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Forges-les-Eaux, 7 juin–11 juin 2004

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F. Bethuel G. Orlandi D. Smets

Résumé

Nous étudions l'équation de Ginzburg-Landau parabolique sur l'espace tout entier, plus particulièrement lorsqu'une des échelles caractéristiques tend vers zéro. Notre seule hypothèse sur la donnée initiale est une borne naturelle sur l'énergie. En comparaison avec le cas des données préparées, notre hypothèse laisse place à de nouveaux phénomènes, en particulier la présence de différents modes pour l'énergie, dont nous étudions l'interaction. Le cas de la dimension 2 d'espace est qualitativement différent et requiert une analyse séparée.

Abstract

We discuss the asymptotics of the parabolic Ginzburg-Landau equation in dimension $N \geq 2$. Our only asumption on the initial datum is a natural energy bound. Compared to the case of "well-prepared" initial datum, this induces possible new energy modes which we analyze, and in particular their mutual interaction. The two dimensional case is qualitatively different and requires a separate treatment.

1. Introduction

The asymptotic analysis for Ginzburg-Landau evolution equations has been broadly investigated in the last decade. The purpose of these notes is to review some recent results obtained in collaboration with Fabrice Bethuel and Giandomenico Orlandi.

Our main focus will be the parabolic complex Ginzburg-Landau equation

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = -\dfrac{1}{\varepsilon^2}\nabla_u V(u_\varepsilon) & \text{on } \mathbb{R}^N \times (0,+\infty), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

for functions $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^d$, $N \geq 1$, $d \geq 1$, and V represents a non-convex smooth non-negative potential on $\mathbb{R}^d$. Here $\varepsilon > 0$ denotes a small parameter (a characteristic length), and we are specially interested in the asymptotic limit $\varepsilon \to 0$.

This equation corresponds to the heat-flow for the Ginzburg-Landau energy

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} e_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon^2} \qquad \text{for } u : \mathbb{R}^N \to \mathbb{R}^d.$$

The set

$$\Sigma = \{y \in \mathbb{R}^d\,,\ V(y) = 0\},$$

which we assume to be non-void, is sometimes called the vacuum manifold in the physical literature and plays an important role in the asymptotic analysis. Indeed, since the potential is non-negative, it achieves its infimum on Σ, and therefore the motion law forces u_ε to take values close to Σ for small ε as time evolves, and in appropriate energy regimes. This however cannot be true uniformly on space-time since the initial data u_ε^0 may not be uniformly close to Σ. We will call defects the points where u_ε is far from Σ. As time evolves these defects will disappear. An important aspect of our discussion will be to show that the defects related to the topology of Σ survive up to a time which is independent of ε, whereas the non-topological ones essentially have a life-span which shrinks with ε. For that reason the topology of Σ will enter directly in the discussion.

The energy $\mathcal{E}_\varepsilon$ has been introduced in the early fifties by Ginzburg and Landau in order to describe phase transitions in condensed matter Physics (more precisely, at low temperature). The nature of the predicted defects (e.g. points, lines, walls) depends crucially on d and Σ (see [23]). Among the many variants of Ginzburg-Landau functionals, there are in particular those including electromagnetic effects, as for instance in superconductivity. Related models have been developed in particle physics (as for examples, Yang-Mills-Higgs theory).

Here we will focus only on the case $N \geq 2$ and $d = 2$ (i.e. u complex-valued).[1] Moreover we assume that the potential is given by

$$V(u) = \frac{(1-|u|^2)^2}{4}.$$

Note that in this case $\Sigma = S^1$, where S^1 is the unit circle in $\mathbb{R}^2$.

