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A Berry–Esseen bound of order $\frac{1}{\sqrt{n}}$ for martingales

Une borne de Berry–Esseen d'ordre $\frac{1}{\sqrt{n}}$ pour les martingales

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Abstract. Renz [13] has established a rate of convergence $1/\sqrt{n}$ in the central limit theorem for martingales with some restrictive conditions. In the present paper a modification of the methods, developed by Bolthausen [2] and Grama and Haeusler [6], is applied for obtaining the same convergence rate for a class of more general martingales. An application to linear processes is discussed.

Résumé. Renz [13] a établi un taux de convergence $1/\sqrt{n}$ dans le théorème de la limite centrale pour les martingales avec certaines conditions restrictives. Dans le présent article, une modification des méthodes, développées par Bolthausen [2] et Grama et Haeusler [6], est appliquée pour obtenir le même taux de convergence pour une classe de martingales plus générales. Une application aux processus linéaires est discutée.

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1. Introduction and main result

For $n \in \mathbf{N}$, let $(\xi_i, \mathscr{F}_i)_{i=0,\dots,n}$ be a finite sequence of martingale differences defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$, where $\xi_0 = 0$ and $\{\emptyset, \Omega\} = \mathscr{F}_0 \subseteq \cdots \subseteq \mathscr{F}_n \subseteq \mathscr{F}$ are increasing σ-fields. Denote

$$X_0 = 0, \quad X_k = \sum_{i=1}^{k} \xi_i, \ k = 1, \dots, n.$$

Then $X = (X_k, \mathscr{F}_k)_{k=0,\dots,n}$ is a martingale. Denote by $\langle X \rangle$ the conditional variance of X:

$$\langle X \rangle_0 = 0, \quad \langle X \rangle_k = \sum_{i=1}^{k} \mathbf{E}\big[\xi_i^2 \big| \mathscr{F}_{i-1}\big], \ k = 1, \dots, n.$$

Define

$$D(X_n) = \sup_{x \in \mathbf{R}} \big| \mathbf{P}(X_n \le x) - \Phi(x) \big|,$$

where $\Phi(x)$ is the distribution function of the standard normal random variable. Denote by $\xrightarrow{\mathbf{P}}$ the convergence in probability as $n \to \infty$. According to the martingale central limit theorem, the "conditional Lindeberg condition"

$$\sum_{i=1}^{n} \mathbf{E}\big[\xi_i^2 \mathbf{1}_{\{|\xi_i| \ge \epsilon\}} \big| \mathscr{F}_{i-1}\big] \xrightarrow{\mathbf{P}} 0, \quad \text{for each } \epsilon > 0,$$

and the "conditional normalizing condition" $\langle X \rangle_n \xrightarrow{\mathbf{P}} 1$ together implies asymptotic normality of X_n, that is, $D(X_n) \to 0$ as $n \to \infty$.

The convergence rate of $D(X_n)$ has attracted a lot of attentions. For instance, Bolthausen [2] proved that if $|\xi_i| \le \epsilon_n$ for a number ϵ_n and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \le c\epsilon_n^3 n \log n$, where, here and after, c is an absolute constant not depending on ϵ_n and n. El Machkouri and Ouchti [3] improved the factor $\epsilon_n^3 n \log n$ in Bolthausen's bound to $\epsilon_n \log n$ under the following more general condition

$$\mathbf{E}\big[|\xi_i|^3 \big| \mathscr{F}_{i-1}\big] \le \epsilon_n \mathbf{E}\big[\xi_i^2 \big| \mathscr{F}_{i-1}\big] \quad a.s. \text{ for all } i = 1, 2, \dots, n.$$

For more related results, we refer to Ouchti [12] and Mourrat [11]. Recently, Fan [4] proved that if there exist a positive constant ρ and a number ϵ_n, such that

$$\mathbf{E}\big[|\xi_i|^{2+\rho} \big| \mathscr{F}_{i-1}\big] \le \epsilon_n^{\rho} \mathbf{E}\big[\xi_i^2 \big| \mathscr{F}_{i-1}\big] \quad a.s. \text{ for all } i = 1, 2, \dots, n,$$

and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \le c_\rho \widehat{\epsilon}_n$, where

$$\widehat{\epsilon}_n = \begin{cases} \epsilon_n^{\rho}, & \text{if } \rho \in (0,1), \\ \epsilon_n |\log \epsilon_n|, & \text{if } \rho \ge 1, \end{cases}$$

and c_ρ is a constant depending only on ρ. Fan [4] also showed that this Berry–Esseen bound is optimal. In particular, if $\epsilon_n \asymp 1/\sqrt{n}$, then we have $\epsilon_n |\log \epsilon_n| \asymp (\log n)/\sqrt{n}$. Thus, we cannot obtain the classical convergence rate $1/\sqrt{n}$ for general martingales.

However, the convergence rate $1/\sqrt{n}$ for martingales is possible to be attained with some additional restrictive conditions. For instance, Renz [13] proved that if there exists a constant $\rho > 0$ such that

$$\mathbf{E}[\xi_i^2 | \mathscr{F}_{i-1}] = 1/n, \quad \mathbf{E}[\xi_i^3 | \mathscr{F}_{i-1}] = 0 \quad \text{and} \quad \mathbf{E}\big[|\xi_i|^{3+\rho} \big| \mathscr{F}_{i-1}\big] \le c n^{-(3+\rho)/2}, \quad \text{a.s.,} \tag{1}$$

then it holds

$$D(X_n) = O\bigg(\frac{1}{\sqrt{n}}\bigg). \tag{2}$$

He also showed that this result is not true for $\rho = 0$. More martingale Berry–Esseen bounds of convergence rate $1/\sqrt{n}$ can be found in Bolthausen [2] and Kir'yanova and Rotar [10].

In this paper we are interested in extending (2) to a class of more general martingales. The following theorem is our main result.

