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Partial Differential Equations / *Équations aux dérivées partielles*

Effective transmission conditions for second-order elliptic equations on networks in the limit of thin domains

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Abstract. We consider star-shaped tubular domains consisting of a number of non intersecting semi-infinite strips of small thickness that are connected by a central region of diameter proportional to the thickness of the strips. At the thin-domain limit, the region reduces to a network of half-lines with the same end point (junction). We show that the solutions of uniformly elliptic partial differential equations set on the domain with Neumann boundary conditions converge, in the thin-domain limit, to the unique solution of a second-order partial differential equation on the network satisfying an effective Kirchhoff-type transmission condition at the junction. The latter is found by solving an "ergodic"-type problem at infinity obtained after a first-order blow up at the junction.

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1. Introduction

The aim of the paper is to study the asymptotic behavior of solutions of uniformly elliptic partial differential equation (pde for short) set on thin tubular domains around a fixed network with one junction. In the thin-domain limit, we obtain a pde on the network coupled with a

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nonlinear effective transmission condition at the junction. In analogy to the classical electrostatic theory, we are referring to this coupling as a (nonlinear) Kirchhoff condition. The results are easily generalized to networks and thin domains with multiple junctions and other boundary conditions.

Since the limiting function is not smooth at the junction (the first derivatives are not continuous), the problem can be thought as a singular perturbation one, the transmission condition providing the necessary balance for the derivatives.

The Kirchhoff conditions can be identified easily when considering linear divergence form equations in thin domains by a simple integration by parts.

This argument, however, fails when dealing with nonlinear elliptic equations and a new approach is needed. Since the sought after condition involves derivatives of the limiting solution at the junction, it is natural to use a first-order blow up at the origin. This leads to a problem in an unscaled domain. The derivatives at the origin become linear growth conditions at infinity. The effective transmission condition then arises as a compatibility condition of the linear growths at infinity in order for the blown-up problem to have a solution. The analysis gives rise to a novel ergodic problem.

The thin domain limit for a class of convex first-order Hamilton–Jacobi equations was studied by Achdou and Tsou [1]. A related problem for a linear pde which is small stochastic anisotropic and possibly degenerate perturbation of Hamiltonian flows has been by studied Ishii and Souganidis [5], where we refer to other previous works.

Organization of the paper

The paper is organized as follows. In the next section, we describe the general setting and assumptions, introduce the problem and state the results. Section 3 is devoted to the proof of the ergodic problem that yields the condition at the junction. In Section 4 we prove the convergence result.

2. The setting, assumptions and results

The setting

We consider a star-shaped network consisting of $k > 1$ straight edges

$$\mathscr{G}_i = \mathbb{R}_+\zeta_i,$$

where, for $i = 1,\dots,k$, ζ_i is a unit vector in $\mathbb{R}^d$, $\zeta_i \neq \zeta_j$ if $i \neq j$ and $\mathbb{R}_+ = (0,\infty)$. To fixe the ideas, each $\mathscr{G}_i$ should be thought as the positive half line in the ζ_i direction.

We denote by $\zeta_i^\perp$ the R^{d-1}-plane which is orthogonal to ζ_i. A point $x \in \mathbb{R}^d \setminus \{0\}$ is written as

$$x = x_i\zeta_i + x_i' \quad \text{with} \quad x_i \in \mathbb{R}_+ \quad \text{and} \quad x_i' \in \zeta_i^\perp.$$

The network $\mathscr{G}$ and its interior $\mathscr{G}^0$ are respectively

$$\mathscr{G} = \{0\} \cup \bigcup_{i=1}^k \mathscr{G}_i \quad \text{and} \quad \mathscr{G}^0 = \bigcup_{i=1}^k \mathscr{G}_i.$$

Moreover, for $\rho > 0$,

$W_\rho \subset \mathbb{R}^d$ is the open neighborhood of 0 given by

$$W_\rho = \left\{x \in \mathbb{R}^d : x\cdot\zeta_i < \rho \ \text{ for all } \ i = 1,\dots,k\right\}.$$

Finally, for each $i = 1,\dots,k$, ω_i is the $\mathbb{R}^{d-1}$–unit ball in $\zeta_i^\perp$ centered at the origin.

Let Σ be an open connected subset of $\mathbb{R}^d$ with smooth boundary such that $0 \in \Sigma$ and

$$\Sigma \setminus \overline{W}_1 = \bigcup_{i=1}^{k} Z_i,$$

where, for each $i = 1, \dots, k$, Z_i is the open half strip (tube) around $\mathscr{G}_i$ given by

$$Z_i = \left\{ z = z_i \zeta_i + z_i' : z_i > 1 \text{ and } z_i' \in \omega_i \right\};$$

note that the choice of W_1 instead of W_{r_0}, for some $r_0 > 0$, is made only to simplify the presentation.

The set

$$\widetilde{K}_0 = \Sigma \cap W_1$$

can be thought as small dilation of the junction zone around the origin.

For each $\epsilon > 0$, the thin domain Σ_ϵ around $\mathscr{G}$ is

$$\Sigma_\epsilon = \epsilon \Sigma,$$

and, for each $i = 1, \dots, k$,

$$Z_{i,\epsilon} = \epsilon Z_i = \left\{ z = z_i \zeta_i + z_i' : z_i > \epsilon \text{ and } z_i' \in \epsilon \omega_i \right\}.$$

To set up the problem, we will see partition of unity $(\eta_i)_{i=1,\dots,k}$ subordinate to $\mathscr{W}_1$ and $Z_1, \dots, Z_k$, that is, for each $i = 1, \dots, k$, $\eta_i \in C_c^\infty(\overline{\Sigma}; [0,\infty))$ are such that

$$\sum_{i=1}^{k} \eta_i = 1 \quad \text{and} \quad \eta_i = 1 \text{ in } \overline{Z}_i. \tag{1}$$