With this choice of potential, $(\mathrm{PGL})_\varepsilon$ writes

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1-|u_\varepsilon|^2) & \text{on } \mathbb{R}^N \times (0,+\infty), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N\,. \end{cases}$$

It is well known that $(\mathrm{PGL})_\varepsilon$ is well-posed for initial datas in H^1_{loc} with finite Ginzburg-Landau energy $\mathcal{E}_\varepsilon(u_\varepsilon^0)$. Moreover, we have the energy identity

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot,T_2)) + \int_{T_1}^{T_2}\int_{\mathbb{R}^N} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 (x,t)\,dx\,dt = \mathcal{E}_\varepsilon(u_\varepsilon(\cdot,T_1)) \quad \forall\, 0 \leq T_1 \leq T_2\,. \tag{1}$$

We assume that the initial condition u_ε^0 verifies the bound

$$(H_0) \qquad \mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0 |\log \varepsilon|,$$

[1] The scalar case $d = 1$ is mostly understood, it is also often refered to as the Allen-Cahn equation, see [12, 24] and the references therein. Whereas the two theories bear some resemblance, important features discussed later are not present in the scalar case.

where M_0 is a fixed positive constant. The coefficient is related to the minimal energy cost needed for defects creation. Notice that, in view of (1), we have

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot,T)) \leq \mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0|\log\varepsilon| \qquad \text{for all } T \geq 0. \tag{2}$$

In order to analyze the asymptotic properties of solutions to $(\mathrm{PGL})_\varepsilon$ we may consider at least two kinds of objects.

The first ones describe the topological defects of u_ε: the jacobian Ju_ε, defined as the 2-form

$$Ju_\varepsilon = du_\varepsilon^1 \wedge du_\varepsilon^2 .$$

Although this may not be obvious at first glance, they are bounded in suitable norms independently of ε and therefore do not need any kind of renormalization. It can be shown that in the asymptotic limit $\varepsilon \to 0$ they concentrate (up to a subsequence) on a codimension 2 rectifiable set in $\mathbb{R}^N \times \mathbb{R}^+$, called the vorticity set. This fact is not related to the equation $(\mathrm{PGL})_\varepsilon$, but due only to the energy bound (2) and properties of the functional $\mathcal{E}_\varepsilon$. The limiting Jacobian J_* is a bounded vector measure on $\mathbb{R}^N \times \mathbb{R}^+$, as well as its restriction J_*^t on each time slice $\mathbb{R}^N \times \{t\}$. We will not study in details the structure of J_* here (see e.g. [1, 16]).

The second objects are the renormalized energy densities given by the Radon measures μ_ε, defined on $\mathbb{R}^N \times [0,+\infty)$,

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\, dx\, dt,$$

and of their time slices μ_ε^t, defined on $\mathbb{R}^N \times \{t\}$,

$$\mu_\varepsilon^t = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\, dx,$$

so that in particular $\mu_\varepsilon = \mu_\varepsilon^t\, dt$. In view of assumption (H_0) and (2), μ_ε is a bounded measure, independently of ε. We may therefore assume, up to a subsequence $\varepsilon_n \to 0$, that there exists a Radon measure μ_* defined on $\mathbb{R}^N \times [0,+\infty)$ such that

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{as measures.}$$

In view of the semi-decreasing property of the measures μ_ε^t (see [12, 7]), passing possibly to a further subsequence, we may also assume that

$$\mu_\varepsilon^t \rightharpoonup \mu_*^t \qquad \text{as measures on } \mathbb{R}^N \times \{t\}, \text{ for all } t \geq 0.$$

In the asymptotic limit $\varepsilon \to 0$, there is a simple relation between the quantities introduced so far, namely

$$||J_*|| \leq |\mu_*|\,, \qquad\qquad ||J_*^t|| \leq |\mu_*^t| \quad \text{for any } t > 0\,. \tag{3}$$

Moreover these bounds are sharp.

The evolution of μ_*^t is easier to analyze than that of J_*^t [2]. Indeed, it is possible to derive directly equations governing the motion of μ_*^t, using $(\mathrm{PGL})_\varepsilon$, whereas this is not clear for J_*^t. The structure of μ_*^t can be summarized as follows.

[2] For the case of the Schrödinger dynamics, it seems instead that the Jacobians are easier to deal with

Theorem 1 ([5]). *There exist a subset* Σ_μ *in* $\mathbb{R}^N \times \mathbb{R}_*^+$, *and a smooth real-valued function* Φ_* *defined on* $\mathbb{R}^N \times \mathbb{R}_*^+$ *such that the following properties hold.*

i) Σ_μ *is closed in* $\mathbb{R}^N \times \mathbb{R}_*^+$ *and for any compact subset* $\mathcal{K} \subset \mathbb{R}^N \times \mathbb{R}_*^+ \setminus \Sigma_\mu$

$$|u_{\varepsilon_n}(x,t)| \to 1 \qquad \text{uniformly on } \mathcal{K} \text{ as } n \to +\infty.$$

ii) For any $t > 0$, $\Sigma_\mu^t \equiv \Sigma_\mu \cap \mathbb{R}^N \times \{t\}$ *satisfies*

$$\mathcal{H}^{N-2}(\Sigma_\mu^t) \leq KM_0.$$

iii) The function Φ_* *satisfies the heat equation on* $\mathbb{R}^N \times \mathbb{R}_*^+$.