Theorem 1. *Assume that there exist some numbers $\rho \in (0,+\infty), \epsilon_n \in (0,\frac{1}{2}]$ and $\delta_n \in [0,\frac{1}{2}]$ such that for all $1 \le i \le n$,*

$$\left|\langle X\rangle_n - 1\right| \le \delta_n^2, \tag{3}$$

$$\mathbf{E}\left[\xi_i^3 \middle| \mathscr{F}_{i-1}\right] = 0 \tag{4}$$

and

$$\mathbf{E}\left[|\xi_i|^{3+\rho} \middle| \mathscr{F}_{i-1}\right] \le \epsilon_n^{1+\rho}\mathbf{E}\left[\xi_i^2 \middle| \mathscr{F}_{i-1}\right] \quad a.s. \tag{5}$$

Then

$$D(X_n) \le c_\rho(\epsilon_n + \delta_n),$$

where c_ρ depends only on ρ. In addition, it holds $c_\rho = O(\rho^{-1}), \rho \to 0$.

Notice that under the conditions of Renz [13], the conditions of Theorem 1 are satisfied with $\delta_n = 0$ and $\epsilon_n \asymp 1/\sqrt{n}$. Thus Theorem 1 extends Renz's result to a class of more general martingales.

Thanks to the additional condition (4), the Berry–Esseen bound (6) improves the bound of Fan [4] by replacing $\epsilon_n|\log\epsilon_n|$ with ϵ_n.

Relaxing the condition (3), we have the following analogue estimation of Fan (cf. [4, (26)]).

Theorem 2. *Assume that there exist some numbers $\rho \in (0,+\infty)$ and $\epsilon_n \in (0,\frac{1}{2}]$ such that for all $1 \le i \le n$,*

$$\mathbf{E}\left[\xi_i^3 \middle| \mathscr{F}_{i-1}\right] = 0$$

and

$$\mathbf{E}\left[|\xi_i|^{3+\rho} \middle| \mathscr{F}_{i-1}\right] \le \epsilon_n^{1+\rho}\mathbf{E}\left[\xi_i^2 \middle| \mathscr{F}_{i-1}\right] \quad a.s.$$

Then, for all $p \ge 1$,

$$D(X_n) \le c_\rho\epsilon_n + c_p\left(\mathbf{E}\left[\left|\langle X\rangle_n - 1\right|^p\right] + \mathbf{E}\left[\max_{1\le i\le n}|\xi_i|^{2p}\right]\right)^{1/(2p+1)}, \tag{6}$$

where c_ρ and c_p depend only on ρ and p, respectively.

It is easy to see that when $p \to \infty$,

$$\left(\mathbf{E}\left[\left|\langle X\rangle_n - 1\right|^p\right]\right)^{1/(2p+1)} \to \|\langle X\rangle_n - 1\|_\infty^{1/2},$$

which coincides with δ_n of Theorem 1.

2. Application

We first extend Theorem 1 to triangular arrays with infinity many terms in each line. For $n \in \mathbf{N}$, let $(\xi_{n,i}, \mathscr{F}_{n,i})_{i=-\infty}^n$ be a sequence of martingale differences defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$, where the adapted filtration is $\{\emptyset, \Omega\} = \mathscr{F}_{-\infty} \subset \cdots \subset \mathscr{F}_{n,n-1} \subset \mathscr{F}_{n,n} \subset \mathscr{F}$. Denote $X_{n,k} = \sum_{i=-\infty}^k \xi_{n,i}, k \le n$. Then $(X_{n,k}, \mathscr{F}_{n,k})_{k=-\infty}^n$ is a martingale. Let $\langle X\rangle_{n,k} = \sum_{i=-\infty}^k \mathbf{E}[\xi_{n,i}^2|\mathscr{F}_{n,i-1}], k \le n$. In particular, denote $X_n := X_{n,n}$ and $\langle X\rangle_n := \langle X\rangle_{n,n}$.

With some slight modification on the proof, Theorem 1 still holds in this new setting. Now we apply Theorem 1 with this new setting to the partial sum of linear processes. Let $(\varepsilon_i)_{i\in\mathbf{Z}}$ be a sequence of identically distributed martingale differences adapted to the filtration $(\mathscr{F}_i)_{i\in\mathbf{Z}}$. We consider the causal linear process in the form

$$Y_k = \sum_{j=-\infty}^{k} a_{k-j}\varepsilon_j, \tag{7}$$

where the martingale differences have finite variance and the sequence of real coefficients satisfies $\sum_{i=0}^\infty a_i^2 < \infty$. Without loss of generality, let the variance of the martingale difference to be 1. We say the linear process has long memory if $\sum_{i=0}^{\infty}|a_i| = \infty$. In this case, we assume that $a_0 = 1$ and

$$a_i = \ell(i)i^{-\alpha},\ i > 0, \text{ with } 1/2 < \alpha < 1. \tag{8}$$

Here $\ell(\cdot)$ is a slowly varying function. On the other hand, we say the linear process has short memory if $\sum_{i=0}^{\infty}|a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i \neq 0$. The third case is $\sum_{i=0}^{\infty}|a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i = 0$.

The long memory linear processes covers the well-known fractional ARIMA processes (cf. Granger and Joyeux [7]; Hosking [9]), which play an important role in financial time series modeling and application. As a special case, let $0 < d < 1/2$ and B be the backward shift operator with $B\varepsilon_k = \varepsilon_{k-1}$ and consider

$$Y_k = (1-B)^{-d}\varepsilon_k = \sum_{i=0}^{\infty} a_i\varepsilon_{k-i}, \quad \text{where } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

For this example we have $\lim_{n\to\infty} a_n/n^{d-1} = 1/\Gamma(d)$. Note that these processes have long memory because $\sum_{j=0}^{\infty}|a_j| = \infty$.

