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On the canonical solution of $\bar\partial$ on polydisks

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Abstract. We observe that the recent result of Chen–McNeal [6] implies that the canonical solution operator satisfies Sobolev estimates with a loss of $n-2$ derivatives on the polydisk Δ^n and particularly is exact regular on Δ^2.

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1. Introduction

This note is motivated by the following $\bar\partial$ question on the bidisk Δ^2 raised in [12].

Question 1. *For any $f \in W^{1,2}_{(0,1)}\left(\Delta^2\right)$ with $\bar\partial f = 0$, can one find a solution $u \in W^{1,2}\left(\Delta^2\right)$ such that $\bar\partial u = f$?*

The solution of this question will lead to the closed range property of $\bar\partial$ on the high dimensional annuli domain. Although Question 1 is already answered in the affirmative by Chakrabarti–Laurent–Shaw in [3] by the powerful L^2-Čech cohomology theory, this note provides the canonical solution with Sobolev estimates.

Recently, Chen–McNeal defined a $\bar\partial$ solution operator T on product domains in [6, 7] using Cauchy transform and derived L^p estimates. We give a brief statement of Chen–McNeal's results and readers are referred to [6] (also [7]) for details. Let $D = D_1 \times \cdots \times D_n$ be a product of piecewise C^1 smooth bounded domains in $\mathbb{C}$. Write

$$\emptyset \neq J = \left\{j_1, \ldots, j_l\right\} \subset \{1, \ldots, n\} \quad \text{with} \quad 1 \leq j_1 < \cdots < j_l \leq n.$$

For $f = \sum_j f_j d\bar z_j \in C^\infty_{(0,1)}(D)$, denote $f_J^{J^c} = \frac{\partial^{l-1} f_{j_1}}{\partial \bar z_{j_2} \cdots \partial \bar z_{j_l}}$ with other variables z_j fixed for all $j \not\in J$. For those $(0,1)$-forms f on D such that $f_J^{J^c} \in L^1(D)$ for $\emptyset \neq J \subset \{1, \ldots, n\}$, Chen–McNeal solution operator

$$Tf = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} \mathscr{C}^J \left(f_J^{J^c}\right)$$

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is introduced in [6], where $\mathscr{C}^J$ is the multi-Cauchy transform is defined as

$$\mathscr{C}^J(u) = -\frac{1}{\pi^l}\int_{D_{j_1}\times\ldots\times D_{j_l}} \frac{u(z)}{\left(z_{j_1}-w_{j_1}\right)\cdots\left(z_{j_l}-w_{j_l}\right)} d A\left(z_{j_1}\right)\cdots d A\left(z_{j_l}\right),$$

for $u \in L^1(D)$ [6]. In particular, if $f \in W^{n-1,1}_{(0,1)}(D)$, Tf can be defined and the following L^p estimates is a special case of the result proved by Chen–McNeal (cf. [6, Corollary 2.17]). Note that when $n = 2$, the result is proved in [7].

Theorem 2 (Chen–McNeal). *Let $p > 1$ and $f \in W^{n-1,p}_{(0,1)}(D)\cap Ker(\bar\partial)$. Then $u = Tf$ solves $\bar\partial u = f$ and satisfies*

$$\|Tf\|_{0,p} \lesssim \|f\|_{n-1,p}.$$

We apply L^p estimate of T by Chen–McNeal to the polydisk Δ^n in $\mathbb{C}^n$. Let $K = T - \mathscr{B}\circ T$, where $\mathscr{B}$ is the classical Bergman projection on Δ^n. We observe that Theorem 2 easily implies the following Theorem 3.

Theorem 3. *Let $p \in (1,\infty)$, $k \geq n-1$. For $f \in W^{k,p}_{(0,1)}(\Delta^n)$ with $\bar\partial f = 0$, $u = Kf$ solves the $\bar\partial$ equation $\bar\partial u = f$ and satisfies the Sobolev estimates*

$$\|Kf\|_{k+2-n,p} \lesssim \|f\|_{k,p}.$$

When $p = 2$, K is the canonical solution operator. Namely, for $f \in L^2_{(0,1)}(\Delta^n)$, $u = Kf$ provides a solution to $\bar\partial u = f$ with minimal L^2 norm. A direct consequence for $n = p = 2$ answers Shaw's question.