iv) For each $t > 0$, *the measure* μ_*^t *can be exactly decomposed as*

$$\mu_*^t = \frac{|\nabla \Phi_*|^2}{2}\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t, \tag{4}$$

where $\Theta_*(\cdot,t)$ *is a bounded function.*

v) There exists a positive function η *defined on* $\mathbb{R}_*^+$ *such that, for almost every* $t > 0$, *the set* Σ_μ^t *is (N-2)-rectifiable and*

$$\Theta_*(x,t) = \Theta_{N-2}(\mu_*^t, x) = \lim_{r \to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-2} r^{N-2}} \geq \eta(t),$$

for $\mathcal{H}^{N-2}$ *a.e.* $x \in \Sigma_\mu^t$.

Remark 1. Theorem 1 remains valid also for $N = 2$. In that case Σ_μ^t is therefore a finite set.

In view of the decomposition (4), μ_*^t can be split into two parts. A diffuse part $|\nabla \Phi_*|^2/2$, and a concentrated part

$$\nu_*^t = \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t.$$

By iii), the diffuse part is governed by the heat equation. Our next theorem focuses on the evolution of the concentrated part ν_*^t as time varies.

Theorem 2 ([5]). *The family* $(\nu_*^t)_{t>0}$ *is a mean curvature flow in the sense of Brakke [7].*

Comment. We recall that there exists a classical notion of mean curvature flow for smooth compact embedded manifolds. In this case, the motion corresponds basically to the gradient flow for the area functional. It is well known that such a flow exists for small times (and is unique), but develops singularities in finite time. Under the assumption that the initial measure is concentrated on a smooth manifold, a conclusion similar to ours has been obtained first on a formal level by Pismen and Rubinstein [20], and then rigorously by Jerrard and Soner [15] and Lin [18], in the time interval where the classical solution exists, that is only before the appearance of singularities. Asymptotic behavior (for convex bodies) and formation of singularities have been extensively studied in particular by Huisken (see [10, 11] and the references therein). Brakke [7] introduced a weak formulation which allows to encompass singularities and makes sense for (rectifiable) measures. Whereas it allows to handle a large class of objects, an important and essential flaw of Brakke's formulation is that there is never uniqueness. Even though non uniqueness is presumably an intrinsic property of mean curvature flow when singularities appear, a major part of non uniqueness in Brakke's formulation is non intrinsic, and therefore allows for weird solutions. A stronger notion of solution will be discussed in Theorem 3.

More precise definitions of the above concepts can be found in [5].

The proof of Theorem 2 relies both on the measure theoretic analysis of Ambrosio and Soner [2], and on the analysis of the structure of μ_*, in particular the statements in Theorem 1. In [2], Ambrosio and Soner proved the result in Theorem 2 under the additional assumption

$$(AS) \qquad \limsup_{r\to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-2}r^{N-2}} \geq \eta, \qquad \text{for } \mu_*^t\text{-a.e } x,$$

for some constant $\eta > 0$. In view of the decomposition (4), assumption (AS) holds if and only if $|\nabla\Phi_*|^2$ vanishes, i.e. there is no diffuse energy. If $|\nabla\Phi_*|^2$ vanishes, it follows therefore that Theorem 2 can be directly deduced from [2] Theorem 5.1 and statements *iv*) and *v*) in Theorem 1. In the general case where $|\nabla\Phi_*|^2$ does not vanish, their argument has to be adapted, however without major changes. Indeed, one of the important consequences of our analysis is that the concentrated and diffuse energies do not interfere in the original time scale.

We now come back to the already mentioned difficulty related to Brakke's weak formulation, namely the strong non-uniqueness. To overcome this difficulty, Ilmanen [13] introduced the stronger notion of enhanced motion, which applies to a slightly smaller class of objects, but has much better uniqueness properties (see [13]). In this direction we prove the following.