The partial sum $S_n = \sum_{k=1}^{n} Y_k$ of causal linear process (7) can be written as $S_n = \sum_{i=-\infty}^{n} b_{n,i}\varepsilon_i$, where $b_{n,i} = \sum_{j=0}^{n-i} a_j$ for $0 < i \le n$, and $b_{n,i} = \sum_{j=1-i}^{n-i} a_j$ for $i \le 0$. The variance of S_n is $B_n^2 = \operatorname{var}(S_n) = \sum_{i=-\infty}^{n} b_{n,i}^2$. Now let $X_{n,k} = \sum_{i=-\infty}^{k} b_{n,i}\varepsilon_i/B_n$. Then $X_n = X_{n,n} = S_n/B_n$ and $\langle X\rangle_n = \sum_{i=-\infty}^{n} b_{n,i}^2\mathbf{E}[\varepsilon_i^2|\mathscr{F}_{i-1}]/B_n^2$. If we assume $|\langle X\rangle_n - 1| \le \delta_n^2$ for some $\delta_n \in [0, \frac{1}{2}]$, $\mathbf{E}[\varepsilon_i^3|\mathscr{F}_{i-1}] = 0$ and $\mathbf{E}[|\varepsilon_i|^{3+\rho}|\mathscr{F}_{i-1}] \le d_\rho^{1+\rho}\mathbf{E}[\varepsilon_i^2|\mathscr{F}_{i-1}]$ a.s. for all $i \in \mathbf{Z}$ and some constant d_ρ, then, by Theorem 1,

$$\sup_{x\in\mathbf{R}} |\mathbf{P}(S_n/B_n \le x) - \Phi(x)| \le c_\rho(\epsilon_n + \delta_n),$$

where $\epsilon_n = d_\rho \sup_{i\le n}|b_{n,i}|/B_n$.

In the case that $\sum_{i=0}^{\infty}|a_i| < \infty$, $\sup_{i\le n}|b_{n,i}| \le \sum_{i=0}^{\infty}|a_i| < \infty$ and it is well known that B_n^2 has order n. Hence ϵ_n has order $1/\sqrt{n}$ in this case. In the long memory case $\sum_{i=0}^{\infty}|a_i| = \infty$, if we assume (8), B_n^2 has order $n^{3-2\alpha}\ell^2(n)$ (e.g., Wu and Min [14]) and $\sup_{i\le n}|b_{n,i}|$ has order $n^{1-\alpha}\ell(n)$ (see Beknazaryan et al. [1] for upper bound and Fortune et al. [5] for lower bound in the case $d = 1$). Hence in this case ϵ_n also has order $1/\sqrt{n}$. In either case the Berry–Esseen bound has order $1/\sqrt{n}$ if $\delta_n = O(n^{-1/2})$. In particular, if we in addition assume that the innovations $(\varepsilon_i)_{i\in\mathbf{Z}}$ are independent, then $\delta_n = 0$ and the Berry–Esseen bound $\sup_{x\in\mathbf{R}}|\mathbf{P}(S_n/B_n \le x) - \Phi(x)|$ has order $1/\sqrt{n}$. Here the condition $\mathbf{E}[\varepsilon_i^3|\mathscr{F}_{i-1}] = 0$ is needed to have the Berry–Esseen bound of order $1/\sqrt{n}$. We cannot have this order from the result of Fan [4].

3. Proofs of theorems

3.1. *Preliminary lemmas*

In the proofs of theorems, we need the following technical lemmas. The first two lemmas can be found in Fan [4, Lemmas 3.1 and 3.2].

Lemma 3. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s|\mathscr{F}_{i-1}] \le \epsilon_n^{s-2}\mathbf{E}[\xi_i^2|\mathscr{F}_{i-1}],$$

then, for any $t \in [3, s)$,

$$\mathbf{E}[|\xi_i|^t|\mathscr{F}_{i-1}] \le \epsilon_n^{t-2}\mathbf{E}[\xi_i^2|\mathscr{F}_{i-1}].$$

Lemma 4. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s|\mathscr{F}_{i-1}] \le \epsilon_n^{s-2}\mathbf{E}[\xi_i^2|\mathscr{F}_{i-1}],$$

then

$$\mathbf{E}[\xi_i^2|\mathscr{F}_{i-1}] \le \epsilon_n^2.$$

The next two technical lemmas are due to Bolthausen (cf. [2, Lemmas 1 and 2]).

Lemma 5. *Let X and Y be random variables. Then*
$$\sup_u \left|\mathbf{P}\big(X \le u\big) - \Phi(u)\right| \le c_1 \sup_u \left|\mathbf{P}\big(X+Y \le u\big) - \Phi(u)\right| + c_2 \left\|\mathbf{E}[Y^2|X]\right\|_\infty^{1/2},$$
where c_1 and c_2 are two positive constants.

Lemma 6. *Let $G(x)$ be an integrable function on $\mathbf{R}$ of bounded variation $\|G\|_V$, X be a random variable and $a, b \neq 0$ are real numbers. Then*
$$\mathbf{E}\left[G\left(\frac{X+a}{b}\right)\right] \le \|G\|_V \sup_u \left|\mathbf{P}\big(X \le u\big) - \Phi(u)\right| + \|G\|_1 |b|,$$
where $\|G\|_1$ is the $L_1(\mathbf{R})$ norm of $G(x)$.

In the proof of Theorem 2, we also need the following lemma of El Machkouri and Ouchti [3].

