INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Song Yang and Xun Yu

Rational cubic fourfolds in Hassett divisors

Volume 358, issue 2 (2020), p. 129-137

Published online: 15 June 2020

https://doi.org/10.5802/crmath.4

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2020, 358, nº 2, p. 129-137
https://doi.org/10.5802/crmath.4

Algebraic Geometry, Hodge Structure / *Géométrie algébrique, Structure de Hodge*

Rational cubic fourfolds in Hassett divisors

Cubiques rationnelles de dimension 4 dans les diviseurs de Hassett

Song Yang [a] **and Xun Yu** [a]

[a] Center for Applied Mathematics, Tianjin University, Weijin Road 92, Tianjin 300072, China.

E-mails: syangmath@tju.edu.cn, xunyu@tju.edu.cn.

Abstract. We prove that every Hassett's Noether–Lefschetz divisor of special cubic fourfolds contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in the moduli space of smooth cubic fourfolds.

Résumé. Nous prouvons que chaque diviseur de Hassett–Noether–Lefschetz de cubiques spéciales de dimension 4 contient une union de trois sous-variétés paramétrant des cubiques rationnelles de dimension 4, de codimension deux dans l'espace de modules des cubiques lisses de dimension 4.

2020 Mathematics Subject Classification. 14C30, 14E08, 14M20.

Funding. This work is partially supported by the National Natural Science Foundation of China (Grant No. 11701413, No. 11701414, No. 11831013).

Manuscript received 19th January 2020, accepted 28th January 2020.

1. Introduction

The rationality problem of smooth cubic fourfolds is one of the most widely open problems in algebraic geometry; we refer to the survey [12] for a comprehensive progress. It has been known that all smooth cubic surfaces are rational since the 19th century. In 1972, Clemens–Griffiths [8] proved that all smooth cubic threefolds are nonrational. For smooth cubic fourfolds, however, the situation is very mysterious. It is expected that a very general smooth cubic fourfold should be nonrational (cf. [10, 11]). Until now, many examples of smooth rational cubic fourfolds are known, but the existence of a smooth nonrational cubic fourfold is still unknown.

Using Hodge theory and lattice theory, Hassett [11] introduced the notion of *special cubic fourfolds* (see Definition 3). Simultaneously, Hassett [11, Theorem 1.0.1] gave a countably infinite list of irreducible divisors $\mathscr{C}_d$ of special cubic fourfolds in the moduli space $\mathscr{C}$ of smooth cubic fourfolds and showed that $\mathscr{C}_d$ is nonempty if and only if $d > 6$ and $d \equiv 0, 2 \pmod 6$. Such a nonempty $\mathscr{C}_d$ is called a *Hassett's Noether–Lefschetz divisor* (for short a *Hassett divisor*).

Currently, there exist two popular point of views toward the rationality of smooth cubic fourfolds and both have associated $K3$ surfaces:

- Hassett's Hodge-theoretic result ([11, Theorem 5.1.3]): a smooth cubic fourfold X has a Hodge-theoretically associated $K3$ surface if and only if its moduli point $[X] \in \mathscr{C}_d$ for some *admissible value* d (i.e., $d > 6$, $d \equiv 0,2 \pmod 6$, $4 \nmid d$, $9 \nmid d$ and $p \nmid d$ for any odd prime $p \equiv 2 \pmod 3$));
- Kuznetsov's derived categorical conjecture ([16, Conjecture 1.1]): a smooth cubic fourfold X is rational if and only if its Kuznetsov component $\mathrm{Ku}(X)$ is derived equivalent to a $K3$ surface (i.e., $\mathrm{Ku}(X)$ is called *geometric*), where $\mathrm{Ku}(X)$ is the right orthogonal to the exceptional collection $\{\mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\}$ in the bounded derived category of coherent sheaves on X.