Theorem 3 ([5]). *Let $\mathcal{M}_0$ be any given integer multiplicity (N-2)-current without boundary, with bounded support and finite mass. There exists a sequence $(u_\varepsilon^0)_{\varepsilon>0}$ and an integer multiplicity (N-1)-current $\mathcal{M}$ in $\mathbb{R}^N \times \mathbb{R}^+$ such that*

$$i)\ \partial\mathcal{M} = \mathcal{M}_0\,, \qquad ii)\ \mu_*^0 = \pi|\mathcal{M}_0|\,,$$

and the pair $\left(\mathcal{M}, \frac{1}{\pi}\mu_*^t\right)$ is an enhanced motion in the sense of Ilmanen [13].

Remark 2. Our result is actually a little stronger than the statement of Theorem 3. Indeed, we prove that **any** sequence u_ε^0 satisfying $Ju_\varepsilon^0 \rightharpoonup J_*^0$ and $\mu_*^0 = |J_*^0|$ gives rise to an Ilmanen motion[3]

For an enhanced motion, if μ_*^0 is a smooth co-dimension 2 manifold without boundary and if the classical mean curvature flow for μ_*^0 exists and is smooth up to time T, then μ_*^t coincides with this flow for $t \leq T$.

[3] Recall that Ju_ε^0 denotes the Jacobian of u_ε^0.

We now restrict our attention to the case $N = 2$. In view of Theorem 1, we may write

$$\Sigma_\mu = \cup_{t>0} \cup_{i=1}^{l} b_i(t) \qquad in \quad \mathbb{R}^2 \times \mathbb{R}_*^+,$$

and for a.e. $t \geq 0$,

$$\mu_*^t = \frac{|\nabla \Phi_*|^2}{2}(.,t)dx + \nu_*^t, \qquad \text{where} \qquad \nu_*^t = \sum_{i=1}^{l} \sigma_i(t)\delta_{b_i(t)},$$

and

$$\text{either } \sigma_i(t) \geq \eta_0 \qquad \text{or} \qquad \sigma_i(t) = 0. \tag{5}$$

Theorem 4 ([6]). *The points $b_i(t)$ do not move, i.e.*

$$b_i(t) = b_i \qquad \forall\, t > 0, \tag{6}$$

and the functions $\sigma_i(t)$ are non-increasing.

This last statement is consistent with Theorem 2: indeed, points have essentially zero mean curvature.

In order to observe vortex motion in dimension $N = 2$, it is necessary to introduce an accelerated time scale. Evidence for the last assertion was first provided on a formal level in [19, 20, 9], and then rigorously in the case of "well-prepared" data in [17, 14, 25, 22]. In particular, such well-prepared data have well defined vortices of degree $+1$ or -1, and the diverging part of the energy is entirely provided by those vortices. In this framework, it is shown that in the accelerated time $t = |\log \varepsilon| s$ vortices evolve according to a simple ordinary differential equation up to the first collision time.

Our purpose here is to describe similarly the asymptotics in dimension $N = 2$ and in the accelerated time scale, relaxing completely the assumption on the well-preparedness, i.e. assuming only (H_0).

A typical initial datum for which we wish to understood the evolution is given by

$$u_\varepsilon^0(z) = \exp(i\varphi_\varepsilon^0(z)) \prod_{i=1}^{l} f(\frac{|z-a_i|}{\varepsilon}) \left(\frac{z-a_i}{|z-a_i|}\right)^{d_i} \quad \text{on } \mathbb{R}^2, \tag{7}$$

where f is a smooth non negative function on $\mathbb{R}^+$ such that $f(0) = 0$, $f \equiv 1$ outside of a compact set, $d_i \in \mathbb{Z}$ with $\sum_i d_i = 0$, and the phase φ_ε^0 verifies the bound

$$\|\nabla \varphi_\varepsilon^0\|_{L^2(\mathbb{R}^2)}^2 \leq C|\log \varepsilon|.$$

Our analysis shows that, in contrast with the higher dimensional case and with existing results on the two-dimensional case, the phase and the vortices[4] do actually interact in the accelerated time scale $t = |\log \varepsilon| s$. This phenomenon is related to persistence of low frequency oscillations in the phase, leading to an additional and somewhat unexpected drift term acting on vortices. This phenomenon would not be observed on a fixed bounded domain.[5]

[4] the linear and topological modes described above.