Lemma 7. *Let X and Y be two random variables. Then, for $p \ge 1$,*
$$D(X+Y) \le 2D(X) + 3\left\|\mathbf{E}\big[Y^{2p}|X\big]\right\|_1^{1/(2p+1)}. \tag{9}$$

3.2. *Proof of Theorem 1*

By Lemma 3, we only need to consider the case of $\rho \in (0,1]$. We follow the method of Grama and Haeusler [6]. Let $T = 1 + \delta_n^2$. We introduce a modification of the conditional variance $\langle X\rangle_n$ as follows:
$$V_k = \langle X\rangle_k \mathbf{1}_{\{k<n\}} + T\mathbf{1}_{\{k=n\}}. \tag{10}$$
It is easy to see that $V_0 = 0, V_n = T$, and that $(V_k, \mathscr{F}_k)_{k=0,\dots,n}$ is a predictable process. Set
$$\gamma = \epsilon_n + \delta_n.$$
Let c_* be some positive and sufficient large constant. Define the following non-increasing discrete time predictable process
$$A_k = c_*^2\gamma^2 + T - V_k, \quad k = 1, \dots, n. \tag{11}$$
Obviously, we have $A_0 = c_*^2\gamma^2 + T$ and $A_n = c_*^2\gamma^2$. In addition, for $u, x \in \mathbf{R}$, and $y > 0$, denote
$$\Phi_u(x,y) = \Phi\left(\frac{u-x}{\sqrt{y}}\right). \tag{12}$$
Let $\mathscr{N} = \mathscr{N}(0,1)$ be a standard normal random variable, which is independent of X_n. Using a smoothing procedure, by Lemma 5, we deduce that
$$\begin{aligned}\sup_u \left|\mathbf{P}\big(X_n \le u\big) - \Phi(u)\right| &\le c_1 \sup_u \left|\mathbf{P}\big(X_n + c_*\gamma\mathscr{N} \le u\big) - \Phi(u)\right| + c_2\gamma\\ &= c_1 \sup_u \left|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \Phi(u)\right| + c_2\gamma\\ &\le c_1 \sup_u \left|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\right|\\ &\quad + c_1 \sup_u \left|\mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] - \Phi(u)\right| + c_2\gamma\\ &= c_1 \sup_u \left|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\right|\\ &\quad + c_1 \sup_u \left|\Phi\left(\frac{u}{\sqrt{c_*^2\gamma^2+T}}\right) - \Phi(u)\right| + c_2\gamma.\end{aligned} \tag{13}$$
It is obvious that
$$\left|\Phi\left(\frac{u}{\sqrt{c_*^2\gamma^2+T}}\right) - \Phi(u)\right| \le c_3 \left|\frac{1}{\sqrt{c_*^2\gamma^2+T}} - 1\right| \le c_4\gamma. \tag{14}$$

Returning to (13), we get

$$\sup_u \left|\mathbf{P}\big(X_n \le u\big) - \Phi(u)\right| \le c_1 \sup_u \left|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\right| + c_5\gamma. \tag{15}$$

By a simple telescoping, we know that

$$\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] = \mathbf{E}\left[\sum_{k=1}^{n} \big(\Phi_u\big(X_k, A_k\big) - \Phi_u\big(X_{k-1}, A_{k-1}\big)\big)\right]. \tag{16}$$

Taking into account the fact that

$$\frac{\partial^2}{\partial x^2}\Phi_u(x,y) = 2\frac{\partial}{\partial y}\Phi_u(x,y),$$

we get

$$\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] = J_1 + J_2 - J_3, \tag{17}$$

where

$$\begin{aligned} J_1 = \mathbf{E}\Bigg[\sum_{k=1}^{n}\bigg(&\Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x}\Phi_u(X_{k-1}, A_k)\xi_k \\ &- \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\xi_k^2 - \frac{1}{6}\frac{\partial^3}{\partial x^3}\Phi_u(X_{k-1}, A_k)\xi_k^3\bigg)\Bigg], \end{aligned} \tag{18}$$

$$J_2 = \frac{1}{2}\mathbf{E}\left[\sum_{k=1}^{n}\frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\big(\triangle\langle X\rangle_k - \triangle V_k\big)\right], \tag{19}$$

$$J_3 = \mathbf{E}\left[\sum_{k=1}^{n}\left(\Phi_u(X_{k-1}, A_{k-1}) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial y}\Phi_u(X_{k-1}, A_k)\triangle V_k\right)\right], \tag{20}$$

where $\triangle\langle X\rangle_k = \langle X\rangle_k - \langle X\rangle_{k-1}$.

Now, we need to give some estimates of J_1, J_2 and J_3. To this end, we introduce some notations. Denote by ϑ_i some random variables satisfying $0 \le \vartheta_i \le 1$, which may represent different values at different places. For the rest of the paper, φ stands for the density function of the standard normal random variable.

Control of J_1. For convenience's sake, let $T_{k-1} = (u - X_{k-1})/\sqrt{A_k},\ k = 1, 2, \ldots, n$. It is easy to see that

$$\begin{aligned} B_k =: &\Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x}\Phi_u(X_{k-1}, A_k)\xi_k \\ &- \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\xi_k^2 - \frac{1}{6}\frac{\partial^3}{\partial x^3}\Phi_u(X_{k-1}, A_k)\xi_k^3 \\ = &\Phi\left(T_{k-1} - \frac{\xi_k}{\sqrt{A_k}}\right) - \Phi(T_{k-1}) + \Phi'(T_{k-1})\frac{\xi_k}{\sqrt{A_k}} \\ &- \frac{1}{2}\Phi''(T_{k-1})\left(\frac{\xi_k}{\sqrt{A_k}}\right)^2 + \frac{1}{6}\Phi'''(T_{k-1})\left(\frac{\xi_k}{\sqrt{A_k}}\right)^3. \end{aligned}$$

To estimate the right hand side of the last equality, we distinguish two cases.

Case 1: $|\xi_k/\sqrt{A_k}| \le 2 + |T_{k-1}|/2$. By a four-term Taylor expansion, it is obvious that if $|\xi_k/\sqrt{A_k}| \le 1$, then

$$\begin{aligned} \big|B_k\big| &= \left|\frac{1}{24}\Phi^{(4)}\left(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\right)\left|\frac{\xi_k}{\sqrt{A_k}}\right|^4\right| \\ &\le \left|\Phi^{(4)}\left(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\right)\right|\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho}. \end{aligned}$$