It is important to notice that Kuznetsov's conjecture implies that a very general cubic fourfold is not rational, since for a very general cubic fourfold its Kuznetsov component can not be geometric. Addington–Thomas [2, Theorem 1.1] showed that for a smooth cubic fourfold X if $\mathrm{Ku}(X)$ is geometric then $[X] \in \mathscr{C}_d$ for some admissible d, and conversely for any admissible value d, the set of cubic fourfolds $[X] \in \mathscr{C}_d$ for which $\mathrm{Ku}(X)$ is geometric is a Zariski open dense subset; see also Huybrechts [13] for the twisted version and a further study. Recently, based on Bridgeland stability conditions on $\mathrm{Ku}(X)$ constructed in [5, Theorem 1.2], Bayer–Lahoz–Macrì–Nuer–Perry–Stellari [4, Corollary 29.7] proved that for any admissible value d, $\mathrm{Ku}(X)$ is geometric for *every* $[X] \in \mathscr{C}_d$. So we now know that for a smooth cubic fourfold X its Kuznetsov component $\mathrm{Ku}(X)$ is geometric if and only if $[X] \in \mathscr{C}_d$ for some admissible value d. Then one can restate Kuznetsov's conjecture as the following equivalent form.

Conjecture 1. *A smooth cubic fourfold X is* rational *if and only if $[X] \in \mathscr{C}_d$ for some admissible value d.*

The first three admissible values are $14, 26, 38$. Every cubic fourfold in $\mathscr{C}_{14}$ is rational [7, 9]; see also [21, Theorem 2] for a different proof. Based on Kontsevich–Tschinkel [15, Theorem 1], Russo–Staglianò [21, Theorems 4, 7] finally showed that every cubic fourfold in $\mathscr{C}_{26}$ and $\mathscr{C}_{38}$ is rational; see also [20] for the construction of explicit birational maps. So far "if " part of Conjecture 1 has been confirmed only for the three Hassett divisors $\mathscr{C}_{14}, \mathscr{C}_{26}, \mathscr{C}_{38}$. Thus finding rational cubic fourfolds in other Hassett divisors is of interest. The main result of this paper is the following.

Theorem 2 (=Theorem 9). *Every Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

The idea of the proof is simple: we first show any two Hassett divisors intersect by Theorem 7, which is of independent interest (for considerations of the intersections among Hassett divisors, see [2, 3, 7, 10] etc.), and finally we consider the intersections $\mathscr{C}_d \cap \mathscr{C}_{14}$, $\mathscr{C}_d \cap \mathscr{C}_{26}$ and $\mathscr{C}_d \cap \mathscr{C}_{38}$ for every Hassett divisor $\mathscr{C}_d$.

After completing this paper, Russo–Staglianò [22] announced the rationality of every cubic fourfold in $\mathscr{C}_{42}$. We remark that our method used for the proof of Theorem 2 also works in this case (in particular, it can be shown that the four intersections $\mathscr{C}_d \cap \mathscr{C}_{14}$, $\mathscr{C}_d \cap \mathscr{C}_{26}$, $\mathscr{C}_d \cap \mathscr{C}_{38}$, $\mathscr{C}_d \cap \mathscr{C}_{42}$ are mutually distinct).

Throughout this paper, we work over the complex number field $\mathbb{C}$.

Acknowledgements

We would like to thank Professors Keiji Oguiso and Paolo Stellari for useful conversations. We also would like to thank the referee for helpful comments.

2. Lattice and Hodge theory for cubic fourfolds

In this section, we collect some known results on Hodge structures and lattices associated with smooth cubic fourfolds. We refer to [6, 11, 12, 14] for more detailed discussions, especially for the Hodge-theoretic aspect, and to [19, 23] for the basics of abstract lattice theory.

The cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by $\mathbb{P}(H^0(\mathbb{P}^5, \mathscr{O}(3))) \cong \mathbb{P}^{55}$. Moreover, the smooth cubic hypersurfaces form a Zariski open dense subset $\mathscr{U} \subset \mathbb{P}^{55}$. Then the moduli space of smooth cubic fourfolds is the quotient space

$$\mathscr{C} := \mathscr{U} // \mathrm{PGL}(6, \mathbb{C})$$

which is a 20-dimensional quasi-projective variety.