[5] One may wonder if it is physically relevant to work on the whole of $\mathbb{R}^2$. For the related Gorkov-Eliashberg equation for superconductivity, the physical domain has to be rescaled by a factor diverging with ε, which allows the same long-range interaction phenomenon.

The second point we wish to emphasize is that our analysis is **not** restricted by the occurrence of collisions. On the other hand, our results provide only a weak form of regularity for motion of vortices: in particular the motion of multiple degree vortices, with possible splittings and recombinations, remains a delicate open issue. A first step in this direction is provided by Theorem 7, where we describe the evolution of clusters of vortices of total degree zero. We show complete annihilation after a time proportional to the square of the confinement radius. In particular, vortices of degree zero are excluded except at a finite number of occurrences, which correspond to collisions. Even in the case of well-prepared data, this provides some new information, and also answers an open question raised by Jerrard and Soner ([14], Remark 2.2).

In the accelerated time, we set

$$\mathfrak{u}_\varepsilon(z,s) = u_\varepsilon(z, s|\log\varepsilon|).$$

Our first result establishes some compactness and rigidity for $\mathfrak{u}_\varepsilon$.

Theorem 5 ([6]). *There exist a function $\vec{c} : \mathbb{R}^+_* \to \mathbb{R}^2$, and for each $s>0$, a finite set $\{a_i(s)\}_{1\le i\le l(s)}$ of $\mathbb{R}^2$ and $l(s)$ integers $d_i(s) \in \mathbb{Z}$, such that, for a subsequence $\varepsilon_n \to 0$,*

$$\mathfrak{u}_{\varepsilon_n} \times \nabla \mathfrak{u}_{\varepsilon_n}(z,s) \to w_* \times \nabla w_*(z,s) + \vec{c}(s) \qquad \text{as } n \to +\infty, \tag{8}$$

and $|\mathfrak{u}_{\varepsilon_n}| \to 1$, uniformly on every compact set $K \subset \mathbb{R}^2 \times \mathbb{R}^+_* \setminus \Sigma_\mathfrak{v}$. Here, we have set

$$w_*(z,s) = \prod_{i=1}^{\ell(s)} \left(\frac{z - a_i(s)}{|z - a_i(s)|} \right)^{d_i(s)},$$

and

$$\Sigma_\mathfrak{v} = \cup_{s>0} \Sigma^s_\mathfrak{v} = \cup_{s>0} \cup_{i=1}^{l(s)} \{a_i(s)\}.$$

Moreover, there exist constants l_0, d_0 and c_0 depending only on M_0 such that for every $s > 0$,

$$l(s) \le l_0, \quad |d_i(s)| \le d_0, \quad \text{and} \quad |\vec{c}(s)| \le \tfrac{c_0}{\sqrt{s}}.$$

In the original time scale, there is no compactness for the functions due to possible wild oscillations in the phase. After times of the order of $|\log\varepsilon|$, these oscillations have been damped to order one.

In the special case of well-prepared data, similar results have been established, up to collision time, in [14, 17]: in their case, however, the additional term $\vec{c}$ is not observed. This new term is related to possible divergence of energy in the phase, and more precisely to (extremely) low frequency terms. Here is an explicit example of initial datum giving rise to a non-zero term $\vec{c}$: take u_0^ε as in (7) and

$$\varphi^0_\varepsilon(z) = \sqrt{|\log\varepsilon|} e^{-\frac{|z-a(\varepsilon)|^2}{4|\log\varepsilon|}},$$

where $a(\varepsilon) = \sqrt{|\log\varepsilon|}\vec{e}_1$. Using the explicit evolution of Gaussians by the heat equation, an elementary computation leads to the formula[6] $\vec{c}(s) = \frac{1}{2(1+s)^2}\exp(-\frac{1}{4(1+s)})\vec{e}_1$.

Clearly, the set $\Sigma_\mathfrak{v}$ in Theorem 5 contains the trajectory of vortices. Our next result provides some regularity properties for $\Sigma_\mathfrak{v}$.