If $|\xi_k/\sqrt{A_k}| > 1$, by a three-term Taylor expansion, then

$$\begin{aligned}
\left|B_k\right| &\le \frac{1}{2}\left(\left|\Phi'''\left(T_{k-1}-\vartheta\frac{\xi_k}{\sqrt{A_k}}\right)\right|+\left|\Phi'''(T_{k-1})\right|\right)\left|\frac{\xi_k}{\sqrt{A_k}}\right|^3\\
&\le \left|\Phi'''\left(T_{k-1}-\vartheta'\frac{\xi_k}{\sqrt{A_k}}\right)\right|\left|\frac{\xi_k}{\sqrt{A_k}}\right|^3\\
&\le \left|\Phi'''\left(T_{k-1}-\vartheta'\frac{\xi_k}{\sqrt{A_k}}\right)\right|\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho},
\end{aligned}$$

where

$$\vartheta'=\begin{cases}\vartheta, & \text{if } \left|\Phi'''\big(T_{k-1}-\vartheta\frac{\xi_k}{\sqrt{A_k}}\big)\right|\ge|\Phi'''(T_{k-1})|,\\ 0, & \text{if } \left|\Phi'''\big(T_{k-1}-\vartheta\frac{\xi_k}{\sqrt{A_k}}\big)\right|<|\Phi'''(T_{k-1})|.\end{cases}$$

Using the inequality $\max\{|\Phi'''(t)|,|\Phi''''(t)|\}\le\varphi(t)(2+t^4)$, we find that

$$\begin{aligned}
\left|B_k\mathbf{1}_{\{|\xi_k/\sqrt{A_k}|\le 2+|T_{k-1}|/2\}}\right| &\le \varphi\left(T_{k-1}-\vartheta_1\frac{\xi_k}{\sqrt{A_k}}\right)\left(2+\left(T_{k-1}-\vartheta_1\frac{\xi_k}{\sqrt{A_k}}\right)^4\right)\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho}\\
&\le g_1(T_{k-1})\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho},
\end{aligned}\tag{21}$$

where

$$g_1(z)=\sup_{|t-z|\le 2+|z|/2}\varphi(t)(2+t^4).$$

Case 2: $|\xi_k/\sqrt{A_k}|>2+|T_{k-1}|/2$**.** It is obvious that, for $|\triangle x|>1+|x|/2$,

$$\begin{aligned}
&\left|\Phi(x-\triangle x)-\Phi(x)+\Phi'(x)\triangle x-\frac{1}{2}\Phi''(x)(\triangle x)^2+\frac{1}{6}\Phi'''(x)(\triangle x)^3\right|\\
&\qquad\le\left(\left|\frac{\Phi(x-\triangle x)-\Phi(x)}{|\triangle x|^3}\right|+|\Phi'(x)|+|\Phi''(x)|+|\Phi'''(x)|\right)|\triangle x|^3\\
&\qquad\le\left(8\left|\frac{\Phi(x-\triangle x)-\Phi(x)}{(2+|x|)^3}\right|+|\Phi'(x)|+|\Phi''(x)|+|\Phi'''(x)|\right)|\triangle x|^3\\
&\qquad\le\left(\frac{\widetilde{c}}{(2+|x|)^3}+|\Phi'(x)|+|\Phi''(x)|+|\Phi'''(x)|\right)|\triangle x|^3\\
&\qquad\le\frac{\widehat{c}}{(2+|x|)^3}|\triangle x|^3\\
&\qquad\le\frac{\widehat{c}}{(2+|x|)^3}|\triangle x|^{3+\rho}.
\end{aligned}$$

Hence, we have

$$\left|B_k\mathbf{1}_{\{|\xi_k/\sqrt{A_k}|>2+|T_{k-1}|/2\}}\right|\le g_2(T_{k-1})\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho},\tag{22}$$

where

$$g_2(z)=\frac{\widehat{c}}{(2+|z|)^3}.$$

Denote

$$G(z)=g_1(z)+g_2(z).$$

Combining (21) and (22) together, we get

$$|B_k|\le G(T_{k-1})\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho}.\tag{23}$$

Therefore,

$$\left|J_1\right| = \left|\mathbf{E}\left[\sum_{k=1}^{n} B_k\right]\right| \le \mathbf{E}\left[\sum_{k=1}^{n} G(T_{k-1})\left|\frac{\xi_k}{\sqrt{A_k}}\right|^{3+\rho}\right]. \tag{24}$$

Next, we consider conditional expectation of $|\xi_k|^{3+\rho}$. By condition (5), we get

$$\mathbf{E}\left[|\xi_k|^{3+\rho}\middle|\mathscr{F}_{k-1}\right] \le \epsilon_n^{1+\rho} \triangle \langle X\rangle_k, \tag{25}$$

where $\triangle\langle X\rangle_k = \langle X\rangle_k - \langle X\rangle_{k-1}$ and we know that

$$\triangle \langle X\rangle_k = \triangle V_k = V_k - V_{k-1},\ 1 \le k < n,\ \triangle\langle X\rangle_n \le \triangle V_n, \tag{26}$$

then

$$\mathbf{E}\left[|\xi_k|^{3+\rho}\middle|\mathscr{F}_{k-1}\right] \le \epsilon_n^{1+\rho} \triangle V_k. \tag{27}$$

By (24) and (27), we obtain

$$|J_1| \le R_1 := \epsilon_n^{1+\rho}\left[\sum_{k=1}^{n} \frac{G(T_{k-1})}{A_k^{(3+\rho)/2}} \triangle V_k\right]. \tag{28}$$

To estimate R_1, we introduce the time change τ_t as follow: for any real $t \in [0,T]$,

$$\tau_t = \min\{k \le n : V_k \ge t\},\ \text{ where } \min\emptyset = n. \tag{29}$$

Obviously, for any $t \in [0,T]$, the stopping time τ_t is predictable. In addition, $(\sigma_k)_{k=1,\dots,n+1}$ (with $\sigma_1 = 0$) stands for the increasing sequence of moments when the increasing and stepwise function $\tau_t, t \in [0,T]$, has jumps. It is easy to see that $\triangle V_k = \int_{[\sigma_k,\sigma_{k+1})} \mathrm{d}t$, and that $k = \tau_t$ for $t \in [\sigma_k, \sigma_{k+1})$. Since $\tau_T = n$, we have

$$\sum_{k=1}^{n} \frac{G(T_{k-1})}{A_k^{(3+\rho)/2}} \triangle V_k = \sum_{k=1}^{n} \int_{[\sigma_k,\sigma_{k+1})} \frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}} \mathrm{d}t = \int_0^T \frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}} \mathrm{d}t. \tag{30}$$