Let X be a smooth cubic fourfold. Then the cohomology $H^*(X, \mathbb{Z})$ is torsion-free and the Hodge numbers for the middle cohomology of X are as follows:

$$0 \quad 1 \quad 21 \quad 1 \quad 0.$$

The Hodge–Riemann bilinear relations imply that $H^4(X, \mathbb{Z})$ is a unimodular lattice under the intersection form $(.)$ of signature $(21, 2)$. Furthermore, as abstract lattices, [11, Proposition 2.1.2] implies the middle cohomology and the primitive cohomology

$$L := E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0} \simeq H^4(X, \mathbb{Z})$$

$$L^0 := (h^2)^{\perp} \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2 \simeq H^4_{\mathrm{prim}}(X, \mathbb{Z})$$

where the square of the hyperplane class h is given as $h^2 = (1, 1, 1) \in I_{3,0}$ of which the intersection form is given by the identity matrix of rank 3, $A_2 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ the hyperbolic plane, E_8 is the unimodular positive definite even lattice of rank 8. Note that L^0 is an even lattice.

Definition 3 (Hassett [11]). *A smooth cubic fourfold X is called* special *if it contains an algebraic surface not homologous to a complete intersection.*

The integral Hodge conjecture holds for smooth cubic fourfolds ([25, Theorem 18] or see also [4, Corollary 29.8] for a new proof). Thus, a smooth cubic fourfold X is *special* if and only if the rank of the positive definite lattice

$$A(X) := H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

is at least 2.

Definition 4 (Hassett [11]). *A* labelling *of a special cubic fourfold consists of a positive definite rank two saturated (i.e. the quotient group $A(X)/K$ is torsion free) sublattice*

$$K \subset A(X) \ \textit{such that} \ h^2 \in K,$$

and its discriminant d is the determinant of the intersection form on K.

In [11], Hassett defined $\mathscr{C}_d$ as the set of special cubic fourfolds X with labelling of discriminant d. Moreover, Hassett [11, Theorem 1.0.1] showed that $\mathscr{C}_d \subset \mathscr{C}$ is an irreducible divisor and is nonempty if and only if

$$d > 6 \quad \text{and} \quad d \equiv 0, 2 \pmod 6. \tag{$\star$}$$

The following proposition is a generalization of [11, Theorems 1.0.1].

Proposition 5 ([12, Proposition 12 and p. 43]). *Fix a positive definite lattice M of rank $r \geq 2$ admitting a saturated embedding*

$$M \subset L \ \textit{such that} \ h^2 \in M.$$

We denote by $\mathscr{C}_M \subset \mathscr{C}$ the smooth cubic fourfolds X admitting algebraic classes with this lattice structure

$$h^2 \in M \subset A(X) \subset L.$$

Then $\mathscr{C}_M$ has codimension $r-1$ and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$, provided $\mathscr{C}_M$ is nonempty. Moreover, $\mathscr{C}_M$ is nonempty if and only if there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6, where $K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $K_6 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

This proposition is crucial for our purpose, so we sketch a proof for the convenience of readers.

Sketch of proof. Suppose $\mathscr{C}_M$ is nonempty. If $K_6 \subset M$ is a sublattice with $h^2 \in K_6$, then there is a smooth cubic fourfold X such that $A(X) \cap \langle h^2 \rangle^{\perp}$ contains an element r with $(r.r) = 2$ and this contradicts Voisin [24, Section 4, Proposition 1]; furthermore, Hassett [11, Theorem 4.4.1] excludes the case when $K_2 \subset M$ is a sublattice with $h^2 \in K_2$.