The assumptions

For $i = 1, \dots, k$, we consider the maps $F_i \in C(\mathscr{S}^d, \overline{\Sigma}; \mathbb{R})$ and $\overline{F}_i \in C(\mathscr{S}^d, \overline{\Sigma}; \mathbb{R})$, where $\mathscr{S}^d$ is the space of $d \times d$ symmetric matrices in $\mathbb{R}^d$, and we assume that

$$F_i : \mathscr{S}^d \times \overline{\Sigma} \to \mathbb{R} \quad \text{is uniformly elliptic and Lipschitz continuous,} \tag{2}$$

and

$$\begin{cases} \overline{F}_i \text{ is uniformly elliptic, Lipschitz continuous, and} \\ 1\text{–positively homogeneous in the first argument,} \end{cases} \tag{3}$$

and there exists $C > 0$ such that, uniformly in $(A, x) \in \mathscr{S}^d \times \overline{\Sigma}$,

$$|F_i(A, x) - \overline{F}_i(A, x)| \le C. \tag{4}$$

Note that, since the homogeneity of $\overline{F}_i$ implies that $\overline{F}_i(0, \cdot) = 0$, it follows from (4) that

$$|F_i(0, \cdot)| \le C. \tag{5}$$

An example of F_i and $\overline{F}_i$ are the classical Isaacs nonlinearities given by

$$F_i(A, x) = \inf_\alpha \sup_\beta \left[-\operatorname{tr}\left[a_i^{\alpha,\beta}(x) A \right] - f_i^{\alpha,\beta}(x) \right],$$

and

$$\overline{F}_i(A, x) = \inf_\alpha \sup_\beta \left[-\operatorname{tr}\left[a_i^{\alpha,\beta}(x) A \right] \right],$$

with $a_i^{\alpha,\beta} \in \mathscr{S}^d$ and $f_i^{\alpha,\beta} \in \mathbb{R}$ satisfy the necessary conditions needed for (2), (3) and (4) to hold.

Recall that any uniformly elliptic and Lipschitz continuous function has a max-min representation. In order, however, to have (4) we need to assume that

$$\sup_\alpha \sup_\beta \left| f_i^{\alpha,\beta}(x) \right| \le C.$$

The problem

We are interested in the behavior, as $\epsilon \to 0$, of the solution u_ϵ of the boundary value problem

$$F\left(D^2 u_\epsilon, x, \frac{x}{\epsilon}\right) + u_\epsilon = 0 \text{ in } \Sigma_\epsilon \qquad \frac{\partial u_\epsilon}{\partial n} = 0 \text{ on } \partial\Sigma_\epsilon, \tag{6}$$

with $F : \mathscr{S}^d \times \overline{\Sigma} \times \overline{\Sigma} \to \mathbb{R}$ defined by

$$F(A, x, y) = \textstyle\sum_{i=1}^k \eta_i(y) F_i(A, x). \tag{7}$$

It is immediate from (1) and (2) and with C as in (5) that

$$F \text{ is uniformly elliptic and Lipschitz continuous and } |F(0,\cdot)| \leq C. \tag{8}$$

For future use, we also introduce $\overline{F} : \mathscr{S}^d \times \overline{\Sigma} \times \overline{\Sigma} \to \mathbb{R}$ defined by

$$\overline{F}(A, x, y) = \sum_{i=1}^k \eta(y) \overline{F}_i(A, x). \tag{9}$$

It is again easy to see from (1), (2), (3) and (4) that

$$\begin{cases} \overline{F} \text{ is uniformly elliptic, Lipschitz continuous, and 1-positively homogeneous} \\ \text{in the first argument, and} \\ \qquad |F - \overline{F}| \leq C \quad \text{on} \quad \mathscr{S}^d \times \overline{\Sigma} \times \overline{\Sigma}. \end{cases} \tag{10}$$

We remark that the nonlinearity F in (4) is only an example. The general picture the reader should keep in mind is that we have elliptic equations on each of the $\mathscr{G}_i$ which are extended to Z_i and are patched (this is the role of the partition of unity) smoothly in a neighborhood of 0 to give an F on Σ. The dependence on x/ϵ is necessary in order to deal with the fact that near $\mathscr{O}$ all the variables interact.

There are, of course, other ways than (7) to do this. Indeed, we may consider F satisfying (8) and admitting a "uniform recession" function, that is, a uniformly elliptic $\overline{F} \in C^{0,1}(\mathscr{S}^d \times \overline{\Sigma} \times \overline{\Sigma})$ such that, for some $C > 0$ and $\epsilon > 0$ small,

$$\left| \epsilon F\left(\frac{1}{\epsilon} A, \epsilon x, x\right) - \overline{F}(A, 0, x) \right| \leq C\epsilon. \tag{11}$$

Before we continue with the statement of the results, we mention that it is possible to study more general boundary value problems with equations like

$$F\left(D^2 u_\epsilon, Du_\epsilon, u_\epsilon, \frac{x}{\epsilon}, x\right) = 0 \text{ in } \Sigma_\epsilon$$

and different boundary conditions (oblique, state constraints, etc.).

To explain the key ideas and keep the exposition simple, in this note we study the simplest possible case, that is, (6).

The results

It is well known that, given (8) and for $\epsilon > 0$, (6) has a unique (viscosity) solution $u_\epsilon \in C^{0,1}(\Sigma_\epsilon)$ with bounds independent of ϵ. Thus, along subsequences $\epsilon \to 0$, the u_ϵ's converge locally uniformly in $\overline{\Sigma}_\epsilon$ to $\overline{u} \in C^{0,1}(\mathscr{G})$. The aim is to characterize $\overline{u}$ as the unique solution of a pde on the network. The main issue is to understand the equation at the junction.

To state the main result, we introduce some additional notation. For each $i = 1, \ldots, k$, let $\mathscr{F}_i : \mathbb{R} \times \overline{\mathscr{G}}_i \to \mathbb{R}$ be defined by

$$\mathscr{F}_i(z, x_i) = F_i(z\zeta_i \otimes \zeta_i, x_i\zeta_i). \tag{12}$$

It is immediate from (2) that, for each $i = 1,\dots,k$,

$$\begin{cases} \mathscr{F}_i \in C(\mathbb{R}\times\overline{\mathscr{G}}_i) & \text{is uniformly elliptic, Lipschitz continuous and} \\ \text{bounded in } [-R,R]\times[0,\infty) \text{ for each } R>0. \end{cases} \tag{13}$$

In what follows, we say that a Lipschitz continuous map $G:\mathbb{R}^k \to \mathbb{R}$ is strictly decreasing, if there exists $\nu > 0$ such that, for all $i = 1,\dots,k$,

$$\nu \le -G_{z_i}. \tag{14}$$

The convergence result is stated next.