[6]In order to keep this paper of reasonable size we will not work out the details here.

Theorem 6 ([6]). *The set* $\Sigma_{\mathfrak{v}}$ *is closed in* $\mathbb{R}^2 \times \mathbb{R}^+_*$ *and of locally finite two-dimensional parabolic Hausdorff measure. Moreover, there exists* $\alpha > 0$ *depending only on* M_0 *such that for each* $s > 0$,

$$\Sigma_{\mathfrak{v}} \subset \cup_{i=1}^{l(s)} \mathcal{P}(a_i(s), s), \tag{9}$$

where, for $(z, s) \in \mathbb{R}^2 \times \mathbb{R}^+_*$, $\mathcal{P}(z, s)$ *denotes the parabolic cone defined by*

$$\mathcal{P}(z, s) = \{(z', s') \in \mathbb{R}^2 \times \mathbb{R}^+ \ s.t. \ |s' - s| \geq \alpha |z' - z|^2\}.$$

In the case of well-prepared initial data, with $d_i = \pm 1$, it is known from [17, 14] that the points $a_i(s)$ evolve according to the motion law

$$\frac{d}{ds} a_i(s) = -2\nabla_{a_i}\Big(\sum_{j \neq i} d_i \log(a_i - a_j)\Big),$$

up to the first collision time. For initial data of the form (7) and with $d_i = \pm 1$ for all i, the motion law for the vortices would be given, similarly, by

$$\frac{d}{ds} a_i(s) = -2\nabla_{a_i}\Big(\sum_{j \neq i} d_i \log(a_i - a_j)\Big) + d_i \vec{c}(s)^{\perp}. \tag{10}$$

In particular, in this range, the set $\Sigma_{\mathfrak{v}}$ is a disjoint finite union of smooth curves. We therefore strongly believe that Theorem 6 is not optimal, and that in the general case $\Sigma_{\mathfrak{v}}$ is a finite union of smooth curves, with possible branching corresponding to collisions and splitting of vortices of multiple degree. As a consequence, such a set would be one-dimensional rectifiable, whereas we only obtained a bound on the two-dimensional parabolic Hausdorff measure. However, to improve Theorem 6 and go beyond the parabolic scaling, one will need some way to describe the evolution of the vortex cores.[7]

Our next theorem settles the question of annihilation.[8] The constant κ needs to be understood as a confinement factor.

Theorem 7 ([6]). *Let* $s_0 > 0$, $R > 0$ *and* $a \in \mathbb{R}^2$. *Assume that* $\sum_{a_i(s_0) \in B(a,R)} d_i(s_0) = 0$ *and that for some* $0 < \kappa < 1$

$$\Sigma_{\mathfrak{v}}^{s_0} \cap B(a, R) \subset B(a, \kappa R). \tag{11}$$

There exists positive constants κ_0, K_1 *and* K_2 *depending only on* M_0 *such that, if* $\kappa \leq \kappa_0$ *then*

$$\Sigma_{\mathfrak{v}}^{s_0} \cap B(a, \frac{R}{2}) = \emptyset,$$

for every $s \in [s_0 + K_1 \kappa^2 R^2, s_0 + K_2 R^2]$.

[7] This can be done in some specific cases, for instance we believe that our method would allow us to handle the case $|d_i| \leq 1$, but that the general case presumably does not have a simple answer. Indeed, splitting of multiple degree vortices involves discussions related to stable and unstable manifolds, and the resulting behavior is therefore very sensitive to the initial datum.

[8] Related results are announced for [23] based on different type of aguments.

Theorem 7 has several consequences, both of global and local nature. First, if at some time s_0 all vortices $a_i(s_0)$ are contained in a ball of radius R, and of total degree zero[9], then at a later time $s_0 + CR^2$ they have completely disappeared and w_* is constant. On a local level, combining Theorem 7 with further elements in the analysis, we complete the description of w_* and $\Sigma_{\mathfrak{v}}$ by the following

Theorem 8 ([6]). *The topological degrees $d_i(s)$ are non zero except for a finite number of times.*[10]

As previously mentioned, the above results allow to give an answer to Remark 2.2 in [14][11], concerning collision for a prepared datum with two vortices of degree $+1$ and -1, for instance

$$u_\varepsilon^0(z) = f(\frac{z-1}{\varepsilon})f(\frac{z+1}{\varepsilon})\frac{(z-1)}{|z-1|}\left(\frac{(z+1)}{|z+1|}\right)^{-1}.$$

In view of [14], it is known that the solution has two vortices a_i, $i = -1, 1$ given by $a_i(s) = (-1)^i\sqrt{1-2s}$. In particular, these two vortices will collide at time $S = \frac{1}{2}$. They disappear after this collision time, as a consequence of Theorem 7, and w_* is constant afterward.