Conversely, suppose that there exists no rank two sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6. Since the signature of L is $(21,2)$ and $M \subset L$ is positive definite, by a standard argument, one can always find $\omega \in L \otimes_{\mathbb{Z}} \mathbb{C}$ such that

$$(\omega.\omega) = 0, \ (\omega.\overline{\omega}) < 0 \text{ and } L \cap \omega^{\perp} = M.$$

According to the description of the image of the period map for cubic fourfolds (Laza [17, Theorem 1.1] and Looijenga [18, Theorem 4.1]), one has a smooth cubic fourfold X and an isometry $\phi : H^4(X,\mathbb{Z}) \xrightarrow{\simeq} L$ mapping the square of the hyperplane class to $h^2 \in L$ and a generator of $H^{3,1}(X)$ to ω. Thus $M = A(X)$ and hence $\mathscr{C}_M$ contains $[X]$ and nonempty. □

In the rest of the text, we will frequently use the following lemma to check the nonemptyness condition in the Proposition 5.

Lemma 6. *Let $M \subset L$ be a positive definite saturated sublattice and $h^2 \in M$. Then the following three conditions are equivalent:*

(i) *there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6;*
(ii) *there exists no $r \in M$ such that $(r.r) = 2$ (i.e., M does not represent 2);*
(iii) *for any $0 \neq x \in M$, $(x.x) \geq 3$.*

In particular, if M satisfies one of the three equivalent conditions, then $\emptyset \neq \mathscr{C}_M \subset \mathscr{C}_{M'}$ for any saturated sublattice $M' \subset M \subset L$ such that $h^2 \in M'$.

Proof. First of all, (ii) ⇒ (i) is clear since both K_2 and K_6 represent 2.

Secondly, (i) ⇒ (ii). Suppose that there exists $r \in M$ such that $(r.r) = 2$. We denote by $K \subset M$ the sublattice generated by h^2 and r. Hence, the Gram matrix of K with respect to the basis (h^2, r) is

$$\begin{pmatrix} (h^2.h^2) & (h^2.r) \\ (r.h^2) & (r.r) \end{pmatrix} = \begin{pmatrix} 3 & a \\ a & 2 \end{pmatrix}.$$

Replacing r by $-r$ if necessary, we may and will assume $a \geq 0$. Since K is positive definite, we have $a^2 < 6$ and thus $a = 0,1,2$. If $a = 0$ (resp. 2), then K is isometric to K_6 (resp. K_2), contradiction. If $a = 1$, then $h^2 - 3r \in (h^2)^{\perp} = L^0$ and $((h^2-3r).(h^2-3r)) = 15$, an odd number, contradicting to the fact L^0 is even.

Finally, clearly (iii) implies (ii). Conversely, we show (ii) implies (iii). By hypothesis, we may assume that there is $r \in M$ with $(r.r) = 1$. Then let $K \subset M$ be the sublattice generated by h^2 and r. Hence, the Gram matrix of K with respect to the basis (h^2, r) is

$$\begin{pmatrix} 3 & a \\ a & 1 \end{pmatrix}$$

where $a = (h^2.r)$. Replacing r by $-r$ if necessary, we may and will assume $a \geq 0$. Since K is positive definite, we have $a^2 < 3$ and thus $a = 0,1$. If $a = 0$, then $r \in (h^2)^{\perp} = L^0$ and $(r.r) = 1$, an odd number, contradicting to the fact L^0 is even. If $a = 1$, then K is isometric to K_2 and K represents 2, contradiction. □

3. Intersections of Hassett divisors

In this section, we prove Theorem 2 (=Theorem 9) and discuss some related results (Theorem 7 and Theorem 13).

Firstly, we setup some notations for latter use. Let

$$L = E_8^{\oplus 2} \oplus U_1 \oplus U_2 \oplus I_{3,0},$$

where U_1 and U_2 are two copies of U. The standard basis of U consists of isotropic elements e, f with $(e.f) = 1$. We denote the standard basis of U_i by e_i, f_i, $i = 1, 2$, and denote by h^2 the element $(1,1,1) \in I_{3,0} \subset L$.