Theorem 1. *Assume* (1), (2), (3), (4) *and* (7) *and let* $\overline{F} \in C(\mathscr{S}^d \times \overline{\Sigma} \times \overline{\Sigma})$ *be as in* (9). *There exists a Lipschitz continuous and strictly decreasing with respect to each of its argument* $G:\mathbb{R}^k \to \mathbb{R}$, *which depends on* $(F_i)_{i=1,\dots,k}$, $(\eta_i)_{i=1,\dots,k}$ *and* Σ, *such that, if* $\overline{u} \in \bigcap_{i=1}^k C^{2,1}(\overline{\mathscr{G}_i}) \cap C^{0,1}(\mathscr{G})$ *is the unique solution of*

$$\mathscr{F}_i(\overline{u}_{x_ix_i}, x_i) + \overline{u} = 0 \ \text{ in } \mathscr{G}_i \ \text{ for each } i = 1,\dots,k \quad \text{and} \quad G(u_{x_1}(0^+),\dots,u_{x_k}(0^+)) = 0, \tag{15}$$

then, as $\epsilon \to 0$ *and locally uniformly,* $u_\epsilon \to \overline{u}$.

There are two novelties in Theorem 1. The first, which is more conceptual, is the identification of the nonlinear coupling G. This is the topic of Theorem 2 and is motivated below. The second, which is technical, is the proof of the convergence, which involves the construction of unusual and new super-and sub-solutions of (6), which are motivated by a formal blow up argument explained next. The fact that limits of the u_ϵ's satisfy the claimed equations in each $\mathscr{G}_i$ is a routine consequence of the stability of solutions. The wellposedness of (15) follows from recent work of the authors [6].

To explain the appearance of G, we begin with the simple linear example

$$-\Delta u_\epsilon + u_\epsilon = 0 \ \text{ in } \ \Sigma_\epsilon \qquad \frac{\partial u_\epsilon}{\partial n} = 0 \ \text{ on } \ \partial\Sigma_\epsilon, \tag{16}$$

ignoring the fact that in this trivial setting it is clear that $u_\epsilon \equiv 0$.

Fix $R > 0$ and consider the truncated sets

$$\Sigma_\epsilon^R = \left\{x \in \Sigma_\epsilon : x_i < R \ \text{ for all i=1,}\dots\text{, k}\right\} \quad \text{and} \quad \Gamma_{\epsilon,i}^R = \{x \in Z_{\epsilon,i} : x_i = R\}. \tag{17}$$

Integrating (16) by parts over Σ_ϵ^R and using the boundary condition we find

$$\sum_{i=1}^K \int_{\Gamma_{\epsilon,i}^R} \frac{\partial u_\epsilon}{\partial n}\,\mathrm{d}S(y) + \int_{\Sigma_\epsilon^R} u_\epsilon\,\mathrm{d}x = 0. \tag{18}$$

Assume next, that, as $\epsilon \to 0$, $u_\epsilon \to \overline{u}$ uniformly in $\overline{\Sigma_\epsilon^R}$, $Du_\epsilon \to D\overline{u}$ locally uniformly in $\overline{\Sigma_\epsilon^R} \setminus \{0\}$, and, for simplicity, that, for $i = 1,\dots,k$, there exist $a_1,\dots,a_k \in (0,\infty)$ such that

$$|\Gamma_{\epsilon,i}^R| = a_i |\Gamma_{\epsilon,1}^R|.$$

Then, dividing (18) by $|\Gamma_{\epsilon,1}^R|$, and letting $\epsilon, R \to 0$ leads to the classical Kirchhoff transmission condition

$$\sum_{i=1}^K \overline{a}_i u_{x_i}(0^+) = 0. \tag{19}$$

Integration by parts is, of course, useless in the nonlinear setting of Theorem 1. To explain how G arises, we argue again formally, assuming that, as $\epsilon \to 0$, $u_\epsilon \to \overline{u}$ locally uniformly in Σ_ϵ. Since we are interested in the behavior of the first derivatives of the limit at the origin, it is rather natural that to assert that, near 0,

$$u_\epsilon(x) \approx \overline{u}(0) + \epsilon v\left(\frac{x}{\epsilon}\right), \tag{20}$$

where a $v : \overline{\Sigma} \to \mathbb{R}$ is a "corrector"-type function that must behave at infinity like the derivatives of $\bar{u}$ at the origin. Note that although we borrow terms from the theory homogenization like corrector and later ergodic problem, the ansatz is really about blowing up at the origin.

For each $i = 1, \dots, k$, let

$$p_i = \bar{u}_{x_i}(0^+).$$

It follows that v must satisfy, uniformly for $x_i^\perp$ with $|x_i^\perp| \le 1$,

$$\lim_{x_i \to \infty} \frac{v(x_i, x_i^\perp)}{x_i} = p_i. \tag{21}$$

Inserting (20) in (6) and using the definition of F in $\widetilde{K}_0$, we find

$$F\left(\frac{1}{\epsilon} D^2 v\left(\frac{x}{\epsilon}\right), x, \frac{x}{\epsilon}\right) + \bar{u}(0) + \epsilon v\left(\frac{x}{\epsilon}\right) \approx 0.$$

Hence, on Σ, we must have

$$\epsilon F\left(\frac{1}{\epsilon} D^2 v(y), \epsilon y, y\right) \approx 0,$$

Then, (10) yields that v must satisfy

$$\overline{F}(D^2 v, 0, y) = 0 \text{ in } \Sigma_1 \quad \text{and} \quad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Sigma_1. \tag{22}$$

The issue, of course, is whether (22) admits a solution v satisfying (21) for each $i = 1, \dots, k$.

It turns out that the existence of such v requires a compatibility condition on the p_i's, that is, given $p_1, \dots, p_{k-1}$, there exists a unique p_k such that (22) has a solution v satisfying (21). The relationship among the p_i' cam be thought as an ergodic condition at infinity.