Although they did not appear in the previous statements in dimension $N = 2$, the Radon measures $\mathfrak{v}_\varepsilon^s$ defined for $s \geq 0$ on $\mathbb{R}^2 \times \{s\}$ by

$$\mathfrak{v}_\varepsilon^s(x) = \frac{e_\varepsilon(\mathfrak{u}_\varepsilon(x,s))}{|\log \varepsilon|}\,dx$$

are central in the proofs, as their equivalents were in the higher dimensional case. The following important result establishes that their asymptotic limits do exist.

Theorem 9 ([6]). *Assume (H_0) and (H_1) hold. There exist a sequence $\varepsilon_n \to 0$ and, for each $s \geq 0$, a measure $\mathfrak{v}_*^s$ on $\mathbb{R}^2 \times \{s\}$ such that*

$$\mathfrak{v}_{\varepsilon_n}^s \rightharpoonup \mathfrak{v}_*^s \qquad \text{as } n \to \infty, \quad \text{for every } s \geq 0. \tag{12}$$

In view of assumption (H_0) and the energy inequality $\|\mathfrak{v}_\varepsilon^s\| \leq M_0$, $\forall s \geq 0$, for **fixed s** it is straightforward to find a sequence $\varepsilon_n \to 0$ such that $\mathfrak{v}_{\varepsilon_n}^s$ converges as $n \to +\infty$. The main difficulty in Theorem 9 is to find a sequence ε_n for which the convergence holds for **all** positive times. Clearly, convergence in (12) requires some specific property for the family $(\mathfrak{v}_\varepsilon^s)_{0<\varepsilon<1}$, which may be interpreted as a regularity in time. In the original time scale, the equivalent of the result described in Theorem 9 was stated above as a direct consequence of a semi-decreasing property. In contrast, in the accelerated time scale, the proof is much less direct, and is obtained at a late stage of our analysis.

Finally, our last result described here relates the points $a_i(s)$ with the measures $\mathfrak{v}_*^s$.

[9] This is not always the case under assumption (H_0). Take as initial datum u_ε^0 with a $+1$ vortex at the origin and a -1 vortex at a distance of order ε^{-1}. Then $l(s) = 1$ for all s, $a_1(s) = 0$, $d_1(s) = 1$ and $w_*(z) = z/|z|$.

[10] These events will be called in the sequel the extinction times.

[11] The method described allows to treat collisions of total degree zero. However collisions with total non zero degree are not excluded, and are not treated here.

Theorem 10 ([6]). *For every* $s > 0$,

$$\mathfrak{v}_*^s = \sum_{i=1}^{l(s)} \theta_i(s) \delta_{a_i(s)}$$

for some non negative densities $\theta_i(s)$ *satisfying, for a.e.* $s > 0$,

$$\textit{either} \qquad \theta_i(s) \geq \eta_0 \qquad \textit{or} \qquad \theta_i(s) = 0,$$

where $\eta_0 > 0$ *is a universal constant.*

In order to conclude, we would like to emphasize once more that our work has left aside the difficult question of the precise dynamics for $N = 2$ in the general setting considered here. As mentioned, this would require a further understanding of high multiplicity vortices, and in particular the mechanism of their splittings and possible recombinations. The case $d_i = \pm 1$ is much simpler, we intend to establish rigorously the motion law (10) in a different place. The general case is still a challenge to us.

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Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie,
4 place Jussieu BC 187, 75252 Paris,
France & Institut Universitaire de France.

bethuel@ann.jussieu.fr

Dipartimento di Informatica, Università di Verona, Strada le Grazie, 37134 Verona, Italy.
orlandi@sci.univr.it

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu BC 187, 75252 Paris, France.
smets@ann.jussieu.fr