Corollary 4. *The canonical solution operator K on Δ^2 is exact regular. Namely, given $k \geq 0$, for any $f \in W^{k,2}_{(0,1)}\left(\Delta^2\right)$ with $\bar\partial f = 0$,*

$$\|Kf\|_{k,2} \lesssim \|f\|_{k,2}. \tag{1}$$

Remark 5. As pointed out by the referee, it follows from [4, Theorem 1.2] that the canonical solution operator on Δ^2 maps $W^{2k,2}_{(0,1)}\left(\Delta^2\right)$ to $W^{k,2}\left(\Delta^2\right)$ continuously for any $k \geq 0$.

Remark 6. By the non-compactness of the $\bar\partial$-Neumann operator on Δ^2 (cf. [9]) and thus the non-compactness of the canonical solution operator K (cf. [14, Proposition 4.2]), the Sobolev estimates in Corollary 4 is optimal in the sense that given any $\epsilon > 0$, there does not exist a constant $C_\epsilon > 0$, such that $\|Kf\|_{\epsilon,2} \leq C_\epsilon \|f\|_{0,2}$ for all $f \in L^2_{(0,1)}\left(\Delta^2\right)\cap Ker\left(\bar\partial\right)$ on Δ^2. Nevertheless, it would be interesting to know if the Sobolev estimates in Theorem 3 is optimal.

2. Proof of Theorem 3

The proof of Theorem 3 is a combination of Sobolev estimates of three operators: Beurling transform, Bergman projection and Chen–McNeal solution operator in [6, 7].

2.1. *Cauchy transform on the planar domain*

Let U be bounded domain in $\mathbb{C}$ and $B(w,R)$ be the ball centered at w with radius R. For $f \in C^\infty_c(U)$, recall that Cauchy transformation on U is

$$(\mathscr{C}f)(w) := -\frac{1}{\pi}\int_U \frac{f(z)}{z-w}\, dA(z),$$

and the Beurling transform (or Hilbert transform) on U is

$$(\mathscr{H}f)(w) := -\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{U\setminus B(w,\epsilon)} \frac{f(z)}{(z-w)^2}\, dA(z).$$

Theorem 7. *Let U be bounded domain in $\mathbb{C}$ and $f \in L^p(U)$. Then*

(1) *$\mathscr{C} : L^p(U) \to L^p(U)$ is bounded for any $p > 1$ and*

$$\frac{\partial(\mathscr{C} f)}{\partial \overline{w}} = f$$

holds in the distribution sense.

(2) *$\mathscr{H} : L^p(U) \to L^p(U)$ is bounded for any $p > 1$ and*

$$\mathscr{H} f = \frac{\partial(\mathscr{C} f)}{\partial w}$$

holds in the distribution sense.

This result is well known in complex analysis (cf. [1,10]; a stronger result of Part (1) is also given in [6]).

It immediately follows from Theorem 7 that $\mathscr{C}$ is bounded from $L^p(U)$ to $W^{1,p}(U)$ for any $1 < p < \infty$. However, it is not that clear for general Sobolev spaces.

Corollary 8. *Let U be a bounded domain in $\mathbb{C}$. For $p > 1, k \in \mathbb{N}$, if $\mathscr{H}$ is bounded from $W^{k,p}(U)$ to $W^{k,p}(U)$, then $\mathscr{C} : W^{k,p}(U) \to W^{k+1,p}(U)$ is bounded.*

Proof. The case of $k = 0$ follows from Theorem 7. Consider $k \geq 1$. Let $D^\alpha = \frac{\partial^{k+1}}{\partial w^a \partial \overline{w}^b}$ be a (mixed) partial derivative in w and $\overline{w}$ of order $k+1$. For any $f \in W^{k,p}(U)$, if $b \geq 1$, then

$$D^\alpha(\mathscr{C} f) = \frac{\partial^k}{\partial w^a \partial \overline{w}^{b-1}} f$$

and the thus

$$\left\| D^\alpha(\mathscr{C} f) \right\|_{0,p} \leq \|f\|_{k,p}.$$

Otherwise,

$$\frac{\partial^{k+1}}{\partial w^{k+1}}(\mathscr{C} f) = \frac{\partial^k}{\partial w^k} \mathscr{H} f$$

and thus

$$\left\| \frac{\partial^{k+1}}{\partial w^{k+1}}(\mathscr{C} f) \right\|_{0,p} \leq \|\mathscr{H} f\|_{k,p} \lesssim \|f\|_{k,p} \text{ by the assumption on } \mathscr{H}.$$

□

The proof illustrates the idea in the proof of the main Theorem 3. As one can see, the Sobolev regularity of $\mathscr{H}$ plays a crucial role and it still remains open that what is the minimal boundary condition on the planar domain to assert the boundedness of $\mathscr{H}$ from $W^{k,p}(U)$ to $W^{k,p}(U)$ for all $p > 1, k \in \mathbb{N}$ (cf. [11] and references therein for the recent study on this subject). The following result is proved in [11, Example 1.4].