The result is formulated in the following theorem. To avoid confusion with the nonlinearities F and $\overline{F}$ earlier, we rewrite the problem as

$$\begin{cases} \widehat{F}(D^2 \hat{v}, x) = 0 \text{ in } \Sigma, \quad \frac{\partial \hat{v}}{\partial n} = 0 \text{ on } \partial\Sigma, \\ \lim_{x_i \to \infty} \frac{\hat{v}(x_i, x_i')}{x_i} = p_i \text{ uniformly in } x_i' \in \overline{\omega}_i \text{ and } i = 1, \dots, k. \end{cases} \tag{23}$$

Theorem 2. *Assume that $\widehat{F} \in C^{0,1}(\mathscr{S}^d \times \Sigma)$ is uniformly elliptic, Lipschitz continuous and $\widehat{F}(0, \cdot) = 0$ on $\overline{\Sigma}$. There exists a unique up to a multiplicative constant, Lipschitz continuous, and strictly decreasing with respect to each argument $G : \mathbb{R}^k \to \mathbb{R}$ such that (23) has a unique up to constants solution $v \in C^1(\Sigma) \cap C^{0,1}(\overline{\Sigma})$ if and only if $G(p_1, \dots, p_k) = 0$.*

The proof of Theorem 2 is based on solving an ergodic-type problem with Neumann conditions in a truncated domain followed by a delicate analysis of what happens as the truncation is removed. This is the place where the compatibility condition arises.

We present next two simple examples to give a flavor of what is behind Theorem 2 and to emphasize the way G depends on $\widehat{F}$ and Σ_1.

We begin with a two-dimensional problem with $\zeta_1 = (1, 0)$ and $\zeta_2 = (-1, 0)$ and Σ the tube around the x-axis with cross section radius 1. We write x for the horizontal coordinate and y for the vertical. Finally we denote by $p_\pm$ the slopes of v at $\pm\infty$. Note that, in order to write a single equation, in the tube Z_2 we have changed from $x > 0$ to $x < 0$. This is reflected in the change of sign in the $x \to -\infty$ limit.

We are interested in a solution $\hat{v} \in C^1(\Sigma) \cap C^{0,1}(\overline{\Sigma})$ of

$$\widehat{F}(D^2 \hat{v}, x) = 0 \text{ in } \Sigma, \quad \frac{\partial \hat{v}}{\partial n} = 0 \text{ on } \partial\Sigma, \quad \lim_{x \to \pm\infty} \frac{\hat{v}(x, y)}{x} = \pm p_\pm \text{ uniformly in } y. \tag{24}$$

We assume that $\widehat{F}$ is independent of y and uniformly elliptic, Lipschitz continuous, and 1−positively homogeneous, hence $\widehat{F}(0, x) = 0$.

It follows that $\widehat{v} = \widehat{v}(x, y) = \widehat{v}(x)$ and the equation in (24) is now

$$\widehat{F}\left(\begin{pmatrix} \widehat{v}_{xx} & 0 \\ 0 & 0 \end{pmatrix}, x\right) = 0.$$

Since, in view of the uniform ellipticity, $\widehat{F}$ is strictly decreasing in $\widehat{v}_{xx}$ and $\widehat{F}(0, x) = 0$, it follows that we must have $\widehat{v}_{xx} = 0$. Hence, $\widehat{v}$ must be linear and the slopes must satisfy the Kirchhoff condition becomes $-p_+ - p_- = 0$.

The next example shows how G may depend on the domain. We work again with $d = 2$, and, consider, for $\lambda \in [0,\infty) \setminus \{1\}$, the boundary value problem

$$-v_{xx} - \lambda v_{yy} = 0 \text{ in } \mathbb{R} \times (-1, 1) \ u_y(\cdot, \pm 1) = 0, \quad \lim_{x\to\pm\infty} \frac{u(x, y)}{x} = \pm p_\pm \text{ uniformly in } y \in [-1, 1].$$

An argument similar to the one above yields again the Kirchhoff condition $p_+ - p_- = 0$.

We choose next $\zeta_1 = (0, 1)$ and $\zeta_2 = (1, 0)$. The resulting domain Σ is a deformation of the one in the previous example. We consider the same equation as above. Similar considerations lead to the transmission condition $p_1 - \lambda p_2 = 0$.

3. The proof of Theorem 2

Like with many ergodic-type problems, the proof consists of three main steps. The first is to identify and solve an approximate problem. The second is to obtain bounds, which are independent of the approximation, that allow for the passage in the limit to obtain the "ergodic constant". The last is to show the uniqueness of the latter.

In preparation for the proof, we recall that

$$\widehat{F} : \mathscr{S}^d \times \overline{\Sigma} \to \mathbb{R} \quad \text{is uniformly elliptic, Lipschitz continuous, and } \widehat{F}(0, \cdot) = 0 \text{ on } \overline{\Sigma}. \tag{25}$$

and we set up some more notation.

For $R > 1$ and each $i = 1, \ldots, k$, Z_i^R is the truncated tube $Z_i^R = \{x \in Z_i : x_i < R\}$, its outer boundary $\Gamma_i^R = \{x \in Z_i : x_i = R\}$, and the truncated domain

$$\Sigma^R = \{x \in \Sigma : x_i < R \text{ for all } i = 1, \ldots, k\}.$$

We consider the approximate problem

$$\widehat{F}(D^2 v_R, y) = 0 \text{ in } \Sigma^R, \qquad \frac{\partial v_R}{\partial n} = 0 \text{ on } \partial\Sigma \cap \overline{\Sigma^R}, \qquad \frac{\partial v_R}{\partial n} = p_i \text{ on } \Gamma_i^R \quad \text{for } i = 1, \ldots, k. \tag{26}$$

Before we begin with the analysis of (26), we state as separate lemmata two observations which will be used in several places in this section. The proof of the first is straightforward consequence of the maximum principle and Hopf's lemma, hence we omit it.

Lemma 3. *Assume $R > 1$ and (25) and let v_R be a solution of (26). Then,*

$$\inf_{\overline{\Sigma}^R} v_R = \min_{1\le i\le K} \inf_{\Gamma_i^R} v_R \quad \text{and} \quad \sup_{\overline{\Sigma}^R} v_R = \min_{1\le i\le K} \sup_{\Gamma_i^R} v_R \tag{27}$$

and

$$[1, R] \ni \rho \to \inf_{\overline{\Sigma}^\rho} v_R \text{ is nonincreasing} \quad \text{and} \quad [1, R] \ni \rho \to \sup_{\overline{\Sigma}^\rho} v_R \text{ is nondecreasing.} \tag{28}$$

Lemma 4. *Assume $R > 1$ and (25), let v_R be a solution of (26) and fix $i \in \{1, \ldots, k\}$. Then, for every $z, \widehat{z} \in Z_i^R$ such that $z_i < \widehat{z}_i$,*

$$\inf_{w\in\zeta_i^\perp} v_R(z_i, w) + p_i(\widehat{z}_i - z_i) \le v_R(\widehat{z}) \le \sup_{w\in\zeta_i^\perp} v_R(z_i, w) + p_i(\widehat{z}_i - z_i). \tag{29}$$

Proof. Since the arguments are similar, we only show the lower bound.