We will use the following theorem, an interesting result for itself, to prove Theorem 9.

Theorem 7. *Any two Hassett divisors intersect, i.e.,* $\mathscr{C}_{d_1} \cap \mathscr{C}_{d_2} \neq \emptyset$ *for any two integers* d_1 *and* d_2 *satisfying* ($\star$). *Moreover, there exists a smooth cubic fourfold* X *and a codimension-two subvariety* $\mathscr{C}_{A(X)} \subset \mathscr{C}$ *such that* $[X] \in \mathscr{C}_{A(X)} \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ *and* $A(X)$ *is a rank 3 lattice with discriminant* $d_1 d_2/3$, *except if both* d_1 *and* d_2 *are* $\equiv 2 \pmod 6$, *in which case the discriminant is* $(d_1 d_2 - 1)/3$.

Proof. By definition, an integer d satisfies ($\star$) if $d > 6$ and $d \equiv 0, 2 \pmod 6$. Therefore, the proof is divided into three cases:

Case 1: $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 0 \pmod 6$**.** Suppose $d_1 = 6n_1$, $d_2 = 6n_2$ and $n_1, n_2 \geq 2$. We consider the rank 3 lattice

$$M := \langle h^2, \alpha_1, \alpha_2 \rangle \subset L$$

generated by h^2, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2n_1 & 0 \\ 0 & 0 & 2n_2 \end{pmatrix}.$$

Therefore, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. In addition, for any nonzero $v = xh^2 + y\alpha_1 + z\alpha_2 \in M$, where x, y, z are integers, we have

$$(v.v) = 3x^2 + 2n_1 y^2 + 2n_2 z^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers x, y, z is nonzero. Hence, the embedding $M \subset L$ satisfies Lemma 6 (iii). Thus, by Lemma 6 and Proposition 5, $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\operatorname{disc}(A(X)) = d_1 d_2/3$. Moreover, we consider the sublattices

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

with discriminant d_1, and

$$K_{d_2} := \langle h^2, \alpha_2 \rangle \subset M$$

with discriminant d_2. Clearly, both K_{d_1} and K_{d_2} are saturated sublattices of M. Applying Lemma 6 and Proposition 5 again, we obtain $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we want.

Case 2: $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 2 \pmod 6$**.** Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1, \alpha_2 + (0,0,1) \rangle \subset L$$

where $(0,0,1) \in I_{3,0}$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1, \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2n_1 & 0 \\ 1 & 0 & 2n_2+1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. Furthermore, for any nonzero $v = xh^2 + y\alpha_1 + z(\alpha_2 + (0,0,1)) \in M$, we get

$$(v.v) = 2x^2 + 2n_1y^2 + 2n_2z^2 + (x+z)^2 \geq 3$$

since $n_1 \geq 2$, $n_2 \geq 1$ and at least one of the integers x, y, z is nonzero. Hence, by Lemma 6 and Proposition 5, we conclude that $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\mathrm{disc}(A(X)) = d_1d_2/3$. Similarly, we consider the sublattices:

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

of discriminant d_1, and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

of discriminant d_2. Again Lemma 6 and Proposition 5 imply $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted.

Case 3: $d_1 \equiv 2 \pmod 6$ and $d_2 \equiv 2 \pmod 6$. Assume $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1) \rangle \subset L$$

here $(0,1,0) \in I_{3,0}$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2n_1+1 & 0 \\ 1 & 0 & 2n_2+1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. For any nonzero $v = xh^2 + y(\alpha_1 + (0,1,0)) + z(\alpha_2 + (0,0,1)) \in M$, we obtain

$$(v.v) = x^2 + 2n_1y^2 + 2n_2z^2 + (x+y)^2 + (x+z)^2 \geq 3$$

since $n_1, n_2 \geq 1$ and at least one of the integers x, y, z is nonzero. Hence, Lemma 6 and Proposition 5 concludes that $\mathscr{C}_M \subset \mathscr{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathscr{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\mathrm{disc}(A(X)) = (d_1d_2 - 1)/3$. Moreover, we consider

$$K_{d_1} := \langle h^2, \alpha_1 + (0,1,0) \rangle \subset M$$

with discriminant d_1 and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

with discriminant d_1. By Lemma 6 and Proposition 5, we obtain $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and $[X] \in \mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. As a consequence, $[X] \in \mathscr{C}_M \subset \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted. This finishes the proof of Theorem 7. □