Let $a_t = c_*^2\gamma^2 + T - t$. Because of $\triangle V_{\tau_t} \le 2\epsilon_n^2 + 2\delta_n^2$ (cf. Lemma 4), we know that

$$t \le V_{\tau_t} = V_{\tau_t-1} + \triangle V_{\tau_t} \le t + 2\epsilon_n^2 + 2\delta_n^2, \quad t \in [0,T]. \tag{31}$$

Assume $c_* \ge 2$, then we have

$$\frac{1}{2}a_t \le A_{\tau_t} = c_*^2\gamma^2 + T - V_{\tau_t} \le a_t, \quad t \in [0,T]. \tag{32}$$

Note that $G(z)$ is symmetric and is non-increasing in $z \ge 0$. The last bound implies that

$$R_1 \le 2^{(3+\rho)/2}\epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{(3+\rho)/2}} \mathbf{E}\left[G\left(\frac{u - X_{\tau_t-1}}{a_t^{1/2}}\right)\right] \mathrm{d}t. \tag{33}$$

Note also that $G(z)$ is a symmetric integrable function of bounded variation. By Lemma 6, it is obvious that

$$\mathbf{E}\left[G\left(\frac{u - X_{\tau_t-1}}{a_t^{1/2}}\right)\right] \le c_6 \sup_z \left|\mathbf{P}\left(X_{\tau_t-1} \le z\right) - \Phi(z)\right| + c_7\sqrt{a_t}. \tag{34}$$

Because of $c_* \ge 2, V_{\tau_t-1} = V_{\tau_t} - \triangle V_{\tau_t},\ V_{\tau_t} \ge t$ and $\triangle V_{\tau_t} \le 2\epsilon_n^2 + 2\delta_n^2$, we obtain

$$V_n - V_{\tau_t-1} = V_n - V_{\tau_t} + \triangle V_{\tau_t} \le 2\epsilon_n^2 + 2\delta_n^2 + T - t \le a_t. \tag{35}$$

Therefore

$$\begin{aligned}
\mathbf{E}\left[\left(X_n - X_{\tau_t-1}\right)^2\middle|\mathscr{F}_{\tau_t-1}\right] &= \mathbf{E}\left[\sum_{k=\tau_t}^{n} \mathbf{E}[\xi_k^2|\mathscr{F}_{k-1}]\middle|\mathscr{F}_{\tau_t-1}\right] \\
&= \mathbf{E}\left[\langle X\rangle_n - \langle X\rangle_{\tau_t-1}\middle|\mathscr{F}_{\tau_t-1}\right] \\
&\le \mathbf{E}[V_n - V_{\tau_t-1}|\mathscr{F}_{\tau_t-1}] \\
&\le a_t.
\end{aligned}$$

Then, by Lemma 5, we deduce that for any $t \in [0, T]$,

$$\sup_z \left|\mathbf{P}\big(X_{\tau_t - 1} \le z\big) - \Phi(z)\right| \le c_8 \sup_z \left|\mathbf{P}\big(X_n \le z\big) - \Phi(z)\right| + c_9\sqrt{a_t}. \tag{36}$$

Combining (28), (33), (34) and (36) together, we get

$$|J_1| \le c_{10}\epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{(3+\rho)/2}} \mathrm{d}t \sup_z \left|\mathbf{P}\big(X_n \le z\big) - \Phi(z)\right| + c_{11}\epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{1+\rho/2}} \mathrm{d}t. \tag{37}$$

Taking some elementary computations, it follows that

$$\int_0^T \frac{1}{a_t^{(3+\rho)/2}} \mathrm{d}t = \int_0^T \frac{1}{(c_*^2\gamma^2 + T - t)^{(3+\rho)/2}} \mathrm{d}t \le \frac{2}{c_*^{1+\rho}(1+\rho)\gamma^{1+\rho}} \tag{38}$$

and

$$\int_0^T \frac{1}{a_t^{1+\rho/2}} \mathrm{d}t = \int_0^T \frac{1}{(c_*^2\gamma^2 + T - t)^{1+\rho/2}} \mathrm{d}t \le \frac{2}{c_*^{\rho}\rho\gamma^{\rho}}. \tag{39}$$

This yields

$$\left|J_1\right| \le \frac{c_{12}}{c_*^{1+\rho}} \sup_z \left|\mathbf{P}\big(X_n \le z\big) - \Phi(z)\right| + \frac{c_{\rho,1}\epsilon_n}{\rho}. \tag{40}$$

Control of J_2. Since $0 \le \triangle V_k - \triangle\langle X\rangle_k \le 2\delta^2 \mathbf{1}_{\{k=n\}}$, we have

$$|J_2| \le \mathbf{E}\left[\frac{1}{2A_n}\left|\varphi'(T_{n-1})(\triangle V_n - \triangle\langle X\rangle_n)\right|\right].$$

Denote $\widetilde{G}(z) = \sup_{|z-t|\le 1}|\varphi'(t)|$, and then $|\varphi'(z)| \le \widetilde{G}(z)$ for any real z. Since $A_n = c_*^2\gamma^2$, then we get the following estimation:

$$|J_2| \le \frac{1}{c_*^2}\mathbf{E}\big[\widetilde{G}(T_{n-1})\big].$$

Note that $\widetilde{G}$ is non-increasing in $z \ge 0$, and thus it has bounded variation on $\mathbf{R}$. By Lemma 6, we get

$$|J_2| \le \frac{c_{13}}{c_*^2} \sup_z \left|\mathbf{P}\big(X_{n-1} \le z\big) - \Phi(z)\right| + c_{*,2}(\epsilon_n + \delta_n). \tag{41}$$