The properties of $\widehat{F}$ imply that $v(\widehat{z}) = \inf_{\Gamma_i^{z,\zeta_i}} v_R + p_i(\widehat{z}_i - z_i)$ solves

$$\widehat{F}(D^2 v, \widehat{z}) = 0 \text{ in } Z_i^R \setminus \overline{Z_i^{z_i}}.$$

Moreover, it is immediate that

$$\frac{\partial v}{\partial n} = 0 \text{ on } \partial Z_i^R \cap \partial\Sigma, \quad \frac{\partial v}{\partial n} = -p_i \text{ on } \Gamma_i^R \quad \text{and} \quad v_R \geq v \text{ on } \Gamma_i^{z_i}.$$

The comparison principle then yields the lower bound in (29) for all $\widehat{z} \in \overline{Z}_i^R \setminus Z_i^{z_i}$. □

The next result, which establishes the existence of solutions of (26) provided the p_i's satisfy a condition should be thought as an "ergodic"-type problem in Σ^R.

Theorem 5. *Fix $R > 1$ and assume* (25). *Then, for any $p_1, \dots, p_{k-1} \in \mathbb{R}$, there exists a unique $p_k = p_k^R$, which is a strictly increasing with respect to each p_i with $i \in \{1, \dots, k-1\}$, such that* (26) *has a unique up to additive constants solution $v_R \in C^{0,1}(\overline{\Sigma}^R)$.*

Proof. Fix $p_1, \dots, p_k \in \mathbb{R}$, and, for $\delta > 0$, consider the approximate problem

$$\begin{cases} \delta v_{R,\delta} + \widehat{F}(D^2 v_{R,\delta}, y) = 0 \text{ in } \Sigma^R, \\ \frac{\partial v_{R,\delta}}{\partial n} = 0 \text{ on } \partial\Sigma \cap \overline{\Sigma^R} \quad \text{and} \quad \frac{\partial v_{R,\delta}}{\partial n} = -p_i \text{ on } \Gamma_i^R \text{ for } i = 1, \dots, k, \end{cases} \tag{30}$$

which, in view of the classical viscosity theory (see, for example, Crandall, Ishii and Lions [4]) has a unique solution $v_{R,\delta}$. It follows from (25) (see, for example, Barles, da Lio, Lions and Souganidis [2]) that the function $\widehat{v}_{R,\delta} = v_{R,\delta} - v_{R,\delta}(0)$ and the constant$-\delta v_{R,\delta}(0)$ converge, as $\delta \to 0$, to a unique $\widehat{v}_R \in C^{0,1}(\overline{\Sigma}^R)$ and $\widehat{c}_R = \widehat{c}_R(p_1, \dots, p_k) \in \mathbb{R}$ satisfying the boundary value problem

$$\begin{cases} \widehat{F}(D^2 \widehat{v}_R, y) = \widehat{c}_R \text{ in } \Sigma^R, \\ \widehat{v}_R(0) = 0, \quad \frac{\partial \widehat{v}_R}{\partial n} = 0 \text{ on } \partial\Sigma \cap \overline{\Sigma^R} \quad \text{and} \quad \frac{\partial \widehat{v}_R}{\partial n} = -p_i \text{ on } \Gamma_i^R \text{ for } i = 1, \dots, k. \end{cases} \tag{31}$$

It follows from the strong maximum (see, for example, Trudinger [7]) that

(i) $\widehat{c}_R$ is strictly decreasing with respect to its arguments, and
(ii) if $p_1, \dots, p_{k-1}$ are fixed and p_k is large (resp. small), then $\widehat{c}_R(p_1, \dots, p_k)$ is negative (resp. positive).

Since $\widehat{c}_R(p_1, \dots, p_{k-1}, p_k)$ is strictly decreasing in p_k, there exists a unique $p_k^R = \Phi_R(p_1, \dots, p_{k-1})$, with $\Phi_R : \mathbb{R}^{k-1} \to \mathbb{R}$ strictly decreasing. Then the solution of (31) actually solves (26), and $\widehat{c}_R(p_1, \dots, p_k) = \Phi_R(p_1, \dots, p_{k-1}) - p_R^k = 0$. □

We proceed now with the proof of the main result.

Proof of Theorem 2. We first show that there exists $G : \mathbb{R}^k \to \mathbb{R}$ such that, if $G(p_1, \dots, p_k) = 0$, then (23) has a solution $v \in C^1(\Sigma) \cap C(\overline{\Sigma})$. The uniqueness of G and $\widehat{v}$ up to a multiplicative and additive constants respectily is a simple consequence of the maximum principle.

The claim will follow if we establish enough estimates to show that, as $R \to \infty$ and up to some normalizations, $\widehat{v}_R$ and $\widehat{c}_R$ converge respectively to a solution of and the compatibility condition in (23). The bounds needed for this convergence are the core of the proof.

We remark that, without loss of generality, we may assume that there exists $C > 0$ such that, along at least a subsequence denoted the same way as the full family, $R \to \infty$,

$$p_k^R \geq -C. \tag{32}$$

Indeed, if, as $R \to \infty$, $p_k^R \to -\infty$, then, for some $C > 0$ and all $R \to \infty$, we have $p_k^R \leq C$. Then we modify the argument below replacing inf by sup.

We work with the subsequence along which (32) holds, but, for simplicity, we do not repeat this fact.

Next we fix some $R_0 \in (1,R)$. Since $\widehat{v}_R$ is unique up to a constant, in what follows we assume without any loss of generality that

$$\inf_{\Sigma^{R_0}} \widehat{v}_R = 0. \tag{33}$$

First we prove, using Lemma 4, (33) and the Harnack inequality, that the $\widehat{v}_R$'s are uniformly bounded in $\overline{\Sigma_{R_1}}$ for all $R_1 \in (1,R_0)$ with the bound depending only on $R_0 - R_1$ and the ellipticity constants of $\widehat{F}$. We use this bound and again Lemma 4 to show that p_k^R are actually bounded uniformly in R. With this information, we employ Lemma 4 again to prove that the $\widehat{v}_R$'s are bounded uniformly in R for all compact subsets of $\overline{\Sigma_R}$. The classical elliptic regularity theory then yields the bounds necessary to obtain that, along subsequences, the $\widehat{v}_R$'s converge locally uniformly in $\overline{\Sigma}$ to some $\widehat{v} \in C^1(\Sigma)$ which solves

$$\widehat{F}(D^2\widehat{v}, y) = 0 \quad \text{and} \quad \frac{\partial \widehat{v}}{\partial n} = 0 \quad \text{on} \quad \partial\Sigma.$$

To conclude, we need show $\widehat{v}$ also satisfies the limiting growth at infinity.

