error γ^A of Proposition 1 leads to the result. We refer to [4] for details.

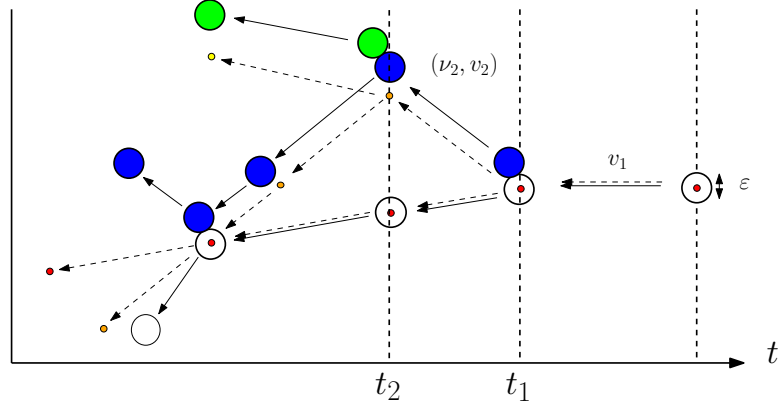


Figure 3.1: The BBGKY Duhamel formula is represented with plain arrows, whereas the Boltzmann one corresponds to the dashed arrows. At time t , the particle with label 1 in the BBGKY hierarchy is a ball of radius ε centered at position x_1 and the particle in the Boltzmann hierarchy is depicted as a point located at $x_1^0 = x_1$. At time t_1 the second particle is added and at time t_2 the third. Both hierarchies are coupled, but a small error in the particle positions of order ε can occur at each collision. In this figure, a recollision between the first and the second particle of the BBGKY pseudo-trajectories occurs and after this recollision the Boltzmann and the BBGKY hierarchies are no longer close to each other.

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