Then, by Lemma 5, we deduce that

$$\sup_z \left|\mathbf{P}\big(X_{n-1} \le z\big) - \Phi(z)\right| \le c_{14} \sup_z \left|\mathbf{P}\big(X_n \le z\big) - \Phi(z)\right| + c_{15}\epsilon_n. \tag{42}$$

This yields

$$|J_2| \le \frac{c_{16}}{c_*^2} \sup_z \left|\mathbf{P}\big(X_n \le z\big) - \Phi(z)\right| + c_{\rho,2}(\epsilon_n + \delta_n). \tag{43}$$

Control of J_3. By a two-term Taylor expansion, it follows that

$$|J_3| = \frac{1}{8}\mathbf{E}\left[\sum_{k=1}^n \frac{1}{(A_k - \vartheta_k \triangle A_k)^2}\varphi'''\left(\frac{u - X_{k-1}}{\sqrt{A_k - \vartheta_k \triangle A_k}}\right)(\triangle A_k)^2\right].$$

Note that $c_* \ge 2, \triangle A_k \le 0$ and, by Lemma 4, $|\triangle A_k| = \triangle V_k \le 2\epsilon_n^2 + 2\delta_n^2$. We obtain

$$A_k \le A_k - \vartheta_k \triangle A_k \le c_*^2\gamma^2 + T - V_k + 2\epsilon_n^2 + 2\delta_n^2 \le 2A_k. \tag{44}$$

Denote $\widehat{G}(z) = \sup_{|t-z|\le 2}|\varphi'''(t)|$. Then $\widehat{G}(z)$ is symmetric, and is non-increasing in $z \ge 0$. Using (44), we get

$$|J_3| \le (2\epsilon_n^2 + 2\delta_n^2)\mathbf{E}\left[\sum_{k=1}^n \frac{1}{A_k^2}\widehat{G}\left(\frac{T_{k-1}}{\sqrt{2}}\right)\triangle V_k\right]. \tag{45}$$

By an argument similar to that of (40), we get

$$\begin{aligned}|J_3| &\le \frac{c_{17}(2\epsilon_n^2+2\delta_n^2)}{c_*^2\gamma^2}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+\frac{2c_{18}(2\epsilon_n^2+2\delta_n^2)}{c_*\gamma}\\ &\le \frac{c_{19}}{c_*^2}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+\frac{4c_{18}(\epsilon_n+\delta_n)^2}{c_*\gamma}\\ &\le \frac{c_{19}}{c_*^2}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+c_{\rho,3}(\epsilon_n+\delta_n).\end{aligned}\tag{46}$$

Combining (17), (40), (43) and (46) together, we get

$$\big|\mathbf{E}\big[\Phi_u\big(X_n,A_n\big)\big]-\mathbf{E}\big[\Phi_u\big(X_0,A_0\big)\big]\big|\le\frac{c_{20}}{c_*^{1+\rho}}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+\frac{\widehat{c}_\rho}{\rho}(\epsilon_n+\delta_n),$$

By (15), we know that

$$\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|\le\frac{c_{21}}{c_*^{1+\rho}}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+\frac{\widetilde{c}_\rho}{\rho}(\epsilon_n+\delta_n),$$

from which, choosing $c_*^{1+\rho}=\max\{2c_{21},2^{1+\rho}\}$, we get

$$\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|\le\frac{2\widetilde{c}_\rho(\epsilon_n+\delta_n)}{\rho}.\tag{47}$$

3.3. *Proof of Theorem 2*

Following the method of Bolthausen [2], we enlarge the sequence $(\xi_i,\mathscr{F}_i)_{1\le i\le n}$ to $\big(\widehat{\xi}_i,\widehat{\mathscr{F}}_i\big)_{1\le i\le N}$ such that $\big\langle\widehat{X}\big\rangle_N:=\sum_{i=1}^N\mathbf{E}\big[\widehat{\xi}_i^2|\widehat{\mathscr{F}}_{i-1}\big]=1$ a.s., and then apply Theorem 1 to the enlarged sequence. Consider the stopping time

$$\tau=\sup\{k\le n:\langle X\rangle_k\le 1\}.\tag{48}$$

Assume that $0\le\varepsilon\le\epsilon_n$. Let $r=\big\lfloor\frac{1-\langle X\rangle_\tau}{\varepsilon^2}\big\rfloor$, where $\lfloor x\rfloor$ denotes the "integer part" of x. It is easy to see that $r\le\big\lfloor\frac{1}{\varepsilon^2}\big\rfloor$. Set $N=n+r+1$. Let $(\zeta_i)_{i\ge1}$ be a sequence of independent Rademacher random variables, which is independent of the martingale differences $(\xi_i)_{1\le i\le n}$. Consider the random variables $\big(\widehat{\xi}_i,\widehat{\mathscr{F}}_i\big)_{1\le i\le N}$ defined as follows:

$$\widehat{\xi}_i=\begin{cases}\xi_i\ \text{ a.s.,} & \text{if } i\le\tau,\\ \varepsilon\zeta_i\ \text{ a.s.,} & \text{if } \tau+1\le i\le\tau+r,\\ \big(1-\langle X\rangle_\tau-r\varepsilon^2\big)^{1/2}\zeta_i\ \text{ a.s.,} & \text{if } i=\tau+r+1,\\ 0\ \text{ a.s.,} & \text{if } \tau+r+1\le i\le N,\end{cases}$$

and $\widehat{\mathscr{F}}_i=\sigma\big(\widehat{\xi}_1,\widehat{\xi}_2,\dots,\widehat{\xi}_i\big)$.