Theorem 9. *$\mathscr{H}$ is bounded from $W^{k,p}(\Delta)$ to $W^{k,p}(\Delta)$ on the unit disk Δ for all $p > 1, k \in \mathbb{N}$.*

2.2. *Bergman projection*

The following result is well known and the key of the proof is a holomorphic integration by parts. The proof is implicitly contained in [2, 5, 13] and it also follows from combining Fubini theorem with [8, Theorem 2.12].

Proposition 10. *The Bergman projection $\mathscr{B} : W^{k,p}(\Delta^n) \to W^{k,p}(\Delta^n)$ is bounded for any $p > 1$, $k \geq 0$.*

Proof. In [8], Edholm and McNeal proved the one-dimensional case. Namely, for any $f \in W^{k,p}(\Delta)$,

$$\int_\Delta \left| \frac{\partial^k}{\partial w^k} \int_\Delta \frac{1}{(1-w\bar{z})^2} f(z)\,dA(z) \right|^p dA(w) \lesssim \sum_{l=0}^{k} \sum_{a+b=l} \int_\Delta \left| \frac{\partial^l}{\partial w^a \partial \overline{w}^b} f(w) \right|^p dA(w). \tag{2}$$

For the higher dimensional case, let α be a multi-index with $|\alpha| \le k$. For $f \in W^{k,p}(\Delta^n)$, we have

$$\begin{aligned}
&\int_{\Delta^n} \left| \frac{\partial^\alpha}{\partial w^\alpha} \big((\mathscr{B} f)(w)\big) \right|^p dV(w) \\
&\qquad = \int_{\Delta^n} \left| \frac{\partial^{\alpha_1}}{\partial w_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial w_n^{\alpha_n}} \int_{\Delta^n} \frac{1}{(1-w_1\bar{z}_1)^2 \cdots (1-w_n\bar{z}_n)^2} f(z)\,dV(z) \right|^p dV(w) \\
&\qquad = \int_{\Delta^n} \left| \frac{\partial^{\alpha_1}}{\partial w_1^{\alpha_1}} \int_\Delta \frac{1}{(1-w_1\bar{z}_1)^2} \frac{\partial^{\alpha_2}}{\partial w_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial w_n^{\alpha_n}} \right. \\
&\qquad\quad \left. \int_{\Delta^{n-1}} \frac{1}{(1-w_2\bar{z}_2)^2 \cdots (1-w_n\bar{z}_n)^2} f(z)\,dV(z_2,\ldots,z_n)\,dA(z_1) \right|^p dV(w) \\
&\qquad \lesssim \sum_{l=0}^{\alpha_1} \sum_{a+b=l} \int_{\Delta^n} \left| \frac{\partial^l}{\partial w_1^a \partial \overline{w}_1^b} \frac{\partial^{\alpha_2}}{\partial w_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial w_n^{\alpha_n}} \right. \\
&\qquad\quad \left. \int_{\Delta^{n-1}} \frac{1}{(1-w_2\bar{z}_2)^2 \cdots (1-w_n\bar{z}_n)^2} f(w_1,z_2,\ldots,z_n)\,dV(z_2,\ldots,z_n) \right|^p dV(w).
\end{aligned}$$

The last inequality follows from the Fubini Theorem and (2) applied to the integration in z_1 and w_1. Repeating this process for $z_2, \cdots, z_n$, we have the following estimate:

$$\begin{aligned}
&\int_{\Delta^n} \left| \frac{\partial^\alpha}{\partial w^\alpha} \big((\mathscr{B} f)(w)\big) \right|^p dV(w) \\
&\lesssim \sum_{l_1=0}^{\alpha_1} \cdots \sum_{l_n=0}^{\alpha_n} \sum_{a_1+b_1=l_1} \cdots \sum_{a_n+b_n=l_n} \int_{\Delta^n} \left| \frac{\partial^{l_1}}{\partial w_1^{a_1}\,\partial \overline{w}_1^{b_1}} \cdots \frac{\partial^{l_n}}{\partial w_n^{a_n}\,\partial \overline{w}_n^{b_n}} f(w_1,\cdots,w_n) \right|^p dV(w).
\end{aligned}$$