We continue with the bound on $\overline{\Sigma_{R_1}}$ for all $R_1 \in (1,R_0)$. It follows from Lemma 4 and (32) that, for each $i = 1,\ldots,K$,

$$\inf_{\Gamma_i^{R_0}} \widehat{v}_R \geq \inf_{\Gamma_i^{R_1}} \widehat{v}_R + p_i(R_0 - R_1) \geq \inf_{\Gamma_i^{R_1}} \widehat{v}_R + C(R_0 - R_1). \tag{34}$$

Then Lemma 3 yields

$$\inf_{\overline{\Sigma^{R_0}}} \widehat{v}_R \geq \inf_{\overline{\Sigma^{R_1}}} \widehat{v}_R - C(R_0 - R_1),$$

and, in view of (33),

$$\inf_{\overline{\Sigma^{R_1}}} \widehat{v}_R \leq C(R_0 - R_1). \tag{35}$$

Since $\widehat{v}_R$ is a nonnegative solution of a homogenous uniformly elliptic pde in Σ^{R_0}, the Harnack inequality and (35) imply that, for each $R_1 \in (1,R_0)$, there exists C_{R_1}, which depends on $R_0 - R_1$ and the ellipticity constants of $\widehat{F}$, such that

$$\sup_{\overline{\Sigma^{R_1}}} \widehat{v}_R \leq C_{R_1}. \tag{36}$$

We show next that the p_k^R's are also bounded from above. Indeed, fix $R_2, R_1 \in (1,R_0)$ such that $R_2 < R_1$. Then, using (33), (36) and Lemma 4, we find

$$\inf_{\Gamma_K^{R_1}} \widehat{v}_R \geq \inf_{\Gamma_K^{R_2}} \widehat{v}_R + p_k^R(R_1 - R_2) \geq p_k^R(R_1 - R_2).$$

It follows that there exists $C > 0$, which depends on (32) and (36), such that

$$p_k^R \leq C. \tag{37}$$

We show next that the $\widehat{v}_R$'s are bounded uniformly in R in compact subsets of $\overline{\Sigma}$.

Applying once more Lemma 4 and the bounds already obtained for $R_1 < R_0$, we find that, for each $i = 1,\ldots,k$, any $R_2 \in (R_0,R)$, all (x_i, x_i') with $x_i \geq R_2$ and some uniform $C > 0$ coming from (36),

$$p_i x_i + C \geq \widehat{v}_R(x_i, x_i') \geq p_i x_i - C. \tag{38}$$

Note that, (38) not implies the claimed local uniform bound but also establishes the asymptotic behavior as $x_i \to \infty$.

We conclude with the other direction of the claim, that is, if there exists a unique up to additive constants solution of (26), then we must have $G(p_1,\ldots,p_k) = 0$.

We argue by contradiction. The arguments are similar, so we assume, for definiteness, that $G(p_1,\dots,p_k)>0$. Since G is strictly decreasing, there exist $\widetilde{p}_i > p_i$ such that

$$G(\widetilde{p}_1,\dots,\widetilde{p}_k)=0. \tag{39}$$

Let $\widetilde{v}$ be the solution of (32) with asymptotic slopes $\widetilde{p}_1,\dots,\widetilde{p}_k$ which exists in view of (39) and the first part of the ongoing proof.

Then $\widehat{v}-\widetilde{v}$ must have a maximum in Σ, which is not possible in view of the properties of $\widehat{F}$ and the maximum priciple. □

4. The proof of Theorem 1

For the proof relies on a novel construction of super-and sub-solutions of (6) near the junction which uses the properties of the nonlinear Kirchhoff condition.

Proof of Theorem 1. The assumptions on F and the theory of uniformly elliptic equations yield that the family $(u_\epsilon)_{\epsilon>0}$ is bounded in $\overline{\Sigma_\epsilon}$, uniformly in ϵ, and precompact in $C^{0,\alpha}_{\mathrm{loc}}(\overline{\Sigma_\epsilon})$. It follows that, along subsequences $\epsilon_n\to 0$, u_{ϵ_n} converge locally uniformly to some $\overline{u}\in C^{0,\alpha}(\mathscr{G})$. We prove that $\overline{u}$ is a solution of (15). Then, since (15) has a unique solution, it follows that the whole family converges and the result is proven.

We show next that any limit $\overline{u}$ is a sub-solution of (15). Since the super-solution property follows similarly, we omit the details. In what follows we denote the sequence by ϵ.

Assume that, for some smooth $\phi\in\left(\bigcap_{i=1}^k C^{2,1}(\overline{\mathscr{G}_i})\right)\cap C^{0,1}(\mathscr{G})$, $\overline{u}-\phi$ attains a strict local maximum $\overline{x}$ in $\mathscr{G}\cap\overline{B_{r_0}(\overline{x})}$ for some $r_0>0$ and, without any loss of generality, $\overline{u}(\overline{x})=\phi(\overline{x})=0$.

If $\overline{x}=\overline{x_i}\in\overline{\mathscr{G}_i}\setminus\{0\}$ for some $i=1,\dots,k$, the conclusion follows in a straightforward manner from the stability properties of the viscosity solutions and the definition of $\mathscr{F}_i$ modulo a small additional argument needed to study Neumann boundary condition of (6). For the convenience of the reader we sketch this argument next.

To fix the ideas, we assume that $i=1$ and $\zeta_1=e_1=(1,0,\dots,0)$ and write x and y instead of x_1 and $x_i^\perp$. The map $(x,y)\to u_\epsilon(x,y)-\phi(x)$ attains a maximum $\overline{\Sigma}_\epsilon\cap\overline{B_{r_0}(\overline{x})}$ at some (x_ϵ,y_ϵ) near $(\overline{x}_1,0)$. The only difficulty arises if $|y_\epsilon|=\epsilon/2$.