Clearly, $\big(\widehat{\xi}_i,\widehat{\mathscr{F}}_i\big)_{1\le i\le N}$ still forms a martingale difference sequence with respect to the enlarged filtration. Then $\widehat{X}_k=\sum_{i=1}^k\widehat{\xi}_i$, $k=0,\dots,N$, with $\widehat{X}_0=0$, is also a martingale. Moreover, it holds that $\big\langle\widehat{X}\big\rangle_N=1$, $\mathbf{E}\big[\widehat{\xi}_i^3\big|\widehat{\mathscr{F}}_{i-1}\big]=0$ and

$$\mathbf{E}\big[\big|\widehat{\xi}_i\big|^{3+\rho}\big|\widehat{\mathscr{F}}_{i-1}\big]\le\epsilon_n^{1+\rho}\mathbf{E}\big[\widehat{\xi}_i^2\big|\widehat{\mathscr{F}}_{i-1}\big],\qquad\text{a.s.}$$

By Theorem 1, we have

$$D\big(\widehat{X}_N\big)\le\frac{c_\rho\epsilon_n}{\rho}.\tag{49}$$

Using Lemma 7, we obtain that

$$D(X_n)\le 2D\big(\widehat{X}_N\big)+3\big\|\mathbf{E}\big[\big|X_n-\widehat{X}_N\big|^{2p}\big|\widehat{X}_N\big]\big\|_1^{1/(2p+1)}\le\frac{2c_\rho\epsilon_n}{\rho}+3\big(\mathbf{E}\big[\big|\widehat{X}_N-X_n\big|^{2p}\big]\big)^{1/(2p+1)}.\tag{50}$$

Since τ is a stopping time and

$$\widehat{X}_N - X_n = \sum_{i=\tau+1}^{N} \big(\widehat{\xi}_i - \xi_i\big), \qquad \text{where put } \xi_i = 0 \text{ for } i > n, \tag{51}$$

$(\widehat{\xi}_i - \xi_i, \widehat{\mathscr{F}}_i)_{i\geq\tau+1}$ still forms a martingale difference sequence. Applying Theorem 2.11 of Hall and Heyde [8], we get

$$\begin{aligned}\mathbf{E}\big[\big|\widehat{X}_N - X_n\big|^{2p}\big] &\leq \mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\big|\widehat{X}_i - X_i\big|^{2p}\Big] \\ &\leq c_p\bigg(\mathbf{E}\bigg[\bigg|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\widehat{\xi}_i - \xi_i\big)^2\big|\widehat{\mathscr{F}}_{i-1}\big]\bigg|^p\bigg] + \mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\big|\widehat{\xi}_i - \xi_i\big|^{2p}\Big]\bigg).\end{aligned} \tag{52}$$

As $\mathbf{E}[\xi_i\widehat{\xi}_i|\widehat{\mathscr{F}}_{i-1}] = 0$ for all $i \geq \tau+1$, we have

$$\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\widehat{\xi}_i - \xi_i\big)^2\big|\widehat{\mathscr{F}}_{i-1}\big] = \sum_{i=\tau+1}^{N}\mathbf{E}\big[\widehat{\xi}_i^2\big|\widehat{\mathscr{F}}_{i-1}\big] + \sum_{i=\tau+1}^{n}\mathbf{E}\big[\xi_i^2\big|\widehat{\mathscr{F}}_{i-1}\big] = 1 - 2\langle X\rangle_\tau + \langle X\rangle_n.$$

Noting that $1-\mathbf{E}[\xi_{\tau+1}^2|\mathscr{F}_\tau] \leq \langle X\rangle_\tau$. Consequently, using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p+|b|^p)$, $p\geq 1$, and Jensen's inequality, we derive that

$$\begin{aligned}\bigg|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\widehat{\xi}_i - \xi_i\big)^2\big|\widehat{\mathscr{F}}_{i-1}\big]\bigg|^p &\leq \big|\langle X\rangle_n - 1 + 2\mathbf{E}\big[\xi_{\tau+1}^2\big|\mathscr{F}_\tau\big]\big|^p \\ &\leq 2^{2p-1}\big(\big|\langle X\rangle_n - 1\big|^p + \big|\mathbf{E}\big[\xi_{\tau+1}^2\big|\mathscr{F}_\tau\big]\big|^p\big) \\ &\leq 2^{2p-1}\big(\big|\langle X\rangle_n - 1\big|^p + \mathbf{E}\big[|\xi_{\tau+1}|^{2p}\big|\mathscr{F}_\tau\big]\big).\end{aligned} \tag{53}$$

Taking expectations on both sides of the last inequality, we deduce that

$$\begin{aligned}\mathbf{E}\bigg[\bigg|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\widehat{\xi}_i - \xi_i\big)^2\big|\widehat{\mathscr{F}}_{i-1}\big]\bigg|^p\bigg] &\leq 2^{2p-1}\big(\mathbf{E}\big[\big|\langle X\rangle_n - 1\big|^p\big] + \mathbf{E}\big[|\xi_{\tau+1}|^{2p}\big]\big) \\ &\leq 2^{2p-1}\Big(\mathbf{E}\big[\big|\langle X\rangle_n - 1\big|^p\big] + \mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]\Big).\end{aligned} \tag{54}$$

Similarly, using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p+|b|^p)$, $p\geq 1$,

$$\begin{aligned}\mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\big|\widehat{\xi}_i - \xi_i\big|^{2p}\Big] &\leq 2^{2p-1}\mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\big(|\xi_i|^{2p} + \big|\widehat{\xi}_i\big|^{2p}\big)\Big] \\ &\leq 2^{2p-1}\Big(\mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big] + \varepsilon^{2p}\Big).\end{aligned} \tag{55}$$

Combining (52), (54) and (55) together, we obtain

$$\mathbf{E}\big[\big|\widehat{X}_N - X_n\big|^{2p}\big] \leq \widehat{c}_p\Big(\mathbf{E}\big[\big|\langle X\rangle_n - 1\big|^p\big] + \mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big] + \varepsilon^{2p}\Big). \tag{56}$$

Finally, applying the last inequality to (50) and let $\varepsilon \to 0$, then we have

$$D(X_n) \leq \widetilde{c}_\rho\frac{\epsilon_n}{\rho} + \widetilde{c}_p\Big(\mathbf{E}\big[\big|\langle X\rangle_n - 1\big|^p\big] + \mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]\Big)^{1/(2p+1)}.$$

This completes the proof of Theorem 2.

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