The proposition 10 is thus proved. □

2.3. *Consequence of Chen–McNeal solution operator*

Theorem 11. *For any $p > 1, k \ge n-1$ and any $f \in W^{k,p}_{(0,1)}(\Delta^n)$ with $\bar\partial f = 0$, Tf satisfies $\bar\partial T f = f$ with Sobolev estimates*

$$\|Tf\|_{k+2-n,p} \lesssim \|f\|_{k,p}.$$

Proof. By the density it suffices to prove the a priori estimate. Assume $f \in C^\infty(\overline{\Delta^n})$. For any $j \in \{1,\dots,n\}$, we use D_j^k to denote $\frac{\partial^k}{\partial w_j^a \partial \overline{w}_j^b}$ for any a, b with $a+b=k$. Then

$$\begin{aligned}
&\left\|D_j^k T f\right\|_{0,p} \\
&\quad= \left\|\sum_{1\le|J|\le n} D_j^k \mathscr{C}^J\left(f_J^{J^c}\right)\right\|_{0,p} \le \sum_{1\le|J|\le n} \left\|D_j^k \mathscr{C}^J\left(f_J^{J^c}\right)\right\|_{0,p} \\
&\quad= \sum_{j\notin J}\left\|\mathscr{C}^J\left(D_j^k f_J^{J^c}\right)\right\|_{0,p} + \sum_{j\in J}\left\|D_j^k \mathscr{C}^j \mathscr{C}^{J\setminus\{j\}}\left(f_J^{J^c}\right)\right\|_{0,p} \\
&\quad\lesssim \sum_{j\notin J}\left\|\mathscr{C}^J\left(D_j^k f_J^{J^c}\right)\right\|_{0,p} + \sum_{j\in J}\left\|D_j^{k-1}\mathscr{C}^{J\setminus\{j\}}\left(f_J^{J^c}\right)\right\|_{0,p} + \sum_{j\in J}\left\|\frac{\partial^{k-1}}{\partial w_j^{k-1}}\mathscr{H}^j \mathscr{C}^{J\setminus\{j\}}\left(f_J^{J^c}\right)\right\|_{0,p} \\
&\quad\lesssim \sum_{j\notin J}\left\|\mathscr{C}^J\left(D_j^k f_J^{J^c}\right)\right\|_{0,p} + \sum_{a+b\le k-1}\sum_{j\in J}\left\|\frac{\partial^{a+b}}{\partial w_j^a \partial \overline{w}_j^b}\mathscr{C}^{J\setminus\{j\}}\left(f_J^{J^c}\right)\right\|_{0,p} \\
&\quad= \sum_{j\notin J}\left\|\mathscr{C}^J\left(D_j^k f_J^{J^c}\right)\right\|_{0,p} + \sum_{a+b\le k-1}\sum_{j\in J}\left\|\mathscr{C}^{J\setminus\{j\}}\left(\frac{\partial^{a+b}}{\partial w_j^a \partial \overline{w}_j^b} f_J^{J^c}\right)\right\|_{0,p} \\
&\quad\lesssim \|f\|_{n+k-2,p} + \|f\|_{n+k-2,p},
\end{aligned} \tag{3}$$

where the fifth line follows from applying Theorem 9 and the Fubini theorem repeatedly, and the seventh line follow from applying Theorem 7, the Fubini theorem repeatedly, and the definition of $f_J^{J^c}$. The case of the general differential operator follows from the similar argument. □

Now Theorem 3 is a simple corollary of Proposition 10 and Theorem 11.

Proof of Theorem 3.

$$\|Kf\|_{k+2-n,p} \le \|Tf\|_{k+2-n,p} + \|\mathscr{B}(Tf)\|_{k+2-n,p} \lesssim \|Tf\|_{k+2-n,p} \lesssim \|f\|_{k,p}.$$

□

Proof of Corollary 4. When $k=0$, (1) follows from the standard L^2 theory. When k is a positive integer, (1) follows from Theorem 3. For general $k \ge 0$, (1) follows from interpolation. □

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