Remark 8. Note that it has been known for 20 years that $\mathscr{C}_8 \cap \mathscr{C}_{14} \neq \emptyset$ (Hassett [10]) and proved more recently that $\mathscr{C}_8$ intersects every Hassett divisor (Addington–Thomas [2, Theorem 4.1]). It is also shown that $\mathscr{C}_8 \cap \mathscr{C}_{14}$ has five irreducible components ([3, 7]). Moreover, [7, p. 166] has mentioned that $\mathscr{C}_{14}$ intersects many other divisors $\mathscr{C}_d$, however, it is not obvious to see which Hassett divisors intersect with $\mathscr{C}_{14}$.

Consequently, Theorem 7 not only generalizes [2, Theorem 4.1] but also implies that $\mathscr{C}_{14}$ intersects all Hassett divisors. Because of the same reason, we may conclude the main result of the current paper.

Theorem 9 (=Theorem 2). *Every Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

Proof. Applying Theorem 7 to the pairs of integers $(d_1, d_2) = (d, 14)$, $(d, 26)$, $(d, 38)$. Then there exist three smooth cubic fourfolds X_1, X_2 and X_3 such that

$$[X_1] \in \mathscr{C}_{A(X_1)} \subset \mathscr{C}_d \cap \mathscr{C}_{14} \subset \mathscr{C}_d,$$
$$[X_2] \in \mathscr{C}_{A(X_2)} \subset \mathscr{C}_d \cap \mathscr{C}_{26} \subset \mathscr{C}_d,$$
$$[X_3] \in \mathscr{C}_{A(X_3)} \subset \mathscr{C}_d \cap \mathscr{C}_{38} \subset \mathscr{C}_d,$$

where $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$, and $\mathscr{C}_{A(X_3)}$ are subvarieties of codimension-two in $\mathscr{C}$. Here $A(X_1)$, $A(X_2)$ and $A(X_3)$ are three different rank 3 lattices of discriminants:

- if $d \equiv 0 \pmod 6$, then $\mathrm{disc}(A(X_1)) = 14d/3$, $\mathrm{disc}(A(X_2)) = 26d/3$ and $\mathrm{disc}(A(X_3)) = 38d/3$;
- if $d \equiv 2 \pmod 6$, then $\mathrm{disc}(A(X_1)) = (14d-1)/3$, $\mathrm{disc}(A(X_2)) = (26d-1)/3$ and $\mathrm{disc}(A(X_3)) = (38d-1)/3$.

By definition of $\mathscr{C}_{A(X_i)}$ (see Proposition 5), a smooth cubic fourfold $[X] \in \mathscr{C}_{A(X_i)}$ only if there exists a saturated embedding $A(X_i) \subset A(X)$. Since $A(X_1)$, $A(X_2)$ and $A(X_3)$ are rank 3 lattices of different discriminants, it follows that there is no saturated embedding $A(X_i) \subset A(X_j)$ if $i \neq j$. Therefore, $[X_i] \notin \mathscr{C}_{A(X_j)}$ if $i \neq j$ and $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$, and $\mathscr{C}_{A(X_3)}$ are three different subvarieties of codimension-two in $\mathscr{C}$.