To avoid this problem, for $\delta>0$, we look at maxima points $(x_{\epsilon,\delta},y_{\epsilon,\delta})$ of $(x,y)\to u_\epsilon(x,y)-\phi_{\epsilon,\delta}(x,y)$ in $\overline{\Sigma}_\epsilon\cap\overline{B_{r_0}(\overline{x})}$, where $\phi_{\epsilon,\delta}(x,y)=\phi(x)-\delta d(y)$, $d(y)$ being a regularization of the distance function of y to the boundary of $[-\epsilon/2,\epsilon/2]$ such that $d>0$ in $(-1/2,1/2)$ and $d(\pm\epsilon/2)=0$. It is, of course, immediate that, as $\epsilon,\delta\to 0$, $(x_{\epsilon,\delta},y_{\epsilon,\delta})\to(\overline{x}_1,0)$.

If $|y_{\epsilon,\delta}|=\epsilon/2$, then $\frac{\partial\phi_{\epsilon,\delta}}{\partial n}(x_{\epsilon,\delta},y_{\epsilon,\delta})=-\delta$, which, in view of viscosity definition of the Neumann boundary condition, means that we need to use the equation and, hence, we must have

$$F_1\left(\begin{pmatrix}\phi_{xx}(x_{\epsilon,\delta}) & 0\\ 0 & 0(\delta)\end{pmatrix},(x_{\epsilon,\delta},y_{\epsilon,\delta})\right)\le 0; \tag{40}$$

note that in (40) we used that (1) and (7) imply that $F=F_1$ in $Z_{1,\epsilon}$.

A similar inequality holds if $|y_{\epsilon,\delta}|<\epsilon/2$, and, thus, after letting $\epsilon,\delta\to 0$, we find

$$\mathscr{F}_i(\phi_{x_1x_1}(\overline{x}_1),\overline{x}_1)+\overline{u}(\overline{x}_1)\le 0.$$

We turn now to the more difficult case $\overline{x}=0$ and argue by contradiction, that is, we assume that, for some $\sigma>0$,

$$\min\Big(G(p_1,\dots,p_k),\min_{1\le i\le k}[\overline{u}(0)+\mathscr{F}_i(\phi_{x_ix_i}(0+),0)]\Big)=\sigma, \tag{41}$$

where, for $i=1,\dots,k$, $p_i=\phi_{x_i}(0+)$.

Using (41) we construct, for $\epsilon>0$ sufficiently small, a super-solution $w_\epsilon\in C(\overline{\Sigma_\epsilon})$ of (6), such that, as $\epsilon\to 0$, $w_\epsilon\to\phi$ uniformly in $\overline{\Sigma_\epsilon}\cap B_r(0)$ for some $r<r_0$.

Then it follows from the comparison principle that

$$\max_{\Sigma_\epsilon \cap B_r(0)} (u_\epsilon - w_\epsilon) \le \max_{\Sigma_\epsilon \cap \partial B_r(0)} (u_\epsilon - w_\epsilon),$$

and, after letting $\epsilon \to 0$,

$$0 = \overline{u}(0) - \phi(0) \le \max_{\mathscr{G} \cap \partial B_r(0)} (\overline{u} - \phi) < 0,$$

which is contradiction.

For the next argument, it is necessary to extend ϕ to a function $\Phi_\epsilon : \overline{\Sigma}_\epsilon \to \mathbb{R}$ such that

$$\Phi_\epsilon \equiv 0 \text{ in } \overline{\mathscr{W}}_{\frac{\epsilon}{2}} \quad \text{and} \quad \Phi_\epsilon(x) = \phi_i(x_i) \text{ if } x \in \epsilon Z_i. \tag{42}$$

We proceed with the construction of w_ϵ. Since, in view of (41), $G(p_1, \dots, p_k) \ge \sigma > 0$ and G is strictly decreasing with respect to each p_i, there exist $q_1, \dots, q_k$ such that

$$q_i > p_i \quad \text{and} \quad G(q_1, \dots, q_K) = 0.$$

Let $v \in C^1(\overline{\Sigma}_1) \cap C(\overline{\Sigma})$ be the solution of (23) with $v(0) = -M$, where $M > 0$ is to be chosen below, and, for each $i = 1, \dots, k$,

$$\lim_{x_i \to \infty} \frac{v(x_i, x_i')}{x_i} = q_i \quad \text{uniformly for } x_i' \in \overline{\omega}_i.$$

It follows that there exist $\delta > 0$ and a sufficiently large $R_0 > 0$ such that, for each $i = 1, \dots, k$, and uniformly in $x_i' \in \overline{\omega}_i$,

$$v(x_i, x_i') \ge x_i(p_i + \delta) \quad \text{if } x_i > R_0. \tag{43}$$

Hence, for each $i = 1, \dots, k$, all $x = (x_i, x_i') \in \mathscr{W}_{\epsilon R_0} \cap (\mathscr{G}_i \times \overline{\omega}_i)$ and uniformly for $x_i' \in \overline{\omega}_i$,

$$\epsilon v\left(\frac{\epsilon R_0}{\epsilon}, x_i'\right) = \epsilon v(R_0, x_i^{\perp}) \ge \epsilon R_0(p_i + \delta). \tag{44}$$

For each $i = 1, \dots, k$, let ϕ_i be the restriction of ϕ on $\mathscr{G}_i$, and set $C_0 = \max_{1,\dots,k} \|\phi_{i,x_i x_i}\|$. Then, since it is assumed that $\phi(0) = 0$,

$$\phi_i(\epsilon R_0) \le \epsilon R_0 p_i + C_0 \epsilon^2 R_0^2.$$

Using (44) we find that, for sufficiently small ϵ, every $i = 1, \dots, k$ and uniformly on $x_i' \in \overline{\omega}$,

$$\epsilon v\left(\frac{\epsilon R_0}{\epsilon}, x_i'\right) \ge \phi(\epsilon R_0) + \frac{\epsilon R_0 \delta}{2}. \tag{45}$$

Next, for small ϵ and $C_1 = C + \|Dv\| R_0 + 1$, C coming from (10), we consider the solution $\widetilde{v}_\epsilon \in C^{0.1}(\overline{\mathscr{W}_{R_0} \cap \Sigma})$ of

$$\begin{cases} \overline{F}(D^2 \widetilde{v}_\epsilon, \epsilon x, x) = \epsilon C_1 \text{ in } \mathscr{W}_{R_0} \cap \Sigma, \\ \widetilde{v}_\epsilon = v \text{ on } \partial \mathscr{W}_{R_0} \cap \Sigma \quad \text{and} \quad \frac{\partial \widetilde{v}_\epsilon}{\partial n} = 0 \text{ on } \overline{\mathscr{W}_{R_0}} \cap \partial \Sigma, \end{cases} \tag{46}$$

which, in view of the properties of $\overline{F}$, exists.