Moreover, since every smooth cubic fourfold in $\mathscr{C}_{14}$, $\mathscr{C}_{26}$ and $\mathscr{C}_{38}$ is rational ([7, 21]), so every smooth cubic fourfold in $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$ and $\mathscr{C}_{A(X_3)}$ is rational. Therefore, $\mathscr{C}_{A(X_1)}$, $\mathscr{C}_{A(X_2)}$ and $\mathscr{C}_{A(X_3)}$ are three different codimension-two subvarieties which parametrize rational cubic fourfolds. This completes the proof of Theorem 9. □

Our main result also motivates the following natural question:

Question 10. *Suppose that d satisfies $(\star)$ and d is not an admissible value. Does the Hassett divisor $\mathscr{C}_d$ contain a union of countably infinite codimension-two subvarieties in $\mathscr{C}$ parametrizing rational cubic fourfolds?*

The answer to Question 10 has already been known for $\mathscr{C}_8$ and $\mathscr{C}_{18}$ ([1, 10]).

Corollary 11. *The answer to Question 10 is yes if the "if" part of Conjecture 1 holds.*

Returning to Conjecture 1, as a by-product of Theorem 9 (=Theorem 2), we have the following.

Corollary 12. *For every admissible value d, the Hassett divisor $\mathscr{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathscr{C}$.*

To obtain more information about the Hassett divisors, it is of importance to notice that Addington–Thomas [2, Theorem 4.1] showed that for any d satisfying $(\star)$ there exists a cubic fourfold $[X] \in \mathscr{C}_8 \cap \mathscr{C}_d$ such that $[X] \in \mathscr{C}_{d'}$ for some admissible value d'. Even if it is conjectured to be rational, however, it is still unknown whether such a X is rational or not. Using the idea of the proof of Theorem 7 and Theorem 9, we obtain a generalization of [2, Theorem 4.1].

Theorem 13. *If d_1 and d_2 satisfy $(\star)$, then $\mathscr{C}_{14} \cap \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ contains a codimension-three subvariety in $\mathscr{C}$ parametrizing rational cubic fourfolds.*

Proof. Analogously to the proof of Theorem 7, we only need to consider three cases:

Case 1. Given $d_1 = 6n_1$ and $d_2 = 6n_2$ with $n_1, n_2 \geq 2$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1, \alpha_2 \rangle \subset L$$

where $v = (3,1,0) \in I_{3,0} \subset L$, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of M with respect to the basis $(h^2, v, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & 10 & 0 & 0 \\ 0 & 0 & 2n_1 & 0 \\ 0 & 0 & 0 & 2n_2 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. For any nonzero $v = x_1 h^2 + x_2 v + x_3 \alpha_1 + x_4 \alpha_2 \in M$, we have

$$(v.v) = 2(x_1 + 2x_2)^2 + x_1^2 + 2x_2^2 + 2n_1 x_3^2 + 2n_2 x_4^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers x_i is nonzero $(i = 1,2,3,4)$. Hence, Lemma 6 and Proposition 5 conclude that $\mathscr{C}_M$ is nonempty and has codimension 3. In addition, we consider the lattices $K_{14} = \langle h^2, v \rangle$ and $K_{d_i} := \langle h^2, \alpha_i \rangle \subset M$ with discriminant d_i. By Lemma 6 and Proposition 5, we obtain $\mathscr{C}_M \subset \mathscr{C}_{K_{d_1}} = \mathscr{C}_{d_1}$ and also $\mathscr{C}_M \subset \mathscr{C}_{K_{d_2}} = \mathscr{C}_{d_2}$. Consequently, $\emptyset \neq \mathscr{C}_M \subset \mathscr{C}_{14} \cap \mathscr{C}_{d_1} \cap \mathscr{C}_{d_2}$ is what we wanted, since every cubic fourfold in $\mathscr{C}_{14}$ is rational.

Since Case 2 and Case 3 are the same as Case 1, we just give the main ingredients and left the details to the interested readers.

Case 2. Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1, \alpha_2 + (0,0,1) \rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, v \rangle$, $K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$.

Case 2. Given $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, v, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1) \rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, v \rangle$, $K_{d_1} := \langle h^2, \alpha_1 + (0,1,0) \rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$. □

References

[1] N. Addington, B. Hassett, Y. Tschinkel, A. Várilly-Alvarado, "Cubic fourfolds fibered in sextic del Pezzo surfaces", *Am. J. Math.* **141** (2019), no. 6, p. 1479-1500.

[2] N. Addington, R. Thomas, "Hodge theory and derived categories of cubic fourfolds", *Duke Math. J.* **163** (2014), no. 10, p. 1885-1927.

[3] A. Auel, M. Bernardara, M. Bolognesi, A. Várilly-Alvarado, "Cubic fourfolds containing a plane and a quintic del Pezzo surface", *Algebr. Geom.* **1** (2014), no. 2, p. 181-193.

[4] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, P. Stellari, "Stability conditions in families", https://arxiv.org/abs/1902.08184v1, 2019.

[5] A. Bayer, M. Lahoz, E. Macrì, P. Stellari, "Stability conditions on Kuznetsov components", https://arxiv.org/abs/1703.10839v2, 2019.

[6] A. Beauville, R. Donagi, "La variété des droites d'une hypersurface cubique de dimension 4", *C. R. Math. Acad. Sci. Paris* **301** (1985), p. 703-706.

[7] M. Bolognesi, F. Russo, G. Staglianò, "Some loci of rational cubic fourfolds", *Math. Ann.* **373** (2019), no. 1-2, p. 165-190.

[8] C. H. Clemens, P. Griffiths, "The intermediate Jacobian of the cubic threefold", *Ann. Math.* **95** (1972), p. 281-356.

[9] G. Fano, "Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4^0 ordine", *Comment. Math. Helv.* **15** (1943), p. 71-80.

[10] B. Hassett, "Some rational cubic fourfolds", *J. Algebr. Geom.* **8** (1999), no. 1, p. 103-114.

[11] ———, "Special cubic fourfolds", *Compos. Math.* **120** (2000), no. 1, p. 1-23.

[12] ———, "Cubic fourfolds, K3 surfaces, and rationality questions", in *Rationality problems in algebraic geometry*, Lecture Notes in Mathematics, vol. 2172, Springer, 2016, p. 29-66.

[13] D. Huybrechts, "The K3 category of a cubic fourfold", *Compos. Math.* **153** (2017), no. 3, p. 586-620.

[14] ———, "Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories", in *Birational Geometry of Hypersurfaces*, Lecture Notes of the Unione Matematica Italiana, vol. 26, Springer, 2019, p. 165-198.

[15] M. Kontsevich, Y. Tschinkel, "Specialization of birational types", *Invent. Math.* **217** (2019), no. 2, p. 415-432.

[16] A. Kuznetsov, "Derived categories of cubic fourfolds", in *Cohomological and geometric approaches to rationality problems. New Perspectives*, Progress in Mathematics, vol. 282, Birkhäuser, 2010, p. 219-243.

[17] R. Laza, "The moduli space of cubic fourfolds via the period map", *Ann. Math.* **172** (2010), no. 1, p. 673-711.

[18] E. Looijenga, "The period map for cubic fourfolds", *Invent. Math.* **177** (2009), no. 1, p. 213-233.

[19] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications", *Math. USSR, Izv.* **14** (1980), p. 103-167.

[20] F. Russo, G. Staglianò, "Explicit rationality of some cubic fourfolds", https://arxiv.org/abs/1811.03502v1, 2018.

[21] ———, "Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds", *Duke Math. J.* **168** (2019), no. 5, p. 849-865.

[22] ———, "Trisecant Flops, their associated K3 surfaces and the rationality of some Fano fourfolds", https://arxiv.org/abs/1909.01263v2, 2019.

[23] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer, 1973.

[24] C. Voisin, "Théorème de Torelli pour les cubiques de $\mathbb{P}^5$", *Invent. Math.* **86** (1986), p. 577-601.

[25] ———, "Some aspects of the Hodge conjecture", *Jpn. J. Math.* **2** (2007), no. 2, p. 261-296.