The comparison principle and classical arguments from the theory of nonlinear elliptic equations (see, for example, Caffarelli and Cabre [3]) yield $C_2 > 0$, which depends on the ellipticity and Lipschitz constants of $\overline{F}$ and the choice of C_1 in (46), such that

$$0 \le \widetilde{v} - v \le \epsilon C_2 \text{ on } \overline{\mathscr{W}_{R_0} \cap \Sigma}. \tag{47}$$

We choose M sufficiently large so that, for some $\rho \in (1, R_0)$ and sufficiently small ϵ,

$$\widetilde{v} \le 0 \text{ in } \mathscr{W}_\rho \cap \Sigma. \tag{48}$$

Indeed, if L is the Lipshitz constant of v, for $x \in \mathscr{W}_{R_0} \cap \Sigma$, we have

$$\widetilde{v}(x) \le v(x) + \epsilon C_2 \le -M + L|x| + \epsilon C_2,$$

and, if $M \in (L, LR_0)$ and ϵ is sufficiently small, then (48) holds.

Next define $\overline{v}_\epsilon : \overline{\Sigma}_\epsilon \to \mathbb{R}$ by

$$\overline{v}_\epsilon(x) = \begin{cases} \epsilon \widetilde{v}(\frac{x}{\epsilon}) & \text{if } x \in \overline{\Sigma} \setminus \overline{\mathscr{W}_{\epsilon R_0} \cap \Sigma}, \\ \epsilon v(\frac{x}{\epsilon}) & \text{otherwise,} \end{cases} \tag{49}$$

and set

$$w_\epsilon = \min\left[\overline{v}_\epsilon, \widetilde{\phi}\right]. \tag{50}$$

It turns out that, for ϵ small enough, w_ϵ is a super-solution of (6), and

$$\lim_{\epsilon\to 0} \sup_{x\in\mathscr{G}, y\in\Sigma_\epsilon\cap B_\rho(0), |x-y|\le\epsilon} (W_\epsilon(y) - \phi(x)) = 0 \quad \text{for all } \rho \in (0, R_0]. \tag{51}$$

With these two at hand we may conclude the proof of the theorem using the argument presented earlier.

We return to the proofs of the two claims above. The second is a direct consequence of (45), the definition of $\widetilde{v}$ and (48).

The proof of the super-solution property is more complicated. In what follows, we argue as if w_ϵ were smooth and leave it up to the reader to fill in the usual details for the justification in the viscosity sense.

We first observe that, in view of (48) and (42), $w_\epsilon = \overline{v}_\epsilon$ in $\overline{\mathscr{W}}_{\frac{\epsilon}{2}}$, while (45) and (49) imply that $w_\epsilon = \overline{v}_\epsilon$ in $\overline{\Sigma}_1 \setminus \mathscr{W}_{\epsilon R_0}$.

Next observe that (41) and (42) yield that, for ϵ small enough,

$$F\left(D^2\Phi_\epsilon, x, \frac{x}{\epsilon}\right) + \Phi_\epsilon \ge 0 \text{ in } \Sigma_\epsilon \setminus \overline{\mathscr{W}_\epsilon} \quad \text{and} \quad \frac{\partial \Phi_\epsilon}{\partial n} = 0 \text{ on } \partial\Sigma_\epsilon \cap (\Sigma_\epsilon \setminus \overline{\mathscr{W}_\epsilon}), \tag{52}$$

while $\overline{v}_\epsilon$ is a supersolution of

$$F(D^2\overline{v}_\epsilon, x) + \overline{v}_\epsilon \ge 0 \text{ in } \Sigma_\epsilon \cap \mathscr{W}_{\epsilon R_0} \quad \text{and} \quad \frac{\partial \overline{v}_\epsilon}{\partial n} = 0 \text{ on } \partial\Sigma_\epsilon \cap \overline{\mathscr{W}_{\epsilon R_0}}. \tag{53}$$

The first claim follows by an argument similar to the one at the beginning of the proof since it involves only the equation satisfied on each $\mathscr{G}_i$.

For (53), we first observe that the boundary condition is immediate from the analogous property of $\widetilde{v}_\epsilon$, while arguing as if $\overline{v}_\epsilon$ where smooth, we find that in $\mathscr{W}_{\epsilon R_0}$

$$F\left(D^2\widetilde{v}_\epsilon, x, \frac{x}{\epsilon}\right) + \widetilde{v}_\epsilon = F\left(\frac{1}{\epsilon}D^2\widetilde{v}\left(\frac{x}{\epsilon}\right), x, \frac{x}{\epsilon}\right) + \epsilon\widetilde{v}\left(\frac{x}{\epsilon}\right).$$

Hence, in view of (5), we get

$$F\left(D^2\overline{v}_\epsilon, x, \frac{x}{\epsilon}\right) + \overline{v}_\epsilon \ge \overline{F}\left(\frac{1}{\epsilon}D^2\widetilde{v}\left(\frac{x}{\epsilon}\right), x, \frac{x}{\epsilon}\right) + \epsilon\widetilde{v}\left(\frac{x}{\epsilon}\right) - C = \frac{1}{\epsilon}\left(F\left(D^2\widetilde{v}\left(\frac{x}{\epsilon}\right), x, \frac{x}{\epsilon}\right) + \epsilon^2\widetilde{v}\left(\frac{x}{\epsilon}\right) - \epsilon C\right). \tag{54}$$

It follows from (54), always in $\mathscr{W}_{\epsilon R_0}$ and for ϵ small, that

$$F\left(D^2\overline{v}_\epsilon, x, \frac{x}{\epsilon}\right) + \overline{v}_\epsilon \ge \frac{1}{\epsilon}\left(F\left(D^2\widetilde{v}\left(\frac{x}{\epsilon}\right), x, \frac{x}{\epsilon}\right) + \epsilon^2\widetilde{v}\left(\frac{x}{\epsilon}\right) - \epsilon(C + 1 + \|Dv\|)\right) \ge 0, \tag{55}$$

the last inequality coming from (46). □

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